THE RIEMANN HYPOTHESIS FOR THE VIBRATING STRING

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Abstract. The vibrating string as a canonical model of scattering produces a scattering function which is analytic and without zeros in the upper half–plane. The structure of a string is the solution of an inverse problem formulated in Hilbert spaces whose elements are entire functions [2]. The Riemann hypothesis is treated as a characterization of special strings whose scattering function is analytic and without zeros in a half–plane containing the upper half–plane [3]. Examples of such strings in Fourier analysis on skew–fields exhibit analogues of the Euler zeta function whose coefficients are eigenfunctions of self–adjoint operators due to Hecke. A proof of the Riemann hypothesis is immediate for those Hecke zeta functions which have no singularity. For the exceptional Hecke zeta function which has a singularity the proof of the Riemann hypothesis applies the exceptional symmetry of the function. A corollary is a proof of the Riemann hypothesis for the analogues of the Euler zeta function whose coefficients are defined by Dirichlet characters. The proof of the Riemann hypothesis for the Euler zeta function is the proof of the Riemann hypothesis for the exceptional Dirichlet zeta function which has a singularity and which is derived from the exceptional Hecke zeta function which has a singularity.

1. The Inverse Problem for the Vibrating String

The Riemann hypothesis is a conjecture concerning the zeros of a special entire function, not a polynomial, which presumes a relationship to zeros found in polynomials. The Hermite class of entire functions is a class of entire functions which contains the polynomials having a given zero–free half–plane and which maintains the relationship to zeros found in these polynomials.

A nontrivial entire function is said to be of Hermite class if it can be approximated by polynomials whose zeros are restricted to a given half–plane. For applications to the vibrating string the upper half–plane is chosen as the half–plane free of zeros. If an entire function $E(z)$ of z is of Hermite class, then the modulus of $E(x+iy)$ is a nondecreasing function of positive y which satisfies the inequality

$$|E(x-iy)| \leq |E(x+iy)|$$

for every real number x. These necessary conditions are also sufficient. The Hermite class is also known as the Pólya class.

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An entire function of Hermite class which has no zero is the exponential

$$\exp F(z)$$

of an entire function $F(z)$ of z with derivative $F'(z)$ such that the real part of

$$iF'(z)$$

is nonnegative in the upper half–plane. The function

$$iF'(z) = a - ibz$$

is a polynomial of degree less than two by the Poisson representation of functions which are nonnegative and harmonic in the upper half–plane. The constant coefficient a has nonnegative real part and b is nonnegative.

If an entire function $E(z)$ of z is of Hermite class and has a zero w, then the entire function

$$E(z)/(z - w)$$

of z is of Hermite class. A sequence of polynomials $P_n(z)$ exists such that

$$E(z)/P_n(z)$$

is an entire function of Hermite class for every nonnegative integer n and such that

$$E(z) = \lim P_n(z)E_n(z)$$

uniformly on compact subsets of the upper half–plane for entire functions $E_n(z)$ of Hermite class which have no zeros.

An analytic weight function is defined as a function $W(z)$ of z which is analytic and without zeros in the upper half–plane. An entire function of Hermite class is an analytic weight function in the upper half–plane. Hilbert spaces of functions analytic in the upper half–plane were introduced in Fourier analysis by Hardy.

The weighted Hardy space $\mathcal{F}(W)$ is defined as the Hilbert space of functions $F(z)$ of z, which are analytic in the upper half–plane, such that the least upper bound

$$\|F\|^2_{\mathcal{F}(W)} = \sup \int_{-\infty}^{+\infty} |F(x + iy)/W(x + iy)|^2 dx$$

taken over all positive y is finite. The classical Hardy space is obtained when $W(z)$ is identically one. Multiplication by $W(z)$ is an isometric transformation of the classical Hardy space onto the weighted Hardy space with analytic weight function $W(z)$.

An isometric transformation of the weighted Hardy space $\mathcal{F}(W)$ into itself is defined by taking a function $F(z)$ of z into the function

$$F(z)(z - w)/(z - w^-)$$

of z when w is in the upper half–plane. The range of the transformation is the set of elements of the space which vanish at w.

A continuous linear functional on the weighted Hardy space $\mathcal{F}(W)$ is defined by taking a function $F(z)$ of z into its value $F(w)$ at w whenever w is in the upper half–plane. The function

$$W(z)W(w)^{-}/[2\pi i(w^{-} - z)]$$

of z belongs to the space when w is in the upper half–plane and acts as reproducing kernel function for function values at w.

A Hilbert space of functions analytic in the upper half–plane which has dimension greater than one is isometrically equal to a weighted Hardy space if an isometric transformation of the space onto the subspace of functions which vanish at w is defined by taking $F(z)$ into

$$F(z)(z - w)/(z - w^{-})$$

when w is in the upper half–plane and if a continuous linear functional is defined on the space by taking $F(z)$ into $F(w)$ for w of the upper half–plane.

Examples of weighted Hardy spaces in Fourier analysis are constructed from the Euler gamma function. The gamma function is a function $\Gamma(s)$ of s which is analytic in the complex plane with the exception of singularities at the nonpositive integers and which satisfies the recurrence relation

$$s\Gamma(s) = \Gamma(s + 1).$$

An analytic weight function

$$W(z) = \Gamma(s)$$

is defined by

$$s = \tfrac{1}{2} - iz.$$

A maximal dissipative transformation is defined in the weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z + i)$ whenever the functions of z belong to the space.

A relation T with domain and range in a Hilbert space is said to be dissipative if the transformation

$$(T - \lambda^{-})/(T - \lambda)$$

with domain and range in the Hilbert space is contractive for some, and hence every, complex number λ in the right half–plane. The relation T is said to be maximal dissipative if the domain of the contractive transformation is the whole space for some, and hence every, complex number λ in the right half–plane.

Theorem 1. *A maximal dissipative transformation is defined in a weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z + i)$ whenever the functions of z belong to the space if, and only if, the function*

$$W(z + \tfrac{1}{2}i)/W(z - \tfrac{1}{2}i)$$

of z admits an extension which is analytic and has nonnegative real part in the upper half–plane.

Proof of Theorem 1. A Hilbert space $\mathcal{H}$ whose elements are functions analytic in the upper half–plane is constructed when a maximal dissipative transformation is defined in the weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z+i)$ whenever the functions of z belong to the space. The space $\mathcal{H}$ is constructed from the graph of the adjoint of the transformation which takes $F(z)$ into $F(z+i)$ whenever the functions of z belong to the space.

An element

$$F(z) = (F_+(z), F_-(z))$$

of the graph is a pair of analytic functions of z, which belong to the space $\mathcal{F}(W)$, such that the adjoint takes $F_+(z)$ into $F_-(z)$. The scalar product

$$\langle F(t), G(t)\rangle = \langle F_+(t), G_-(t)\rangle_{\mathcal{F}(W)} + \langle F_-(t), G_+(t)\rangle_{\mathcal{F}(W)}$$

of elements $F(z)$ and $G(z)$ of the graph is defined formally as the sum of scalar products in the space $\mathcal{F}(W)$. Scalar self–products are nonnegative in the graph since the adjoint of a maximal dissipative transformation is dissipative.

An element $K(w,z)$ of the graph is defined by

$$K_+(w,z) = W(z)W(w - \tfrac{1}{2}i)^-/[2\pi i(w^- + \tfrac{1}{2}i - z)]$$

and

$$K_-(w,z) = W(z)W(w + \tfrac{1}{2}i)^-/[2\pi i(w^- - \tfrac{1}{2}i - z)]$$

when w is in the half–plane

$$1 < iw^- - iw.$$

The identity

$$F_+(w + \tfrac{1}{2}i) + F_-(w - \tfrac{1}{2}i) = \langle F(t), K(w,t)\rangle$$

holds for every element

$$F(z) = (F_+(z), F_-(z))$$

of the graph. An element of the graph which is orthogonal to itself is orthogonal to every element of the graph.

An isometric transformation of the graph onto a dense subspace of $\mathcal{H}$ is defined by taking

$$F(z) = (F_+(z), F_-(z))$$

into the function

$$F_+(z + \tfrac{1}{2}i) + F_-(z - \tfrac{1}{2}i)$$

of z in the half–plane

$$1 < iz^- - iz.$$

The reproducing kernel function for function values at w in the space $\mathcal{H}$ is the function

$$[W(z+\tfrac{1}{2}i)W(w-\tfrac{1}{2}i)^- + W(z-\tfrac{1}{2}i)W(w+\tfrac{1}{2}i)^-]/[2\pi i(w^- - z)]$$

of z in the half–plane when w is in the half–plane.

Division by $W(z+\frac{1}{2}i)$ is an isometric transformation of the space $\mathcal{H}$ onto a Hilbert space appearing in the Poisson representation of functions which are analytic and have nonnegative real part in the upper half–plane. The function

$$\phi(z) = W(z-\tfrac{1}{2}i)/W(z+\tfrac{1}{2}i)$$

of z admits an analytic extension to the upper half–plane. The function

$$[\phi(z) + \phi(w)^-]/[2\pi i(w^- - z)]$$

of z belongs to the space when w is in the upper half–plane and acts as reproducing kernel function for function values at w. Since multiplication by $W(z+\frac{1}{2}i)$ is an isometric transformation of the space onto $\mathcal{H}$, the elements of $\mathcal{H}$ have analytic extensions to the upper half–plane. The function

$$[W(z+\tfrac{1}{2}i)W(w-\tfrac{1}{2}i)^- + W(z-\tfrac{1}{2}i)W(w+\tfrac{1}{2}i)^-]/[2\pi i(w^- - z)]$$

of z belongs to the space when w is in the upper half–plane and acts as reproducing kernel function for function values at w.

The argument is reversed to construct a maximal dissipative transformation in the weighted Hardy space $\mathcal{F}(W)$ when the function $\phi(z)$ of z admits an extension which is analytic and has nonnegative real part in the upper half–plane. The Poisson representation constructs a Hilbert space whose elements are functions analytic in the upper half–plane and which contains the function

$$[\phi(z) + \phi(w)^-]/[2\pi i(w^- - z)]$$

of z as reproducing kernel function for function values at w when w is in the upper half–plane. Multiplication by $W(z+\frac{1}{2}i)$ acts as an isometric transformation of the space onto a Hilbert space $\mathcal{H}$ whose elements are functions analytic in the upper half–plane and which contains the function

$$[W(z+\tfrac{1}{2}i)W(w-\tfrac{1}{2}i)^- + W(z-\tfrac{1}{2}i)W(w+\tfrac{1}{2}i)^-]/[2\pi i(w^- - z)]$$

of z as reproducing kernel function for function values at w when w is in the upper half–plane.

A transformation is defined in the space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z+i)$ whenever the functions of z belong to the space. The graph of the adjoint is a space of pairs

$$F(z) = (F_+(z), F_-(z))$$

of elements of the space such that the adjoint takes the function $F_+(z)$ of z into the function $F_-(z)$ of z. The graph contains

$$K(w,z) = (K_+(w,z), K_-(w,z))$$

with

$$K_+(w,z) = W(z)W(w - \tfrac{1}{2}i)^-/[2\pi i(w^- + \tfrac{1}{2}i - z)]$$

and

$$K_-(w,z) = W(z)W(w + \tfrac{1}{2}i)^-/[2\pi i(w^- - \tfrac{1}{2}i - z)]$$

when w is in the half–plane

$$1 < iw^- - iw.$$

The elements $K(w,z)$ of the graph span the graph of a restriction of the adjoint. The transformation in the space $\mathcal{F}(W)$ is recovered as the adjoint of the restricted adjoint.

A scalar product is defined on the graph of the restricted adjoint so that an isometric transformation of the graph of the restricted adjoint into the space $\mathcal{H}$ is defined by taking

$$F(z) = (F_+(z), F_-(z))$$

into

$$F_+(z + \tfrac{1}{2}i) + F_-(z - \tfrac{1}{2}i).$$

The identity

$$\langle F(t), G(t)\rangle = \langle F_+(t), G_-(t)\rangle_{\mathcal{F}(W)} + \langle F_-(t), G_+(t)\rangle_{\mathcal{F}(W)}$$

holds for all elements

$$F(z) = (F_+(z), F_-(z))$$

and

$$G(z) = (G_+(z), G_-(z))$$

of the graph of the restricted adjoint. The restricted adjoint is dissipative since scalar self–products are nonnegative in its graph. The adjoint is dissipative since the transformation in the space $\mathcal{F}(W)$ is the adjoint of its restricted adjoint.

The dissipative property of the adjoint is expressed in the inequality

$$\|F_+(t) - \lambda^- F_-(t)\|_{\mathcal{F}(W)} \leq \|F_+(t) + \lambda F_-(t)\|_{\mathcal{F}(W)}$$

for elements

$$F(z) = (F_+(z), F_-(z))$$

of the graph when λ is in the right half–plane. The domain of the contractive transformation which takes the function

$$F_+(z) + \lambda F_-(z)$$

of z into the function

$$F_+(z) - \lambda^- F_-(z)$$

of z is a closed subspace of the space $\mathcal{F}(W)$. The maximal dissipative property of the adjoint is the requirement that the contractive transformation be everywhere defined for some, and hence every, λ in the right half–plane.

Since $K(w, z)$ belongs to the graph when w is in the half–plane

$$1 < iw^- - iw,$$

an element $H(z)$ of the space $\mathcal{F}(W)$ which is orthogonal to the domain satisfies the identity

$$H(w - \tfrac{1}{2}i) + \lambda H(w + \tfrac{1}{2}i) = 0$$

when w is in the upper half–plane. The function $H(z)$ of z admits an analytic extension to the complex plane which satisfies the identity

$$H(z) + \lambda H(z + i) = 0.$$

A zero of $H(z)$ is repeated with period i. Since

$$H(z)/W(z)$$

is analytic and of bounded type in the upper half–plane, the function $H(z)$ of z vanishes everywhere if it vanishes somewhere.

The space of elements $H(z)$ of the space $\mathcal{F}(W)$ which are solutions of the equation

$$H(z) + \lambda H(z + i) = 0$$

for some λ in the right half–plane has dimension zero or one. The dimension is independent of λ.

If τ is positive, multiplication by

$$\exp(i\tau z)$$

is an isometric transformation of the space $\mathcal{F}(W)$ into itself which takes solutions of the equation for a given λ into solutions of the equation with λ replaced by

$$\lambda \exp(\tau).$$

A solution $H(z)$ of the equation for a given λ vanishes identically since the function

$$\exp(-i\tau z)H(z)$$

of z belongs to the space for every positive number τ and has the same norm as the function $H(z)$ of z.

The transformation which takes $F(z)$ into $F(z+i)$ whenever the functions of z belong to the space $\mathcal{F}(W)$ is maximal dissipative since it is the adjoint of its adjoint, which is maximal dissipative.

This completes the proof of the theorem.

An example of an analytic weight function which satisfies the hypotheses of the theorem is obtained when

$$W(z) = \Gamma(\tfrac{1}{2} - iz)$$

since

$$W(z + \tfrac{1}{2}i)/W(z - \tfrac{1}{2}i) = -iz$$

is analytic and has nonnegative real part in the upper half–plane by the recurrence relation for the gamma function. The weight functions which satisfy the hypotheses of the theorem are generalizations of the gamma function in which an identity is replaced by an inequality.

The theorem has another formulation. A maximal dissipative transformation is defined in a weighted Hardy space $\mathcal{F}(W)$ for some real number h by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space if, and only if, the function

$$W(z + \tfrac{1}{2}ih)/W(z - \tfrac{1}{2}ih)$$

of z admits an extension which is analytic and has nonnegative real part in the upper half–plane.

Another theorem is obtained in the limit of small h. A maximal dissipative transformation is defined in a weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $iF'(z)$ whenever the functions of z belong to the space if, and only if, the function

$$iW'(z)/W(z)$$

of z has nonnegative real part in the upper half–plane.

The proof of the theorem is similar to the proof of Theorem 1. A relationship to the Hermite class of entire functions is indicated in another formulation. A maximal dissipative transformation is defined in a weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $iF'(z)$ whenever the functions of z belong to the space if, and only if, the modulus of $W(x+iy)$ is a nondecreasing function of positive y for every real number x.

An Euler weight function is defined as an analytic weight function $W(z)$ such that a maximal dissipative transformation is defined in the weighted Hardy space $\mathcal{F}(W)$ whenever h is in the interval $[-1, 1]$ by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space.

If a nontrivial function $\phi(z)$ of z is analytic and has nonnegative real part in the upper half–plane, a logarithm of the functions is defined continuously in the half–plane with values in the strip of width π centered on the real line. The inequalities

$$-\pi \leq i \log \phi(z)^{-} - i \log \phi(z) \leq \pi$$

are satisfied. A function $\phi_h(z)$ of z which is analytic and has nonnegative real part in the upper half–plane is defined when h is in the interval $(-1, 1)$ by the integral

$$\log \phi_h(z) = \sin(\pi h) \int_{-\infty}^{+\infty} \frac{\log \phi(z-t) dt}{\cos(2\pi i t) + \cos(\pi h)}.$$

An application of the Cauchy formula in the upper half–plane shows that the function

$$\frac{\sin(\pi h)}{\cos(2\pi i z) + \cos(\pi h)} = \int_{-\infty}^{+\infty} \exp(2\pi i t z) \frac{\exp(\pi h t - \exp(-\pi h t)}{\exp(\pi t) - \exp(-\pi t)} \, dt$$

of z is the Fourier transform of a function

$$\frac{\exp(\pi h t) - \exp(-\pi h t)}{\exp(\pi t) - \exp(-\pi t)}$$

of positive t which is square integrable with respect to Lebesgue measure and bounded by h when h is in the interval $(0, 1)$.

The identity

$$\phi_{-h}(z) = \phi_h(z)^{-1}$$

is satisfied. The function

$$\phi(z) = \lim \phi_h(z)$$

of z is recovered in the limit as h increases to one. The identity

$$\phi_{a+b}(z) = \phi_a(z - \tfrac{1}{2} i b) \phi_b(z + \tfrac{1}{2} i a)$$

when a, b, and $a + b$ belong to the interval $(-1, 1)$ is a consequence of the trigonometric identity

$$\frac{\sin(\pi a + \pi b)}{\cos(2\pi i z) + \cos(\pi a + \pi b)}$$
$$= \frac{\sin(\pi a)}{\cos(2\pi i z + \pi b) + \cos(\pi a)} + \frac{\sin(\pi b)}{\cos(2\pi i z - \pi a) + \cos(\pi b)}.$$

An Euler weight function $W(z)$ is defined within a constant factor by the limit

$$iW'(z)/W(z) = \lim \frac{\log \phi_h(z)}{h} = \pi \int_{-\infty}^{+\infty} \frac{\log \phi(z-t) dt}{1 + \cos(2\pi i t)}.$$

The identity

$$W(z + \tfrac{1}{2} i h) = W(z - \tfrac{1}{2} i h) \phi_h(z)$$

applies when h is in the interval $(-1, 1)$. The identity reads

$$W(z + \tfrac{1}{2} i) = W(z - \tfrac{1}{2} i) \phi(z)$$

in the limit as h increases to one.

An Euler weight function $W(z)$ has been constructed which satisfies the identity

$$W(z + \tfrac{1}{2}i) = W(z - \tfrac{1}{2}i)\phi(z)$$

for a given nontrivial function $\phi(z)$ of z which is analytic and has nonnegative real part in the upper half–plane.

If a maximal dissipative transformation is defined in a weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z+i)$ whenever the functions of z belong to the space, then the identity

$$W(z + \tfrac{1}{2}i) = W(z - \tfrac{1}{2}i)\phi(z)$$

holds for a function $\phi(z)$ of z which is analytic and has nonnegative real part in the upper half–plane. The analytic weight function $W(z)$ is the product of an Euler weight function and an entire function which is periodic of period i and has no zeros.

Entire functions of Hermite class are examples of analytic weight functions which are limits of polynomials having no zeros in the upper half–plane. Such polynomials appear in the Stieltjes representation of positive linear functionals on polynomials.

A linear functional on polynomials with complex coefficients is said to be nonnegative if it has nonnegative values on polynomials whose values on the real axis are nonnegative. A positive linear functional on polynomials is a nonnegative linear functional on polynomials which does not vanish identically. A nonnegative linear functional on polynomials is represented as an integral with respect to a nonnegative measure μ on the Baire subsets of the real line. The linear functional takes a polynomial $F(z)$ into the integral

$$\int F(t) d\mu(t).$$

Stieltjes examines the action of a positive linear functional on polynomials of degree less than r for a positive integer r. A polynomial which as nonnegative values on the real axis is a product

$$F(z)F^*(z)$$

of a polynomial $F(z)$ and the conjugate polynomial

$$F^*(z) = F(z^-)^-.$$

If the positive linear functional does not annihilate

$$F(z)F^*(z)$$

for any nontrivial polynomial $F(z)$ of degree less than r, then a Hilbert space exists whose elements are the polynomials of degree less than r and whose scalar product

$$\langle F(t), G(t) \rangle$$

is defined as the action of the positive linear functional on the polynomial

$$G^*(z)F(z).$$

Stieltjes shows that the Hilbert space of polynomials of degree less than r is contained isometrically in a weighted Hardy space $\mathcal{F}(W)$ whose analytic weight function $W(z)$ is a polynomial of degree r having no zeros in the upper half–plane.

Examples of such spaces are applied by Legendre in quadratic approximations of periodic motion and motivate the application to number theory made precise by the Riemann hypothesis.

An axiomatization of the Stieltjes spaces is stated in a general context [2]. Hilbert spaces are examined whose elements are entire functions and which have these properties:

(H1) Whenever an entire function $F(z)$ of z belongs to the space and has a nonreal zero w, the entire function

$$F(z)(z - w^-)/(z - w)$$

of z belongs to the space and has the same norm as $F(z)$.

(H2) A continuous linear functional on the space is defined by taking a function $F(z)$ of z into its value $F(w)$ at w for every nonreal number w.

(H3) The entire function

$$F^*(z) = F(z^-)^-$$

of z belongs to the space whenever the entire function $F(z)$ of z belongs to the space, and it has the same norm as $F(z)$.

An example of a Hilbert space of entire functions which satisfies the axioms is obtained when an entire function $E(z)$ of z satisfies the inequality

$$|E(x - y)| < |E(x + iy)|$$

for all real x when y is positive. A weighted Hardy space $\mathcal{F}(W)$ is defined with analytic weight function

$$W(z) = E(z).$$

A Hilbert space $\mathcal{H}(E)$ which is contained isometrically in the space $\mathcal{F}(W)$ is defined as the set of entire functions $F(z)$ of z such that the entire functions $F(z)$ of z and $F^*(z)$ of z belong to the space $\mathcal{F}(W)$. The entire function

$$[E(z)E(w)^- - E^*(z)E(w^-)]/[2\pi i(w^- - z)]$$

of z belongs to the space $\mathcal{H}(E)$ for every complex number w and acts as reproducing kernel function for function values at w.

A Hilbert space $\mathcal{H}$ of entire functions which satisfies the axioms (H1), (H2), and (H3) is isometrically equal to a space $\mathcal{H}(E)$ if it contains a nonzero element. The proof applies reproducing kernel functions which exist by the axiom (H2).

For every nonreal number w a unique entire function $K(w, z)$ of z exists which belongs to the space and acts as reproducing kernel function for function values at w. The function does not vanish identically since the axiom (H1) implies that some element of the space has a nonzero value at w when some element of the space does not vanish identically. The scalar self–product $K(w, w)$ of the function $K(w, z)$ of z is positive. The axiom (H3) implies the symmetry

$$K(w^-, z) = K(w, z^-)^-.$$

If λ is a nonreal number, the set of elements of the space which vanish at λ is a Hilbert space of entire functions which is contained isometrically in the given space. The function

$$K(w, z) - K(w, \lambda)K(\lambda, \lambda)^{-1}K(\lambda, z)$$

of z belongs to the subspace and acts as reproducing kernel function for function values at λ. The identity

$$\begin{aligned}&[K(w, z) - K(w, \lambda)K(\lambda, \lambda)^{-1}K(\lambda, z)](z - \lambda^-)(w^- - \lambda)\\ &= [K(w, z) - K(w, \lambda^-)K(\lambda^-, \lambda^-)^{-1}K(\lambda^-, z)](z - \lambda)(w^- - \lambda^-)\end{aligned}$$

is a consequence of the axiom (H1).

An entire function $E(z)$ of z exists such that the identity

$$K(w, z) = [E(z)E(w)^- - E^*(z)E(w^-)]/[2\pi i(w^- - z)]$$

holds for all complex z when w is not real. The entire function can be chosen with a zero at λ when λ is in the lower half–plane and with a zero at λ^- when λ is in the upper half–plane. The function is then unique within a constant factor of absolute value one. A space $\mathcal{H}(E)$ exists and is isometrically equal to the given space $\mathcal{H}$.

Examples of Hilbert spaces of entire functions which satisfy the axioms (H1), (H2), and (H3) are constructed from the analytic weight function

$$W(z) = a^{iz}\Gamma(\tfrac{1}{2} - iz)$$

for every positive number a. The space is contained isometrically in the weighted Hardy space $\mathcal{F}(W)$ and contains every entire function $F(z)$ such that the functions $F(z)$ and $F^*(z)$ of z belong to the space $\mathcal{F}(W)$. The space of entire functions is isometrically equal to a space $\mathcal{H}(E)$ whose defining function $E(z)$ is a confluent hypergeometric series [1]. Properties of the space define a class of Hilbert spaces of entire functions.

An Euler space of entire functions is a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) such that a maximal dissipative transformation is defined in the space for every h in the interval $[-1, 1]$ by taking $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space.

Theorem 2. *A maximal dissipative transformation is defined in a Hilbert space $\mathcal{H}(E)$ of entire functions for a real number h by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space if, and only if, a Hilbert space $\mathcal{H}$ of entire functions exists which contains the function*

$$[E(z+\tfrac{1}{2}ih)E(w-\tfrac{1}{2}ih)^- - E^*(z+\tfrac{1}{2}ih)E(w^-+\tfrac{1}{2}ih)]/[2\pi i(w^- - z)]$$
$$+ [E(z-\tfrac{1}{2}ih)E(w+\tfrac{1}{2}ih)^- - E^*(z-\tfrac{1}{2}ih)E(w^- - \tfrac{1}{2}ih)]/[2\pi i(w^- - z)]$$

of z as reproducing kernel function for function values at w for every complex number w.

Proof of Theorem 2. The space $\mathcal{H}$ is constructed from the graph of the adjoint of the transformation which takes $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space. An element

$$F(z) = (F_+(z), F_-(z))$$

of the graph is a pair of entire functions of z, which belong to the space $\mathcal{H}(E)$ such that the adjoint takes $F_+(z)$ into $F_-(z)$. The scalar product

$$\langle F(t), G(t)\rangle = \langle F_+(t), G_-(t)\rangle_{\mathcal{H}(E)} + \langle F_-(t), G_+(t)\rangle_{\mathcal{H}(E)}$$

of elements $F(z)$ and $G(z)$ of the graph is defined as a sum of scalar products in the space $\mathcal{H}(E)$. Scalar self–products are nonnegative since the adjoint of a maximal dissipative transformation is dissipative.

An element

$$K(w,z) = (K_+(w,z), K_-(w,z))$$

of the graph is defined for every complex number w by

$$K_+(w,z) = [E(z)E(w-\tfrac{1}{2}ih)^- - E^*(z)E(w^-+\tfrac{1}{2}ih)]/[2\pi i(w^- + \tfrac{1}{2}ih - z)]$$

and

$$K_-(w,z) = [E(z)E(w+\tfrac{1}{2}ih)^- - E^*(z)E(w^- - \tfrac{1}{2}ih)]/[2\pi i(w^- - \tfrac{1}{2}ih - z)].$$

The identity

$$F_+(w+\tfrac{1}{2}ih) + F_-(w-\tfrac{1}{2}ih) = \langle F(t), K(w,t)\rangle$$

holds for every element

$$F(z) = (F_+(z), F_-(z))$$

of the graph. An element of the graph which is orthogonal to itself is orthogonal to every element of the graph.

A partially isometric transformation of the graph onto a dense subspace of the space $\mathcal{H}$ is defined by taking

$$F(z) = (F_+(z), F_-(z))$$

into the entire function

$$F_+(z + \tfrac{1}{2}ih) + F_-(z - \tfrac{1}{2}ih)$$

of z. The reproducing kernel function for function values at w in the space $\mathcal{H}$ is the function

$$[E(z + \tfrac{1}{2}ih)E(w - \tfrac{1}{2}ih)^- - E^*(z + \tfrac{1}{2}ih)E(w^- + \tfrac{1}{2}ih)]/[2\pi i(w^- - z)]$$
$$+ [E(z - \tfrac{1}{2}ih)E(w + \tfrac{1}{2}ih)^- - E^*(z - \tfrac{1}{2}ih)E(w^- - \tfrac{1}{2}ih)]/[2\pi i(w^- - z)]$$

of z for every complex number w.

This completes the construction of a Hilbert space $\mathcal{H}$ of entire functions with the desired reproducing kernel functions when the maximal dissipative transformation exists in the space $\mathcal{H}(E)$. The argument is reversed to construct the maximal dissipative transformation in the space $\mathcal{H}(E)$ when the Hilbert space of entire functions with the desired reproducing kernel functions exists.

A transformation is defined in the space $\mathcal{H}(E)$ by taking $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space. The graph of the adjoint is a space of pairs

$$F(z) = (F_+(z), F_-(z))$$

such that the adjoint takes the function $F_+(z)$ of z into the function $F_-(z)$ of z. The graph contains

$$K(w, z) = (K_+(w, z), K_-(w, z))$$

with

$$K_+(w, z) = [E(z)E(w - \tfrac{1}{2}ih)^- - E^*(z)E(w^- + \tfrac{1}{2}ih)]/[2\pi i(w^- + \tfrac{1}{2}ih - z)]$$

and

$$K_-(w, z) = [E(z)E(w + \tfrac{1}{2}ih)^- - E^*(z)E(w^- - \tfrac{1}{2}ih)]/[2\pi i(w^- - \tfrac{1}{2}ih - z)]$$

for every complex number w. The elements $K(w, z)$ of the graph span the graph of a restriction of the adjoint. The transformation in the space $\mathcal{H}(E)$ is recovered as the adjoint of its restricted adjoint.

A scalar product is defined on the graph of the restricted adjoint so that an isometric transformation of the graph of the restricted adjoint into the space $\mathcal{H}$ is defined by taking

$$F(z) = (F_+(z), F_-(z))$$

into

$$F_+(z + \tfrac{1}{2}ih) + F_-(z - \tfrac{1}{2}ih).$$

The identity

$$\langle F(t), G(t) \rangle = \langle F_+(t), G_-(t) \rangle_{\mathcal{H}(E)} + \langle F_-(t), G_+(t) \rangle_{\mathcal{H}(E)}$$

holds for all elements

$$F(z) = (F_+(z), F_-(z))$$

of the graph of the restricted adjoint. The restricted adjoint is dissipative since scalar self–products are nonnegative in its graph. The adjoint is dissipative since the transformation in the space $\mathcal{H}(E)$ is the adjoint of its restricted adjoint.

The dissipative property of the adjoint is expressed in the inequality

$$\|F_+(t) - \lambda^- F_-(t)\|_{\mathcal{H}(E)} \leq \|F_+(t) + \lambda F_-(t)\|_{\mathcal{H}(E)}$$

for elements

$$F(z) = (F_+(z), F_-(z))$$

of the graph when λ is in the right half–plane. The domain of the contractive transformation which takes the function

$$F_+(z) + \lambda F_-(z)$$

of z into the function

$$F_+(z) - \lambda^- F_-(z)$$

of z is a closed subspace of the space $\mathcal{H}(E)$. The maximal dissipative property of the adjoint is the requirement that the contractive transformation be everywhere defined for some, and hence every, λ in the right half–plane.

Since $K(w, z)$ belongs to the graph for every complex number w, an entire function $H(z)$ of z which belongs to the space $\mathcal{H}(E)$ and is orthogonal to the domain is a solution of the equation

$$H(z) + \lambda H(z + i) = 0.$$

The function vanishes identically if it has a zero since zeros are repeated periodically with period i and since the function

$$H(z)/E(z)$$

of z is of bounded type in the upper half–plane. The space of solutions has dimension zero or one. The dimension is zero since it is independent of λ.

This completes the proof of the theorem.

The transformation which takes $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space $\mathcal{H}(E)$ is maximal dissipative since it is the adjoint of its adjoint, which is maximal dissipative.

The defining function $E(z)$ of an Euler space of entire functions is of Hermite class since the function

$$E(z - \tfrac{1}{2}ih)/E(z + \tfrac{1}{2}ih)$$

of z is of bounded type and of nonpositive mean type in the upper half–plane when h is in the interval $(0, 1)$. Since the function is bounded by one on the real axis, it is bounded by one in the upper half–plane. The modulus of $E(x + iy)$ is a nondecreasing function of positive y for every real x. An entire function $F(z)$ of z which belongs to the space $\mathcal{H}(E)$ is of Hermite class if it has no zeros in the upper half–plane and if the inequality

$$|F(x - iy)| \leq |F(x + iy)|$$

holds for all real x when y is positive.

A construction of Hilbert spaces of entire functions which satisfy the axioms (H1), (H2), and (H3) appears in the spectral theory of ordinary differential equations of second order which are formally self–adjoint. The spectral theory is advantageously reformulated as a spectral theory of first order differential equations for pairs of scalar functions. The resulting canonical form is the classical model for a vibrating string.

The canonical form for the integral equation is obtained with a continuous function of positive t whose values are matrices

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$$

with real entries such that the matrix inequality

$$m(a) \leq m(b)$$

holds when a is less than b. It is assumed that $\alpha(t)$ is positive when t is positive, that

$$\lim \alpha(t) = 0$$

as t decreases to zero, and that the integral

$$\int_0^1 \alpha(t) d\gamma(t)$$

is finite.

The matrix

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is applied in the formulation of the integral equation. When a is positive, the integral equation

$$M(a, b, z)I - I = z \int_a^b M(a, t, z) dm(t)$$

admits a unique continuous solution

$$M(a, b, z) = \begin{pmatrix} A(a, b, z) & B(a, b, z) \\ C(a, b, z) & D(a, b, z) \end{pmatrix}$$

as a function of b greater than or equal to a for every complex number z. The entries of the matrix are entire functions of z which are self–conjugate and of Hermite class for every b. The matrix has determinant one. The identity

$$M(a, c, z) = M(a, b, z)M(b, c, z)$$

holds when $a \le b \le c$.

A bar is used to denote the conjugate transpose

$$M^- = \begin{pmatrix} A^- & C^- \\ B^- & D^- \end{pmatrix}$$

of a square matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with complex entries and also for the conjugate transpose

$$c^- = (c_+^-, c_-^-)$$

of a column vector

$$c = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

with complex entries. The space of column vectors with complex entries is a Hilbert space of dimension two with scalar product

$$\langle u, v \rangle = v^- u = v_+^- u_+ + v_-^- u_- .$$

When a and b are positive with a less than or equal to b, a unique Hilbert space $\mathcal{H}(M(a,b))$ exists whose elements are pairs

$$F(z) = \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$$

of entire functions of z such that a continuous transformation of the space into the Hilbert space of column vectors is defined by taking $F(z)$ into $F(w)$ for every complex number w and such that the adjoint takes a column vector c into the element

$$[M(a,b,z)IM(a,b,w)^- - I]c/[2\pi(z - w^-)]$$

of the space.

An entire function

$$E(c,z) = A(c,z) - iB(c,z)$$

of z which is of Hermite class exists for every positive number c such that the self–conjugate entire functions $A(c,z)$ and $B(c,z)$ satisfy the identity

$$(A(b,z), B(b,z)) = (A(a,z), B(a,z))M(a,b,z)$$

when a is less than or equal to b and such that the entire functions

$$E(c,z)\exp[\beta(c)z]$$

of z converge to one uniformly on compact subsets of the complex plane as c decreases to zero.

A space $\mathcal{H}(E(c))$ exists for every positive number c. The space $\mathcal{H}(E(a))$ is contained contractively in the space $\mathcal{H}(E(b))$ when a is less than or equal to b. The inclusion is isometric on the orthogonal complement in the space $\mathcal{H}(E(a))$ of the elements which are linear combinations

$$A(a,z)u + B(a,z)v$$

with complex coefficients u and v. These elements form a space of dimension zero or one since the identity

$$v^- u = u^- v$$

is satisfied.

A positive number b is said to be singular with respect to the function $m(t)$ of t if it belongs to an interval (a,c) such that equality holds in the inequality

$$[\beta(c) - \beta(a)]^2 \leq [\alpha(c) - \alpha(a)][\gamma(c) - \gamma(a)]$$

with $m(b)$ unequal to $m(a)$ and unequal to $m(c)$. A positive number is said to be regular with respect to $m(t)$ if it is not singular with respect to the function of t.

If a and c are positive numbers such that a is less than c and if an element b of the interval (a,c) is regular with respect to $m(t)$, then the space $\mathcal{H}(M(a,b))$ is contained isometrically in the space $\mathcal{H}(M(a,c))$ and multiplication by $M(a,b,z)$ is an isometric transformation of the space $\mathcal{H}(M(b,c))$ onto the orthogonal complement of the space $\mathcal{H}(M(a,b))$ in the space $\mathcal{H}(M(a,c))$.

If a and b are positive numbers such that a is less than b and if a is regular with respect to $m(t)$, then the space $\mathcal{H}(E(a))$ is contained isometrically in the space $\mathcal{H}(E(b))$ and an isometric transformation of the space $\mathcal{H}(M(a,b))$ onto the orthogonal complement of the space $\mathcal{H}(E(a))$ in the space $\mathcal{H}(E(b))$ is defined by taking

$$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$$

into

$$\sqrt{2}\ [A(a,z)F_+(z) + B(a,z)F_-(z)].$$

A function $\tau(t)$ of positive t with real values exists such that the function

$$m(t) + Iih(t)$$

of positive t with matrix values is nondecreasing for a function $h(t)$ of t with real values if, and only if, the functions

$$\tau(t) - h(t)$$

and

$$\tau(t) + h(t)$$

of positive t with real values are nondecreasing. The function $\tau(t)$ of t, which is continuous and nondecreasing, is called a greatest nondecreasing function such that

$$m(t) + Ii\tau(t)$$

is nondecreasing. The function is unique within an added constant.

If a and b are positive numbers such that a is less than b, multiplication by

$$\exp(ihz)$$

is a contractive transformation of the space $\mathcal{H}(E(a))$ into the space $\mathcal{H}(E(b))$ for a real number h, if, and only if, the inequalities

$$\tau(a) - \tau(b) \leq h \leq \tau(b) - \tau(a)$$

are satisfied. The transformation is isometric when a is regular with respect to $m(t)$.

An analytic weight function $W(z)$ may exist such that multiplication by

$$\exp(i\tau(c)z)$$

is an isometric transformation of the space $\mathcal{H}(E(c))$ into the weighted Hardy space $\mathcal{F}(W)$ for every positive number c which is regular with respect to $m(t)$. The analytic weight function is unique within a constant factor of absolute value one if the function

$$\alpha(t) + \gamma(t)$$

of positive t is unbounded in the limit of large t. The function

$$W(z) = \lim E(c,z) \exp(i\tau(c)z)$$

can be chosen as a limit uniformly on compact subsets of the upper half–plane.

If multiplication by

$$\exp(i\tau z)$$

is an isometric transformation of a space $\mathcal{H}(E)$ into the weighted Hardy space $\mathcal{F}(W)$ for some real number τ and if the space $\mathcal{H}(E)$ contains an entire function $F(z)$ whenever its product with a nonconstant polynomial belongs to the space, then the space $\mathcal{H}(E)$ is isometrically equal to the space $\mathcal{H}(E(c))$ for some positive number c which is regular with respect to $m(t)$.

A construction of Euler spaces of entire functions is made from Euler weight functions when a hypothesis is satisfied.

Theorem 3. *If for some real number τ a nontrivial entire function $F(z)$ of z exists such that the functions*

$$\exp(i\tau z)F(z)$$

and

$$\exp(i\tau z)F^*(z)$$

of z belong to the weighted Hardy space $\mathcal{F}(W)$ of an Euler weight function $W(z)$, then an Euler space of entire functions exists such that multiplication by $\exp(i\tau z)$ *is an isometric transformation of the space into the weighted Hardy space and such that the space contains every entire function $F(z)$ of z such that the functions*

$$\exp(i\tau z)F(z)$$

and

$$\exp(i\tau z)F^*(z)$$

of z belong to the weighted Hardy space.

Proof of Theorem 3. The set of entire functions $F(z)$ such that the functions

$$\exp(i\tau z)F(z)$$

and

$$\exp(i\tau z)F^*(z)$$

of z belong to the weighted Hardy space is a vector space with scalar product determined by the isometric property of multiplication as a transformation of the space into the weighted Hardy space. The space is shown to be a Hilbert space by showing that a Cauchy sequence of elements $F_n(z)$ of the space converge to an element $F(z)$ of the space.

Since the elements

$$\exp(i\tau z)F_n(z)$$

and

$$\exp(i\tau z)F_n^*(z)$$

of the weighted Hardy space form Cauchy sequences, a function $F(z)$ of z which is analytic separately in the upper half–plane and the lower half–plane exists such that the limit functions

$$\exp(i\tau z)F(z) = \lim \exp(i\tau z)F_n(z)$$

and

$$\exp(i\tau z)F^*(z) = \lim \exp(i\tau z)F_n^*(z)$$

of z belong to the weighted Hardy space. Since

$$|z - z^-|^{\frac{1}{2}} \exp(i\tau z)F(z)/W(z) = \lim |z - z^-|^{\frac{1}{2}} \exp(i\tau z)F_n(z)/W(z)$$

and

$$|z - z^-|^{\frac{1}{2}} \exp(i\tau z) F^*(z)/W(z) = \lim |z - z^-|^{\frac{1}{2}} \exp(i\tau z) F_n^*(z)/W(z)$$

uniformly in the upper half–plane and since the functions

$$\log |F_n(z)/W(z)|$$

and

$$\log |F_n^*(z)/W(z)|$$

of z are subharmonic in the half–plane

$$-1 < iz^- - iz,$$

the convergence of

$$F(z) = \lim F_n(z)$$

is uniform on compact subsets of the complex plane. The limit function $F(z)$ of z is analytic in the complex plane.

A Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) is obtained such that multiplication by $\exp(i\tau z)$ is an isometric transformation of the space into the weighted Hardy space. Since the space contains a nonzero element by hypothesis, it is isometrically equal to a space $\mathcal{H}(E)$.

The space is shown to be an Euler space of entire functions by showing that a maximal dissipative transformation is defined in the space for h in the interval $[-1, 1]$ by taking $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space. The dissipative property of the transformation is a consequence of the dissipative property in the weighted Hardy space.

Maximality is proved by showing that every element of the space is a sum

$$F(z) + F(z + ih)$$

of functions $F(z)$ and $F(z + ih)$ of z which belong to the space.

Since a maximal dissipative transformation exists in the weighted Hardy space, every element of the Hilbert space of entire functions is in the upper half–plane a sum

$$F(z) + F(z + ih)$$

of functions $F(z)$ and $F(z + ih)$ of z such that the functions

$$\exp(i\tau z) F(z)$$

and

$$\exp(i\tau z) F(z + ih)$$

of z belong to the weighted Hardy space. The function $F(z)$ of z admits an analytic continuation to the complex plane. The decomposition applies for all complex z.

The entire function

$$F^*(z) + F^*(z - ih)$$

of z belongs to the Hilbert space of entire functions since the space satisfies the axiom (H3). An entire function $G(z)$ of z exists such that

$$F^*(z) + F^*(z - ih) = G(z) + G(z + ih)$$

and such that the functions

$$\exp(i\tau z)G(z)$$

and

$$\exp(i\tau z)G(z + ih)$$

of z belong to the weighted Hardy space.

Vanishing of the entire function

$$F^*(z) - G(z + ih) = G(z) - F^*(z - ih)$$

of z implies the desired conclusion that the functions $F(z)$ and $F(z + ih)$ of z as well as the functions $G(z)$ and $G(z + ih)$ of z belong to the Hilbert space of entire functions. Vanishing is proved by showing boundedness of the function in the strip

$$-2h < iz^- - iz < 0$$

since the function is periodic of period $2ih$ with modulus which is periodic of period ih.

It can be assumed that the functions

$$\exp(i\tau z)F(z)$$

and

$$\exp(i\tau z)G(z)$$

of z are elements of norm at most one in the weighted Hardy space. The inequalities

$$2\pi|F(z)|^2 \leq |\exp(-i\tau z)W(z)|^2/(iz^- - iz)$$

and

$$2\pi|G(z)|^2 \leq |\exp(-i\tau z)W(z)|^2/(iz^- - iz)$$

apply when z is in the upper half–plane. Since the inequalities

$$2\pi|F^*(z)|^2 \leq |\exp(i\tau z)W^*(z)|^2/(iz - iz^-)$$

and

$$2\pi|G(z + ih)|^2 \leq \exp(2\pi h)|\exp(-i\tau z)W(z + ih)|^2/(2h + iz^- - iz)$$

apply when z is in the strip, the inequality

$$\pi|F^*(z) - G(z + ih)|^2 \leq |\exp(i\tau z)W^*(z)|^2/(iz - iz^-)$$
$$+ \exp(2\pi h)|\exp(-i\tau z)W(z + ih)|^2/(2h + iz^- - iz)$$

applies when z is in the strip.

Boundedness of the entire function

$$F^*(z) - G(z + ih)$$

of z in the complex plane follows from the subharmonic property of the logarithm of its modulus. The entire function is a constant which vanishes because of the identity

$$F^*(z) - G(z + ih) = G(z) - F^*(z - ih).$$

This completes the proof of the theorem.

The hypothesis of the theorem are satisfied by an Euler weight function $W(z)$ which satisfies the identity

$$W(z + \tfrac{1}{2}i) = W(z - \tfrac{1}{2}i)\varphi(z)$$

for a function $\phi(z)$ of z which is analytic and has nonnegative real part in the upper half–plane if $\log \phi(z)$ has nonnegative real part in the upper half–plane. The modulus of $W(x + iy)$ is then a nondecreasing function of positive y for every real x.

Since the weight function can be multiplied by a constant, it can be assumed to have value one at the origin. The phase $\psi(x)$ is defined as the continuous function of real x with value zero at the origin such that

$$\exp(i\psi(x))W(x)$$

is positive for all real x. The phase function is a nondecreasing function of real x which is identically zero if it is constant in any interval.

When the phase function vanishes identically, the modulus of $W(x + iy)$ is a constant as a function of positive y for every real x. The weight function is then the restriction of a self–conjugate entire function of Hermite class. For every positive number τ an entire function

$$F(z) = W(z)\sin(\tau z)/z$$

is obtained such that the functions

$$\exp(i\tau z)F(z)$$

and

$$\exp(i\tau z)F^*(z)$$

of z belong to the space $\mathcal{F}(W)$. No nonzero entire function $F(z)$ exists such that the functions $F(z)$ and $F^*(z)$ belong to the space $\mathcal{F}(W)$.

When the phase function does not vanish identically, an entire function $E(z)$ of Hermite class which has no real zeros exists such that $E(x)$ is real for a real number x if, and only if, $\psi(x)$ is an integral is an integral multiple of π, and then

$$\exp(i\psi(x))E(x)$$

is positive. Such an entire function is unique within a factor of a self–conjugate entire function of Hermite class. The factor is chosen so that the function

$$E(z)/W(z)$$

of z has nonnegative real part in the upper half–plane. The entire functions $E(z)$ and $E^*(z)$ are linearly independent. A nontrivial entire function

$$F(z) = [E(z) - E^*(z)]/z$$

is obtained such that the functions $F(z)$ and $F^*(z)$ of z belong to the space $\mathcal{F}(W)$.

The same conclusions are obtained under a weaker hypothesis.

Theorem 4. *If an Euler weight function $W(z)$ satisfies the identity*

$$W(z + \tfrac{1}{2}i) = W(z - \tfrac{1}{2}i)\phi(z)$$

for a function $\phi(z)$ of z which is analytic and has nonnegative real part in the upper half–plane such that

$$\sigma(z) + \log \phi(z)$$

has nonnegative real part in the upper half–plane for a function $\sigma(z)$ of z which is analytic and has nonnegative real part in the upper half–plane such that the least upper bound

$$\sup \int_{-\infty}^{+\infty} \mathcal{R}\sigma(x + iy)dx$$

taken over all positive y is finite, then for every positive number τ a nontrivial entire function $F(z)$ exists such that the functions

$$\exp(i\tau z)F(z)$$

and

$$\exp(i\tau z)F^*(z)$$

of z belong to the weighted Hardy space $\mathcal{F}(W)$.

Proof of Theorem 4. It can be assumed that the symmetry condition

$$\sigma^*(z) = \sigma(-z)$$

is satisfied since otherwise $\sigma(z)$ can be replaced by $\sigma(z)+\sigma^*(-z)$. When h is in the interval $(0,1)$, the integral

$$\log \phi_h(z) = \sin(\pi h) \int_{-\infty}^{+\infty} \frac{\log \phi(z-t)dt}{\cos(2\pi it)+\cos(\pi h)}$$

defines a function $\phi_h(z)$ of z which is analytic and has nonnegative real part in the upper half–plane such that

$$W(z+\tfrac{1}{2}ih) = W(z-\tfrac{1}{2}ih)\phi_h(z).$$

The integral

$$\mathcal{R}\sigma_h(z) = \sin(\pi h) \int_{-\infty}^{+\infty} \frac{\mathcal{R}\sigma(z-t)dt}{\cos(2\pi it)+\cos(\pi h)}$$

and the symmetry condition

$$\sigma_n^*(z) = \sigma_n(-z)$$

define a function $\sigma_h(z)$ which is analytic and has nonnegative real part in the upper half–plane. The function

$$\sigma_h(z) + \log \phi_h(z)$$

of z has nonnegative real part in the upper half–plane since the function

$$\sigma(z) + \log \phi(z)$$

has nonnegative real part in the upper half–plane by hypothesis.

An analytic weight function $U(z)$ which admits an analytic extension without zeros to the half–plane $iz^- - iz > -1$ is defined within a constant factor by the identity

$$\log U(z+\tfrac{1}{2}ih) - \log U(z-\tfrac{1}{2}ih) = \sigma_h(z)$$

for h in the interval $(0,1)$ and by the symmetry

$$U^*(z) = U(-z).$$

The analytic weight function

$$V(z) = U(z)W(z)$$

has an analytic extension without zeros to the half–plane $iz^- - iz > -1$. The modulus of $U(x+iy)$ and the modulus of $V(x+iy)$ are nondecreasing functions of positive y for every real x.

Since

$$\frac{\partial}{\partial y} \log |U(x+iy)| = \pi \int_{-\infty}^{+\infty} \frac{\mathcal{R}\sigma(x+iy-t)dt}{1+\cos(2\pi it)}$$

the integral

$$\int_{-\infty}^{+\infty} \frac{\partial}{\partial y} \log |U(x+iy)| dx = \int_{-\infty}^{+\infty} \mathcal{R}\sigma(x+iy)dx$$

is a bounded function of positive y. The phase $\psi(x)$ is the continuous, nondecreasing, odd function of real x such that

$$\exp(i\psi(x))U(x)$$

is positive for all real x. Since

$$\frac{\partial}{\partial y} \log |U(x+iy)| = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{d\psi(t)}{(t-x)^2+y^2}$$

when y is positive, the inequality

$$\psi(b) - \psi(a) \leq \sup \int_{-\infty}^{+\infty} \mathcal{R}\sigma(x+iy)dx$$

holds when a is less than b with the least upper bound taken over all positive y.

The remaining arbitrary constant in $U(z)$ is chosen so that the integral representation

$$\log U(z) = \frac{1}{2\pi} \int_0^\infty \log(1 - z^2/t^2) d\psi(t)$$

holds when z is in the upper half–plane with the logarithm of $1 - z^2/t^2$ defined continuously in the upper half–plane with nonnegative values when z is on the upper half of the imaginary axis. The inequality

$$|U(z)| \leq |U(i|z|)$$

holds when z is in the upper half–plane since

$$|1 - z^2/t^2| \leq 1 + z^- z/t^2.$$

If a positive integer r is chosen so that the inequality

$$\int_{-\infty}^{+\infty} \mathcal{R}\sigma(x+iy)dx \leq 2\pi r$$

holds for all positive y, then the function

$$U(z)/(z+i)^r$$

is bounded and analytic in the upper half–plane.

Since the modulus of $V(x+iy)$ is a nondecreasing function of positive y for every real x, there exists for every positive number τ a nontrivial entire function $G(z)$ such that the functions

$$\exp(i\tau z)G(z)$$

and

$$\exp(i\tau z)G^*(z)$$

of z belong to the weighted Hardy space $\mathcal{F}(V)$. Since the entire function

$$G(z) = F(z)P(z)$$

is the product of an entire function $F(z)$ and a polynomial $P(z)$ of degree r, a nontrivial entire function $F(z)$ is obtained such that the functions

$$\exp(i\tau z)F(z)$$

and

$$\exp(i\tau z)F^*(z)$$

of z belong to the weighted Hardy space $\mathcal{F}(W)$.

This completes the proof of the theorem.

A construction of Euler spaces of entire functions results.

Theorem 5. *A Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) and which contains a nonzero element is an Euler space of entire functions if it contains an entire function whenever its product with a nonconstant polynomial belongs to the space and if for some real number τ multiplication by $\exp(i\tau z)$ is an isometric transformation of the space into the weighted Hardy space $\mathcal{F}(W)$ for an Euler weight function $W(z)$ which satisfies the hypotheses of Theorem 4.*

Proof of Theorem 5. The given Hilbert space of entire functions is isometrically equal to a space $\mathcal{H}(E)$ for an entire function $E(z)$ of z which has no real zeros since an entire function belongs to the space whenever its product with a nonconstant polynomial belongs to the space. A dissipative transformations in the space $\mathcal{H}(E)$ is defined by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space since multiplication by $\exp(i\tau z)$ is an isometric transformation of the space into the space $\mathcal{F}(W)$ and since a dissipative transformation is defined in the weighted Hardy space by taking $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space. The maximal dissipative property of the transformation in the weighted Hardy space is applied in a proof of the maximal dissipative property in the Hilbert space of entire functions.

A proof is first given under an additional hypothesis: Assume that the function $\varphi(z)$ of z, which is analytic and has nonnegative real part in the upper half–plane, is bounded in the half–plane. A least positive number κ exists such that the inequality

$$|\phi(z)| \leq \kappa$$

holds when z is in the half–plane. Since the integral representation

$$iW'(z)/W(z) = \pi \int_{-\infty}^{+\infty} \frac{\log \phi(z-t)dt}{1 + \cos(2\pi it)}$$

holds when z is in the upper half–plane, the inequality

$$\frac{\partial}{\partial y} \log |W(x+iy)| \log \kappa$$

holds for all real x when y is positive. The inequality

$$|W(z+ih)/W(z)| \leq \kappa^h$$

holds for z in the upper half–plane when h is positive. The inequality is applied when h is not greater than one. A contractive transformation of the weighted Hardy space $\mathcal{F}(W)$ into itself is defined by taking a function $F(z)$ of z into the function

$$\kappa^{-h} F(z+ih)$$

of z.

The ordering theorem for Hilbert spaces of entire functions applies to spaces which satisfy the axioms (H1), (H2), and (H3) and which contain nonzero elements when multiplication by $\exp(i\tau z)$ is an isometric transformation of the space into the space $\mathcal{F}(W)$ and when the space contains an entire function whose product with a nonconstant polynomial belongs to the space. The theorem states that any two such spaces are comparable: One space is contained properly in the other if they are not equal.

A Hilbert space of entire functions which satisfies the axioms (H1) and (H2) and which contains a nonzero element need not satisfy the axiom (H3). The space is however the isometric image of a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) and which contains a nonzero element under multiplication by $\exp(iaz)$ for some real number a.

A Hilbert space $\mathcal{H}$ of entire functions which satisfies the axioms (H1) and (H2) is defined so that multiplication by $\exp(i\tau z)$ is an isometric transformation of the space into the space $\mathcal{F}(W)$ and so that the space contains a dense set of elements which are functions $F(z+ih)$ of z for functions $F(z)$ of z which belong to the space $\mathcal{H}(E)$.

Since multiplication by $\exp(i\tau z)$ is an isometric transformation of the space $\mathcal{H}(E)$ into the space $\mathcal{F}(W)$, the function

$$E(z)/W(z)$$

of z is of bounded type in the upper half–plane. If τ is the least real number such that multiplication by $\exp(i\tau z)$ is an isometric transformation of the space $\mathcal{H}(E)$ into the space $\mathcal{F}(W)$, the mean type of

$$E(z)/W(z)$$

is equal to τ. Since the function

$$W(z+ih)/W(z)$$

of z is analytic and bounded in the upper half–plane, the function

$$E(z+ih)/W(z)$$

of z is analytic and of bounded type in the upper half–plane and has mean type τ.

The space $\mathcal{H}(E)$ is the closed span of elements which are self–conjugate entire functions

$$F(z) = F^*(z)$$

of z such that the function

$$F(z)/W(z)$$

of z is of bounded type and of mean type τ in the upper half–plane. An example is the function

$$[E(z)E(w)^- - E^*(z)E(w^-)]/[2\pi i(w^- - z)]$$

of z when w is real. An element of the space $\mathcal{H}(E)$ vanishes identically if it is orthogonal to this reproducing kernel function of z for all real numbers w.

If $F(z)$ is an element of the space $\mathcal{H}(E)$ such that the functions

$$F(z)/W(z)$$

and

$$F^*(z)/W(z)$$

of z are of bounded type and of mean type τ in the upper half–plane, then the function

$$G(z) = F(z + ih)$$

of z is an element of the space $\mathcal{H}$ such that the functions

$$G(z)/W(z)$$

and

$$G^*(z)/W(z)$$

of z are of bounded type and of mean type τ in the upper half–plane.

Since the space $\mathcal{H}$ satisfies the axioms (H1) and (H2), a real number a exists such that multiplication by $\exp(iaz)$ is an isometric transformation of the space onto a space which satisfies the axioms (H1), (H2), and (H3). Since the space $\mathcal{H}$ is the closed span of functions $G(z)$ such that the functions

$$G(z)/W(z)$$

and

$$G^*(z)/W(z)$$

of z are of bounded type and of mean type τ in the upper half–plane, the space satisfies the axioms (H1), (H2), and (H3).

The ordering theorem for Hilbert spaces of entire functions is applicable since an entire function belongs to the space $\mathcal{H}(E)$ whenever its product with a nonconstant polynomial belongs to the space and belongs to the space $\mathcal{H}$ whenever its product with a nonconstant polynomial belongs to the space. Either the space $\mathcal{H}$ is contained in the space $\mathcal{H}(E)$ or the space $\mathcal{H}(E)$ is contained in the space $\mathcal{H}$.

If the space $\mathcal{H}$ is contained in the space $\mathcal{H}(E)$, the dissipative transformation which takes a function $F(z)$ of z into the function $F(z+ih)$ of z whenever the functions belong to the space $\mathcal{H}(E)$ is maximal dissipative since it is a continuous transformation of the space into itself. Since the transformation takes the space $\mathcal{H}$ into itself, its restriction is a maximal dissipative transformation in the space $\mathcal{H}$. The space $\mathcal{H}$ is equal to the space $\mathcal{H}(E)$ since it satisfies the axiom (H3).

A similar argument shows that the space $\mathcal{H}(E)$ is isometrically equal to the space $\mathcal{H}$ if it is contained in the space $\mathcal{H}$. The continuous transformation of the space $\mathcal{H}(E)$ into the space $\mathcal{H}$ which takes a function $F(z)$ of z into the function $F(z+ih)$ of z has a continuous inverse which takes the space $\mathcal{H}$ into the space $\mathcal{H}(E)$ since the spaces $\mathcal{H}$ and $\mathcal{H}(E)$ satisfy the axiom (H3).

When the function $\phi(z)$ of z is unbounded in the upper half–plane, a proof of the theorem is given on approximation by the function

$$\phi_n(z) = (1+1/n)\phi(z)/[1+\phi(z)/n]$$

for positive integers n. The function is analytic and bounded by $n+1$ in the upper half–plane and has nonnegative real part in the half–plane.

A unique Euler weight function $W_n(z)$, which agrees at the origin with $W(z)$, exists such that

$$W_n(z+\tfrac{1}{2}i) = W_n(z-\tfrac{1}{2}i)\phi_n(z).$$

The weight function $W_n(z)$ satisfies the hypotheses of Theorem 4 since the weight function $W(z)$ satisfies these hypotheses and since the inequality

$$|\phi(z)| \leq |\phi_n(z)|$$

holds when

$$|\phi_n(z)| \leq 1.$$

For every positive number τ the set of entire functions $F(z)$ such that the functions

$$\exp(i\tau z)F(z)$$

and

$$\exp(i\tau z)F^*(z)$$

belong to the space $\mathcal{F}(W_n)$ is a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) and which contains a nonzero element in the scalar product such that multiplication by $\exp(i\tau z)$ is an isometric transformation of the space into the space $\mathcal{F}(W_n)$. The space is an Euler space of entire functions by Theorem 4.

Additional Euler spaces of entire functions are found since the function $\phi_n(z)$ of z is bounded in the upper half–plane. A Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) and which contains a nonzero element is an Euler space of entire functions if multiplication by $\exp(i\tau z)$ is an isometric transformation of the space into the space $\mathcal{F}(W_n)$ for some positive number τ and if an entire function belongs to the space whenever its product with a nonconstant polynomial belongs to the space.

The spaces are parametrized by positive numbers t so that the space of parameter a is contained in the space of parameter b when a is less than b. The defining function

$$E_n(t,z) = A_n(t,z) - iB_n(t,z)$$

of the space of parameter t has value one at the origin and depends continuously on the parameter. An integral equation is satisfied with respect to a nondecreasing continuous function

$$m_n(t) = \begin{pmatrix} \alpha_n(t) & \beta_n(t) \\ \beta_n(t) & \gamma_n(t) \end{pmatrix}$$

of positive t whose values are matrices with real entries such that $\alpha_n(t)$ is positive when t is positive and has limit zero as t decreases to zero. The integral

$$\int_0^1 \alpha_n(t) d\gamma_n(t)$$

converges.

The greatest nondecreasing function $\gamma_n(t)$ of positive t such that the function

$$m_n(t) + iI\tau_n(t)$$

of t is nondecreasing is normalized so as to have limit zero as t decreases to zero. Multiplication by $\exp(i\tau_n(t)z)$ is an isometric transformation of the set of elements of the space of parameter t whose product by z belongs to the space into the space $\mathcal{F}(W_n)$. If the transformation is not isometric on the whole space, then the parameter t belongs to an interval (t_-, t_+) such that the transformation is isometric on the spaces parametrized by t_- and t_+ but is not isometric on spaces parametrized by elements of the interval. The space of parameter t is an Euler space of entire functions by Theorem 2 since the spaces of parameters t_+ and t_- are Euler spaces of entire functions and since the reproducing kernel functions of the space with parameter t are convex combinations of the reproducing kernel functions of the spaces with parameters t_+ and t_-.

Since the limit

$$\phi(z) = \lim \phi_n(z)$$

holds uniformly for z in every compact subset of the upper half–plane and since the weight functions $W_n(z)$ agree with the weight function $W(z)$ at the origin, the limit

$$W(z) = \lim W_n(z)$$

holds uniformly for z in every compact subset of the larger half–plane

$$-1 < iz^- - iz.$$

For every positive integer n the analytic weight function

$$V_n(z) = U(z)W_n(z)$$

has the property that the modulus of $V_n(x+iy)$ is a nondecreasing function of positive y for every real x. An essentially unique entire function $S_n(z)$ of Hermite class which has no real zeros exists such that

$$S_n(z)/V_n(z)$$

has nonnegative real part when z is in the upper half–plane. Uniqueness is obtained by choosing the positive constant factor in the function so that it agrees with $V_n(z)$ at the origin if $W(z)$, and hence also, $W_n(z)$ has a positive value at the origin. An entire function

$$S(z) = \lim S_n(z)$$

of Hermite class is then obtained as a limit uniformly on compact subsets of the upper half–plane. The function

$$S(z)/V(z)$$

of z has nonnegative real part in the upper half–plane and has value one at the origin.

The Hilbert spaces of entire functions applied in the construction of a vibrating string for the Euler weight functions $W_n(z)$ are given identical parametrizations. For every positive positive number τ a positive number $t_n(\tau)$ exists which parametrizes the Hilbert space of entire functions $F(z)$ such that the functions

$$\exp(i\tau z)F(z)$$

and

$$\exp(i\tau z)F^*(z)$$

of z belong to the space $\mathcal{F}(W)$ and such that multiplication by $\exp(i\tau z)$ is an isometric transformation of the space into the space $\mathcal{F}(W)$. The parametrization is made so that

$$t_n(\tau) = t(\tau)$$

is uniquely determined by τ and is independent of n.

The stationary set of the nondecreasing continuous function $\tau_n(t)$ of positive t is defined as the union of the open intervals on which the function is constant. The set is the union of its connected components. A component which does not bound on the origin is parametrized so that t is a linear function of the trace of $m_n(t)$ in the interval. A component which bounds the origin is parametrized so that t is a linear function of $\alpha_n(t)$ in the interval.

An entire function

$$E(t,z) = \lim E_n(t,z)$$

of Hermite class is obtained as a limit uniformly on compact subsets of the complex plane for every positive number t. An integral equation is satisfied with respect to a nondecreasing function

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$$

of positive t whose entries are continuous functions of t with real values. The increments

$$m(b) - m(a) = \lim[m_n(b) - m_n(a)]$$

are limits of increments for all positive a and b with a less than b. The greatest nondecreasing function $\tau(t)$ of positive t such that the function

$$m(t) + iI\tau(t)$$

of t is nondecreasing is normalized so as to have limit zero as t decreases to zero.

A Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) and which contains a nonzero element is defined by the entire function

$$E(t, z) = A(t, z) - iB(t, z)$$

for every positive number t. The space contains an entire function whenever its product with a nonconstant polynomial belongs to the space. Multiplication by $\exp(i\tau z)$ is a contractive transformation of the space into the space $\mathcal{F}(W)$ which is isometric on elements of the space whose by z belongs to the space. The space is an Euler space of entire functions by Theorem 2 since $E_n(t, z)$ is the defining function of an Euler space of entire functions for every positive integer n.

A Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3), and which contains a nonzero element is an Euler space of entire functions if multiplication by $\exp(i\tau z)$ is an isometric transformation of the space into the space $\mathcal{F}(W)$ for some positive number τ and if the space contains an entire function whenever its product with a nonconstant polynomial belongs to the space since the space is isometrically equal to the space defined by $E(t, z)$ for some positive number t.

This completes the proof of the theorem.

Computable examples of Hilbert spaces of entire functions are constructed from the gamma function. The Euler weight function

$$W_h(z) = \Gamma(h - iz)$$

has special properties when h is positive since

$$W_h^*(z) = W_h(-z)$$

and since a self–adjoint transformation is defined in the weighted Hardy space $\mathcal{F}(W_h)$ by taking $F(z)$ into

$$F(z + i)/(h - iz)$$

whenever the functions of z belong to the space.

A Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) and which contains an entire function whenever its product with a nonconstant polynomial belongs to the space is contained isometrically in the weighted Hardy space if it is contained contractively in the space and if the inclusion is isometric on elements of the space whose product with z belongs to the space.

A defining function $E(z)$ for the space can be chosen with the symmetry

$$E^*(z) = E(-z)$$

since an isometric transformation of the space onto itself is defined by taking $F(z)$ into $F(-z)$.

When $h = \frac{1}{2}$, a self–adjoint transformation in the space is defined by taking $F(z)$ into

$$[F(z+i) - F(-z)]/(\tfrac{1}{2} - iz)$$

whenever the functions of z belong to the space. If

$$E(z) = A(z) - iB(z)$$

for self–conjugate entire functions $A(z)$ and $B(z)$, the identities

$$[A(z+i) - A(-z)]/(\tfrac{1}{2} - iz) = A(z)s - iB(z)r$$

and

$$[B(z+i) - B(-z)]/(\tfrac{1}{2} - iz) = iA(z)p + B(z)s$$

hold for positive numbers p, r, and s such that

$$pr = s^2.$$

The spaces are parametrized by positive numbers t so that the space with parameter b is contained isometrically in the space with parameter a when a is less than b. The defining function $E(t, z)$ of the space with parameter t satisfies the identities with

$$s(t) = 2/t.$$

The function is chosen so that the identities are satisfied with

$$p(t) = \exp(-t)s(t)$$

and

$$r(t) = \exp(t)s(t).$$

The functions $E(t,z)$ depend continuously on t and satisfy the integral equation

$$(A(b,z),B(b,z))I-(A(a,z),B(a,z))I=z\int_a^b (A(t,z),B(t,z))dm(t)$$

when a is less than b with

$$m(t)=\begin{pmatrix}\alpha(t) & \beta(t)\\ \beta(t) & \gamma(t)\end{pmatrix}$$

a nonincreasing matrix–valued function of positive t with differentiable entries such that

$$-t\alpha'(t)=p(t)/s(t)$$

and

$$-t\gamma'(t)=r(t)/s(t)$$

and

$$\beta(t)=0.$$

The analytic weight function

$$W(z)=\Gamma(h-iz)$$

is treated in relation to the contiguous analytic weight functions

$$W_-(z)=\Gamma(h-\tfrac{1}{2}-iz)$$

and

$$W_+(z)=\Gamma(h+\tfrac{1}{2}-iz)$$

when $h-\frac{1}{2}$ is positive.

Contiguity applies also to associated Hilbert spaces of entire functions. Assume that a space $\mathcal{H}(E)$ is contained isometrically in the weighted Hardy space $\mathcal{F}(W)$ and that the space contains an entire function whenever its product with a polynomial belongs to the space. The defining function $E(z)$ of the space is chosen to satisfy the symmetry condition

$$E^*(z)=E(-z)$$

since the symmetry condition

$$W^*(z)=W(-z)$$

is satisfied. The function $E(z)$ is uniquely determined by the requirement that $E(z)$ has value one at the origin. Contiguous spaces $\mathcal{H}(E_-)$ contained isometrically in the weighted Hardy space $\mathcal{F}(W_-)$ and $\mathcal{H}(E_+)$ contained isometrically in the weighted Hardy space $\mathcal{F}(W_+)$ are constructed with analogous properties.

Since multiplication by

$$h-\tfrac{1}{2}-iz$$

is an isometric transformation of the space $\mathcal{F}(W_-)$ onto the space $\mathcal{F}(W_+)$, the spaces $\mathcal{H}(E_-)$ and $\mathcal{H}(E_+)$ are chosen so that the multiplication is an isometric transformation of the space $\mathcal{H}(E_-)$ onto the set of elements of the space $\mathcal{H}(E_+)$ which vanish at $\frac{1}{2}i - ih$. The adjoint transformation of the space $\mathcal{H}(E_+)$ into the space $\mathcal{H}(E_-)$ takes a function $F(z)$ of z into the function

$$[F(z) - L_-(z)F(\tfrac{1}{2}i - ih)]/(h - \tfrac{1}{2} - iz)$$

of z with

$$L_-(z) = K_+(\tfrac{1}{2}i - ih, z)/K_+(\tfrac{1}{2}i - ih, \tfrac{1}{2}i - ih)$$

the constant multiple of the reproducing kernel function for function values at $\frac{1}{2}i - ih$ in the space $\mathcal{H}(E_+)$ which has value one at $\frac{1}{2}i - ih$. The equations

$$\lambda A(z) = [A_+(z) - L_-(z)A_+(\tfrac{1}{2}i - ih)]/(h - \tfrac{1}{2} - iz)$$

and

$$\lambda^{-1} B(z) = [B_+(z) - L_-(z)B_+(\tfrac{1}{2}i - ih)]/(h - \tfrac{1}{2} - iz)$$

apply with

$$\lambda = [1 - L_-(0)A_+(\tfrac{1}{2}i - ih)]/(h - \tfrac{1}{2})$$

a nonzero real number.

A contractive transformation of the space $\mathcal{F}(W_-)$ into the space $\mathcal{F}(W)$ and of the space $\mathcal{F}(W)$ into the space $\mathcal{F}(W_+)$ is defined by taking a function $F(z)$ of z into the function $F(z + \frac{1}{2}i)$ of z. The adjoint transformation of the space $\mathcal{F}(W)$ into the space $\mathcal{F}(W_-)$ takes a function $F(z)$ of z into the function

$$F(z + \tfrac{1}{2}i)/(h - \tfrac{1}{2} - iz)$$

of z. The adjoint transformation of the space $\mathcal{F}(W_+)$ into the space $\mathcal{F}(W)$ takes a function $F(z)$ of z into the function

$$F(z + \tfrac{1}{2}i)/(h - iz)$$

of z.

The spaces $\mathcal{H}(E_-)$ and $\mathcal{H}(E_+)$ are constructed so that a contractive transformation of the space $\mathcal{H}(E_-)$ into the space $\mathcal{H}(E)$ and of the space $\mathcal{H}(E)$ into the space $\mathcal{H}(E_+)$ is defined by taking a function $F(z)$ of z into the function $F(z + \frac{1}{2}i)$ of z and so that the adjoint transformation of the space $\mathcal{H}(E)$ into the space $\mathcal{H}(E_-)$ takes a function $F(z)$ of z into the function

$$[F(z + \tfrac{1}{2}i) - L_-(z)F(i - ih)]/(h - \tfrac{1}{2} - iz)$$

of z.

The transformation of the space $\mathcal{H}(E_-)$ into the space $\mathcal{H}(E)$ takes the function

$$[B_-(z)A_-(w)^- - A_-(z)B_-(w)^-]/[\pi(z - w^-)]$$

of z into the function

$$[B_-(z+\tfrac{1}{2}i)A_-(w)^- - A_-(z+\tfrac{1}{2}i)B_-(w)^-]/[\pi(z+\tfrac{1}{2}i-w^-)]$$

of z for every complex number w. The adjoint transformation of the space $\mathcal{H}(E)$ into the space $\mathcal{H}(E_-)$ takes the function

$$[B(z)A(w)^- - A(z)B(w)^-]/[\pi(z-w^-)]$$

of z into the function

$$\frac{B(z+\frac{1}{2}i)A(w)^- - A(z+\frac{1}{2}i)B(w)^-}{\pi(z+\frac{1}{2}\ i-w^-)(h-\frac{1}{2}-iz)}$$
$$-L_-(z)\ \frac{B(i-ih)A(w)^- - A(i-ih)B(w)^-}{\pi(i-ih-w^-)(h-\frac{1}{2}-iz)}$$

of z for every complex number w.

The identity

$$[B_-(z+\tfrac{1}{2}i)A_-(w)^- - A_-(z+\tfrac{1}{2}i)B_-(w)^-]/[\pi(z+\tfrac{1}{2}i-w^-)]$$
$$= \frac{B(z)A(w+\frac{1}{2}i)^- - A(z)B(w+\frac{1}{2}i)^-}{\pi(z+\frac{1}{2}i-w^-)(h-\frac{1}{2}+iw^-)}$$
$$-L_-(w)^-\ \frac{B(z)A(i-ih)^- - A(z)B(i-ih)^-}{\pi(z+i-ih)(h-\frac{1}{2}+iw^-)}$$

follows by properties of reproducing kernel functions for all complex numbers z and w.

The identity can be written

$$B_-(z+\tfrac{1}{2}i)A_-(w)^- - A_-(z+\tfrac{1}{2}i)B_-(w)^-$$
$$= B(z)[A(w+\tfrac{1}{2}i)^- - L_-(w)^-A(i-ih)^-]/(h-\tfrac{1}{2}+iw^-)$$
$$-A(z)[B(w+\tfrac{1}{2}i)^- - L_-(w)^-B(i-ih)^-]/(h-\tfrac{1}{2}+iw^-)$$
$$+L_-(w)^-[B(z)A(i-ih)^- - A(z)B(i-ih)^-]/(h-1+iz).$$

Since the functions $A(z)$, $B(z)$, and

$$[B(z)A(i-ih)^- - A(z)B(i-ih)^-]/(h-1+iz)$$

of z are linearly independent, the equations

$$A_-(z+\tfrac{1}{2}i) = A(z)P + B(z)R$$
$$-v_-^-\ [B(z)A(i-ih)^- - A(z)B(i-ih)^-]/(h-1+iz)$$

and

$$B_-(z+\tfrac{1}{2}i) = A(z)Q + B(z)S$$
$$+u_-^- \, [B(z)A(i-ih)^- - A(z)B(i-ih)^-]/(h-1+iz)$$

apply for unique complex numbers u_- and v_- and for a unique matrix

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

with complex entries.

Symmetry implies that u_- and the diagonal entries of the matrix are real and that v_- and the off–diagonal entries of the matrix are imaginary. The equations

$$A_-(z)S + B_-(z)R = [A(z+\tfrac{1}{2}\,i) - L_-(z)A(i-ih)]/(h-\tfrac{1}{2}-iz)$$

and

$$A_-(z)Q + B_-(z)P = [B(z+\tfrac{1}{2}\,i) - L_-(z)B(i-ih)]/(h-\tfrac{1}{2}-iz)$$

are satisfied with

$$L_-(z) = A_-(z)u_- + B_-(z)v_-.$$

Consistency of the equations implies the constraints

$$PS = QR$$

and

$$1 = [PA(ih-i) + RB(ih-i)] \, u_- - [QA(ih-i) + SB(ih-i)] \, v_-.$$

The recurrence relations

$$[A_-(z+i) - L_-(z)A_-(\tfrac{3}{2}i-ih)]/(h-\tfrac{1}{2}-iz)$$
$$+ \, [B_-(z)A_-(\tfrac{3}{2}i-ih)^- - A_-(z)B_-(\tfrac{3}{2}i-ih)]v_-/(\tfrac{3}{2}-h-iz)$$
$$= 2P[A_-(z)S + B_-(z)R]$$

and

$$[B_-(z+i) - L_-(z)B_-(\tfrac{3}{2}i-ih)]/(h-\tfrac{1}{2}-iz)$$
$$+ \, [B_-(z)A_-(\tfrac{3}{2}i-ih)^- - A_-(z)B_-(\tfrac{3}{2}i-ih)]u_-/(\tfrac{3}{2}-h-iz)$$
$$= 2S[A_-(z)Q + B_-(z)P]$$

are satisfied.

The recurrence relations

$$[A(z+i) - L(z)A(i-ih)]/(h-iz)$$
$$+ \, [B(z)A(i-ih)^- - A(z)B(i-ih)^-] \, v/(1-h-iz)$$
$$= 2S[A(z)P + B(z)R]$$

and

$$[B(z+i) - L(z)B(i-ih)]/(h-iz)$$
$$+ [B(z)A(i-ih)^- - A(z)B(i-ih)^-]\ u/(1-h-iz)$$
$$= 2P[A(z)Q + B(z)S]$$

are obtained with

$$L(z) = A(z)u + B(z)v$$

where

$$u = Pu_- + Qv_- + B(ih-i)u_-v_-/(h-1)$$

and

$$v = Ru_- + Sv_- + A(ih-i)u_-v_-/(h-1).$$

The entire function $E(t,z)$ of z depends on a positive parameter t and satisfies the differential equations

$$\frac{\partial}{\partial t}\ B(t,z) = zA(t,z)\alpha'(t)$$

and

$$-\frac{\partial}{\partial t}\ A(t,z) = zB(t,z)\gamma'(t)$$

for differentiable functions $\alpha(t)$ and $\gamma(t)$ of positive t. The coefficients $u(t)$ and $v(t)$ and the entries of the matrix

$$\begin{pmatrix} P(t) & Q(t) \\ R(t) & S(t) \end{pmatrix}$$

are differentiable functions of t. The derivatives $\alpha'(t)$ and $\gamma'(t)$ are negative since the space parametrized by b is contained in the space parametrized by a when a is less than b. The parametrization is made so that

$$-t\tau'(t) = 1.$$

The constant of integration in $\tau(t)$ is chosen so that

$$t = \exp(-\tau(t)).$$

Similar constructions are made with subscripts plus and minus. The differential equations

$$-t\alpha'(t) = -iQ(t)/S(t) \text{ and } -t\alpha'_-(t) = -iQ(t)/P(t)$$

and

$$-t\gamma'(t) = iR(t)/P(t) \text{ and } -t\gamma'_-(t) = iR(t)/S(t)$$

and

$$u'(t) = ihv(t)\alpha'(t)$$

and

$$v'(t) = -ihu(t)\gamma'(t)$$

are obtained on differentiation.

The differential equations

$$Q(t)A(t, ih - i)v_-(t) + R(t)B(t, ih - i)u_-(t)$$
$$= S'(t)/S(t) + \tfrac{1}{2}\ t^{-1} = -P'(t)/P(t) - \tfrac{1}{2}\ t^{-1}$$

and

$$P(t)A(t, ih - i)u_-(t) + S(t)B(t, ih - i)v_-(t)$$
$$= R'(t)/R(t) + \tfrac{1}{2}\ t^{-1} = -Q'(t)/Q(t) - \tfrac{1}{2}\ t^{-1}$$

are obtained where

$$t\ P(t)S(t) = 1 = tQ(t)R(t)$$

and

$$1 = P(t)A(t, ih - i)u_-(t) - S(t)B(t, ih - i)v_-(t)$$
$$-Q(t)A(t, ih - i)v_-(t) + R(t)B(t, ih - i)u_-(t).$$

The equations

$$[R(t)/P(t)]' = L(t, ih - i)\ R(t)/P(t)$$

and

$$[S(t)/Q(t)]' = L(t, ih - i)\ S(t)/Q(t)$$

are satisfied.

2. Fourier Analysis on the Complex Skew–Plane

The Hilbert spaces of entire functions constructed from the gamma function apply to Fourier analysis for the complex plane and for the complex skew–plane. The complex skew–plane is a vector space of dimension four over the real numbers which contains the complex plane as a vector subspace of dimension two. The multiplicative structure of the complex plane as a field is generalized as the multiplicative structure of the complex skew–plane as a skew–field. The conjugation of the complex plane is an automorphism which extends as an anti–automorphism of the complex skew–plane.

An element

$$\xi = t + ix + jy + kz$$

of the complex skew–plane has four real coordinates x, y, z, and t. The conjugate element is

$$\xi^- = t - ix - jy - kz.$$

The multiplication table

$$ij = k, jk = i, ki = j$$
$$ji = -k, kj = -i, ki = -j$$

defines a conjugated algebra in which every nonzero element has an inverse. An automorphism of the skew–field is an inner automorphism which is defined by an element with conjugate as inverse. A plane is a maximal commutative subalgebra. Every plane is isomorphic to every other plane under an automorphism of the skew–field. The complex plane is the subalgebra of elements which commute with i.

The topology of the complex skew–plane is derived from the topology of the real line as is the topology of the complex plane. Addition and multiplication are continuous as transformations of the Cartesian product of the space with itself into the space. The topology of the real line is derived from Dedekind cuts. A real number t divides the real line into two open half–lines (t, ∞) and $(-\infty, t)$. The intersection of open half–lines is an open interval (a, b) when it is nonempty and not a half–line. A open subset of the line is a union of open intervals. The topology of the plane is the Cartesian product topology of two Dedekind topologies of two coordinate lines. The topology of the complex skew–plane is the Cartesian product topology of the Dedekind topologies of four coordinate lines.

The canonical measure for the complex skew–plane is derived from the canonical measure for the real line as is the canonical measure for the complex plane. In all cases the canonical measure is defined on the Baire subsets of the space defined as the smallest class of sets containing the open sets and the closed sets and containing countable unions and countable intersections of sets of the class. A measure preserving transformation of the space onto itself is defined by taking ξ into $\xi + \eta$ for every element η of the space. This condition determines the canonical measure within a constant factor.

Multiplication by an element ξ of the space multiplies the canonical measure by $|\xi|$ in the case of the line, by $|\xi|^2$ in the case of the plane, and by $|\xi|^4$ in the case of the skew–plane with $|\xi|$ the nonnegative solution of the equation

$$|\xi|^2 = \xi^- \xi.$$

The modulus $|\xi|$ of ξ defines a metric on the space whose topology is identical with the Dedekind topology. The identity

$$|\xi\eta| = |\xi||\eta|$$

holds for all elements ξ and η of the space.

The canonical measure for the real line is Lebesgue measure, whose normalization is made with respect to the integral elements of the line. The additive group of integral elements inherits a discrete topology. The quotient group is compact. Elements ξ and η of the line are defined as equivalent with respect to the subgroup if their difference $\eta - \xi$ belongs to the subgroup. A fundamental domain for the equivalence relation consists of the elements which are closer to the origin than to any other integral element. Every element of the line is equivalent to an element of the closure of the fundamental domain. Elements of the fundamental domain are equal if they are equivalent. The fundamental domain is the open interval $(-\frac{1}{2}, \frac{1}{2})$. The closure $[-\frac{1}{2}, \frac{1}{2}]$ is compact and has measure one.

The ring of integral elements of the real line has significant multiplicative properties. An ideal of the ring of integral elements is defined as an additive subgroup such that for every element γ of the ring $\beta\gamma$ belongs to the subgroup whenever β belongs to the subgroup. A computation of ideals is made possible by the Euclidean algorithm. If α is an element of the ring and if β is a nonzero element of the ring, then an element γ of the ring exists which satisfies the inequality

$$|\alpha - \beta\gamma| < |\beta|.$$

An ideal of the ring which contains a nonzero element contains a nonzero element β which minimizes the positive integer $|\beta^-\beta|$. Every element α of the ideal is a product

$$\alpha = \beta\gamma$$

with an element γ of the ring.

In a definition due to Gauss the integral elements of the complex plane are the elements whose coordinates are integral elements of the line. An Euclidean algorithm applies to the ring of integral elements of the complex plane. If α is an integral element and if β is a nonzero integral element, then an integral element γ exists which satisfies the inequality

$$|\alpha - \beta\gamma| < |\beta|.$$

An ideal of the ring of integral elements which contains a nonzero element contains an element β which minimizes the positive integer $\beta^-\beta$. Every element α of the ideal is a product

$$\alpha = \beta\gamma$$

with an element γ of the ring.

An application of the Euclidean algorithm for the plane is the representation of a positive integer as the sum of four squares

$$a^2 + b^2 = (a + ib)^-(a + ib)$$

for integers a and b. The representation problem reduces to one for prime numbers by the multiplicative structure of the complex plane. The even prime

$$2 = 1 + 1$$

is a sum of two squares. A prime which is congruent to three modulo four is not a sum of two square since the representation is not possible modulo four. A theorem attributed to Diophantus and proved by Lagrange states that a prime

$$p = a^2 + b^2$$

is a sum of two squares of integers a and b if it is congruent to one modulo four.

In a definition due to Hurwitz the integral elements of the complex skew–plane are the elements whose coordinates are either all integral elements of the line or all half–integral elements of the line. An Euclidean algorithm applies to the ring of integral elements of the complex skew–plane. If α is an integral element and if β is a nonzero integral element, then an integral element γ exists which satisfies the inequality

$$|\alpha - \beta\gamma| < |\beta|.$$

An additive subgroup of the ring of integral elements is said to be a right ideal if it contains the product $\beta\gamma$ of every element β with an integral element γ of the ring. A right ideal of the ring which contains a nonzero element contains a nonzero element β which minimizes the positive integer $\beta^-\beta$. Every element α of the ideal is a product

$$\alpha = \beta\gamma$$

with an element γ of the ring.

An application of the Euclidean algorithm for the skew–plane is the representation of a positive integer as a sum of four squares:

$$a^2 + b^2 + c^2 + d^2 = (d + ia + jb + kc)^-(d + ia + jb + kc).$$

The representation problem reduces to one for prime numbers by the multiplicative structure of the complex skew–plane.

The additive subgroup of integral elements of the complex plane inherits a discrete topology and generates a compact quotient group. Elements ξ and η of the complex plane are defined as equivalent if the difference $\eta - \xi$ belongs to the subgroup. A fundamental domain for the equivalence relation consists of the elements which are closer to the origin than to any other integral element. Every element of the complex plane is equivalent to an element of the fundamental domain. Elements of the fundamental domain are equal if they are equivalent. The canonical measure for the complex plane is normalized so that the fundamental domain has measure one. The canonical measure is the Cartesian product measure of the Lebesgue measures for two coordinate lines.

The additive subgroup of elements of the complex skew–plane with integral coordinates inherits a discrete topology and generates a compact quotient space. Elements ξ and η of the complex skew–plane are defined as equivalent if their difference $\eta - \xi$ belongs to the subgroup. A fundamental domain for the equivalence relation consists of the elements which are closer to the origin than to any other such element. Every element of the complex skew–plane is equivalent to an element of the closure of the fundamental domain. Elements of the fundamental domain are equal if they are equivalent. The canonical measure for the complex plane is normalized so that the fundamental domain has measure one. The canonical measure is two times the Cartesian product measure of the Lebesgue measures of four coordinate lines.

The Fourier transformation for the real line is an isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure into the same space. The Fourier transform of an integrable function $f(\xi)$ of real ξ is the continuous function

$$g(\xi) = \int \exp(2\pi i\xi\eta) f(\eta) d\eta$$

of ξ is defined by integration with respect to the canonical measure. Fourier inversion

$$f(\xi) = \int \exp(-2\pi i\xi\eta) g(\eta) d\eta$$

applies with integration with respect to the canonical measure when the function $g(\xi)$ of ξ is integrable and the function $f(\xi)$ of ξ is continuous.

The Fourier transformation for the complex plane is an isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure into the same space. The Fourier transformation of an integrable function $f(\xi)$ of ξ is the continuous function

$$g(\xi) = \int \exp(\pi i(\xi^-\eta + \eta^-\xi))f(\eta)d\eta$$

of ξ defined by integration with respect to the canonical measure. Fourier inversion

$$f(\xi) = \int \exp(-\pi i(\xi^-\eta + \eta^-\xi))g(\eta)d\eta$$

applies with integration with respect to the canonical measure when the function $g(\xi)$ of ξ is integrable and the function $f(\xi)$ of ξ is continuous.

The Fourier transformation for the complex skew–plane is an isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure onto the same space. The Fourier transform of an integrable function $f(\xi)$ of ξ is the continuous function

$$g(\xi) = \int \exp(2\pi i(\xi^-\eta + \eta^-\xi))f(\eta)d\eta$$

of ξ which is defined by integration with respect to the canonical measure. Fourier inversion

$$f(\xi) = \int \exp(-2\pi i(\xi^-\eta + \eta^-\xi))g(\eta)d\eta$$

applies with integration with respect to the canonical measure when the function $g(\xi)$ of ξ is integrable and the function $f(\xi)$ of ξ is continuous.

The Fourier transformation commutes with the isometric transformations of the Hilbert space onto itself which are defined by taking a function $f(\xi)$ of ξ into the functions $f(\omega\xi)$ and $f(\xi\omega)$ of ξ for every element ω of the complex skew–plane with conjugate as inverse. The Hilbert space decomposes into the orthogonal sum of invariant subspaces for the transformations taking a function $f(\xi)$ of ξ into the function $f(\omega\xi)$ of ξ for every element ω of the complex skew–plane with conjugate as inverse. An invariant subspace is defined for every integer μ as the set of functions $f(\xi)$ of ξ which satisfy the equation

$$f(\xi\omega) = \omega^\mu f(\xi)$$

for every element ω of the complex plane with conjugate as inverse. Such functions are said to have spin μ.

A homomorphism of the multiplicative group of nonzero elements of the complex skew–plane onto the multiplicative group of the positive half–line is defined by taking ξ into $\xi^-\xi$. The identity

$$\int |f(\xi^-\xi)|^2 d\xi = 4\pi \int |f(\xi)|^2 \xi d\xi$$

holds with integration on the left with respect to the canonical measure for the complex skew–plane and with integration on the right with respect to Lebesgue measure for every Baire function $f(\xi)$ of ξ in the positive half–line.

A computation of invariant subspaces is made in Hilbert spaces of homogeneous polynomials defined on the complex skew–plane. A homogeneous polynomial of degree ν is a function $f(\xi)$ of

$$\xi = t + ix + jy + kz$$

of ξ in the complex skew–plane which is a linear combination of monomials

$$x^a y^b z^c t^d$$

whose exponents are nonnegative integers with sum ν. The functions $f(\omega\xi)$ and $f(\xi\omega)$ of ξ in the complex skew–plane are homogeneous polynomials of degree ν for every element ω of the complex skew–plane with conjugate as inverse if the function $f(\xi)$ of ξ in the complex skew–plane is a homogeneous polynomial of degree ν. The set of homogeneous polynomials of degree ν is a Hilbert space of dimension

$$(1 + \nu)(2 + \nu)(3 + \nu)/6$$

with scalar product defined so that the monomials form an orthogonal set and so that

$$\frac{a!b!c!d!}{\nu!}$$

is the scalar self–product of the monomial with exponents a, b, c, and d. Isometric transformations of the space onto itself are defined by taking a function $f(\xi)$ of ξ into the functions $f(\omega\xi)$ and $f(\xi\omega)$ for every element ω of the complex skew–plane with conjugate as inverse.

The set of elements of the complex skew–plane with conjugate as inverse is a compact group which is the kernel of the homomorphism of the multiplicative group of the complex skew–plane onto the positive half–line. The canonical measure for the complex skew–plane, as it acts on the multiplicative group, is the Cartesian product measure of an invariant measure on the compact subgroup and the measure on the positive half–line whose value on a Baire set is the integral

$$\int t dt$$

with respect to Lebesgue measure over the set. Measure preserving transformations of the compact group onto itself are defined by taking ξ into $\omega\xi$ and into $\xi\omega$ for every element ω of the group. The integral

$$\int |f(\xi)|^2 d\xi = 8\pi \|f\|^2$$

with respect to the invariant measure computes the scalar self–product of a function $f(\xi)$ of ξ in the complex skew–plane which is a homogeneous polynomial of degree ν.

The Laplacian

$$\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is applied in the decomposition of the Hilbert space of homogeneous polynomials of degree ν into invariant subspaces. The Laplacian annihilates homogeneous polynomials of degree less than two and takes homogeneous polynomials of greater degree ν into homogeneous polynomials of degree $\nu - 2$. The Laplacian commutes with the transformations which take a function $f(\xi)$ of ξ into the functions $f(\omega\xi)$ and $f(\xi\omega)$ of ξ for every element ω of the complex skew–plane with conjugate as inverse. A homogeneous polynomial of degree ν is said to be harmonic if it is annihilated by the Laplacian. Homogeneous polynomials of degree less than two are harmonic. Homogeneous polynomials of degree ν greater than two are harmonic if, and only if, they are orthogonal to products of $\xi^-\xi$ with homogeneous polynomials of degree $\nu - 2$. The dimension of the space of homogeneous harmonic polynomials of degree ν is

$$(1 + \nu)^2.$$

The space of homogeneous harmonic polynomials of degree ν is the orthogonal sum of subspaces whose elements are of spin μ for integers μ of the same parity as ν such that

$$-\nu \leq \mu \leq \nu.$$

The space of homogeneous harmonic polynomials of order ν and spin μ admits an orthogonal basis of elements which satisfy the identity

$$f(\omega\xi) = \omega^n f(\xi)$$

for every element ω of the complex plane with conjugate as inverse with n an integers having the same parity as ν such that

$$-\nu \leq n \leq \nu.$$

The dimension of the space of homogeneous harmonic polynomials of order ν and spin μ is

$$1 + \nu.$$

If a nontrivial function $f(\xi)$ of ξ in the complex skew–plane is a homogeneous harmonic polynomial of degree ν and spin μ, then the function $f(\omega\xi)$ of ξ in the complex skew–plane is a homogeneous harmonic polynomial of degree ν and spin μ for every element ω of the complex skew–plane with conjugate as inverse. Every homogeneous harmonic polynomial of degree ν and spin μ is a finite linear combination of the functions $f(\omega\xi)$ of ξ in the complex skew–plane for elements ω of the complex skew–plane with conjugate as inverse.

If a nontrivial function $f(\xi)$ of ξ in the complex skew–plane is a homogeneous harmonic polynomial of degree ν and spin μ, then the function $f(\omega\xi)$ of ξ in the complex skew–plane is a homogeneous harmonic polynomial of degree ν and spin μ for every element ω of the complex skew–plane with conjugate as inverse. Every homogeneous harmonic polynomial of degree ν and spin μ is a finite linear combination of the functions $f(\omega\xi)$ of ξ in the complex skew–plane for elements ω of the complex skew–plane with conjugate as inverse.

The complementary space to the complex plane in the complex skew–plane is the set of elements η of the complex skew–plane which satisfy the identity

$$\xi\eta = \eta\xi^-$$

for every element ξ of the complex plane. An element η of the complex skew–plane is skew–conjugate:

$$\eta^- = -\eta.$$

Multiplication on left or right by η is an injective transformation of the complex plane onto the complementary space for every nonzero element η of the complementary space. The transformation is continuous when the complementary space is given the topology inherited from the complex skew–plane. The transformation takes the canonical measure for the complex plane into $\frac{1}{2}\eta^-\eta$ times the measure defined as the canonical measure for the complementary space.

An element of the complex skew–plane is the unique sum $\alpha + \beta$ of an element α of the complex plane and an element β of the complementary space. The topology of the complex skew–plane is the Cartesian product topology of the topology of the complex plane and the topology of the complementary space. The canonical measure for the complex skew–plane is the Cartesian product measure of the canonical measure for the complex plane and the canonical measure for the complex skew–plane.

The Radon transformation for the complex skew–plane is a transformation with closed graph whose domain and range are contained in the Hilbert space of functions which are square integrable with respect to the canonical measure for the complex skew–plane and whose graph contains the pairs $(f(\omega\xi), g(\omega\xi))$ and $(f(\xi\omega), g(\xi\omega))$ of functions of ξ whenever it contains the pair $(f(\xi), g(\xi))$ of functions of ξ for every element ω of the complex skew–plane with conjugate as inverse. The transformation is defined as an integral on elements of its domain which are integrable with respect to the canonical measure.

The Radon transform of an integrable function $f(\xi)$ of ξ in the complex skew–plane is a function $g(\xi)$ of ξ in the complex skew–plane which satisfies the identity

$$g(\omega\xi) = \int f(\omega\xi + \omega\eta)d\eta$$

for every element ω of the complex skew–plane with conjugate as inverse, for almost all elements ξ of the complex plane, with integration with respect to the canonical measure for the complementary space to the complex plane in the complex skew–plane. The inequality

$$\int |g(\omega\xi)|d\xi \leq \int |f(\xi)|d\xi$$

holds for every element ω of the complex skew–plane with conjugate as inverse with integration on the left with respect to the canonical measure for the complex plane and integration on the right with respect to the canonical measure for the complex skew–plane.

The Radon transformation for the complex skew–plane factors the Fourier transformation for the complex skew–plane as a composition with the Fourier transformation for the complex plane. If the Radon transformation takes a function $f(\xi)$ of ξ into a function $g(\xi)$ of ξ and if the function $f(\xi)$ of ξ is integrable with respect to the canonical measure for the complex skew–plane, then the function $g(\omega\xi)$ of ξ is integrable with respect to the canonical measure for the complex plane for every element ω of the complex skew–plane with conjugate as inverse. The Fourier transform of the function $\frac{1}{4}g(\frac{1}{2}\omega\xi)$ of ξ in the complex plane is the restriction to the complex plane of the Fourier transform of the function $f(\omega\xi)$ of ξ in the complex skew–plane.

Spectral analysis of the Radon transformation is made by Laplace transformations. A Laplace transformation of harmonic ϕ is defined for every homogeneous harmonic polynomial $\phi(\xi)$ of degree ν such that the normalization

$$\int |\phi(\xi)|^2 d\xi = \int (\xi^- \xi)^\nu d\xi$$

applies with integration with respect to the canonical measure for the complex skew–plane over the unit disk $\xi^-\xi < 1$ is equal to one. The function

$$\phi(\xi) \exp(2\pi i z \xi^- \xi)$$

of ξ in the complex skew–plane is an eigenfunction of the Radon transformation for the eigenvalue

$$i/z$$

when z is in the upper half–plane.

The domain of the Laplace transformation of harmonic ϕ is the Hilbert space of functions $f(\xi)$ of ξ in the complex skew–plane which are square integrable with respect to the canonical measure and which satisfy the identity

$$\phi(\xi) f(\omega\xi) = \phi(\omega\xi) f(\xi)$$

for every element ω of the complex skew–plane with conjugate as inverse. The Laplace transform of the function $f(\xi)$ of ξ is the analytic function

$$F(z) = \int \phi(\xi)^- f(\xi) \exp(2\pi i z \xi^- \xi) d\xi$$

of z in the upper half–plane defined by integration with respect to the canonical measure.

A function

$$f(\xi) = \phi(\xi) h(\xi^- \xi)$$

of ξ which is in the domain of the Laplace transformation of harmonic ϕ is parametrized by a Baire function $h(t)$ of t in the positive half–line which satisfies the identity

$$\int |f(\xi)|^2 d\xi = 2\pi \int_0^\infty |h(t)|^2 t^{1+\nu} dt$$

with integration on the left with respect to the canonical measure for the complex skew–plane. Since

$$F(z) = 2\pi \int_0^\infty h(t) \exp(2\pi izt) t^{1+\nu} dt,$$

the identity

$$\int_{-\infty}^{+\infty} |F(x+iy)|^2 dx = 64\pi^2 \int_0^\infty |h(t)|^2 \exp(-4\pi yt) t^{2+2\nu} dt$$
$$= 8\pi \int |f(\xi)|^2 \exp(-4\pi y \xi^- \xi)(\xi^- \xi)^{1+\nu} d\xi$$

holds with integration with respect to the canonical measure for the complex skew–plane when y is positive. An analytic function $F(z)$ of z in the upper half–plane is a Laplace transform of harmonic ϕ if, and only if, the integral

$$\int_0^\infty \int_{-\infty}^{+\infty} |F(x+iy)|^2 y^\nu dx dy = 2(4\pi)^{-\nu} \Gamma(1+\nu) \int |f(\xi)|^2 d\xi$$

on the left converges. The identity then holds with integration on the right with respect to the canonical measure for the complex skew–plane.

The Laplace transformation of harmonic ϕ computes the adjoint of the Radon transformation on the Hilbert space which is the domain of the Laplace transformation. The Radon transformation acts as a maximal dissipative transformation on the space. The adjoint takes a function $f(\xi)$ of ξ into a function $g(\xi)$ of ξ when the identity

$$\int \phi(\xi)^- g(\xi) \exp(2\pi iz\xi^- \xi) d\xi = (i/z) \int \phi(\xi)^- f(\xi) \exp(2\pi iz\xi^- \xi) d\xi$$

holds when z is in the upper half–plane with integration with respect to the canonical measure for the complex skew–plane.

The Fourier transform for the complex skew–plane of the function

$$\phi(\xi) \exp(2\pi iz\xi^- \xi)$$

of ξ in the complex skew–plane is the function

$$i^\nu (i/z)^{2+\nu} \phi(\xi) \exp(-2\pi iz^{-1} \xi^- \xi)$$

of ξ in the complex skew–plane when z is in the upper half–plane. Since the Fourier transformation commutes with the transformations which take a function $f(\xi)$ of ξ into the functions $f(\omega\xi)$ and $f(\xi\omega)$ of ξ for every element ω of the complex skew–plane with conjugate as inverse, it is sufficient to make the verification when

$$\phi(t + ix + jy + kz) = (t + ix)^\nu.$$

The Radon transformation for the complex skew–plane reduces the verification to showing that the Fourier transform for the complex plane of the function

$$\xi^\nu \exp(\pi i z \xi^- \xi)$$

of ξ in the complex plane is the function

$$i^\nu (i/z)^{1+\nu} \xi^\nu \exp(-\pi i z^{-1} \xi^- \xi)$$

of ξ in the complex plane. It is sufficient by analytic continuation to make the verification when z lies on the imaginary axis. It remains by a change of variable to show that the Fourier transform of the function

$$\xi^\nu \exp(-\pi \xi^- \xi)$$

of ξ in the complex plane is the function

$$i^\nu \xi^{-\nu} \exp(-\pi \xi^- \xi)$$

of ξ in the complex plane.

The desired identity

$$i^\nu \xi^\nu \exp(-\pi \xi^- \xi) = \sum_{k=0}^\infty \xi^\nu \int \frac{(\pi i \xi^- \xi)^k (\pi i \eta^- \eta)^{\nu+k}}{k!(\nu+k)!} \exp(-\pi \eta^- \eta) d\eta$$

follows since

$$\exp(\pi i(\xi^- \eta + \eta^- \xi)) = \sum_{n=0}^\infty \frac{(\pi i \xi^- \eta + \pi i \eta^- \xi)^n}{n!}$$

and since

$$(\pi i \xi^- \eta + \pi i \eta^- \xi)^n = \sum_{k=0}^\infty \frac{(\pi i \eta^- \xi)^{n-2k} (\pi i \xi^- \xi)^k (\pi i \eta^- \eta)^k}{k!(n-k)!}$$

where

$$\int (\pi i \eta^- \eta)^{\nu+k} \exp(-\pi \eta^- \eta) d\eta = i^{\nu+k} (\nu+k)!$$

and

$$i^\nu \exp(-\pi \xi^- \xi) = \sum_{k=0}^\infty i^{\nu+k} \frac{(\pi i \xi^- \xi)^k}{k!}$$

Integrations are with respect to the canonical measure for the complex plane. Interchanges of summation and integration are justified by absolute convergence.

If a function $f(\xi)$ of ξ in the complex skew–plane is square integrable with respect to the canonical measure and satisfies the identity

$$\phi(\xi) f(\omega \xi) = \phi(\omega \xi) f(\xi)$$

for every element ω of the complex skew–plane with conjugate as inverse, then its Fourier transform is a function $g(\xi)$ of ξ in the complex skew–plane which is square integrable with respect to the canonical measure and which satisfies the identity

$$\phi(\xi)g(\omega\xi) = \phi(\omega\xi)g(\xi)$$

for every element ω of the complex skew–plane with conjugate as inverse. The Laplace transforms of harmonic ϕ are functions $F(z)$ and $G(z)$ of z in the upper half–plane which satisfy the identity

$$G(z) = i^\nu (i/z)^{2+\nu} F(-1/z).$$

3. Fourier Analysis on an r–adic Skew–Plane

An r–adic skew–plane is defined for every positive integer r. An algebraic element is an element

$$t + ix + jy + kz$$

of the complex skew–plane whose coordinates are rational numbers. The algebraic elements form a skew–field with addition and multiplication inherited from the complex skew–plane. The conjugate of an algebraic element is an algebraic element. An integral element is an element whose coordinates are all integers or all halves of odd integers. The conjugated ring of integral elements has the skew–field of algebraic elements as a ring of quotients. The algebraic elements of the complex skew–plane are accepted as algebraic elements of the r–adic skew–plane. The r–adic skew–plane is the completion of the set of algebraic elements in a topology made uniform by the continuity of addition.

The quotient space of the ring of integral elements of the r–adic skew–plane modulo the left ideal generated by a nonzero element γ such that the prime divisors of $\gamma^-\gamma$ are divisors of r contains $(\gamma^-\gamma)^2$ elements. The r–adic topology of the ring of integral elements is defined as the least topology with respect to which the projection into every such quotient space is continuous when the quotient space is given the discrete topology. Addition and multiplication are continuous as transformations of the Cartesian product of the ring with itself into the ring. The completion of the ring is a compact Hausdorff space to which addition and multiplication extend as continuous transformations of the Cartesian product of the completion with itself into the completion. The completion is a conjugated ring which is taken as the definition of the ring of integral elements of the r–adic skew–plane.

The canonical measure for the ring of integral elements is defined as the unique non-negative measure on its Baire subsets such that a measure preserving transformation is defined by taking ξ into $\xi + \eta$ for every element η of the ring and such that the ring has measure one when r is odd and one–half when r is even. If γ is a nonzero algebraic element of the ring such that the prime divisors of $\gamma^-\gamma$ are divisors of r, then the projection of the ring onto the quotient space modulo the left ideal generated by γ takes the canonical measure into the measure whose value on a set is the number of elements of the set divided by $(\gamma^-\gamma)^2$. Multiplication by γ multiplies the canonical measure by a factor of

$$1/(\gamma^-\gamma)^2.$$

The r–adic skew–plane is defined as the ring of quotients of the ring of integral elements with denominators which are such algebraic elements γ. The r–adic topology of the r–adic skew–plane is defined so that the subspace topology of the ring of integral elements coincides with its r–adic topology and so that a continuous transformation of the r–adic skew–plane into itself is defined by taking ξ into $\xi + \eta$ for every element η of the r–adic skew–plane. The canonical measure of the r–adic skew–plane is the unique nonnegative measure on its Baire subsets which agrees with the canonical measure of the ring of integral elements on subsets of the ring and which defines a measure preserving transformation by taking ξ into $\xi + \eta$ for every element η of the r–adic skew–plane.

Conjugation is a continuous transformation of the r–adic skew–plane into itself which is also measure preserving. The p–adic line is a commutative subring whose elements are the self–conjugate elements of the r–adic skew–plane. An element

$$\xi = t + ix + jy + kz$$

of the r–adic skew–plane has coordinates in the r–adic line. The r–adic line is given the subspace topology inherited from the r–adic skew–plane. The r–adic topology of the r–adic skew–plane is the Cartesian product topology of four r–adic lines.

The canonical measure for the r–adic line is the unique nonnegative measure on its Baire subsets such that a measure preserving transformation is defined by taking ξ into $\xi + \eta$ for every element η of the r–adic line and such that the set of integral elements has measure one.

The r–adic line is the field of p–adic numbers when $r = p$ is a prime. The r–adic line is isomorphic to the Cartesian product of p–adic lines taken over the prime divisors p of r. The r–adic topology of the r–adic line agrees with the Cartesian product topology of p–adic topologies of p–adic lines taken over the prime divisors p of r. The canonical measure for the r–adic line is the Cartesian product measure of the canonical measures of p–adic lines taken over the prime divisors p of r.

An integral element of the r–adic line is said to be a unit if it is the inverse of an integral element. An invertible element of the r–adic line is the product of an invertible algebraic element γ and a unit. The choice of γ can be made so that the positive rational number $\gamma^- \gamma$ is a ratio of positive integers whose prime divisors are divisors of r. Multiplication by a unit ξ taking ξ into $\xi\omega$ is a measure preserving transformation of the r–adic skew–plane onto itself. Multiplication by γ taking ξ into $\xi\gamma$ multiplies the canonical measure by a factor of $1/(\gamma^- \gamma)^{\frac{1}{2}}$.

A continuous homomorphism of the group of invertible elements of the r–adic skew–plane onto the group of invertible elements of the r–adic line is defined by taking ξ into $\xi^- \xi$. The homomorphism takes the canonical measure for the r–adic skew–plane into a measure which is absolutely continuous with respect to the canonical measure for the r–adic line.

The homomorphism takes the group of units of the r–adic skew–plane onto the group of units of the r–adic line. The measure of the group of units of the r–adic skew–plane is equal to the product

$$\prod (1 - p^{-2})$$

taken over the prime divisors p of r when r is odd and one–half the product when r is even. The measure of the group of units of the r–adic line is equal to the product

$$\prod (1 - p^{-1})$$

taken over the prime divisors p of r. The quotient is equal to the product

$$\prod (1 + p^{-1})$$

taken over the prime divisors p of r when r is odd and one–half the product when r is even.

The image of the canonical measure for the r–adic skew–plane is the measure which assigns to a Baire subset of the line the integral

$$\prod (1 + p^{-1}) \int \lambda_r(\xi) d\xi$$

over the set with respect to the canonical measure of a continuous function $\lambda_r(\xi)$ of ξ in the r–adic line. The function acts as a homomorphism of the group of invertible elements of the r–adic line onto the subgroup of its rational elements whose numerator and denominator have divisors of r as their prime divisors. The homomorphism takes ξ into

$$\lambda_r(\xi) = 1/\xi$$

for every rational number ξ whose numerator and denominator have divisors of r as their prime divisors. The function vanishes at noninvertible elements of the r–adic line.

The function $\exp(2\pi i \xi)$ of rational numbers ξ admits a unique extension as a continuous function of ξ in the r–adic line. The function acts as a homomorphism of the additive group of the r–adic line into the multiplicative group of complex numbers of absolute value one.

The Fourier transformation for the r–adic line is the unique isometric transformation of the Hilbert space of functions which are square integrable with respect to the canonical measure for the r–adic line onto itself which takes an integrable function $f(\xi)$ of ξ into the continuous function

$$g(\xi) = \int \exp(2\pi i \xi \eta) f(\eta) d\eta$$

of ξ defined by integration with respect to the canonical measure for the r–adic line. Fourier inversion states that the integral

$$f(\xi) = \int \exp(-2\pi i \xi \eta) g(\eta) d\eta$$

with respect to the canonical measure for the r–adic line represents the function $f(\xi)$ of ξ if it is continuous and if the function $g(\xi)$ of ξ is integrable.

The Fourier transformation for the r–adic skew–plane is the unique isometric transformation of the Hilbert space of functions which are square integrable with respect to the canonical measure for the r–adic skew–plane onto itself which takes an integrable function $f(\xi)$ of ξ into the continuous function

$$g(\xi) = \int \exp(2\pi i(\xi^- \eta + \eta^- \xi)) f(\eta) d\eta$$

of ξ defined by integration with respect to the canonical measure for the r–adic skew–plane. Fourier inversion states that the integral

$$f(\xi) = \int \exp(-2\pi i(\xi^- \eta + \eta^- \xi)) g(\eta) d\eta$$

with respect to the canonical measure for the r–adic skew–plane represents the function $f(\xi)$ of ξ if it is continuous and if the function $g(\xi)$ of ξ is integrable.

An r–adic plane is a maximal commutative subring of the r–adic skew–plane which decomposes as a Cartesian product of p–adic planes taken over the prime divisors p of r. A p–adic plane is determined by an integral element ι_p of the complex skew–plane which represents

$$p = \iota_p^- \iota_p.$$

The elements of the p–adic plane are the elements of the p–adic skew–plane which commute with ι_p. An element

$$\xi = \alpha + \iota_p \beta$$

of the p–adic plane has coordinates α and β in the p–adic line. The conjugate

$$\xi^- = \alpha + \iota_p^- \beta$$

of an element ξ of the p–adic plane is an element of the p–adic plane since

$$\iota_p + \iota_p^-$$

is an integer. An element of the p–adic plane is integral if, and only if, its coordinates are integral elements of the r–adic line. An element of the p–adic plane is algebraic if, and only if, its coordinates are rational numbers.

An ideal of the ring of integral elements of the p–adic plane which contains a nonzero element contains ι_p^n for some nonnegative integer n. If n is the least nonnegative integer such that ι_p^n belongs to the ideal, then multiplication by ι_p^n takes the ring of integral elements of the p–adic plane onto the ideal. The quotient ring modulo the ideal contains p^n elements.

The construction of the p–adic plane requires an integral element ι_p of the complex skew–plane which represents

$$p = \iota_p^- \iota_p.$$

Such an element is

$$\iota_p = 1 + i$$

when p is the even prime. The integral algebraic elements of the p–adic plane are then the Gauss integers. The integral element ι_p is otherwise found by applying the Euclidean algorithm.

When p is an odd prime, the elements of the quotient ring of the ring of integral elements of the complex skew–plane modulo the ideal generated by p are represented

$$\xi = t + ix + jy + kz$$

with coordinates in the integers modulo p. The image of ι_p in the quotient ring is a nonzero element ξ such that

$$\xi^- \xi = 0.$$

Such an element can be chosen with

$$z = 0 \text{ and } t = 1.$$

Since there are $(p+1)/2$ squares of integers modulo p, the coordinates x and y can be chosen as integers modulo p such that

$$1 + x^2 = -y^2.$$

A left ideal of the ring of integral elements of the complex skew–plane is defined as the set of elements whose image in the quotient ring belongs to the left ideal generated by ξ. Since the ideal contains a nonzero element, it is generated by a nonzero element ι_p which minimizes the positive integer $\iota_p^- \iota_p$. The element is unique within a left factor of an integral unit of the complex skew–plane.

There are twenty–four integral elements of the complex skew–plane which are inverses of integral elements of the complex skew–plane. The group has a normal subgroup of eight elements. The quotient group is a cyclic group of three elements.

The ring of integral elements of the r–adic plane is a compact Hausdorff space in the subspace topology inherited from the r–adic skew–plane. Addition and multiplication are continuous as transformations of the Cartesian product of the ring with itself into the ring. In the subspace topology inherited from the r–adic skew–plane the r–adic plane is a Hausdorff space which contains the ring of integral elements as an open subset containing the origin. A continuous transformation of the r–adic plane into itself is defined by taking ξ into $\xi + \eta$ for every element η of the r–adic plane.

The canonical measure for the r–adic plane is the unique nonnegative measure on its Baire subsets such that a measure preserving transformation of the space into itself is defined by taking ξ into $\xi + \eta$ for every element η of the space and such that the ring of integral elements has measure one. Multiplication by a unit of the r–adic plane is a measure preserving transformation. If γ is an invertible algebraic element of the r–adic plane such that numerator and denominator of the positive rational number $\gamma^- \gamma$ have prime divisors of r as their prime divisors, then multiplication by γ multiplies the canonical measure by a factor of $1/(\gamma^- \gamma)$.

The conjugation of the r–adic skew–plane acts as a continuous isomorphism of the r–adic plane onto itself. The set of self–conjugate elements of the r–adic plane is the r–adic line. If ι_r is the element of the r–adic plane whose p–adic component is ι_p for every prime divisor p of r, then an element

$$\xi = \alpha + \iota_r \beta$$

of the r–adic plane has coordinates α and β in the r–adic line. The topology of the r–adic plane is the Cartesian product topology of the r–adic topologies of two r–adic lines. The canonical measure for the r–adic plane is the Cartesian product measure of the canonical measures of two r–adic lines.

The complementary space to the r–adic plane in the r–adic skew–plane is the set of elements η of the r–adic skew–plane which satisfy the identity

$$\xi\eta = \eta\xi^-$$

for every element ξ of the r–adic plane. An element of the complementary space is skew–conjugate:

$$\eta^- = -\eta.$$

Multiplication on left or right by an invertible element of the complementary space is an injective transformation of the r–adic plane onto the complementary space. The transformation is continuous when the complementary space is given the topology inherited from the r–adic skew–plane. The canonical measure for the complementary space is defined as the image of the canonical measure for the r–adic plane when r is odd, and one–half the image measure when r is even, under multiplication by a unit of the complementary space.

An element of the r–adic skew–plane is the unique sum $\alpha + \beta$ of an element α of the r–adic plane and an element β of the complementary space. The topology of the r–adic skew–plane is the Cartesian product topology of the topology of the r–adic plane and the topology of the complementary space. The canonical measure for the r–adic skew–plane is the Cartesian product measure of the canonical measure for the r–adic plane and the canonical measure for the complementary space.

An element of the r–adic skew–plane is the unique sum $\alpha + \beta$ of an element α of the r–adic plane and an element β of the complementary space. The topology of the r–adic skew–plane is the Cartesian product topology of the topology of the r–adic plane and the topology of the complementary space. The canonical measure for the r–adic skew–plane is the Cartesian product measure of the canonical measure for the r–adic plane and the canonical measure for the complementary space.

The Fourier transformation for the r–adic plane is the unique isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure for the r–adic plane into itself which takes an integrable function $f(\xi)$ of ξ into the continuous function

$$g(\xi) = \int \exp(\pi i(\xi^-\eta + \eta^-\xi)) f(\eta) d\eta$$

of ξ defined by integration with respect to the canonical measure. Fourier inversion

$$f(\xi) = \int \exp(-\pi i(\xi^-\eta + \eta^-\xi)) g(\eta) d\eta$$

applies with integration with respect to the canonical measure when the function $g(\xi)$ of ξ is integrable and the function $f(\xi)$ of ξ is continuous.

The Fourier transformation for the r–adic skew–plane commutes with the transformations which take a function $f(\xi)$ of ξ into the functions $f(\omega\xi)$ and $f(\xi\omega)$ of ξ for every element ω of the r–adic skew–plane with conjugate as inverse. The Radon transformation for the r–adic skew–plane commutes with the same transformations. The graph of the transformation is a closed subspace of the Cartesian product Hilbert space which contains the pairs $(f(\omega\xi), g(\omega\xi))$ and $(f(\xi\omega), g(\xi\omega))$ of functions of ξ whenever it contains the pair $(f(\xi), g(\xi))$ of functions of ξ. The transformation is defined as an integral on those elements of its domain which are integrable with respect to the canonical measure for the r–adic skew–plane.

The Radon transform of an integrable function $f(\xi)$ of ξ in the r–adic skew–plane is a function $g(\xi)$ of ξ in the r–adic skew–plane defined for almost all ξ by the integral

$$g(\omega\xi) = \int f(\omega\xi + \omega\eta) d\eta$$

with respect to the canonical measure for the complementary space to the r–adic plane in the r–adic skew–plane for every element ω of the r–adic skew–plane with conjugate as inverse. The inequality

$$\int |g(\omega\xi)| d\xi \leq \int |f(\xi)| d\xi$$

holds for every element ω of the r–adic skew–plane with conjugate as inverse with integration on the left with respect to the canonical measure for the r–adic plane and with integration on the right with respect to the canonical measure for the r–adic skew–plane.

The Radon transformation for the r–adic skew–plane factors the Fourier transformation for the r–adic skew–plane as a composition with the Fourier transformation for the r–adic plane. If the Radon transformation takes a function $f(\xi)$ of ξ into a function $g(\xi)$ of ξ and if the function $f(\xi)$ of ξ is integrable with respect to the canonical measure for the r–adic skew–plane, then the function $g(\omega\xi)$ of ξ is integrable with respect to the canonical measure for the r–adic plane for every element ω of the r–adic skew–plane with conjugate as inverse. The restriction to the r–adic plane of the Fourier transform of the function $f(\omega\xi)$ of ξ in the r–adic skew–plane is the Fourier transform of the function $g(\frac{1}{2}\omega\xi)$ of ξ in the r–adic plane when r is odd and of the function $4g(\frac{1}{2}\omega\xi)$ of ξ in the r–adic plane when r is even.

Spectral analysis of the Radon transformation for the r–adic skew–plane is made by the Laplace transformation for the r–adic skew–plane. The domain of the Laplace transformation for the r–adic skew–plane is the Hilbert space of functions $f(\xi)$ of ξ in the r–adic skew–plane which are square integrable with respect to the canonical measure and which satisfy the identity

$$f(\omega\xi) = f(\xi)$$

for every element ω of the r–adic skew–plane with conjugate as inverse.

A function

$$f(\xi) = h(\xi^- \xi)$$

of ξ which belongs to the domain of the Laplace transformation for the r–adic skew–plane is parametrized by a Baire function $h(\xi)$ of ξ in the r–adic plane which satisfies the identity

$$h(\omega\xi) = h(\omega^- \xi)$$

for every element ω of the r–adic plane with conjugate as inverse. The identity

$$\int |f(\xi)|^2 d\xi = \int |h(\xi)|^2 d\xi$$

holds when r is odd and the identity

$$\int |f(\xi)|^2 d\xi = \tfrac{1}{2} \int |h(\xi)|^2 d\xi$$

holds when r is even with integration on the left with respect to the canonical measure for the r–adic skew–plane and with integration on the right with respect to the canonical measure for the r–adic plane.

The Laplace transform of a function $f(\xi)$ of ξ in the r–adic skew–plane is the function $F(\xi)$ of ξ in the r–adic plane which is the Fourier transform for the r–adic plane of the function $h(\xi)$ of ξ in the r–adic plane. The identity

$$F(\omega\xi) = F(\omega^- \xi)$$

holds for every element ω of the r–adic plane with conjugate as inverse.

If a function $f(\xi)$ of ξ in the r–adic skew–plane is square integrable with respect to the canonical measure and satisfies the identity

$$f(\omega\xi) = f(\xi)$$

for every element ω of the r–adic skew–plane with conjugate as inverse, then its Fourier transform is a function $g(\xi)$ of ξ in the r–adic skew–plane which is square integrable with respect to the canonical measure and which satisfies the identity

$$g(\omega\xi) = g(\xi)$$

for every element ω of the r–adic skew–plane with conjugate as inverse. The function $f(\xi)$ of ξ in the r–adic skew–plane is then the Fourier transform of the function $g(\xi)$ of ξ in the r–adic skew–plane.

If functions $f(\xi)$ and $g(\xi)$ of ξ in the r–adic skew–plane are square integrable with respect to the canonical measure and satisfy the identities

$$f(\omega\xi) = f(\xi)$$

and

$$g(\omega\xi) = g(\xi)$$

for every element ω of the r–adic skew–plane with conjugate as inverse, then the functions are Fourier transforms of each other if, and only if, the identity

$$G(\xi) = \lambda_r(\xi^-\xi)^{-1}F(-1/\xi)$$

holds for their Laplace transforms. It is sufficient to verify the identity when the functions $F(\xi)$ and $G(\xi)$ of ξ in the r–adic plane vanish at nonunits, in which case the verification applies properties of characters for the r–adic plane.

A character χ modulo ρ is defined for a positive integer ρ as a function of rational numbers which vanishes at nonintegers and at integers not relatively prime to ρ, which is periodic of period ρ, and which acts as a homomorphism of the multiplicative group of integers modulo ρ relatively prime to ρ into the multiplicative group of complex numbers of absolute value one.

A character modulo ρ is said to be primitive modulo ρ if it does not agree on integers relatively prime to ρ with a character modulo a proper divisor of ρ. The conjugate character of a character χ modulo ρ is the character χ^- modulo ρ with conjugate values:

$$\chi^-(\xi) = \chi(\xi)^-$$

for every rational number ξ. The conjugate character χ^- modulo ρ is primitive modulo ρ when the given character χ is primitive modulo ρ.

A character χ modulo ρ admits a continuous extension to the r–adic line when the prime divisors of ρ are divisors of r. The extension to the r–adic line is a function $\chi(\xi)$ of ξ in the r–adic line which vanishes at nonunits, which is periodic of period ρ, and which acts as a homomorphism of the multiplicative group of units into the multiplicative group of complex numbers of absolute value one.

The Fourier transform of a character χ modulo ρ for the r–adic line is a continuous function of ξ in the r–adic line which vanishes when $\rho\xi$ is nonintegral and which is equal to

$$\epsilon(\chi)p^{-\frac{1}{2}}\chi^-(\rho\xi) = \int \exp(2\pi i\xi\eta)\chi(\eta)d\eta$$

for a complex number $\epsilon(\chi)$ of absolute value one when χ is a primitive character modulo ρ. Integration is with respect to the canonical measure for the r–adic line.

A function $f(\xi)$ of ξ in the r–adic line admits a unique extension as a function $f(\xi)$ of ξ in the r–adic plane which satisfies the identity

$$f(\omega\xi) = f(\omega^-\xi)$$

for every element ω of the r–adic plane with conjugate as inverse. The function of ξ in the r–adic plane is continuous when the function of ξ in the r–adic line is continuous and is a Baire function when the function of ξ in the r–adic line is a Baire function. The identity

$$\prod(1+p^{-1}) \int |f(\xi)|^2\lambda_r(\xi)d\xi = \int |f(\xi)|^2 d\xi$$

holds with the product taken over the prime divisors p of r, with integration on the left with respect to the canonical measure for the r–adic line, and with integration on the right with respect to the canonical measure for the r–adic plane.

A character χ modulo ρ for the r–adic plane is the extension to the r–adic plane of a character χ modulo ρ for the r–adic line which satisfies the identity

$$\chi(\omega\xi) = \chi(\omega^-\xi)$$

for every element ω of the r–adic plane with conjugate as inverse. The Fourier transform for the r–adic plane of a character χ modulo ρ is a continuous function of ξ in the r–adic plane which vanishes when $\rho\xi$ is nonintegral and which is equal to

$$\epsilon(\chi)\rho^{-1}\chi^-(\rho\xi) = \int \exp(\pi i(\xi^-\eta + \eta^-\xi))\chi(\eta)d\eta$$

for the same complex number $\epsilon(\chi)$ of absolute value one when χ is a primitive character modulo ρ. Integration is with respect to the canonical measure for the r–adic plane.

The Hilbert space of functions which are square integrable with respect to the canonical measure for the r–adic plane and which vanish at nonunits is the closed span of characters modulo ρ for positive integers ρ whose prime divisors are divisors of r. It is sufficient to make the computation of Fourier transforms of functions $f(\xi)$ and $g(\xi)$ of ξ which are square integrable with respect to the canonical measure for the r–adic skew–plane and which satisfy the identities

$$f(\omega\xi) = f(\xi)$$

and

$$g(\omega\xi) = g(\xi)$$

for every element ω of the r–adic skew–plane with conjugate as inverse when their Laplace transforms are the functions

$$F(\xi) = \chi(\xi)$$

and

$$G(\xi) = \chi^-(-\xi)$$

of ξ in the r–adic plane for a character χ modulo ρ.

The desired identity reads

$$\int \exp(-2\pi i\xi^-\xi\omega)\chi(-\omega^{-1})d\omega$$
$$= \iint \exp(2\pi i(\xi^-\eta + \eta^-\xi)) \exp(-2\pi i\eta^-\eta\omega)\chi(\omega)d\omega d\eta$$

with integration on the left and inner integration on the right with respect to the canonical measure for the r–adic plane over the set of units ω and with outer integration on the right with respect to the canonical measure for the r–adic skew–plane over the set of elements η

such that $\rho\eta^-\eta$ is integral. The integrations on the right can be interchanged by absolute convergence.

When r is odd, the integral

$$\iint \exp(2\pi i(\xi^-\eta + \eta^-\xi)) \exp(-2\pi i\eta^-\eta\omega)d\eta$$

with respect to the canonical measure for the r–adic skew–plane over the set of elements such that $\rho\eta^-\eta$ is integral can be restricted to the set of integral elements since integration over the set of nonintegral elements makes no net contribution. When r is even, the integral can be restricted to the set of elements η which are either integral or have two integral coordinates and two half–integral coordinates since integration over the set of other elements makes no net contribution.

The computation of Fourier transforms is made by verifying the identity

$$1 = \int \exp(-2\pi i(\xi^-\omega^{-1}\xi - \xi^-\eta - \eta^-\xi + \eta^-\omega\eta))d\eta$$

with integration with respect to the canonical measure for the r–adic skew–plane over the set of integral elements when r is odd and, when r is even, over the set of elements which are either integral or have two integral coordinates and two half–integral coordinates.

The computation of Fourier transforms requires the identity only when ω is a unit of the r–adic line, in which case the equation

$$\omega = \lambda^-\lambda$$

admits a solution λ which is a unit of the r–adic skew–plane. The identity then reduces by changes of variable to the case

$$\omega = 1$$

since multiplication by a unit is a measure preserving transformation which takes integral elements into integral elements and elements with two integral coordinates and two half–integral coordinates into elements with two integral coordinates and two half–integral coordinates.

The identity is satisfied since the integrand is identically one and since integration is over a set of measure one.

4. Fourier Analysis on an r–Adelic Skew–Plane

The r–adelic skew–plane is the Cartesian product of the complex skew–plane and the r–adic skew–plane. An element ξ of the r–adelic skew–plane has a component ξ_+ in the complex skew–plane and a component ξ_- in the r–adic skew–plane. The r–adelic skew–plane is a conjugated ring with coordinate addition and multiplication.

The sum $\xi + \eta$ of elements ξ and η of the r–adelic skew–plane is the element of the r–adelic skew–plane whose component in the complex skew–plane is the sum

$$\xi_+ + \eta_+$$

of components in the complex skew–plane and whose component in the r–adic skew–plane is the sum

$$\xi_- + \eta_-$$

of components in the r–adic skew–plane.

The product $\xi\eta$ of elements ξ and η of the r–adelic skew–plane is the element of the r–adelic skew–plane whose component in the complex skew–plane is the product

$$\xi_+\eta_+$$

of components in the complex skew–plane and whose component in the r–adic skew–plane is the product

$$\xi_-\eta_-$$

of components in the r–adic skew–plane.

The conjugate of an element ξ of the r–adelic skew–plane is the element ξ^- of the r–adelic skew–plane whose component in the complex skew–plane is the conjugate

$$\xi_+^-$$

of the component in the complex skew–plane and whose component in the r–adic skew–plane is the conjugate

$$\xi_-^-$$

of the component in the r–adic skew–plane.

The r–adelic skew–plane is a locally compact Hausdorff space in the Cartesian product topology of the topology of the complex skew–plane and the topology of the r–adic skew–plane. Addition is continuous as a transformation of the Cartesian product of the r–adelic skew–plane with itself into the r–adelic skew–plane. Multiplication by an element of the r–adelic skew–plane is a continuous transformation of the r–adelic skew–plane into itself. Conjugation is a continuous transformation of the r–adelic skew–plane into itself.

The canonical measure for the r–adelic skew–plane is the Cartesian product measure of the canonical measure for the complex skew–plane and the canonical measure for the r–adic skew–plane. The measure is defined on Baire subsets of the r–adelic skew–plane. A measure preserving transformation of the r–adelic skew–plane into itself is defined by taking ξ into $\xi + \eta$ for every element η of the r–adelic skew–plane. Measure preserving transformations of the r–adelic skew–plane into itself are defined by taking ξ into $\omega\xi$ and $\xi\omega$ for every element ω of the r–adelic skew–plane whose component ω_+ in the complex skew–plane has conjugate as inverse and whose component ω_- in the r–adic skew–plane is a unit. If ω is an invertible element of the r–adelic skew–plane, multiplication on left or right by ω multiplies the canonical measure by a factor of

$$(\omega_+^-\omega_+)^2\lambda_r(\omega_-^-\omega_-)^2.$$

The set of noninvertible elements of the r–adelic skew–plane has measure zero.

The Fourier transformation for the r–adelic skew–plane is the isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure for the r–adelic skew–plane which takes an integrable function $f(\xi)$ of ξ into the continuous function

$$g(\xi) = \int \exp(2\pi i(\xi_+^-\eta_+ + \eta_+^-\xi_+))\exp(2\pi i(\xi_-^-\eta_- + \eta_-^-\xi_-))f(\eta)d\eta$$

of ξ defined by integration with respect to the canonical measure. Fourier inversion

$$f(\xi) = \int \exp(-2\pi i(\xi_+^-\eta_+ + \eta_+^-\xi_+))\exp(-2\pi i(\xi_-^-\eta_- + \eta_-^-\xi_-))g(\eta)d\eta$$

applies with integration with respect to the canonical measure when the function $g(\xi)$ of ξ is integrable and the function $f(\xi)$ of ξ is continuous.

The r–adelic plane is defined as the set of elements of the r–adelic skew–plane whose component in the complex skew–plane belongs to the complex plane and whose component in the r–adic skew–plane belongs to the r–adic plane. The r–adelic plane is a maximal commutative subring of the r–adelic skew–plane which is isomorphic to the Cartesian product of the complex plane and the r–adic plane. The conjugation of the r–adelic skew–plane acts as an isomorphism of the r–adelic plane onto itself.

The r–adelic plane is a locally compact Hausdorff space in the topology inherited from the r–adelic skew–plane. The topology of the r–adelic plane is identical with the Cartesian product topology of the topology of the complex plane and the topology of the r–adic plane. Addition is continuous as a transformation of the Cartesian product of the r–adelic plane with itself into the r–adelic plane. Multiplication by an element of the r–adelic plane is a continuous transformation of the r–adelic plane into itself. Conjugation is continuous as a transformation of the r–adelic plane into itself.

The canonical measure for the r–adelic plane is the Cartesian product measure of the canonical measure for the complex plane and the canonical measure for the r–adic plane. The measure is defined on Baire subsets of the r–adelic plane. A measure preserving transformation of the r–adelic plane into itself is defined by taking ξ into $\xi + \eta$ for every element η of the r–adelic plane. A measure preserving transformation of the r–adelic plane into itself is defined by taking ξ into $\omega\xi$ for every element ω of the r–adelic plane whose component ω_+ in the complex plane has conjugate as inverse and whose component ω_- in the r–adic plane is a unit. Multiplication by an invertible element ω of the r–adelic plane multiplies the canonical measure by a factor of

$$\omega_+^-\omega_+\lambda_r(\omega_-^-\omega_-).$$

The set of noninvertible elements has measure zero.

The Fourier transformation for the r–adelic plane is the isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure for the r–adelic plane which takes an integrable function $f(\xi)$ of ξ into the continuous function

$$g(\xi) = \int \exp(\pi i(\xi_+^- \eta_+ + \eta_+^- \xi_+)) \exp(\pi i(\xi_-^- \eta_- + \eta_-^- \xi_-)) f(\eta) d\eta$$

of ξ defined by integration with respect to the canonical measure. Fourier inversion

$$f(\xi) = \int \exp(-\pi i(\xi_+^- \eta_+ + \eta_+^- \xi_+)) \exp(-\pi i(\xi_-^- \eta_- + \eta_-^- \xi_-)) g(\eta) d\eta$$

applies with integration with respect to the canonical measure when the function $g(\xi)$ of ξ is integrable and the function $f(\xi)$ of ξ is continuous.

The complementary space to the r–adelic plane in the r–adelic skew–plane is the set of elements η of the r–adelic skew–plane which satisfy the identity

$$\xi\eta = \eta\xi^-$$

for every element ξ of the r–adelic plane. An element η of the complementary space is skew–conjugate:

$$\eta^- = -\eta.$$

An injective transformation of the r–adelic plane onto the complementary space is defined by taking ξ into $\eta\xi$ for every invertible element η of the complementary space. The transformation is continuous when the complementary space is given the topology inherited from the r–adelic skew–plane. The transformation takes the canonical measure for the r–adelic plane into

$$\tfrac{1}{2}\eta_+^- \eta_+ \lambda_r(\eta_-^- \eta_-)$$

times the canonical measure for the complementary space when r is odd and into

$$\eta_+^- \eta_+ \lambda_r(\eta_-^- \eta_-)$$

times the canonical measure for the complementary space when r is even.

An element of the r–adelic skew–plane is the unique sum $\alpha + \beta$ of an element α of the r–adelic plane and an element β of the complementary space. The topology of the r–adelic skew–plane is the Cartesian product topology of the topology of the r–adelic plane and the topology of the complementary space. The canonical measure for the r–adelic skew–plane is the Cartesian product measure of the canonical measure for the r–adelic plane and the canonical measure for the complementary space.

The Radon transformation for the r–adelic skew–plane is a transformation with closed graph whose domain and range are contained in the Hilbert space of square integrable functions with respect to the canonical measure for the r–adelic skew–plane and whose graph contains the pairs $(f(\omega\xi), g(\omega\xi))$ and $(f(\xi\omega), g(\xi\omega))$ of functions of ξ whenever the graph contains the pair $(f(\xi), g(\xi))$ of functions of ξ for every element ω of the r–adelic skew–plane with conjugate as inverse. The transformation is defined as an integral on elements of its domain which are integrable with respect to the canonical measure.

The Radon transform of an integrable function $f(\xi)$ of ξ in the r–adelic skew–plane is the function $g(\xi)$ of ξ in the r–adelic skew–plane which satisfies the identity

$$g(\omega\xi) = \int f(\omega\xi + \omega\eta)d\eta$$

for almost all elements ξ of the r–adelic plane when ω is an element of the r–adelic skew–plane with conjugate as inverse with integration with respect to the canonical measure for the complementary space to the r–adelic plane in the r–adelic skew–plane. The inequality

$$\int |g(\omega\xi)|d\xi \leq \int |f(\xi)|d\xi$$

holds for every element ω of the r–adelic skew–plane with conjugate as inverse with integration on the left with respect to the canonical measure for the r–adelic plane and with integration on the right with respect to the canonical measure for the r–adelic skew–plane.

The Radon transformation for the r–adelic skew–plane factors the Fourier transformation for the r–adelic skew–plane as a composition with the Fourier transformation for the r–adelic plane. If the Radon transformation takes a function $f(\xi)$ of ξ into a function $g(\xi)$ of ξ and if the function $f(\xi)$ of ξ is integrable with respect to the canonical measure for the r–adelic skew–plane, then the function $g(\omega\xi)$ of ξ in the r–adelic plane is integrable with respect to the canonical measure for the r–adelic plane for every element ω of the r–adelic skew–plane with conjugate as inverse. The Fourier transform of the function $g(\omega\xi)$ of ξ in the r–adelic plane is the restriction to the r–adelic plane of the Fourier transform of the function $2f(2\omega\xi)$ of ξ in the r–adelic skew–plane when r is odd and of the function $f(2\omega\xi)$ of ξ in the r–adelic skew–plane when r is even.

Spectral analysis of the Radon transformation for the r–adelic skew–plane is made by the Laplace transformations for the r–adelic skew–plane. Laplace transforms are functions defined on an r–adelic half–plane which is the Cartesian product of the upper half–plane and the r–adic line. An element $\xi + iy$ of the r–adelic half–plane is determined by a positive number y and an element of ξ of the r–adic line. Laplace transforms are analytic functions of $\xi_+ + iy$ in the upper half–plane for every element ξ_- of the r–adic line.

A Laplace transformation of harmonic ϕ for the r–adelic skew–plane is defined by a homogeneous harmonic polynomial $\phi(\xi)$ of degree ν which is normalized so that the equation

$$\int |\phi(\xi)|^2 d\xi = \int (\xi^- \xi)^\nu d\xi$$

holds with integration on the left with respect to the canonical measure for the complex skew–plane and with integration on the right with respect to the same measure over the unit disk $\xi^- \xi < 1$.

The domain of the Laplace transformation of harmonic ϕ is the Hilbert space of functions $f(\xi)$ of ξ in the r–adelic skew–plane which are square integrable with respect to the canonical measure and which satisfy the identity

$$\phi(\xi_+)f(\omega\xi) = \phi(\omega_+\xi_+)f(\xi)$$

for every element ω of the r–adelic skew–plane with conjugate as inverse.

A function

$$f(\xi) = \phi(\xi_+)h(\xi^-\xi)$$

of ξ in the r–adelic skew–plane which belongs to the domain of the Laplace transformation of harmonic ϕ is parametrized by a function

$$h(\xi) = h(\xi_+, \xi_-)$$

of ξ in the r–adelic line which is extended as a function of ξ_+ in the real line and of ξ_- in the r–adic plane so as to satisfy the identity

$$h(\xi_+, \omega\xi_-) = h(\xi_+, \omega^-\xi_-)$$

for every element ω of the r–adic plane with conjugate as inverse. The identity

$$\int |f(\xi)|^2 d\xi = 2\pi \iint |h(\xi_+, \xi_-)|^2 \xi_+^{1+\nu} d\xi_+ d\xi_-$$

holds when r is odd and the identity

$$\int |f(\xi)|^2 d\xi = \pi \iint |h(\xi_+, \xi_-)|^2 \xi_+^{1+\nu} d\xi_+ d\xi_-$$

holds when r is even with integration on the left with respect to the canonical measure for the r–adelic skew–plane, with outer integration on the right with respect to the canonical measure for the r–adic plane, and with outer integration with respect to Lebesgue measure for the positive half–line.

The Laplace transform of a function $f(\xi)$ of ξ in the r–adelic skew–plane is a function $F(\xi + iy)$ of $\xi + iy$ in the r–adelic half–plane which is defined for positive y as the Fourier transform of the function

$$h(\xi) = h(\xi_+, \xi_-)$$

of ξ_+ in the real line and ξ_- in the r–adic line. For $\xi_+ + iy$ in the upper half–plane the Fourier transform $F(\xi + iy)$ is the restriction of a function which is square integrable with respect to the canonical measure for the r–adic plane. The identity

$$\iint |F(\xi + iy)|^2 d\xi_+ d\xi_- = 64\pi^2 \iint |h(\xi)|^2 \exp(-4\pi y\xi_+) d\xi_+ d\xi_-$$

holds when r is odd and the identity

$$\iint |F(\xi + iy)|^2 d\xi_+ d\xi_- = 32\pi^2 \iint |h(\xi)|^2 \exp(-4\pi y \xi_+) d\xi_+ d\xi_-$$

holds when r is even with integration on the left with respect to the canonical measure for the real line and the canonical measure for the r–adic plane. The identity

$$\iint |F(\xi + iy)|^2 d\xi_+ d\xi_- = 8\pi \int |f(\xi)| 2 \exp(-4\pi y \xi^- \xi)(\xi^- \xi)^{1+\nu} d\xi$$

holds when r is odd and the identity

$$\iint |F(\xi + iy)|^2 d\xi_+ d\xi_- = 4\pi \int |f(\xi)|^2 \exp(-4\pi y \xi^- \xi)(\xi^- \xi)^{1+\nu} d\xi$$

holds when r is even with integration on the left with respect to the canonical measure for the real line and the canonical measure for the r–adic plane and with integration on the right with respect to the canonical measure for the r–adelic skew–plane.

These identities for given positive y imply the identities

$$\int_0^\infty \iint |F(\xi + iy)|^2 y^\nu d\xi_+ d\xi_- dy = 2(4\pi)^{-\nu} \Gamma(1+\nu) \int |f(\xi)|^2 d\xi$$

when r is odd and

$$\int_0^\infty \iint |F(\xi + iy)|^2 y^\nu d\xi_+ d\xi_- dy = (4\pi)^{-\nu} \Gamma(1+\nu) \int |f(\xi)|^2 d\xi$$

when r is even hold with inner integration on the left with respect to the canonical measure for the real line and and the canonical measure for the r–adic plane and with integration on the right with respect to the canonical measure for the r–adelic skew–plane.

A Baire function $F(\xi + iy)$ of $\xi + iy$ in the r–adelic half–plane is a Laplace transform of harmonic ϕ for the r–adelic skew–plane if, and only if, an analytic function of $\xi_+ + iy$ in the upper half–plane results for every ξ_- in the r–adic line and the integral

$$\int_0^\infty \iint |F(\xi + iy)|^2 y^\nu d\xi_+ d\xi_- dy$$

converges with inner integration with respect to the canonical measure for the real line and the canonical measure for the r–adic plane.

Functions $f(\xi)$ and $g(\xi)$ of ξ in the r–adelic half–plane which belong to the domain of the Laplace transformation of harmonic ϕ are Fourier transforms of each other if, and only if, their Laplace transforms are functions $F(\xi + iy)$ and $G(\xi + iy)$ of $\xi + iy$ in the r–adelic half–plane related by the identity

$$G(\xi + iy) = i^\nu \left(\frac{i}{\xi_+ + iy} \right)^{2+\nu} \frac{1}{\lambda_r(\xi_-^- \xi_-)} \, F\left(\frac{-1}{\xi + iy} \right).$$

5. Fourier Analysis on an r–adelic Skew–Plane

Hecke operators are continuous linear transformations of the Hilbert space of square integrable functions with respect to the canonical measure for the product skew–plane, which are of character χ, into itself. A Hecke operator $\Delta(r)$ is defined for every generating positive integer r which is relatively prime to ρ. The transformation takes a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane into the function $g(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane defined by the sum

$$24g(\xi_+, \xi_-) = \sum f(\xi_+\omega, \xi_-\omega)$$

over the integral elements ω of the algebraic skew–plane which represent

$$r = \omega^-\omega.$$

The Hecke operator $\Delta(1)$ acts as the orthogonal projection of the Hilbert space onto the subspace of functions $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane which satisfy the identity

$$f(\xi_+, \xi_-) = f(\xi_+\omega, \xi_-\omega)$$

for every integral element ω of the algebraic skew–plane with integral inverse.

The identity

$$\Delta(m)\Delta(n) = \sum \Delta(mn/k^2)$$

holds for all generating positive integers m and n which are relatively prime to ρ with summation over the common odd divisors k of m and n.

The spectral analysis of Hecke operators is made in Hilbert spaces of finite dimension. A space is defined for every nonnegative integer ν whose elements are function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane, which are of character χ, such that the function $f(\xi_+, \xi_-)$ of ξ_+ in the Dedekind skew–plane is a homogeneous harmonic polynomial of degree ν for every element ξ_- of the adic skew–plane. The function $f(\xi_+, \xi_-)$ of ξ_- in the adic skew–plane has equal values at elements whose p–adic components are equal for every prime divisor p of ρ when ξ_+ is an element of the Dedekind skew–plane. The Hilbert space is defined as a tensor product of the Hilbert space of homogeneous harmonic polynomials of degree ν and a Hilbert space of functions which are square integrable with respect to the canonical measure for the product skew–plane on the set of integral elements of the adic skew–plane.

A Hecke operator $\Delta(r)$ is defined on the tensor product for every generating positive integer r which is relatively prime to ρ. The transformation takes a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane into the function $g(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane defined by the sum

$$24r^{\frac{1}{2}\nu}g(\xi_+, \xi_-) = \sum f(\xi_+\omega, \xi_-\omega)$$

over the integral elements ω of the algebraic skew–plane which represent

$$r = \omega^- \omega.$$

The Hecke operator $\Delta(1)$ acts as the orthogonal projection of the tensor product onto the subspace of functions $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane which satisfy the identity

$$f(\xi_+, \xi_-) = f(\xi_+ \omega, \xi_- \omega)$$

for every integral element ω of the algebraic skew–plane with integral inverse.

The identity

$$\Delta(m)\Delta(n) = \sum \Delta(mn/k^2)$$

holds for all generating positive integers m and n which are relatively prime to ρ with summation over the common odd divisors k of m and n.

The tensor product is the orthogonal sum of invariant subspaces whose elements are defined as eigenfunctions of Hecke operators for given eigenvalues. The kernel of the Hecke operator $\Delta(1)$ is annihilated by every Hecke operator. An invariant subspace which is orthogonal to the kernel of $\Delta(1)$ is contained in the range of $\Delta(1)$. The elements of the invariant subspace are defined as eigenfunctions of $\Delta(r)$ for a given eigenvalue $\tau(r)$ for every generating positive integer r which is relatively prime to ρ. The identity

$$\tau(m)\tau(n) = \sum \tau(mn/k^2)$$

holds for all generating positive integers m and n which are relatively prime to ρ with summation over the common odd divisors k of m and n.

The zeta function characteristic of an invariant subspace is the Dirichlet series

$$\zeta(s) = \sum \tau(n) n^{-s}$$

defined as a sum over the generating positive integers n which are relatively prime to ρ.

The number of zeta functions associated with a primitive character modulo ρ is equal to the product

$$\rho \prod (1 + p^{-1})$$

taken over the odd prime divisor p of ρ.

A Laplace kernel κ associated with the zeta function is an element of the tensor–product which is an eigenfunction of the Hecke operator $\Delta(r)$ for the eigenvalue $\tau(r)$ for every generating positive integer r which is relatively prime to ρ and whose scalar self–product is equal to the canonical measure of the set of integral elements of the adic skew–plane which are invertible modulo ρ.

A Laplace kernel for the product skew–plane is applied in the definition of a Laplace transformation with kernel κ for the product skew–plane. The domain of the transformation is the Hilbert space of functions $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane which are square integrable with respect to the canonical measure, which are of character χ, and which satisfy the identity

$$\kappa(\xi_+, \xi_-) f(\omega_+ \xi_+, \omega_- \xi_-) = \kappa(\omega_+ \xi_+, \omega_- \xi_-) f(\xi_+, \xi_-)$$

for every element ω_+ of the Dedekind skew–plane with conjugate as inverse and every element ω_- of the adic skew–plane with conjugate as inverse.

Elements of the domain of the Laplace transformation with kernel κ are eigenfunctions of the Hecke operator $\Delta(r)$ for the eigenvalue $\tau(r)$ for every generating positive integer r which is relatively prime to ρ.

The Hilbert space functions which are square integrable with respect to the canonical measure for the product skew–plane and which are of character χ is the orthogonal sum of invariant subspaces whose elements are determined as eigenfunctions of Hecke operators for given eigenvalues. An invariant subspace which is orthogonal to the kernel of $\Delta(1)$ is determined by a zeta function

$$\zeta(s) = \sum \tau(n) n^{-s}$$

for the tensor product space as the set of elements which are eigenfunctions of $\Delta(r)$ for the eigenvalue $\tau(r)$ for every generating positive integer r which is relatively prime to ρ. The invariant subspace is spanned by the domains of Laplace transformations whose kernels are eigenfunctions of the Hecke operators for the same eigenvalues.

The Laplace transform with kernel κ of a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane is the bounded continuous function

$$F(z, \xi) = 2 \int \kappa(\eta_+, \eta_-)^- f(\eta_+, \eta_-) \chi(\eta_-^- \eta_-) \exp(\pi i z \eta_+^- \eta_+) \exp(-\pi i \xi \eta_-^- \eta_-) d\eta$$

of z in the upper half–plane and ξ in the adic line which is defined by integration with respect to the canonical measure when the integral is absolutely convergent. The identity

$$(1 + 2\nu) \int \int_0^\infty \int_{-\infty}^{+\infty} |F(x + iy, \xi)|^2 y^\nu dx dy d\xi = 8(2\pi)^{-\nu} \Gamma(1 + \nu) \int |f(\xi_+, \xi_-)|^2 d\xi$$

holds with integration on the left with respect to the Laplace measure and with integration on the right with respect to the canonical measure. The Laplace transformation with kernel κ is defined so as to preserve the identity.

A function $F(z, \xi)$ of z in the upper half–plane and ξ in the adic line which belongs to the range of the Laplace transformation is analytic as a function of z for every element ξ of the adic line, vanishes when the p–adic component of $2\rho\xi$ is nonintegral for some prime divisor p of ρ, and satisfies the identity

$$F(z, \xi\omega) = F(z, \xi) \chi(\omega)^-$$

for every integral element ω of the adic line whose p–adic component is invertible modulo p for every prime divisor p of ρ and has integral inverse for every other generating prime p.

The domain of the Laplace transformation with kernel κ is an invariant subspace for the Radon transformation. The adjoint of the Radon transformation is a maximal dissipative transformation in the Hilbert space which takes a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane into the function $g(\xi_+, \xi_-)$ in the product skew plane defined by the identity

$$G(z, \xi) = (i/z)|\xi|F(z, \xi)$$

for almost all elements ξ of the adic line with respect to the Laplace measure when z is in the upper half–plane. The integrals

$$F(z, \xi) = \int \kappa(\eta_+, \eta_-)^- f(\eta_+, \eta_-)\chi(\eta_-^- \eta_-) \exp(\pi i z \eta_+^- \eta_+) \exp(-\pi i \xi \eta_-^- \eta_-) d\eta$$

and

$$G(z, \xi) = \int \kappa(\eta_+, \eta_-)^- g(\eta_+, \eta_-)\chi(\eta_-^- \eta_-) \exp(\pi i z \eta_+^- \eta_+) \exp(-\pi i \xi \eta_-^- \eta_-) d\eta$$

with respect to the canonical measure for the product skew–plane are interpreted as Laplace transforms when they are not absolutely convergent.

A maximal dissipative transformation $-iH$ is defined as the inverse of the adjoint of the Radon transformation in the Hilbert space of functions of character χ which are square integrable with respect to the canonical measure for the product skew–plane.

A symplectic transformation T which has the Hilbert space of square integrable functions of character χ as an invariant subspace is associated with a symplectic matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for the algebraic skew–plane with self–conjugate entries which is compatible with the constraints modulo ρ. The domain of the Laplace transformation with kernel κ is an invariant subspace.

The transformation takes a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane into a function $g(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane when the identity

$$G(z, \xi) = (Cz + D)^{-2-\nu}|C\xi + D|^2 F((Cz + D)^{-1}(Az + B), (C\xi + D)^{-1}(A\xi + B))$$

holds for almost all elements ξ of the adic line with respect to the Laplace measure when z is in the upper half–plane. The integrals

$$F(z, \xi) = \int \kappa(\eta_+, \eta_-)^- f(\eta_+, \eta_-)\chi(\eta_-^- \eta_-) \exp(\pi i z \eta_+^- \eta_+) \exp(-\pi i \xi \eta_-^- \eta_-) d\eta$$

and

$$G(z, \xi) = \int \kappa(\eta_+, \eta_-)^- g(\eta_+, \eta_-)\chi(\eta_-^- \eta_-) \exp(\pi i z \eta_+^- \eta_+) \exp(-\pi i \xi \eta_-^- \eta_-) d\eta$$

with respect to the canonical measure are interpreted as Laplace transforms when they are absolutely convergent.

The theta function determined by a zeta function

$$\zeta(s) = \sum \tau(n) n^{-s}$$

is a function $\theta(z, \xi)$ of z in the upper half–plane and invertible elements ξ of the adic line which is analytic as a function of z for every ξ and which satisfies the identity

$$\theta(z, \xi) = \theta(z\lambda, \xi\lambda)$$

for every nonzero element λ of the algebraic line whose Dedekind modulus is equal to its adic modulus.

The theta function appears in a simplification of the sum

$$\sum \kappa(\eta_+\lambda, \eta_-\lambda) \exp(\pi i z \eta_+^- \eta_+ \lambda^- \lambda) \exp(-\pi i \xi \eta_-^- \eta_- \lambda^- \lambda)$$

over the nonzero elements λ of the algebraic skew–plane whose Dedekind modulus is equal to its adic modulus such that $\eta_-\lambda$ is integral. Since the sum remains unchanged when η_+ is replaced by $\eta_+\lambda$ and η_- is replaced by $\eta_-\lambda$ for a nonzero element λ of the algebraic skew–plane whose Dedekind modulus is equal to its adic modulus, the computation of the sum for invertible elements η_- of the adic skew–plane reduces to the case in which η_- is integral and has integral inverse.

When η_- is integral and has integral inverse, the sum is taken over the integral elements λ of the algebraic skew–plane such that $\lambda^-\lambda$ is positive and relatively prime to ρ. A partial sum in which

$$\lambda^- \lambda = n$$

for a generating positive integer n which is relatively prime to ρ is computed using the definition of the Hecke operator $\Delta(n)$. The sum is a product

$$\kappa(\eta_+, \eta_-) \theta(z \eta_+^- \eta_+, \xi \eta_-^- \eta_-)$$

with the theta function

$$\theta(z, \xi) = \sum n^{\frac{1}{2}\nu} \tau(n) \exp(\pi i n z) \exp(-\pi i n \xi)$$

defined as a sum over the generating positive integers n which are relatively prime to ρ when ν is positive or ρ is not one. When ν is zero and ρ is one, the theta function

$$\theta(z, \xi) = 1 + \sum \tau(n) \exp(\pi i n z) \exp(-\pi i n \xi)$$

is defined as a sum over the generating positive integers n with a contribution for the integer zero.

The function $\theta(z, \xi)$ of invertible elements ξ of the adic line admits a unique continuous extension as a function $\theta(z, \xi)$ of elements ξ of the adic line for every element z of the upper half–plane. The identity

$$\theta(z, \xi) = \theta(z + t, \xi + t)$$

holds for every element t of the algebraic line. The function

$$\theta(z) = \theta(z, 0)$$

of z in the upper half–plane is represented as a sum

$$\theta(z) = \sum n^{\frac{1}{2}\nu} \tau(n) \exp(\pi i n z)$$

over the generating positive integers n which are relatively prime to ρ when ν is positive or ρ is not one. The function

$$\theta(z) = 1 + \sum \tau(n) \exp(\pi i n z)$$

of z in the upper half–plane is represented as a sum over the generating positive integers n with a contribution for the integer zero when ν is zero and ρ is one.

The theta function satisfies a functional identity when all primes are generating primes. A computation of Fourier transforms is made for application of the Poisson summation formula. With a primitive character χ modulo ρ is associated the conjugate character χ^- which is primitive modulo ρ. With a zeta function

$$\zeta(s) = \sum \tau(n) n^{-s}$$

for the character χ is associated the conjugate zeta function

$$\zeta(s^-)^- = \sum \tau(n)^- n^{-s}$$

for the conjugate character χ^-. With a Laplace kernel κ of character χ for the zeta function $\zeta(s)$ is associated a Laplace kernel $\kappa^\wedge$ of character χ^- associated with the zeta function $\zeta(s^-)^-$. Assume that z is in the upper half–plane and that ξ is an invertible element of the adic line. An integral element ω of the algebraic plane is chosen which represents

$$2 = \omega^- w.$$

When ρ is even, the Fourier transform of the function of $\eta = (\eta_+, \eta_-)$ in the product skew–plane which is equal to

$$\kappa(\eta_+, \eta_-) \exp(\pi i z \eta_+^- \eta_+) \exp(-\pi i \xi \eta_-^- \eta_-)$$

when η_- is integral and which vanishes otherwise is the function $\eta = (\eta_+, \eta_-)$ in the product skew–plane which is equal to

$$4\rho^{-2}(i/z)^{2+\nu}|\xi|^2\kappa^\wedge(\eta_+, \tfrac{1}{2}\rho\eta_-)\exp(-\pi i z^{-1}\eta_+^-\eta_+)\exp(\pi i \xi^{-1}\eta_-^-\eta_-)$$

when $\frac{1}{2}\rho\eta_-$ is integral and which vanishes otherwise.

When ρ is odd, the Fourier transform of the function of $\eta = (\eta_+, \eta_-)$ in the product skew–plane which is equal to

$$\kappa(\eta_+, \eta_-)\exp(\pi i z\eta_+^-\eta_+)\exp(-\pi i\xi\eta_-^-\eta_-)$$

when η_- is integral and which vanishes otherwise is the function of $\eta = (\eta_+, \eta_-)$ in the product skew–plane which is equal to

$$2\rho^{-2}(i/z)^{2+\nu}|\xi|^2\kappa^\wedge(\eta_+, \rho\omega^{-1}\eta_-)\exp(-\pi i z^{-1}\eta_+^-\eta_+)\exp(\pi i\xi^{-1}\eta_-^-\eta_-)$$

when $\rho\omega^{-1}\eta_-$ is integral and which vanishes otherwise.

When ρ is even, the Poisson formula yields the functional identity

$$\begin{aligned}&\kappa(1,1)\theta(2\rho^{-1}z, 2\rho^{-1}\xi)\\ &= \kappa^\wedge(2\rho^{-1}, 1)(2\rho^{-1})^{2+\nu}(i/z)^{2+\nu}|\xi|^2\theta^\wedge(-2\rho^{-1}z^{-1}, -2\rho^{-1}\xi^{-1}).\end{aligned}$$

When ρ is odd, the Poisson formula yields the functional identity

$$\begin{aligned}&\kappa(1,1)\theta(\rho^{-1}z, \rho^{-1}\xi)\\ &= \kappa^\wedge(\omega\rho^{-1}, 1)|\omega\rho^{-1}|^{2+\nu}(i/z)^{2+\nu}|\xi|^2\theta(-2\rho^{-1}z^{-1}, -2\rho^{-1}\xi^{-1}).\end{aligned}$$

The Fourier transformation for the product skew–plane is a symplectic transformation associated with the symplectic matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

for the algebraic skew–plane. A generalization of the Poisson formula applies when the Fourier transformation is replaced by a symplectic transformation associated with an admissible symplectic matrix for the algebraic skew–plane. An example is the matrix

$$\begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$$

for every integer n. The associated symplectic transformation takes a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane into the function

$$\exp(2\pi i n\xi_+^-\xi_+)\exp(-2\pi i n\xi_-^-\xi_-)f(\xi_+, \xi_-)$$

of $\xi = (\xi_+, \xi_-)$ in the product skew–plane.

The modular group is the set of matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with integer entries and determinant one. A submodular group is the set of matrices in the modular group whose diagonal entries have equal parity and whose off–diagonal entries have equal parity. The submodular group is generated by the matrices

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

which satisfy no relations other than the one implied by the Fourier transformation, whose fourth power is the identity.

The signature for the submodular group is the homomorphism of the subgroup onto the fourth roots of unity which has value one on

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

and which has value i on

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

An element

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

of the submodular group has signature one if, and only if, its diagonal entries are congruent to one modulo four and its off–diagonal entries are even.

A symplectic transformation for which the Poisson summation formula applies is associated with every matrix in the submodular group. The symplectic transformation associated with a matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

of signature one in the submodular group takes functions of character χ into functions of character for a primitive character χ modulo ρ when the subdiagonal entry is divisible by 2ρ. A theta function $\theta(z, \xi)$ associated with the character χ satisfies the identity

$$\theta(z, \xi) = \chi(D)(Cz + D)^{-2-\nu}|C\xi + D|^2 \theta((Cz + D)^{-1}(Az + B), (C\xi + D)^{-1}(A\xi + B))$$

when ν is positive or ρ is not one. The identity

$$1 + \theta(z, \xi) = (Cz + D)^{-2} |C\xi + D|^2 [1 + \theta((Cz + D)^{-1}(Az + B), (C\xi + D)^{-1}(A\xi + B))]$$

holds when ν is zero and ρ is one.

The analytic function $\theta(z)$ of z in the upper half–plane satisfies the identity

$$\theta(z) = \chi(D)(Cz + D)^{-2-\nu} \theta((Cz + D)^{-1}(Az + B))$$

for every matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

of signature one in the submodular group which has subdiagonal entry divisible by 2ρ.

An analytic function $F(z)$ of z in the upper half–plane is said to be a submodular form of order ν associated with a primitive character χ modulo ρ if the function

$$(Cz + D)^{-2-\nu} F((Cz + D)^{-1}(Az + B))$$

is represented by a power series in

$$\exp(\pi i z)$$

for every matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

in the submodular group and if the identity

$$F(z) = \chi(D)(Cz + D)^{-2-\nu} F((Cz + D)^{-1}(Az + B))$$

holds whenever the matrix has signature one and has subdiagonal entry divisible by 2ρ.

A submodular form of order ν associated with a primitive character χ modulo ρ is a linear combination of analytic functions obtained from theta functions of order ν associated with the character. The proof is given by showing that the dimension of the space of submodular forms of order ν associated with a primitive character modulo ρ is not greater than the number of theta functions of order ν associated with the character.

Elements z and w of the upper half–plane are considered equivalent modulo ρ if

$$w = (Cz + D)^{-1}(Az + B)$$

for a matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

of signature one in the submodular group whose subdiagonal entry is divisible by 2ρ.

A fundamental region modulo ρ is a connected open subset of the upper half–plane such that every element of the upper half–plane is equivalent to an element of the closure of the set and such that equivalent elements of the set are equal.

A fundamental region modulo ρ is constructed as the interior of the union of closures of fundamental regions modulo one.

Considerations of symmetry are applied in the construction of a fundamental region modulo one. Elements z and w of the upper half–plane are said to be symmetric modulo one if z and $-w^-$ are equivalent modulo one. The elements of the upper half–plane which are not self–symmetric modulo one form an open set which is the union of connected components. A fundamental region modulo one is constructed as the interior of the union of the closures of two symmetric components when the interior is connected.

An example of a symmetric component is the set of elements z of the upper half–plane which satisfy the inequalities

$$0 < z + z^- < 2$$

and

$$(2z - 1)^-(2z - 1) > 1.$$

Another example is the set of elements z of the upper half–plane which satisfy the inequalities

$$-2 < z + z^- < 0$$

and

$$(2z + 1)^-(2z + 1) > 1.$$

A fundamental region modulo one is the union of the two components with the imaginary axis.

The boundary line

$$z + z^- = -2$$

is mapped onto the boundary line

$$z + z^- = 2$$

by taking z into

$$z + 2.$$

The boundary circle

$$(2z + 1)^-(2z + 1) = 1$$

is mapped onto the boundary circle

$$(2z - 1)^-(2z - 1) = 1$$

by taking z into

$$z/(1 + 2z).$$

A fundamental region modulo ρ is constructed as the interior of the union of the closure of symmetric components modulo one. The fundamental region is compactified by taking its closure in the complex plane and by supplying an element at the upper end of the imaginary axis.

Analytic structure is supplied at the infinite element by requiring a power series expansion in

$$\exp(\pi iz).$$

Analytic structure is supplied at finite elements of the boundary by requiring analyticity of the mapping which takes z into

$$(Cz + D)^{-1}(Az + B)$$

whenever

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is an element of signature one of the submodular group whose subdiagonal entry is divisible by 2ρ.

The dimension of the space of submodular forms of order ν associated with a primitive character modulo ρ is less than or equal to the maximal number of inequivalent zeros of a nontrivial submodular form of order ν associated with the character in the compactification of the fundamental region.

The number of inequivalent zeros in the closure of a fundamental region modulo ρ is the sum of the number of inequivalent zeros in the closures of symmetric components contained in the region. The number of zeros of a nontrivial submodular form $F(z)$ in a symmetric component is a Cauchy integral

$$(2\pi i)^{-1} \int F(z)^{-1} F'(z) dz$$

taken counterclockwise over the boundary of the component. Since there may be zeros on the boundary and since some boundary elements lie outside the upper half–plane, the Cauchy integral is interpreted as a limit of Cauchy integrals over the boundaries of regions which are contained in the symmetric component and which contain the zeros of the function in the component.

The boundary of a symmetric component is divided into three arcs by the three elements of the boundary which lie outside the upper half–plane. Every arc of the boundary of a fundamental region is an arc of the boundary of a symmetric component inside the region and of a symmetric component outside the region. An arc of the boundary of the fundamental region is paired with another arc of the boundary whose elements are equivalent modulo ρ. A matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

of signature one in the submodular group exists which has subdiagonal entry divisible by 2ρ such that the mapping z into

$$(Cz + D)^{-1}(Az + B)$$

takes a bounding component inside the region onto a bounding component outside the region. The contribution to the Cauchy integral of each bounding arc depends on the function. The sum of the contributions of two equivalent arcs is independent of the function.

The number of inequivalent zeros of a nontrivial submodular form of order ν associated with a primitive character modulo ρ is independent of the function. The number is equal to the product of

$$1 + \nu$$

and the number of fundamental regions modulo one contained in a fundamental region modulo ρ.

This completes the proof that all submodular forms of order ν associated with a primitive character modulo ρ are derived from theta functions of order ν associated with the character since the number of fundamental regions modulo one contained in a fundamental region modulo ρ is equal to the product

$$\rho \prod (1 + p^{-1})$$

taken over the odd prime divisors p of ρ.

The Euler product for a zeta function

$$\zeta(s) = \sum \tau(n) n^{-s}$$

is a consequence of the identity

$$\tau(m)\tau(n) = \sum \tau(mn/k^2)$$

which holds for all generating positive integers m and n which are relatively prime to ρ with summation over the common odd divisors k of m and n. The Euler product

$$\zeta(s)^{-1} = \prod (1 - \tau(p)p^{-s} + p^{-2s})$$

when ρ is even and

$$\zeta(s)^{-1} = (1 - \tau(2)2^{1-s}) \prod (1 - \tau(p)p^{-s} + p^{-2s})$$

when ρ is odd and is taken over the odd generating primes p which are not divisors of ρ.

A zeta function

$$\zeta(s) = \sum \tau(n) n^{-s}$$

of order ν associated with a primitive character χ modulo ρ satisfies a functional identity when all primes are generating primes. The functional identity for the zeta function is obtained from the functional identity for the theta function. The zeta function admits an analytic extension to the complex plane when ν is positive or ρ is not one. When ν is zero and ρ is one, the zeta function admits an analytic extension to the complex plane except for a simple pole at two.

When ρ is even, the analytic extension of the function

$$(2\pi/\rho)^{-\frac{1}{2}\nu-s}\Gamma(\tfrac{1}{2}\nu+s)\zeta(s)$$

and the analytic extension of the function obtained from

$$(2\pi/\rho)^{-\frac{1}{2}\nu-s}\Gamma(\tfrac{1}{2}\nu+s)\zeta(s)$$

on replacing s by $2-s$ are linearly dependent.

When ρ is odd, the analytic extension of the function

$$(2^{\frac{1}{2}}\pi/\rho)^{-\frac{1}{2}\nu-s}\Gamma(\tfrac{1}{2}\nu+s)\zeta(s)$$

and the analytic extension of the function obtained from

$$(2^{\frac{1}{2}}\pi/\rho)^{-\frac{1}{2}\nu-s}\Gamma(\tfrac{1}{2}\nu+s)\zeta(s^-)^-$$

on replacing s by $2-s$ are linearly dependent. The analytic extensions are equal when ν is zero and ρ is one.

A Laplace transformation is defined by a theta function. Associated with the theta function $\theta(z,\xi)$ of z in the upper half–plane and ξ in the adic line is the conjugate theta function

$$\theta^*(z,\xi)=\theta(-z^-,-\xi)^-$$

of z in the upper half–plane and ξ in the adic line.

The domain of the Laplace transformation is the Hilbert space of functions $f(\xi_+,\xi_-)$ of $\xi=(\xi_+,\xi_-)$ in the product skew–plane which are of character χ and satisfy the identity

$$\kappa(\xi_+,\xi_-)f(\omega_+\xi_+,\omega_-\xi_-)=\kappa(\omega_+\xi_+,\omega_-\xi_-)f(\xi_+,\xi_-)$$

for every element ω_+ of the Dedekind skew–plane with conjugate as inverse and every element ω_- of the adic skew–plane with conjugate as inverse, which satisfy the identity

$$f(\xi_+,\xi_-)=f(\xi_+\lambda,\xi_-\lambda)$$

for every nonzero element λ of the algebraic skew–plane such that $\lambda^-\lambda$ is a ratio of generating positive integers which are relatively prime to ρ, and whose product with $\xi_+^-\xi_+)^{-1}$ is square integrable with respect to the canonical measure for the product skew–plane over the set of elements whose adic component is integral and has integral inverse.

The Laplace space of order ν and character χ is a Hilbert space which is applied in the description of the range of the Laplace transformation. The elements of the space are functions $F(z,\xi)$ of z in the upper half–plane and invertible elements ξ of the adic line such that the function $F(z,\xi)$ of z is analytic for every invertible element ξ of the adic line, such that $F(z,\xi)$ vanishes when the p–adic component of $2\rho\xi$ is nonintegral for some prime divisor p of ρ, such that the identity

$$F(z,\xi\omega) = F(z,\xi)\chi(\omega)^-$$

holds for every integral element ω of the adic line whose p–adic component has integral inverse for every generating prime p which does not divide ρ when the p–adic component of $2\rho\xi$ is integral for every prime divisor p of ρ, such that the identity

$$F(z,\xi) = F(z\lambda,\xi\lambda)$$

holds for every positive element λ of the algebraic line which is a ratio of generating positive integers relatively prime to ρ, and such that the integral

$$\int\int_0^\infty\int_{-\infty}^{+\infty}|F(x+iy,\xi)|^2y^{2+\nu}dxdyd\xi$$

with respect to the Laplace measure over the set of integral elements of the adic line with integral inverse is finite.

If ν is positive or if ρ is not one, the Laplace transform of a function $f(\xi_+,\xi_-)$ of $\xi=(\xi_+,\xi_-)$ in the product skew–plane is defined as an integral when the function is square integrable with respect to the canonical measure for the product skew–plane over the set of elements whose adic component is integral and has integral inverse. The Laplace transform is the function

$$F(z,\xi) = 2\int\kappa(\eta_+,\eta_-)^-f(\eta_+,\eta_-)\chi(\eta_-^-\eta_-)\theta^*(z\eta_+^-\eta_+,\xi\eta_-^-\eta_-)d\eta$$

of z in the upper half–plane and invertible elements ξ of the adic line which is defined by integration with respect to the canonical measure for the product skew–plane over the set of elements whose adic component is integral and has integral inverse.

If ν is zero and ρ is one, the Laplace transform of a function $f(\xi_+,\xi_-)$ of $\xi=(\xi_+,\xi_-)$ in the product skew–plane is defined as an integral when the function is square integrable with respect to the canonical measure for the product skew–plane over the set of elements whose adic component is integral and has integral inverse. The identity

$$\theta^*(z,\xi) = \theta(z,\xi)$$

is then satisfied. The Laplace transform is the function

$$F(z,\xi) = 2\int f(\eta_+,\eta_-)[\theta(z\eta_+^-\eta_+,\xi\eta_-^-\eta_-)-1]d\eta$$

of z in the upper half–plane and invertible elements ξ of the adic line defined by integration with respect to the canonical measure for the product skew–plane over the set of elements whose adic component is integral and has integral inverse.

In all cases a function $F(z, \xi)$ of z in the upper half–plane and invertible elements ξ of the adic line is a Laplace transform if, and only if, the function

$$G(z, \xi) = \sum n^{\frac{1}{2}\nu} \tau(n) F(zn, \xi n) \chi(n)^-$$

of z in the upper half–plane and invertible elements ξ of the adic line, which is defined as a sum over the generating positive integers which are relatively prime to ρ, belongs to the Laplace space of order ν and character χ. The identity

$$(1+2\nu) \int \int_0^\infty \int_{-\infty}^{+\infty} |G(x+iy, \xi)|^2 y^{2+\nu} dx dy = 8(2\pi)^{-2-\nu} \Gamma(3+\nu) \int |f(\xi_+, \xi_-)|^2 (\xi_+^- \xi_+)^{-2} d\xi$$

holds with integration on the left with respect to the Laplace measure over the set of integral elements of the adic line with integral inverse and with integration on the right with respect to the canonical measure for the product skew–plane over the set of elements whose adic component is integral and has integral inverse. The identity defines the Laplace transformation when the transformation is not defined by an absolutely convergent integral.

A Radon transformation is defined on the space of functions $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane which satisfy the constraints modulo ρ, which satisfy the identity

$$f(\xi_+, \xi_-) = f(\xi_+ \lambda, \xi_- \lambda)$$

for every nonzero element λ of the algebraic skew–plane such that $\lambda^- \lambda$ is a ratio of generating positive integers which are relatively prime to ρ, and which are integrable with respect to the canonical measure for the product skew–plane over the set of elements whose adic component is integral and has integral inverse.

The Radon transformation takes a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane into a function $g(\xi_+, \xi_-)$ in the product skew–plane when the identity

$$g(\omega_+ \xi_+, \omega_- \xi_-) = \int f(\omega_+ \xi_+ + \omega_+ \eta_+, \omega_- \xi_- + \omega_- \eta_-) d\eta$$

holds for almost all elements $\xi = (\xi_+, \xi_-)$ of the product plane for every element ω_+ of the Dedekind skew–plane with conjugate as inverse and for every element ω_- of the adic skew–plane with conjugate as inverse with integration with respect to the canonical measure for the complementary space to the product plane in the product skew–plane over the set of elements whose adic component is integral and has integral inverse.

The function $g(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane satisfies the constraints modulo ρ and satisfies the identity

$$g(\xi_+, \xi_-) = g(\xi_+ \lambda, \xi_- \lambda)$$

for every nonzero element λ of the algebraic skew–plane such that $\lambda^- \lambda$ is a ratio of generating positive integers which are relatively prime to ρ. The inequality

$$\int |g(\omega_+ \xi_+, \omega_- \xi_-)| d\xi \leq \int |f(\xi_+, \xi_-)| d\xi$$

holds for every element ω_+ of the Dedekind skew–plane with conjugate as inverse and every element ω_- of the adic skew–plane with conjugate as inverse with integration on the left with respect to the canonical measure for the product plane over the set of elements whose adic component is integral and has integral inverse and with integration on the right over the canonical measure for the product skew–plane over the set of elements whose adic component is integral and has integral inverse.

The adjoint of the Radon transformation is a maximal dissipative transformation in the domain of the Laplace transformation when ν is positive or ρ is not one. The adjoint of the Radon transformation takes a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane into a function $g(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane when the identity

$$\int \kappa(\eta_+, \eta_-)^- g(\eta_+, \eta_-) \chi(\eta_-^- \eta_-) \theta^*(z \eta_+^- \eta_+, \xi \eta_-^- \eta_-) d\eta$$
$$= (i/z)|\xi| \int \kappa(\eta_+, \eta_-)^- f(\eta_+, \eta_-) \chi(\eta_-^- \eta_-) \theta^*(z \eta_+^- \eta_+, \xi \eta_-^- \eta_-) d\eta$$

holds when z is in the upper half–plane for almost all invertible elements ξ of the adic line with respect to the Laplace measure. The integrals with respect to the canonical measure for the product skew–plane over the set of elements whose adic component is integral and has integral inverse are interpreted as Laplace transforms when they are not absolutely convergent.

A relation T with domain and range in a Hilbert space is said to be nearly maximal dissipative if

$$(T - \lambda^-)(T - \lambda)^{-1}$$

is a contractive transformation of a closed subspace of the Hilbert space of codimension at most one into the Hilbert for some, and hence every, complex number λ in the right half–plane.

The adjoint of the Radon transformation is a nearly maximal dissipative transformation in the domain of the Laplace transformation when ν is zero and ρ is one. The adjoint of the Radon transformation takes a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane into a function $g(\xi_+, \xi_-)$ in the product skew–plane when the identity

$$\int g(\eta_+, \eta_-)[\theta(z \eta_+^- \eta_+, \xi \eta_-^- \eta_-) - 1] d\eta$$
$$= (i/z)|\xi| \int f(\eta_+, \eta_-)[\theta(z \eta_+^- \eta_+, \xi \eta_-^- \eta_-) - 1] d\eta$$

holds when z is in the upper half–plane for almost all invertible elements ξ of the adic line. The integrals with respect to the canonical measure for the product skew–plane over the set of elements whose adic component is integral and has integral inverse are interpreted as Laplace transforms when they are not absolutely convergent.

The range of the Laplace transformation is the set of analytic functions $F(z)$ of z in the upper half–plane such that the function

$$G(z) = \sum n^{\frac{1}{2}\nu} \tau(n) n^2 F(nz)$$

of z in the upper half–plane belongs to the reduced Laplace space of order ν. Summation is over the generating positive integers n which are relatively prime to ρ.

When ν is positive or ρ is not one, the quantized Laplace transform of a function $f(\xi)$ of ξ in the Dedekind skew–plane is the analytic function

$$F(z) = \int (\xi^- \xi) \phi(\xi)^- f(\xi) \theta^*(z\xi^- \xi) d\xi$$

of z in the upper half–plane which is defined as an integral with respect to the canonical measure for the Dedekind skew–plane when the integral is absolutely convergent. When ν is zero and ρ is one, the quantized Laplace transform of the function $f(\xi)$ of ξ in the Dedekind skew–plane is the analytic function

$$F(z) = \int (\xi^- \xi) f(\xi) [\theta(z\xi^- \xi) - 1] d\xi$$

of z in the upper half–plane which is defined as an integral with respect to the same measure. The identity

$$(1 + 2\nu) \int_0^\infty \int_{-\infty}^{+\infty} |G(x + iy)|^2 y^{2+\nu} dx dy = 2(2\pi)^{-\nu-2} \Gamma(3 + \nu) \int |f(\xi)|^2 d\xi$$

holds with integration on the right with respect to the canonical measure for the Dedekind skew–plane. The quantized Laplace transformation is defined so as to maintain the identity.

A maximal dissipative transformation is defined in the domain of the quantized Laplace transformation when ν is positive or ρ is not one. The transformation takes a function $f(\xi)$ of ξ in the Dedekind skew–plane into a function $g(\xi)$ of the Dedekind skew–plane when the identity

$$\int (\xi^- \xi) \phi(\xi)^- g(\xi) \theta^*(z\xi^- \xi) d\xi = (i/z) \int (\xi^- \xi) \phi(\xi)^- f(\xi) \theta^*(z\xi^- \xi) d\xi$$

holds for z in the upper half–plane. The integrals with respect to the canonical measure for the Dedekind skew–plane are interpreted as reduced Laplace transforms when they are not absolutely convergent.

A nearly maximal dissipative transformation is defined in the domain of the quantized Laplace transformation when ν is zero and ρ is one. The transformation takes a function

$f(\xi)$ of ξ in the Dedekind skew–plane into a function $g(\xi)$ of ξ in the Dedekind skew–plane when the identity

$$\int (\xi^-\xi)g(\xi)[\theta(z\xi^-\xi) - 1]d\xi = (i/z) \int (\xi^-\xi)f(\xi)[\theta(z\xi^-\xi) - 1]d\xi$$

holds for z in the upper half–plane. The integrals with respect to the canonical measure for the Dedekind skew–plane are interpreted as reduced Laplace transforms when they are not absolutely convergent.

The Mellin transformation defined by a theta function is an application of the quantized Laplace transformation defined by the theta function. The quantized Laplace transform of a function $f(\xi)$ of ξ in the Dedekind skew–plane is the analytic function

$$g(z) = \int (\xi^-\xi)\phi(\xi)^- f(\xi)\theta^*(z\xi^-\xi)d\xi$$

of z in the upper half–plane when v is positive or ρ is not one and is the function

$$g(z) = \int (\xi^-\xi)f(\xi)[\theta(z\xi^-\xi) - 1]d\xi$$

of z in the upper half–plane when ν is zero and ρ is one. The integral with respect to the canonical measure for the Dedekind skew–plane is interpreted as a Laplace transform when it is not absolutely convergent. The Mellin transform is the analytic function

$$F(z) = \int_0^\infty g(it)t^{\frac{1}{2}\nu+1-iz}dt$$

of z in the upper half–plane which is defined when for some positive number a the function $f(\xi)$ of ξ in the Dedekind skew–plane vanishes in the neighborhood

$$\xi^-\xi < a$$

of the origin.

Since the analytic function

$$W(z) = \pi^{-\frac{1}{2}\nu-2+iz}\Gamma(\tfrac{1}{2}\nu + 2 - iz)\zeta(2 - iz)$$

of z in the upper half–plane admits the integral representation

$$W(z) = (\xi^-\xi)^{\frac{1}{2}\nu+2-iz} \int_0^\infty \theta(it)t^{\frac{1}{2}\nu+1-iz}dt$$

when ν is positive or ρ is not one and the integral representation

$$W(z) = (\xi^-\xi)^{2-iz} \int_0^\infty [\theta(it) - 1]t^{1-iz}dt$$

when ν is zero and ρ is one, the identity

$$F(z)/W(z) = \int \phi(\xi)^- f(\xi)(\xi^-\xi)^{-\frac{1}{2}\nu-1+iz} d\xi$$

holds when z is in the upper half–plane with integration with respect to the canonical measure. The identity

$$(1+2\nu)\int_{-\infty}^{+\infty} |F(x+iy)/W(x+iy)|^2 dx = 16\pi^2 \int |f(\xi)|^2 (\xi^-\xi)^{-2y} d\xi$$

holds when y is positive with integration on the right with respect to the canonical measure for the Dedekind skew–plane.

The function $W(z)$ of z is analytic and without zeros in the upper half–plane. An analytic function $F(z)$ of z in the upper half–plane is the Laplace transform of a function which vanishes when $\xi^-\xi < a$ if, and only if, the least upper bound

$$\sup a^{2y} \int_{-\infty}^{+\infty} |F(x+iy)/W(x+iy)|^2 dx$$

taken over all positive y is finite.

The weighted Hardy space $\mathcal{F}(W)$ is the set of Mellin transforms of functions $f(\xi)$ of $|xi$ in the Dedekind skew–plane which vanish when

$$\xi^-\xi < 1.$$

If ν is positive or if ρ is not one, a maximal dissipative transformation is defined in the space when h is in the interval $[0,1]$ by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space. If ν is zero and ρ is one, a nearly maximal dissipative transformation is defined in the space when h is in the interval $[0,1]$ by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space. When v is zero and ρ is one, an isometric transformation of the space onto itself is defined by taking $F(z)$ into $F(-z)$.

A zeta function of order ν and character χ is a function

$$\zeta(s) = \sum \tau(n) n^{-s}$$

which has an analytic extension to the complex plane when ν is positive or ρ is not one and which has an analytic extension to the complex plane with the exception of a simple pole at two when ν is zero and ρ is one. The zeta function has no zeros in the half–plane

$$\mathcal{R}s > \tfrac{3}{2}.$$

Examples of zeta functions associated with a primitive character modulo ρ are constructed from zeta functions associated with the character modulo one. A Laplace kernel associated with the character modulo one is a function

$$\phi(\xi_+)$$

of $\xi = (\xi_+, \xi_-)$ in the product skew–plane which is determined by a homogeneous harmonic polynomial ϕ of degree ν in the Dedekind component of ξ. The zeta function

$$\sum \tau(n) n^{-s}$$

is a sum over the generating positive integers n.

If χ is a primitive character modulo ρ for a generating positive integer ρ, a Laplace kernel κ associated with the character χ is defined by

$$\kappa(\xi_+, \xi_-) = \phi(\xi_+)\chi(\xi_-^- \xi_-)$$

when the adic component of $\xi = (\xi_+, \xi_-)$ is integral. The corresponding zeta function

$$\sum \tau(n)\chi(n) n^{-s}$$

is a sum over the generating positive integers n which are relatively prime to ρ.

Computable examples of zeta functions are obtained when ν is zero since the homogeneous harmonic polynomial ϕ is a constant. The zeta function

$$\sum \tau(n) n^{-s}$$

associated with the character modulo one has coefficient $\tau(n)$ equal to the sum of the odd divisors of n for every generating positive integer n.

Dirichlet zeta functions appear when all primes are generators of adic topology. The Dirichlet zeta function

$$\zeta_\chi(s) = \sum \chi(n) n^{-s}$$

defined by a primitive character χ modulo ρ is a sum over all positive integers n. The Euler product

$$\zeta_\chi(s)^{-1} = \prod (1 - \chi(p) p^{-s})$$

is taken over the primes p. Sum and product define the Dirichlet zeta function in the half–plane

$$\mathcal{R}s > 1.$$

The Dirichlet zeta function admits an analytic extension to the complex plane when ρ is not one. The functional identity states that the analytic extension of the function

$$(\rho/\pi)^{\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta_\chi(s)$$

of s and the function obtained on replacing s by $1 - s$ and σ by σ^- are linearly dependent when χ is an even character. The analytic extension of the function

$$(\rho/\pi)^{\frac{1}{2}s+\frac{1}{2}}\Gamma(\tfrac{1}{2}s + \tfrac{1}{2})\zeta_\chi(s)$$

of s and the function obtained on replacing s by $1-s$ and χ by χ^- are linearly dependent when χ is an odd character.

The Euler zeta function is the Dirichlet zeta function when ρ is one. The Euler zeta function admits an analytic extension to the complex plane with the exception of a simple pole at one. The Euler functional identity states that the analytic extension of the function

$$\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta_\chi(s)$$

of s and the function obtained on replacing s by $1-s$ are equal. The conjugate character χ^- is identical with χ since χ is identically one on integral elements of the adic line.

The Euler duplication formula for the gamma function

$$2^s\Gamma(\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}s+\tfrac{1}{2}) = 2\sqrt{\pi}\;\Gamma(s)$$

$\pi^{\frac{1}{2}}$ is acceptable if $\sqrt{\pi}$ is awkward is applied in relating the functional identities for Dirichlet zeta functions to the functional identities for Hecke zeta functions of order zero.

The identity

$$\sum \chi(n)\tau(n)n^{-s} = (1-\chi(2)2^{1-s})\zeta_\chi(s)\zeta_\chi(s-1)$$

expresses a zeta function of order zero associated with a primitive character χ modulo ρ in terms of the Dirichlet zeta function associated with the character. The Dirichlet zeta function is the Euler zeta function when ρ is one.

The Dirichlet zeta function has no zeros in the half–plane

$$\mathcal{R}s > \tfrac{1}{2}.$$

The Euler zeta function has no zeros in the half–plane.

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