A RESULT ON REAL POLYNOMIALS WITH REAL CRITICAL POINTS

J. E. BROWN AND V. F. POWELL

Abstract. Let $p(z) = (z - z_1)(z - z_2)\cdots(z - z_n)$ be a polynomial whose zeros z_k all lie in the closed unit disk $|z| \le 1$. It is known that there exists $\delta_n > 0$ such that if a zero z_j satisfies $|z_j| \le \delta_n$, then the disk $|z - z_j| \le 1$ contains a critical point of $p(z)$. The Sendov Conjecture asserts that $\delta_n = 1$ for all $n > 2$. We improve the known values of δ_n for the case when $p(z)$ has only real critical points and real coefficients.

1. Introduction and Main Results

Let $\mathcal{P}(n)$ denote the class of all monic polynomials of degree n with zeros z_k which all lie in the closed unit disk $|z| \le 1$:

$$p(z) = \prod_{k=1}^{n} (z - z_k), \quad z_k \in \mathbb{C}, \quad |z_k| \le 1\,. \tag{1}$$

Since the zeros are bounded, there is no loss of generality to assume they are bounded by 1 and since we are concerned only with zeros and critical points we assume the polynomial is monic. The derivative of $p(z)$ has the form

$$p'(z) = n \prod_{j=1}^{n-1} (z - \zeta_j)\,. \tag{2}$$

Rolle's Theorem fails for complex functions as the example $f(z) = e^{\pi i z} - 1$ shows. This function has zeros at $z = 0$ and $z = 2$ but there are no zeros of its derivative $f'(z) = \pi i e^{\pi i z}$ along the interval [0, 2]. In fact this derivative never vanishes. There is however a simple analogue of Rolle's Theorem which holds for complex polynomials. The well-known *Gauss-Lucas Theorem* asserts that all the critical points of $p(z)$ lie in the closed convex hull of its zeros.

Suppose $p(z)$ has the form (1) and that, say, $z_1 = 0$, then by the Gauss-Lucas Theorem we see that all criticial points of $p(z)$ must lie in the unit disk $|z| \le 1$ and thus in particular shows that the disk $|z - z_1| \le 1$ contains a critical point of $p(z)$. A natural question then arises, namely does this result remain true if we perturb z_1 ? More precisely, we ask:

2000 *Mathematics Subject Classification.* Primary 30D50; Secondary 30 D99.

Key words and phrases. Real polynomials, critical points, Sendov Conjecture.

Question: Does there exist a $\delta_n > 0$ such that if $p(z) = \prod_{k=1}^{n}(z - z_k) \in \mathcal{P}(n)$ and if $|z_j| \le \delta_n$ for some $j = 1, 2, \cdots n$, then the disk $|z - z_j| \le 1$ must contain a critical point of $p(z)$?

This question is closely related to a famous unsolved conjecture:

SENDOV CONJECTURE (1962): *If $p(z) = \prod_{k=1}^{n}(z - z_k)$ with $|z_k| \le 1$ for $k = 1, 2 \cdots n$, then each of the disks $|z - z_k| \le 1$ must contain a critical point of $p(z)$. The polynomial $p(z) = z^n - 1$ shows that "1" is best possible.*

The Sendov Conjecture is true if and only if the answer to the question is yes and $\delta_n = 1$. For a brief survey of this conjecture see [4] or see [6] for a much more extensive survey.

The Sendov Conjecture has been verified for polynomials of degree 8 or less [2]. However the full validity of the Sendov Conjecture is unknown even for polynomials with real critical points. Hence we will consider the question for the subclass $\mathcal{P}_{\mathbb{R}}(n)$ which consists of polynomials in $\mathcal{P}(n)$ which have only *real* critical points and real coefficients.

There are some known results related to the question for $n \ge 4$. Phelps and Rodriguez [5] estimated δ_n as the root in $0 < x \le 1$ of the equation

$$(1 + x^2)(1 + x)^{n-3} - n = 0\,.$$

Their result however applies only for polynomials in $\mathcal{P}(n)$ which are extremal for the Sendov Conjecture. Brown([1]) improved this result but only for polynomials extremal for the Sendov Conjecture for the subclass $\mathcal{P}_{\mathbb{R}}(n)$. His estimate for δ_n is the root in $0 < x \le 1$ of the equation

$$2x(1 + x^2)(1 + x)^{n-4} - n = 0\,.$$

Our result improves both estimates of δ_n and the polynomial need not be extremal for the Sendov Conjecture. Before stating the main result we introduce some notation. For convenience we let

$$M_0(x) = \frac{2x}{1 + x^2} \tag{3}$$

and

$$M_1(x) = \begin{cases} 1\,, & \text{if } n \text{ is even} \\ \dfrac{1 + 4x + x^2}{2(1 + x + x^2)}, & \text{if } n \text{ is odd} \end{cases}\,. \tag{4}$$

Define δ_n to be the largest number $x \in (0, 1]$ such that

$$\Phi(x) \le \frac{1}{1 + x - x^2} \quad \text{for} \quad 0 \le x \le \delta_n \le 1\,. \tag{5}$$

where

$$\Phi(x) = \left(\frac{M_0(x)}{n - x[n-3+M_1(x)+M_0(x)]} \right)^{\frac{1}{n-1}} \tag{6}$$

Using the values of δ_n defined as above, we can now state our main result:

Theorem 1. *Let* $p(z) = \prod_{k=1}^{n}(z-z_k)$ *be a polynomial in* $\mathcal{P}_{\mathbb{R}}(n)$ *of degree* $n > 2$. *If* $|z_j| \le \delta_n$ *for some* $j = 1, 2, \cdots n$, *then the disk* $|z - z_j| \le 1$ *must contain a critical point of* $p(z)$.

We can compare our results with those known earlier for a few values of n :

	Phelps & Rodriguez	*Brown*	*Theorem* 1
δ_9	0.4060	0.4913	0.5282
δ_{10}	0.3649	0.4491	0.4794
δ_{11}	0.3319	0.4144	0.4432
δ_{12}	0.3050	0.3852	0.4101
δ_{13}	0.2825	0.3604	0.3842
δ_{14}	0.2634	0.3389	0.3601
δ_{15}	0.2469	0.3202	0.3406

Since the estimates of Brown [1] only applied to polynomials in $\mathcal{P}_{\mathbb{R}}(n)$ which are extremal for the Sendov Conjecture, our estimates are slightly better here and substantially better than those of Phelps and Rodriguez [5]. The results in [1] and [5] used special properties of extremal polynomials which we do not use here and still obtain better results.

2. Preliminary Results

Let $P(z) \in \mathcal{P}(n)$ be a polynomial of the form

$$P(z) = (z-a)\prod_{k=1}^{n-1}(z - Z_k), \text{ where } 0 < a < 1 \tag{7}$$

with

$$P'(z) = n\prod_{j=1}^{n-1}(z-\zeta_j)\,.$$

For $j = 1, 2, \cdots, n-1$ and $k = 1, 2, \cdots, n-1$, we define

$$\gamma_j = \frac{\zeta_j - a}{a\zeta_j - 1}$$

and

$$w_k = \frac{Z_k - a}{aZ_k - 1}\,. \tag{8}$$

We require the following results:

Lemma 1. *If* $|\gamma_j| \le \dfrac{1}{1+a-a^2}$ *for some value of* $j = 1, 2, \cdots, n-1$, *then the disk* $|z-a| \le 1$ *contains a critical point of* $P(z)$, *namely* ζ_j.

Proof: By hypothesis we have $|\gamma_j| = \left|\dfrac{\zeta_j - a}{a\zeta_j - 1}\right| \le \dfrac{1}{1+a-a^2}$ and hence $\zeta_j - a = \rho e^{i\theta}(a\zeta_j - 1)$ from which we get $\zeta_j = (a-\rho e^{i\theta})/(1-a\rho e^{i\theta})$, where $0 \le \rho \le \dfrac{1}{1+a-a^2}$ and $0 \le \theta < 2\pi$. It follows that

$$|\zeta_j - a| \le \frac{\rho(1-a^2)}{1-a\rho} \le 1\,. \qquad \square$$

From the above result we will eventually need an estimate on $|\gamma_j|$. The following result was first proved by Joyal [3]. We include the proof here for completeness.

Lemma 2. *If* $A = \prod_{k=1}^{n-1} |w_k|$ *and* $B = -\sum_{k=1}^{n-1} w_k$, *then*

$$\prod_{j=1}^{n-1} |\gamma_j| = \frac{A}{|n+aB|}\,.$$

Proof: Suppose $P(z)$ is a polynomial of the form (7). Let L be the linear fractional transformation $z = L(w) = \dfrac{w-a}{aw-1}$ and let $\lambda = -(a^2-1)^{-n}\prod_{k=1}^{n-1}(aw_k - 1)$. We see that the rational function $P(L(w))$ has zeros at $w = 0$ and $w = w_k$ $(k = 1, 2, \cdots, n-1)$ and then after a simple calculation we obtain $\lambda P(L(w)) = P_0(w)(aw-1)^{-n}$, where $P_0(w) = w^n + b_{n-1}w^{n-1} + \cdots + b_1 w$. Since the zeros of $P(L(w))$ are w_k, we have

$$b_1 = (-1)^{n-1}\prod_{k=1}^{n-1} w_k \qquad \text{and} \qquad b_{n-1} = -\sum_{k=1}^{n-1} w_k\,.$$

It follows that

$$\lambda\frac{dP(L(w))}{dw} = \lambda P'(L(w))\frac{dz}{dw} = \left\{-a(aw-1)^{-n-1}\right\} D_{\frac{1}{a}}P_0(w) \tag{9}$$

where $'$ denotes differentiation with respect to z and $D_{\frac{1}{a}}P_0(w)$ is the *polar derivative* of $P_0(w)$ with respect to $\frac{1}{a}$:

$$D_{\frac{1}{a}}P_0(w) = nP_0(w) + \left(\frac{1}{a} - w\right)p_0'(w) = \left(b_{n-1} + \frac{n}{a}\right)\left[w^{n-1} + \cdots + \frac{b_1}{n + ab_{n-1}}\right].$$

Now from (9), the zeros of $D_{\frac{1}{a}}P_0(w)$ are precisely the zeros of $P'(L(w))$, i.e., when $L(w) = \zeta_j$ or $w = \gamma_j$. Hence we get

$$(-1)^{n-1}\prod_{j=1}^{n-1}\gamma_j = \frac{b_1}{n + ab_{n-1}}$$

and the result now follows. □

For the special class $\mathcal{P}_{\mathbb{R}}(n)$, we can prove the following:

Lemma 3. *If* $p(z) = \prod_{k=1}^{n}(z - z_k) \in \mathcal{P}_{\mathbb{R}}(n)$ *and* $|z_j - 1| \leq 1$, *for some* $j = 1, 2, \cdots, n$, *then the disk* $|z - z_j| \leq 1$ *contains a critical point of* $p(z)$.

Proof: Let $p(z) = \prod_{k=1}^{n}(z - z_k) \in \mathcal{P}_{\mathbb{R}}(n)$ and let $\Omega = \{z : |z-1| \leq 1,\ |z| \leq 1\}$. We first assert that if $z_j \in \Omega$ then the disk $|z - z_j| \leq 1$ contains the interval $[0,1]$. To verify this, let $f(x) = |z_j - x|^2$ and note that since $z_j \in \Omega$, if $0 \leq x \leq 1$ then

$$f(x) \leq \max\{f(0), f(1), f(\Re e\{z_j\})\}.$$

Note that $f(0) \leq 1$, $f(1) = |z_j - 1|^2 \leq 1$ and $f(\Re e\{z_j\}) = (\Im m\{z_j\})^2 \leq 1$. Thus $f(x) \leq 1$ for all $0 \leq x \leq 1$ and this proves the assertion.

<u>*Case 1*</u>: Suppose there exists a zero, say, z_1, with $|z_1| = 1$ and $\Re e\, z_1 \geq 0$. It is well-known (see [4] for example) that if $|z_1| = 1$, then there must be a critical point x_0 in the disk $\left|z - \frac{z_1}{2}\right| \leq \frac{1}{2}$. Since $\Re e\, z_1 \geq 0$ and all critical points are real, we conclude that $x_0 \in [0,1]$. The above assertion proves the lemma in this case.

<u>*Case 2*</u>: Suppose there are no zeros of $p(z)$ on the semicircle $|z| = 1$ and $\Re e\{z\} \geq 0$. Then there exists a unique ϵ, where $0 < \epsilon < 1$, such that the shifted polynomial

$$p^*(z) = p(z - \epsilon) = \prod_{k=1}^{n}(z - z_k^*) \in \mathcal{P}_{\mathbb{R}}(n)$$

has zeros $z_k^* = z_k + \epsilon$ with the property that there is a zero, say, z_1^*, which satisfies $|z_1^*| = 1$ and $\Re e\{z_1^*\} > 0$. Apply Case 1 to the zero z_1^* to conclude that there must be a critical point of $p^*(z)$, say $x_0^* = x_0 + \epsilon$, where x_0 is a critical point of $p(z)$, such that $x_0^* \in [0,1]$. Since $z_j \in \Omega$ the shifted zero $z_j^* = z_j + \epsilon$ also belongs to Ω and hence by the above assertion the disk $|z - z_j^*| \leq 1$ contains the critical point $x_0^* \in [0,1]$, i.e., $|x_0^* - z_j^*| \leq 1$. Now this yields $|x_0 - z_j| = |x_0^* - z_j^*| \leq 1$ and the proof of the lemma is complete. □

3. Proof of Main Result

Let $p(z)$ be an arbitrary polynomial in $\mathcal{P}_{\mathbb{R}}(n)$. Hence we have

$$p(z) = \prod_{k=1}^{n} (z - z_k), \text{ where } |z_k| \le 1 \text{ for } k = 1, 2, \cdots, n$$

and

$$p'(z) = n \prod_{j=1}^{n-1} (z - x_j), \text{ with } x_j \in \mathbb{R} \text{ for } j = 1, 2, \cdots, n\,.$$

Observe that since all critical points must lie in the unit disk, we have $-1 \le x_j \le 1$ for $j = 1, , 2, \cdots, n - 1$.

Assume that $|z_j| \le \delta_n$, for some fixed j, where δ_n is defined as before, namely the largest number $x \in (0, 1]$ such that

$$\Phi(x) \le \frac{1}{1 + x - x^2} \quad \text{for} \quad 0 \le x \le \delta_n \le 1$$

where $\Phi(x)$ is defined in (6). Note that if $|z_j| = 0$, then the theorem holds by the Gauss-Lucas Theorem, while if $|z_j| = 1$ then there is a critical point in $|z - \frac{z_j}{2}| \le \frac{1}{2}$ as remarked earlier. Hence we may assume that $0 < |z_j| < 1$. By relabeling, if necessary, assume $j = n$. Thus we have

$$z_n = ae^{i\theta} \tag{10}$$

with $0 < a \le \delta_n$. Note that if $p(z)$ belongs to $\mathcal{P}_{\mathbb{R}}(n)$ then so do $\overline{p(\overline{z})}$ and $(-1)^n p(-z)$. Hence, without loss of generality, we may assume that z_n lies in the first quadrant and so $0 \le \theta \le \frac{\pi}{2}$. Note also that if $0 \le \theta \le \frac{\pi}{3}$, then $|z_n - 1| \le 1$ and so by Lemma 3 the disk $|z - z_n| \le 1$ contains a critical point of $p(z)$. Thus we may henceforth suppose that

$$\frac{\pi}{3} < \theta \le \frac{\pi}{2} \tag{11}$$

and

$$0 < a \le \delta_n\,. \tag{12}$$

Because the coefficients of $p(z)$ are real, the non-real zeros of $p(z)$ must occur in conjugate pairs. If in addition $p(z)$ has odd degree, then at least one zero must be real. Thus since $z_n \notin \mathbb{R}$, then we must have, say, $z_{n-1} = ae^{-i\theta}$ and so

$$p(z) = (z - ae^{i\theta})(z - ae^{-i\theta}) \prod_{k=1}^{n-2} (z - z_k)\,.$$

Consider the rotated polynomial

$$P(z) = e^{-in\theta} p(ze^{i\theta}) = \prod_{k=1}^{n} (z - Z_k) = (z - a) \prod_{k=1}^{n-1} (z - Z_k)\,,$$

with derivative

$$P'(z) = n \prod_{j=1}^{n-1} (z - \zeta_j)$$

where $Z_k = z_k e^{-i\theta}$ and $\zeta_j = x_j e^{-i\theta}$. Now by (8), we clearly have the estimate $|w_k| \leq 1$ for all $k = 1, 2, \cdots, n-1$, but we can obtain better estimates.

First observe that it can be shown that

$$|w_{n-1}| = \left| \frac{Z_{n-1} - a}{aZ_{n-1} - 1} \right| = \left| \frac{ae^{-2i\theta} - a}{a^2 e^{-2i\theta} - 1} \right| \leq \frac{2a}{1+a^2} = M_0(a) \tag{13}$$

and that

$$\Re e\{w_{n-1}\} = \Re e \left\{ \frac{ae^{-2i\theta} - a}{a^2 e^{-2i\theta} - 1} \right\} \leq \frac{2a}{1+a^2} = M_0(a) \, . \tag{14}$$

These estimates are best possible. We can use these to estimate the values of A and B in Lemma 2. Clearly we have

$$A = \prod_{k=1}^{n-1} |w_k| \leq |w_{n-1}| \leq M_0(a) \, . \tag{15}$$

Since $B = -\sum_{k=1}^{n-1} w_k$, we get

$$-\Re e\{B\} = \sum_{k=1}^{n-1} \Re e\{w_k\} \leq (n-3) + \Re e\{w_{n-2}\} + \Re e\{w_{n-1}\}$$

and hence

$$-\Re e\{B\} \leq (n-3) + \Re e\{w_{n-2}\} + M_0(a)$$

If n is even, then $\Re e\{w_{n-2}\} \leq 1$; however when n is odd we can do better. Since at least one zero of $p(z)$ must be real, say z_{n-2}, it follows that the rotated zero must be $Z_{n-2} = re^{-i\theta}$, where $-1 \leq r \leq 1$ and $\frac{\pi}{3} < \theta \leq \frac{\pi}{2}$. Observe that

$$\Re e\{w_{n-2}\} = \Re e \left\{ \frac{re^{-i\theta} - a}{are^{-i\theta} - 1} \right\} = \frac{a(1+r^2) - rx(1+a^2)}{1 + a^2 r^2 - 2arx} = H(x, r) \, ,$$

where $x = \cos\theta \in [0, \frac{1}{2})$. From this we see that $\dfrac{\partial H}{\partial x} < 0$ if $0 < r \leq 1$ and $\dfrac{\partial H}{\partial x} > 0$ if $-1 \leq r < 0$. Calculations then give

$$H(x, r) \leq \begin{cases} \dfrac{2a}{1+a^2} & , \text{ if } 0 < r \leq 1 \\[2ex] \dfrac{1 + 4a + a^2}{2(1 + a + a^2)} & , \text{ if } -1 \leq r \leq 0 \end{cases}$$

It is clear that $\dfrac{1+4a+a^2}{2(1+a+a^2)} \leq \dfrac{2a}{1+a^2}$ and hence if we define $M_1(a)$ as

$$M_1(a) = \begin{cases} 1 \quad , & \text{if } n \text{ is even} \\ \dfrac{1+4a+a^2}{2(1+a+a^2)}, & \text{if } n \text{ is odd} \end{cases} ,$$

then we obtain the estimate

$$-\Re e\{B\} \leq (n-3) + M_1(a) + M_0(a)\,. \tag{16}$$

Without loss of generality $|\gamma_1| \leq |\gamma_j|$ for $j = 1, 2, \cdots, n-1$. Using the estimates (15) and (16), from Lemma 2 we obtain

$$\begin{aligned} |\gamma_1|^{n-1} &\leq \prod_{j=1}^{n-1} |\gamma_j| = \frac{A}{|n+aB|} \\ &\leq \frac{A}{n + a\Re e\{B\}} \\ &\leq \left(\frac{M_0(a)}{n - a[n-3+M_1(a)+M_0(a)]} \right) = \Phi(a)^{n-1}\,, \end{aligned}$$

where $\Phi(x)$ is defined by (6). Now by assumption,

$$\Phi(a) \leq \frac{1}{1+a-a^2}$$

and hence we obtain

$$|\gamma_1| \leq \frac{1}{1+a-a^2}\,.$$

Apply Lemma 1 to conclude that the disk $|z-a| \leq 1$ contains the critical point ζ_1 of the (rotated) polynomial $P(z)$, i.e., $|\zeta_1 - a| \leq 1$. Hence we get $|x_1 - z_n| \leq 1$ and so the disk $|z - z_n| \leq 1$ contains the critical point x_1 and this completes the proof of the theorem. □

4. Remarks

The interpretation of our results in terms of force fields can be traced back to Gauss. This is pointed out for example in the article by Marden [4] and in the article by Walsh [7] where he writes:

“ *If there are plotted in the plane of the complex variable the roots of a polynomial $f(z)$ and the roots of the derived polynomial $f'(z)$, there are interesting geometric relations between the two sets of points. It was shown by Gauss that the roots of $f'(z)$ are the positions of equilibrium in the force field due to equal particles situated at each root of $f(z)$, if each particle repels with a force equal to the inverse distance. The derivative vanishes not only at the positions of equilibrium but also at the multiple roots of $f(z)$.*”

Indeed, suppose a point charge is located at $z_0 \in \mathbb{C}$. If the force F exerted on a test charge located at $z \in \mathbb{C}$ is inversely proportional to its distance from z_0, then

$$F = k\overline{\left\{\frac{1}{z - z_0}\right\}}. \tag{17}$$

Now suppose that a unformly charged infinitely long wire is placed perpendicular to the complex plane $\mathbb{C}$ and passes through the point $z_0 \in \mathbb{C}$. By Coulomb's Law it can be shown that the force exerted on a test charge at z is given by (17). This is because the wire has symmetric contributions of force from above and below the complex plane and hence the resulting force is planar. Moreover, instead of the force being inversely proportional to the *square* of the distance, it can be shown that it is inversely proportional to the distance. Consequently any complex polynomial $p(z) = (z - z_1)(z - z_2)\cdots(z - z_n)$ gives rise to a force field with point charges at z_k where the force exerted on a test charge is inversely proportional to the distance from the point charges. And conversely, given any such force field (with positive integer charges) there corresponds a complex polynomial. In addition since

$$\overline{\frac{p'(z)}{p(z)}} = \overline{\left\{\sum_{k=1}^{n} \frac{1}{z - z_k}\right\}},$$

the equilibrium points in the force field correspond to the critical points of $p(z)$ (unless the critical point is also a zero of $p(z)$). Thus we may interpret our result physically, namely if all the point charges of this type of force field lies inside or on the unit circle, then for every point charge sufficiently close to the origin there must be an equilibrium point within unit distance of it (unless the point charge has a charge greater than 1, in this case the zero corresponding to the point charge has multiplicity larger than 1 and is also a critical point).

Acknowledgement. *The authors thank the referee for several helpful comments. The second author would like to thank the MARC-AIM summer research program at Purdue University for their hospitality and generous support.*

References

[1] J. E. Brown, *On The Sendov Conjecture For Polynomials With Real Critical Points*, AMS Contemporary Math. **252**(1999), 49-62.

[2] J. E. Brown and G. P. Xiang, *Proof of the Sendov Conjecture For Polynomials of Degree at Most Eight*, J. Math. Anal. Appl. **232**(1999), 272-292.

[3] A. Joyal, *On the Zeros of a Polynomial and its Derivative*, J. Math. Anal. Appl. **25**(1969), 315-317.

[4] M. Marden, *Conjectures On The Critical Points of a Polynomial*, Amer. Math. Monthly **90**(1980), 267-276 .

[5] D. Phelps and R. Rodriguez, *Some Properties of Extremal Polynomials for the Ilieff Conjecture*, Kodai Math. Sem. Report **24**(1972), 172-175.

[6] Bl. Sendov, *Hausdorff Geometry of Polynomials*, East Jour. Approximations **7**, Number 2 (2001), 123-178.

[7] J. L. Walsh, *On the Location of the Roots of the Derivative of a Polynomial*, Comptes Rendus du Congrès International des Mathématiciens, Strasbourg 1920, 339-342.

Current address: Department of Mathematics Purdue University, West Lafayette, IN 47907 and Fayetteville State University, Fayetteville, NC