INTEGRAL TRANSFORMS OF FUNCTIONS WITH RESTRICTED DERIVATIVES

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Abstract. In this paper we show that functions whose derivatives lie in a half-plane are preserved under the Pommerenke, Chandra-Singh, Libera, Alexander and Bernardi Integral Transforms. We determine precisely how these transforms act on such functions. We prove that if the derivative of a function lies in a convex region then the derivative of its Pommerenke, Chandra-Singh, Libera, Alexander and Bernardi Transforms lie in a strickly smaller convex region which can be determined. We also consider iterates of these transforms. Best possible results are obtained.

1. Introduction

Let $A(\mathbb{D})$ denote the class of functions f which are analytic in the unit disk $\mathbb{D}$ and normalized by $f(0) = 0$ and $f'(0) = 1$. The classical family of univalent functions in $A(\mathbb{D})$ is denoted by S. The following are well-known integral transforms on $A(\mathbb{D})$:

$$\mathbf{A}f(z) = \int_0^z \frac{f(\zeta)}{\zeta}\, d\zeta \quad \text{(Alexander Transform [1])}$$

$$\mathbf{L}f(z) = \frac{2}{z}\int_0^z f(\zeta)\, d\zeta \quad \text{(Libera Transform [9])}$$

$$\mathbf{B}_c f(z) = (c+1)\int_0^1 t^{c-1} f(tz)\, dt\ , c > -1 \quad \text{(Bernardi Transform [2])}\,.$$

The Alexander and Libera Transforms are special cases of the Bernardi Transform with $c = 0$ and $c = 1$, respectively.

Biernacki [3] claimed that the Alexander Transform preserved the class S, however a counterexample to this was constructed by Krzyż and Lewandowski

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[8]. Campbell and Singh [4] later showed that S is not preserved under the Libera Transform either. Hence it was of interest to determine which subclasses of S and, more generally, of $A(\mathbb{D})$ are preserved under these and other transforms. It is known that the subclasses of S consisting of convex, starlike and close-to-convex functions (denoted by K, S^* and C, respectively) are each preserved under the Alexander and Libera transforms and also under the Bernardi Transform for $c = 0, 1, 2, \cdots$ (see [2] for example). Ruscheweyh and Sheil-Small [13] also proved these same results using the theory of convolutions.

Another interesting integral transform was first introduced by Pommerenke [11]:

$$\mathbf{P}f(z) = \int_0^z \frac{f(z_1\zeta) - f(z_2\zeta)}{z_1\zeta - z_2\zeta}\, d\zeta\,, \tag{1.1}$$

for fixed $|z_1| \le 1$ and $|z_2| \le 1$. He proved that if $f \in C(\alpha)$ for $0 \le \alpha \le 1$, the class of strongly close-to-convex functions of order α (i.e., $|\arg\{f'(z)/h'(z)\}| \le \pi\alpha/2$ for some convex function h), then $\mathbf{P}f \in C(\alpha)$. Note that $\alpha = 0$ and $\alpha = 1$ correspond to the class of convex and close-to-convex functions, respectively. Recall that a function f is close-to-convex of order α if $\Re e\{f'(z)/h'(z)\} > \alpha$.

Later, and apparently unaware of this result, Chandra and Singh [5] introduced a special case of the transform (1.1) defined by

$$\mathbf{P}_{\nu_1,\nu_2}f(z) = \frac{1}{e^{i\nu_1} - e^{i\nu_2}} \int_0^z \frac{f(te^{i\nu_1}) - f(te^{i\nu_2})}{t}\, dt, \tag{1.2}$$

where $0 \le \nu_1 < \nu_2 < 2\pi$ and proved that convex, starlike and close-to-convex functions of order α as well as strongly close-to-convex functions of order α are all preserved under the transform $\mathbf{P}_{\nu_1,\nu_2}$. Since integral transforms tend to smooth functions these results are not too surprising. In this paper we shall study these transforms on classes of functions in $A(\mathbb{D})$ with restricted derivatives.

A function $f \in A(\mathbb{D})$ is said to be of *bounded turning* of order β, where $0 \le \beta < 1$, if $\Re e\{f'(z)\} > \beta$ for all $z \in \mathbb{D}$. We denote this class by R_β. By the Noshiro-Warshawski Theorem we know that R_β is a subclass of S and is in fact a subclass of close-to-convex functions (see Duren [6]). It is easy to see that the Bernardi Transform maps R_β into R_β:

$$\Re e\{(\mathbf{B}_c f)'(z)\} = (c+1)\int_0^1 t^c\, \Re e\{f'(tz)\}\, dt > (c+1)\int_0^1 t^c\, \beta\, dt = \beta\,.$$

Moreover, it is also known for example that if $f \in R_0$ then $\mathbf{A}f \in S^*$ (see [14]).

Ponnusamy and Ronning [12] generalized R_β and studied the Bernardi Transform of functions in $A(\mathbb{D})$ whose derivatives lie in an arbitrary half-plane. They define this class of functions as

$$\mathcal{P}_\beta = \{f \in A(\mathbb{D}) \,:\, \exists \alpha \in \mathbb{R},\ \Re e[e^{i\alpha}(f'(z) - \beta)] > 0,\, \forall z \in \mathbb{D}\}\,,$$

where $\beta \in \mathbb{R}$, and proved a number of sharp results including finding the largest $\beta = \beta(c, \gamma)$ such that if $f \in \mathcal{P}_\beta$, then its Bernardi Transform $\mathbf{B}_c f(z)$ is starlike of order γ, generalizing the result in [14]. We should point out that unlike R_β, the class $\mathcal{P}_\beta$ may contain nonunivalent functions as can be shown by the function $f(z) = z + z^2$ which belongs to every $\mathcal{P}_\beta$ for $\beta < -1$, but does not belong to S.

We define the class of functions R^α_β as follows:

$$R^\alpha_\beta = \{f \in A(\mathbb{D}) \,:\, \Re e[e^{i\alpha}(f'(z) - \beta)] > 0,\, \forall z \in \mathbb{D}\}\,. \tag{1.3}$$

It is clear that if $f \in R^\alpha_\beta$ then $f'(0) = 1$ and so necessarily we must have

$$(1-\beta)\cos\alpha > 0\,. \tag{1.4}$$

Note that for a fixed β, we have $R^\alpha_\beta \subset \mathcal{P}_\beta$. As above, it is easy to see that the Bernardi Transform also maps R^α_β into R^α_β. It is natural to ask if the class R^α_β is preserved under the Chandra-Singh Transform (1.2) and more generally the Pommerenke Transform (1.1). We prove that this is indeed the case and also show that all these transforms actually map R^α_β into strictly smaller subclasses which can be determined.

We can now state our main results.

Theorem 1. *Let* $\alpha, \beta \in \mathbb{R}$ *satisfy* (1.4). *If* $f \in R^\alpha_\beta$, *then*

(*a*) $\mathbf{P}f \in R^\alpha_{\beta_\mathbf{P}}$, *where*

$$\mathbf{P}f(z) = \int_0^z \frac{f(z_1\zeta) - f(z_2\zeta)}{z_1\zeta - z_2\zeta}\, d\zeta \quad (z_1, z_2 \in \overline{\mathbb{D}}\,),$$

$$\beta_\mathbf{P} = (2\beta - 1) + (1-\beta)\left(\frac{3+\delta}{2+2\delta}\right) \tag{1.5}$$

and $\delta = \max\left\{\min\{|z_1|, |z_2|\}, \dfrac{|z_1+z_2|}{2}\right\}$

(b) $\mathbf{P}_{\nu_1,\nu_2} f \in R^{\alpha}_{\beta_*}$, *where*

$$\mathbf{P}_{\nu_1,\nu_2} f(z) = \frac{1}{e^{i\nu_1} - e^{i\nu_2}} \int_0^z \frac{f(te^{i\nu_1}) - f(te^{i\nu_2})}{t}\, dt \quad (0 \le \nu_1 < \nu_2 < 2\pi),$$

$$\beta_* = (2\beta - 1) + (1-\beta)\left(\frac{\nu}{\sin\nu}\right) \tag{1.6}$$

and $\nu = \frac{1}{2}\min\{(\nu_2 - \nu_1), 2\pi - (\nu_2 - \nu_1)\} \in (0, \frac{\pi}{2}]$.
This result is best possible.

(c) $\mathbf{B}_c f \in R^{\alpha}_{\beta_c}$, *for* $c = 0, 1, 2, \cdots$, *where*

$$\mathbf{B}_c f(z) = (c+1)\int_0^1 t^{c-1} f(tz)\, dt,$$

$$\beta_c = (2\beta - 1) + (1-\beta)\gamma_c \tag{1.7}$$

with $\gamma_0 = \log 4$,

$$\gamma_c = 2(c+1)(-1)^c\left[\log 2 - \sum_{k=1}^{c} \frac{(-1)^{k+1}}{k}\right],\ c = 1, 2, \cdots \tag{1.8}$$

and $1 < \gamma_c < 2$. *This result is best possible.*

Remark 1. If both z_1 and z_2 lie on $|z| = 1$, then the Pommerenke Transform (1.1) reduces to the Chandra-Singh Transform (1.2). Consequently, without loss of generality, we shall henceforth assume when referring to the Pommerenke Transform that at most one of z_1 and z_2 lies on $|z| = 1$. Thus we then have $0 \le \delta < 1$.

The proof of this main theorem is given in the next section. We first state and prove some applications.

Corollary 1. *If* $f \in R^{\alpha}_{\beta}$, *then*

(*i*) $\mathbf{P} f \in R^{\alpha}_{\beta_{\mathbf{P}}} \subset R^{\alpha}_{\beta}$, *where* $\beta_{\mathbf{P}}$ *is given by* (1.5).

(*ii*) $\mathbf{P}_{\nu_1,\nu_2} f \in R^{\alpha}_{\beta_*} \subset R^{\alpha}_{\beta}$, *where* β_* *is given by* (1.6).

(*iii*) $\mathbf{B}_c f \in R^{\alpha}_{\beta_c} \subset R^{\alpha}_{\beta}$, *for* $c = 0, 1, 2, \cdots$, *where* β_c *is given by* (1.7).

Proof. Let α and β be fixed and let

$$\beta^{**} = (2\beta - 1) + M(1-\beta)\,,$$

where $M > 1$ is fixed. We assert that $R^{\alpha}_{\beta^{**}} \subset R^{\alpha}_{\beta}$. The corollary then follows because if $f \in R^{\alpha}_{\beta}$ then from the theorem in each of the cases (i)-(iii) we simply let $M = \dfrac{3+\delta}{2+2\delta}, \dfrac{\nu}{\sin\nu}, \gamma_c$, respectively, to conclude that the corresponding transform F belongs to $R^{\alpha}_{\beta^{**}}$.

To prove our assertion that $R^{\alpha}_{\beta^{**}} \subset R^{\alpha}_{\beta}$ we consider cases. Suppose $F \in R^{\alpha}_{\beta^{**}}$ and recall that $(1-\beta)\cos\alpha > 0$.

Case 1: $-\infty < \beta < 1$. In this case we have $\cos\alpha > 0$ and we obtain

$$\beta^{**} = (2\beta - 1) + M(1-\beta) > \beta\,.$$

Since $F \in R^{\alpha}_{\beta^{**}}$, i.e., $\Re e\{e^{i\alpha}[F'(z) - \beta^{**}]\} > 0$, we obtain

$$\Re e\{e^{i\alpha}F'(z)\} > \beta^{**}\cos\alpha > \beta\cos\alpha,$$

which implies that $F \in R^{\alpha}_{\beta}$.

Case 2: $1 < \beta < \infty$. Here $\cos\alpha < 0$ and observe that $\beta^{**} < \beta$. Thus we have $\Re e\{e^{i\alpha}F'(z)\} > \beta^{**}\cos\alpha > \beta\cos\alpha$ and hence $F \in R^{\alpha}_{\beta}$. ∎

In the above result, these transforms map R^{α}_{β} into strickly smaller subclasses and, since the values given by (1.6) and (1.7) are best possible, the Chandra-Singh and Bernardi Transforms do not map R^{α}_{β} into any class smaller than the corresponding $R^{\alpha}_{\beta^{**}}$.

If the derivative of an arbitrary function in $A(\mathbb{D})$ lies in a region, then one might expect the region in which the derivative of its integral transform lies should be related. We obtain the following result:

Theorem 2. *Let $f \in A(\mathbb{D})$ and let F be its Pommerenke, Chandra-Singh or Bernardi Transform with $c = 0, 1, 2, \cdots$. If $\Delta(f) = \{f'(z) : z \in \mathbb{D}\}$ lies in a convex region Ω, then $\Delta(F) = \{F'(z) : z \in \mathbb{D}\}$ also lies in Ω.*

Proof. Note that $f \in R^{\alpha}_{\beta}$ if and only if $f(rz)/r \in R^{\alpha}_{\beta}$ for any $0 < r < 1$. Hence, without loss of generality, we may assume that $\Omega \subset \mathbb{C}$ is bounded. Furthermore, we may assume that Ω is a convex polygonal region. Consequently it is sufficient to prove the theorem when Ω is a bounded convex polygonal region with m sides. Necessarily we have $1 \in \Omega$.

Let $f \in A(\mathbb{D})$ and suppose that $\Delta(f) = \{f'(z) : z \in \mathbb{D}\} \subset \Omega$. Assume first that $\partial\Omega$ contains no horizontal segments. Because $\overline{\Omega}$ may be obtained as the intersection of m closed half-planes, each containing 1, it follows that

$$f \in \bigcap_{j=1}^{m} R_{\beta_j}^{\alpha_j}$$

for suitable choices of α_j and β_j, each satisfying $(1-\beta_j)\cos\alpha_j > 0$. To see this, we let L_j be the line bounding a side of Ω, β_j its intersection with the real axis and μ_j $(0 < \mu_j < \pi)$ the angle L_j makes with the positive real axis. If $\beta_j > 1$, choose $\alpha_j = \frac{3\pi}{2} - \mu_j$; while if $\beta_j < 1$, set $\alpha_j = \frac{\pi}{2} - \mu_j$. Hence $f \in R_{\beta_j}^{\alpha_j}$ for each j and by Corollary 1 the same holds for F. Thus $F \in \bigcap_{j=1}^{m} R_{\beta_j}^{\alpha_j}$ and so we conclude that $\Delta(F) \subset \Omega$.

If a side of $\overline{\Omega}$ is a horizontal segment then we construct a larger convex polygonal region containing all non-horizontal sides of Ω but replace each horizontal side by two non-horizontal sides as follows. Let $0 < \epsilon < 1$ and define the convex set $\Omega(\epsilon)$ to be bounded by all the lines bounding $\overline{\Omega}$ except the horizontal lines. Each horizontal line is to be replaced by two intersecting lines. In particular, if say Ω is bounded by the horizontal line L_h through the vertices $\omega_1 = a + i\lambda$ and $\omega_2 = b + i\lambda$ with $a < b$ and $\lambda > 0$, then instead of bounding $\Omega(\epsilon)$ by L_h, we bound it by the two lines $L_h^{(1)}$ and $L_h^{(2)}$ which pass through the pair ω_1 and $\omega_\epsilon = \frac{b+a}{2} + i[\lambda + \epsilon(b-a)]$ and the pair ω_2 and ω_ϵ, respectively. With this construction, it is clear that $\Omega \subset \Omega(\epsilon)$ for all $0 < \epsilon < 1$ and that $\Omega(\epsilon)$ has no horizontal lines bounding it. A similar construction holds for $\lambda < 0$. Apply the above argument to $\Omega(\epsilon)$ and let $\epsilon \to 0$ to complete the proof of the theorem. ∎

Remark 2. It should be pointed out that by Corollary 1, since the transforms maps R_β^α strickly into itself, we actually have $\Delta(F) \subset \Omega' \subset \Omega$, where Ω' is a convex region strickly inside Ω. The convex region Ω' can be determined, once Ω is known.

Finally we consider iterates of integral transforms. Because these integral transforms map R_β^α into strickly smaller subclasses the following result obtains:

Theorem 3. *If f is any arbitrary function in R_β^α and $\mathbf{T}f$ is its Pommerenke, Chandra-Singh or Bernardi Transform with $c = 0, 1, 2, \cdots$, then*

$$\lim_{n\to\infty} \mathbf{T}^{(n)} f(z) = z\,,$$

where $\mathbf{T}^{(n)} = \mathbf{T} \circ \mathbf{T} \cdots \circ \mathbf{T}$ is the n^{th} iterate of $\mathbf{T}$ and the convergence is uniform on compact subsets in $\mathbb{D}$.

We shall also prove this theorem in the next section.

2. Proof of the Main Results

We begin with a few preliminaries about the class R_β^α. Assume throughout that α and β are fixed and satisfy (1.4).

It is clear that the function K defined by

$$K(z) = e^{-i\alpha}[Az + B\log(1-z)], \tag{2.1}$$

where

$$\begin{aligned} A &= -\lambda\cos\alpha + i\sin\alpha \\ B &= -(1+\lambda)\cos\alpha \\ \lambda &= 1-2\beta \end{aligned} \tag{2.2}$$

belongs to the class R_β^α and so it is nonempty. The class R_β^α is convex: if $f, g \in R_\beta^\alpha$ then $tf + (1-t)g \in R_\beta^\alpha$ for all $0 \le t \le 1$. It is also rotationally invariant: $f \in R_\beta^\alpha$ if and only if $e^{-i\mu} f(e^{i\mu} z) \in R_\beta^\alpha$ for $\mu \in \mathbb{R}$.

The Carathéodory class $\mathfrak{P}$ consists of all functions p which are analytic in $\mathbb{D}$ with $\Re e\, p(z) > 0$ and normalized by $p(0) = 1$. Observe that $g \in R_\beta^\alpha$ if and only if

$$p(z) = \frac{e^{i\alpha}(g'(z) - \beta) - i(1-\beta)\sin\alpha}{(1-\beta)\cos\alpha} \tag{2.3}$$

belongs to $\mathfrak{P}$. From this and the distortion theorems for $p \in \mathfrak{P}$ (see [6] or [7] for example), we see that if $g \in R_\beta^\alpha$, then $|g'(z)|$ and hence $|g(z)|$ are bounded on all compact sets in $\mathbb{D}$ and so the normalization for functions in R_β^α makes it a compact family.

The extreme points of the Carathéodory class $\mathfrak{P}$ are well-known [7]:

$$\mathcal{E}(\mathfrak{P}) = \left\{ \frac{1+xz}{1-xz} \,:\, |x| = 1 \right\}. \tag{2.4}$$

From (2.3) and (2.4) it follows that the extreme points for the class R_β^α are precisely

$$\mathcal{E}(R_\beta^\alpha) = \{\overline{x}K(xz) \,:\, |x| = 1\} \tag{2.5}$$

where K is defined by (2.1) and (2.2).

We will make use of the following result which is essentially due to Marx [10].

Lemma 1. *If* $H(\theta,\mu) = \Im m\left\{-e^{-i\theta}\log\left(\frac{1-e^{i(\theta+\mu)}}{1-e^{i(\theta-\mu)}}\right)\right\}, 0 \le \theta \le \pi$ *and* $0 \le \mu \le \pi$, *then*

$$\min_{0\le\theta\le\pi} H(\theta,\mu) = \begin{cases} \mu & , 0 \le \mu \le \frac{\pi}{2} \\ \pi - \mu & , \frac{\pi}{2} < \mu \le \pi \end{cases} .$$

<u>Proof</u>. Observe that if $\theta \neq \mu$ then

$$H(\theta,\mu) = \frac{\sin\theta}{2}\log\left(\frac{1-\cos(\theta+\mu)}{1-\cos(\theta-\mu)}\right) - \gamma\cos\theta\,,$$

where

$$\gamma = \begin{cases} \mu & , 0 \le \mu < \theta \le \pi \\ \mu - \pi & , 0 \le \theta < \mu \le \pi \end{cases} .$$

After a calculation we obtain

$$\frac{\partial H}{\partial\theta} = \left(\frac{\cos\theta}{2}\right)\log\left(\frac{1-\cos(\theta+\mu)}{1-\cos(\theta-\mu)}\right) + \frac{\sin\theta\sin\mu}{\cos\theta-\cos\mu} + \gamma\sin\theta\,.$$

A further calculation leads to the following:

(2.6)

$$\begin{aligned} \frac{\partial}{\partial\mu}\left(\frac{\partial H}{\partial\theta}\right) &= \frac{\sin\theta}{(\cos\theta-\cos\mu)^2}\left(2\cos\theta\cos\mu - \cos^2\theta - 1\right) \\ &\le -\frac{(\sin\theta)\,(1-|\cos\theta|)^2}{(\cos\theta-\cos\mu)^2}\,. \end{aligned}$$

Consequently for fixed $0 \le \theta_0 \le \pi$, the function $\dfrac{\partial H}{\partial\theta}$ is nonincreasing with μ.

Suppose first that $0 \le \theta_0 < \mu \le \pi$. Then we see that

$$\frac{\partial H}{\partial\theta}(\theta_0,\mu) \ge \frac{\partial H}{\partial\theta}(\theta_0,\pi) = 0$$

and so for $0 \le \theta < \mu \le \pi$, we see that H is an nondecreasing function of θ and thus

$$H(\theta, \mu) \ge H(0, \mu) = \pi - \mu\,.$$

Next, if $0 \le \mu < \theta_0 \le \pi$ then

$$\frac{\partial H}{\partial \theta}(\theta_0, \mu) \le \frac{\partial H}{\partial \theta}(\theta_0, 0) = 0\,.$$

In this case, H is a nonincreasing function of θ and hence for $0 \le \mu < \theta \le \pi$ we get

$$H(\theta, \mu) \ge H(\pi, \mu) = \mu\,.$$

Thus if $\theta \ne \mu$ then $H(\theta, \mu) \ge \min\{\mu,\, (\pi - \mu)\}$ and the function is unbounded as $\theta \to \mu$. This proves the lemma. ▮

It should be pointed out that there is a typo in formula (65) in Marx[10]. It should read:

$$\frac{\partial}{\partial \phi}\left(4 \sin\phi\, \frac{\partial p(\phi, \theta)}{\partial \theta}\right) = \frac{(2\sin\theta)(2\cos\theta\cos\phi - \cos^2\theta - 1)}{(\cos\theta - \cos\phi)^2}\,.$$

Fortunately, his conclusion that the function on the left is nonpositive still holds as our (2.6) shows.

Lemma 2. *If* $\Phi(\zeta_1, \zeta_2) = \dfrac{1}{\zeta_2 - \zeta_1} \log\left(\dfrac{1 - \zeta_1}{1 - \zeta_2}\right)$ *and* $\zeta_1, \zeta_2 \in \overline{\mathbb{D}}$ $(\zeta_1 \ne \zeta_2)$, *then*

$$\Re e\, \Phi(\zeta_1, \zeta_2) \ge \frac{3 + \delta}{4 + 4\delta}$$

where $\delta = \max\left\{\min\{|\zeta_1|, |\zeta_2|\},\ \dfrac{|\zeta_1 + \zeta_2|}{2}\right\}$.

Proof. Let $\omega(t) = \zeta_1 + (\zeta_2 - \zeta_1)t,\ 0 \le t \le 1$, be the line segment from ζ_1 to ζ_2 in $\overline{\mathbb{D}}$. It follows that $|\omega(t)| \le \delta$ for $0 \le t \le \frac{1}{2}$ or $\frac{1}{2} \le t \le 1$. To see this, suppose say $\delta = |\zeta_1|$ then

$$\left|\omega\left(\frac{1}{2}\right)\right| = \frac{|\zeta_1 + \zeta_2|}{2} \le |\zeta_1| = |\omega(0)| = \delta$$

and hence $|\omega(t)| \leq \delta$ for $0 \leq t \leq \frac{1}{2}$. The proof of the other cases follows similarly. Using this we conclude that

$$\begin{aligned}\Re e\,\Phi(\zeta_1,\zeta_2) &= \Re e\left\{\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\frac{1}{1-z}\,dz\right\}\\ &= \Re e\int_0^1\frac{1}{1-\omega(t)}\,dt\\ &\geq \int_0^1\frac{1}{1+|\omega(t)|}\,dt\\ &\geq \frac{1}{2}\left(\frac{1}{2}\right)+\frac{1}{2}\left(\frac{1}{1+\delta}\right)=\frac{3+\delta}{4+4\delta}. \quad \blacksquare\end{aligned}$$

We can now prove the main results.

Proof of Theorem 1. We consider each transform separately.

(a): Let $F = \mathbf{P}f$. Now for fixed $z_0 \in \mathbb{D}$ we have

$$\Re e\{e^{i\alpha}F'(z_0)\} = \Re e\left\{e^{i\alpha}\left[\frac{f(z_1z_0)-f(z_2z_0)}{z_1z_0-z_2z_0}\right]\right\}.$$

The linear functional $L(f) = e^{i\alpha}\left[\dfrac{f(z_1z_0)-f(z_2z_0)}{z_1z_0-z_2z_0}\right]$ attains its minimum real part over the set of extreme points of R_β^α. (This follows for example from Thm 4.5, p.44, in [7] by observing that $-\min\Re e\{L(f)\} = \max\Re e\{J(f)\}$, where $J(f) = -L(f)$.) It follows that

$$\Re e\{e^{i\alpha}F'(z_0)\} \geq \min_{|x|=1}\Re e\left\{e^{i\alpha}\left[\frac{K(xz_1z_0)-K(xz_2z_0)}{xz_1z_0-xz_2z_0}\right]\right\},$$

where K is given by (2.1). Using (2.1) and (2.2) we obtain

$$\Re e\{e^{i\alpha}F'(z_0)\} \geq \min_{|x|=1}\Re e\left\{A+B\frac{1}{(xz_1z_0-xz_2z_0)}\log\left(\frac{1-xz_1z_0}{1-xz_2z_0}\right)\right\}.$$

The above function is analytic in the variable $z = xz_0$ and hence by the minimum principle and Lemma 2 we may conclude that

$$\begin{aligned}
\Re e\{e^{i\alpha}F'(z_0)\} > &\quad (2\beta-1)\cos\alpha + 2(1-\beta)(\cos\alpha)\min_{|z|=1}\Re e\left\{\Phi(\{zz_1\},\{zz_2\})\right\} \\
\geq &\quad (2\beta-1)\cos\alpha + 2(1-\beta)(\cos\alpha)\left(\frac{3+\delta}{4+4\delta}\right) \\
= &\quad \beta_{\mathbf{P}}\cos\alpha\,.
\end{aligned}$$

Thus for any $z_0 \in \mathbb{D}$, we get $\Re e\{e^{i\alpha}[F'(z_0)-\beta_{\mathbf{P}}]\} > 0$ and hence $F = \mathbf{P}f \in R^{\alpha}_{\beta_{\mathbf{P}}}$.

(b): Let $F = \mathbf{P}_{\nu_1,\nu_2}f$. Note that the function $F \in R^{\alpha}_{\beta_*}$ if and only if $G(z) = e^{-i\mu}F(e^{i\mu}z) \in R^{\alpha}_{\beta_*}$ for any $\mu \in \mathbb{R}$. Hence we see that

(2.7)
$$G(z) = \frac{1}{e^{i(\nu_1+\mu)} - e^{i(\nu_2+\mu)}}\int_0^z \frac{f(se^{i(\nu_1+\mu)}) - f(se^{i(\nu_2+\mu)})}{s}\,ds\,.$$

If $\nu = \frac{(\nu_2-\nu_1)}{2}$ then setting $\mu = -\frac{(\nu_1+\nu_2)}{2}$ in (2.8) gives

$$G(z) = \frac{1}{e^{i\nu} - e^{-i\nu}}\int_0^z \frac{f(se^{i\nu}) - f(se^{-i\nu})}{s}\,ds\,.$$

On the other hand if $\nu = \pi - \frac{(\nu_2-\nu_1)}{2}$, set $\mu = \pi - \frac{(\nu_1+\nu_2)}{2}$ to obtain the same form of $G(z)$. Thus it is sufficient to show that if $f \in R^{\alpha}_{\beta}$, then $G \in R^{\alpha}_{\beta_*}$ where

$$G(z) = \frac{1}{e^{i\nu} - e^{-i\nu}}\int_0^z \frac{f(se^{i\nu}) - f(se^{-i\nu})}{s}\,ds \tag{2.8}$$

with $0 < \nu \leq \frac{\pi}{2}$ and

$$\beta_* = (2\beta-1) + (1-\beta)\left(\frac{\nu}{\sin\nu}\right)\,.$$

For fixed $0 < \nu \leq \frac{\pi}{2}$ we see from (2.8) that

$$\Re e\{\, e^{i\alpha}G'(z)\} = \Re e\left\{\frac{e^{i\alpha}}{2i\sin\nu}\left[\frac{f(ze^{i\nu}) - f(ze^{-i\nu})}{z}\right]\right\}\,.$$

Now fix $z_0 \in \mathbb{D}$ and consider the linear functional on $A(\mathbb{D})$ defined by

$$L(f) = \frac{e^{i\alpha}}{2i\sin\nu}\left\{\frac{f(z_0e^{i\nu}) - f(z_0e^{-i\nu})}{z_0}\right\}\,.$$

The minimum real part of L is achieved at an extreme point of R^{α}_{β}. Hence we have

$$\Re e\{e^{i\alpha}G'(z_0)\} \geq \min_{|x|=1} \Re e\left\{L(\overline{x}K(xz))\right\},$$

where K is given by (2.1). A calculation shows that

$$\{L(\overline{x}K(xz))\} = A + \frac{B}{2i\sin\nu}\left\{\frac{1}{xz_0}\log\left(\frac{1-e^{i\nu}xz_0}{1-e^{-i\nu}xz_0}\right)\right\}.$$

This is an analytic function of $\omega = xz_0$. Using (2.2) it follows from the minimum principle and symmetry that

$$\Re e\{e^{i\alpha}G'(z_0)\} \geq \min_{|x|=1} \Re e\left\{L(\overline{x}K(xz))\right\} > (2\beta - 1)\cos\alpha + \frac{(1-\beta)\cos\alpha}{\sin\nu}\left[\min_{0\leq\theta\leq\pi} H(\theta,\nu)\right]$$

where

$$H(\theta,\nu) = \Im m\left\{-e^{-i\theta}\log\left(\frac{1-e^{i(\theta+\nu)}}{1-e^{i(\theta-\nu)}}\right)\right\}.$$

We may now apply Lemma 1 with $\mu = \nu$ and $0 < \nu \leq \frac{\pi}{2}$ to see that

$$\Re e\{e^{i\alpha}G'(z_0)\} \geq (2\beta - 1)\cos\alpha + (1-\beta)\cos\alpha\left(\frac{\nu}{\sin\nu}\right) = \beta_*\cos\alpha.$$

Hence $\Re e\{e^{i\alpha}[G'(z_0) - \beta_*]\} > 0$ for any $z_0 \in \mathbb{D}$ and so $G \in R^{\alpha}_{\beta_*}$.

To show that β_* is best possible, consider the function $f = K$ given by (2.1) and let $z = -r$. A calculation gives

$$\Re e\{e^{i\alpha}G'(-r)\} = \left[(2\beta - 1) + \frac{(1-\beta)}{\sin\nu}\Im m\left\{\frac{1}{r}\log\left(\frac{1+re^{i\nu}}{1+re^{-i\nu}}\right)\right\}\right]\cos\alpha$$

and hence

$$\lim_{r\to 1} \Re e\{e^{i\alpha}[G'(-r) - \beta_*]\} = 0.$$

(c): Let $F = \mathbf{B}_c$. For $z_0 \in \mathbb{D}$ arbitrary but fixed, the linear functional $L(f) = (c+1)\int_0^1 e^{i\alpha}t^c f'(tz_0)\,dt$ assumes its minimum real part over the set of extreme points of R^{α}_{β} and hence

$$\Re e\{e^{i\alpha}F'(z_0)\} \geq \min_{|x|=1} \Re e\left\{(c+1)\int_0^1 e^{i\alpha}t^c K'(xtz_0)\,dt\right\} \tag{2.9}$$

where K is given by (2.1) and (2.2). Next, by the minimum principle, we see that

$$\begin{aligned}\min_{|x|=1}\int_0^1 \Re e\left\{\frac{t^c}{1-txz_0}\right\}dt &> \min_{-\pi<\theta\leq\pi}\int_0^1 \Re e\left\{\frac{t^c}{1-te^{i\theta}}\right\}dt\\ &\geq \int_0^1 \frac{t^c}{1+t}\,dt\\ &= (-1)^c\left[\log 2-\sum_{k=1}^{c}\frac{(-1)^{k+1}}{k}\right].\end{aligned}$$

Using (2.9) and this estimate we obtain after a calculation

$$\begin{aligned}\Re\{e^{i\alpha}F'(z_0)\} &> \left\{(2\beta-1)+2(1-\beta)(c+1)(-1)^c\left[\log 2-\sum_{k=1}^{c}\frac{(-1)^{k+1}}{k}\right]\right\}(\cos\alpha)\\ &= \{(2\beta-1)+(1-\beta)\gamma_c\}(\cos\alpha)\\ &= \beta_c\cos\alpha\,.\end{aligned}$$

(If $c=0$, then $\int_0^1 \Re e\left\{\frac{1}{1-te^{i\theta}}\right\}dt \geq \log 2$ and from (2.9) we get the above result with $\gamma_0=\log 4$.) Thus we get $\Re e\{e^{i\alpha}[F'(z_0)-\beta_c]\}>0$ and hence we conclude that $F\in R^{\alpha}_{\beta_c}$. Because

$$\frac{1}{2(c+1)}=\int_0^1\frac{t^c}{2}\,dt<\int_0^1\frac{t^c}{1+t}\,dt<\int_0^1 t^c\,dt=\frac{1}{c+1}$$

and

$$\int_0^1\frac{t^c}{1+t}\,dt=(-1)^c\left[\log 2-\sum_{k=1}^{c}\frac{(-1)^{k+1}}{k}\right]=\frac{\gamma_c}{2(c+1)}$$

we must have

$$1<\gamma_c<2\,.$$

To show that β_c is best possible we consider the function $f=K$ given by (2.1) and let $z=-r$:

$$\begin{aligned}\Re e\{e^{i\alpha}F'(-r)\} &= (c+1)\int_0^1\Re e\left\{e^{i\alpha}K'(-tr)\right\}dt\\ &= \left[(2\beta-1)+2(1-\beta)(c+1)\int_0^1\frac{t^c}{1+rt}\,dt\right]\cos\alpha\,.\end{aligned}$$

From this and (1.7) we let $r \to 1$ to obtain that $\Re e\{e^{i\alpha}[F'(-r) - \beta_c]\} \to 0$ and hence β_c is best possible. This completes the proof of the theorem. ∎

Proof of Theorem 3. Fix α and β satisfying (1.4). By Corollary 1, we conclude that each of the transforms $\mathbf{P}, \mathbf{P}_{\nu_1,\nu_2}$ or $\mathbf{B}_c$ (for $c = 0, 1, 2, \cdots$) map R^α_β into $R^\alpha_{\beta^{**}}$, where

$$\beta^{**} = (2\beta - 1) + M(1-\beta) = \beta(2-M) + (M-1) \tag{2.10}$$

and $M = \dfrac{3+\delta}{2+2\delta}$, $M = \dfrac{\nu}{\sin\nu}$ or $M = \gamma_c$, respectively. Recall that $0 \le \delta < 1$ (see Remark 1) and $0 < \nu \le \frac{\pi}{2}$. Consequently in each case we have $1 < M < 2$. Now let

$$\begin{aligned} F_0 &= f \\ F_1 &= \mathbf{T}f \\ &\vdots \\ F_n &= (\mathbf{T} \circ \mathbf{T} \cdots \circ \mathbf{T})f\,, \end{aligned}$$

where $\mathbf{T}$ is the Pommerenke Transform $\mathbf{P}$, the Chandra-Singh Transform $\mathbf{P}_{\nu_1,\nu_2}$ or the Bernardi Transform $\mathbf{B}_c$. For convenience, set $x = (2 - M)$ in (2.10). From Corollary 1, we may apply an induction argument to show that $F_n \in R^\alpha_{\beta(n)} \subset R^\alpha_\beta$ where

$$\beta(n) = \beta x^n + 1 - x^n\,.$$

Let $\epsilon > 0$ be given. It suffices to show that $|F_n(z) - z| < \epsilon$ for all $|z| \le r < 1$ and all $n > N(\epsilon)$. Since $F_n \in R^\alpha_{\beta(n)}$, it follows from (2.3) that

$$F'_n(z) = e^{-i\alpha}\{p(z) - 1\}(1 - \beta(n))\cos\alpha + 1 \tag{2.11}$$

for some $p \in \mathfrak{P}$. Using (2.11) and the estimate $|p(re^{i\theta})| \le (1+r)/(1-r)$ for any $p \in \mathfrak{P}$, we obtain the following:

$$\begin{aligned} |F_n(z) - z| &= \left|\int_0^z [e^{-i\alpha}\{p(\zeta) - 1\}(1-\beta(n))\cos\alpha]\,d\zeta\right| \\ &= \left|ze^{-i\alpha}(1-\beta(n))\cos\alpha \int_0^1 \{p(tz) - 1\}\,dt\right| \\ &\le r(1-\beta(n))\cos\alpha \int_0^1 \left\{\frac{2}{1-rt}\right\}\,dt \\ &= x^n\left\{-2(1-\beta)(\cos\alpha)\log(1-r)\right\}\,. \end{aligned}$$

Hence, since $0 < x < 1$, by choosing n sufficiently large we obtain the desired estimate and this completes the proof of the theorem. ∎

3. Remarks

(1) Our results show that the Pommerenke, Chandra-Singh and Bernardi Transforms map R^α_β into strickly smaller classes. It is not too difficult to see that these transforms map K, S^* and C *into* smaller classes but these subclasses are not given explicitly as we have for R^α_β. It is known however that the Alexander Transform maps S^* one-to-one and onto K i.e., $f \in S^*$ if and only if $\mathbf{A}f \in K$. This is in fact Alexander's original theorem in [1].

(2) The search for invariant subclasses under these transforms stemmed from the fact that S was not preserved under $\mathbf{L}$ or $\mathbf{A}$. The Chandra-Singh Transform does not preserve S either. In fact, simply consider the spirallike function in S given in [8]:

$$f(z) = \frac{z}{(1-iz)^{1-i}} ,$$

where the principal branch of $(1-iz)^{1-i}$ is chosen. If we let $\nu_1 = 0$ and $\nu_2 = \pi$ and apply (1.2) to this f, then

$$F(z) = \mathbf{P}_{0,\pi} f(z) = \frac{1}{2}\left\{e^{i\,\mathrm{Log}(1-iz)} - e^{i\,\mathrm{Log}(1+iz)}\right\} .$$

A check shows that for all $k \in \mathbb{N}$, we get $F(z_k) = 0$ where

$$z_k = i\left(\frac{1-e^{-2\pi k}}{1+e^{-2\pi k}}\right) .$$

This shows that the Chandra-Singh Transform of the univalent function f is of *infinite* valence.

References

[1] Alexander, J. W., *Functions Which Map the Interior of the Unit disk Upon Simple Regions*, Ann. of Math. **17**(1915), 12-22.

[2] Bernardi, S.D., *Convex and Starlike Univalent Functions*, Trans. Amer. Math. Soc. **135**(1969), 429-466.

[3] Biernacki, M., *Sur l'intégrale des fonctions univalentes*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **8**(1960), 29-34.

[4] Campbell, D.M. and Singh, V., *Valence Properties of the Solution of a Differential Equation*, Pacific J. Math. **84**(1979), 29-33.

[5] Chandra, S. and Singh, P., *An Integral Transformation of Some Classes of regular Univalent Functions*, Indian J. Pure Appl. Math. **6**(1975), 1270-1275.

[6] P. L. Duren, *Univalent Functions*, Springer-Verlag, Berlin and New York, 1983.

[7] D. J. Hallenbeck and T. H. MacGregor, *Linear Problems and Convexity Techniques in Geometric Function Theory*, Monographs and Studies in Mathematics 22, Pitman, 1984.

[8] Krzyż, J. and Lewandowski, Z., *On the Integral of Univalent Functions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **11**(1963), 447-448.

[9] Libera, R.J., *Some Classes of Regular Univalent Functions*, Proc. Amer. Math. Soc. **16**(1965), 755-758.

[10] Marx, A., *Untersuchungen über Schlichte Abbildungen*, Math. Ann. **107**(1933), 40-67.

[11] Pommerenke, Ch., *On Close-to-Convex Analytic Functions*, Trans. Amer. Math. Soc. **114**(1965), 176-186.

[12] Ponnusamy, S. and Ronning, F., *Integral Transforms of Functions With the Derivative in a Halfplane*, Israel J. Math. **114**(1999), 177-188.

[13] Ruscheweyh, St. and Sheil-Small, T., *Hadamard Products of Schlicht Functions and the Pólya-Schoenberg Conjecture*, Comment. Math. Helv. **48**(1973), 119-135.

[14] Singh, R. and Singh, S., *Starlikeness and Convexity of Certain Integrals*, Ann. Univ. Mariae Curie-Sklod. Sect. A **35**(1981), 145-148.

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