

# A CARLEMAN TYPE THEOREM FOR PROPER HOLOMORPHIC EMBEDDINGS

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## **&1. Introduction**

We denote by  $\mathbf{C}$  the field of complex numbers and by  $\mathbf{R}$  the field of real numbers. To motivate our main result we recall the Carleman approximation theorem [4,11]: *For each continuous function  $\lambda: \mathbf{R} \rightarrow \mathbf{C}$  and positive continuous function  $\eta: \mathbf{R} \rightarrow (0, \infty)$  there exists an entire function  $f$  on  $\mathbf{C}$  such that  $|f(t) - \lambda(t)| < \eta(t)$  for all  $t \in \mathbf{R}$ .* If  $\lambda$  is smooth, we can also approximate its derivatives by those of  $f$ . A more general result was proved by Arakelian [2] (see [14] for a simple proof).

Let  $\mathbf{C}^n$  be the complex Euclidean space of dimension  $n$ . Our main result is an extension of Carleman's theorem to *proper holomorphic embeddings* of  $\mathbf{C}$  into  $\mathbf{C}^n$  for  $n > 1$ :

**1.1 Theorem.** *Let  $n > 1$  and  $r \geq 0$  be integers. Given a proper embedding  $\lambda: \mathbf{R} \hookrightarrow \mathbf{C}^n$  of class  $C^r$  and a continuous positive function  $\eta: \mathbf{R} \rightarrow (0, \infty)$ , there exists a proper holomorphic embedding  $f: \mathbf{C} \hookrightarrow \mathbf{C}^n$  such that*

$$|f^{(s)}(t) - \lambda^{(s)}(t)| < \eta(t), \quad t \in \mathbf{R}, \quad 0 \leq s \leq r.$$

*If in addition  $T = \{t_j\} \subset \mathbf{R}$  is discrete, there exists  $f$  as above such that*

$$f^{(s)}(t) = \lambda^{(s)}(t), \quad t \in T, \quad 0 \leq s \leq r.$$

**Definition.** *Two proper holomorphic embeddings  $f, g: \mathbf{C} \hookrightarrow \mathbf{C}^n$  are said to be  $\text{Aut}\mathbf{C}^n$ -equivalent if  $\Phi \circ f = g$  for some holomorphic automorphism  $\Phi$  of  $\mathbf{C}^n$ .*

**1.2 Corollary.** *For each  $n > 1$  the set of  $\text{Aut}\mathbf{C}^n$ -equivalence classes of proper holomorphic embeddings  $\mathbf{C} \hookrightarrow \mathbf{C}^n$  is uncountable.*

For  $n \geq 3$  the corollary is due to Rosay and Rudin [16]. The corollary follows from theorem 1.1 and a result of Rosay and Rudin [15] to the effect that for each  $n > 1$  there exist uncountably many discrete sets in  $\mathbf{C}^n$  which are pairwise inequivalent under the group of holomorphic automorphisms of  $\mathbf{C}^n$ . Theorem 1.1 provides for each discrete set  $E = \{e_k: k = 1, 2, 3, \dots\} \subset \mathbf{C}^n$  a proper holomorphic embedding  $f_E: \mathbf{C} \hookrightarrow \mathbf{C}^n$  such that

$f_E(k) = e_k$  for all  $k = 1, 2, 3, \dots$  (For  $n \geq 3$  such embeddings were constructed in [16].) Clearly the embeddings corresponding to inequivalent discrete sets are inequivalent.

In this context we recall that the first construction of proper holomorphic embeddings  $\mathbf{C} \hookrightarrow \mathbf{C}^2$  which are inequivalent to the standard embedding  $\zeta \rightarrow (\zeta, 0)$  by automorphisms of  $\mathbf{C}^2$  can be found in [8]. On the other hand, it is well known that all polynomial embeddings  $\mathbf{C} \hookrightarrow \mathbf{C}^2$  are equivalent to the standard embedding by polynomial automorphisms of  $\mathbf{C}^2$  [1,18].

We emphasize that, in theorem 1.1, one cannot expect in general to *extend*  $\lambda$  to a holomorphic embedding of  $\mathbf{C}$  into  $\mathbf{C}^n$ . If  $\lambda$  is real-analytic, it will extend holomorphically to some open set in  $\mathbf{C}$ , but in general not to all of  $\mathbf{C}$ ; and even if  $\lambda$  extends to all of  $\mathbf{C}$ , the extension need not be a proper map into  $\mathbf{C}^n$ . So the best we can do in general is to approximate  $\lambda$  by a proper holomorphic embedding as in theorem 1.1.

The original motivation for theorem 1.1 was the question, communicated to us by R. Narasimhan, as to whether there exist proper holomorphic embeddings  $f: \mathbf{C} \rightarrow \mathbf{C}^2$  such that  $f(\mathbf{C})$  is a nontrivial knot in  $\mathbf{C}^2$ , i.e.,  $\mathbf{C}^2 \setminus f(\mathbf{C})$  is not homeomorphic to  $\mathbf{C}^2 \setminus (\mathbf{C} \times \{0\})$ , the complement of the embedding  $\zeta \mapsto (\zeta, 0)$ . Unfortunately we have not been able to answer this question even with the aid of theorem 1.1.

In order to place theorem 1.1 in context we recall some recent developments on embedding Stein manifolds in  $\mathbf{C}^n$  from [3,7,8,9]. (For Stein manifolds and other topics in several complex variables mentioned here we refer the reader to Hörmander [12].) In these papers it was shown that a Stein manifold  $M$  which admits a proper holomorphic embedding in  $\mathbf{C}^n$  for some  $n > 1$  also admits an embedding  $f: M \hookrightarrow \mathbf{C}^n$  whose image  $f(M)$  contains a given discrete subset  $E \subset \mathbf{C}^n$  (with prescribed finite order jet of  $f(M)$  at any point of  $E$ ), and such that  $f(M)$  intersects the image of any entire map  $g: \mathbf{C}^d \rightarrow \mathbf{C}^n$  of rank  $d = n - \dim M$  at infinitely many points [7, theorem 5.1]. This implies in particular that  $f(M)$  cannot be mapped into any hyperplane in  $\mathbf{C}^n$  by an automorphism of  $\mathbf{C}^n$ . In this context we recall that any Stein manifold  $M$  embeds in  $\mathbf{C}^n$  for  $n > (3 \dim M + 1)/2$  according to Eliashberg and Gromov [5].

In light of these results it is a natural question whether one can in some sense prescribe the embedding  $M \hookrightarrow \mathbf{C}^n$  on a positive dimensional submanifold  $N \subset M$  (at least approximately). Our result provides an affirmative answer in the simplest case when  $M = \mathbf{C}$  and  $N = \mathbf{R} \times \{i0\} \subset \mathbf{C}$ . The details of our construction are considerable even in this simplest case, and the full scope of the method remains to be seen. If  $\lambda$  is of class  $\mathcal{C}^\infty$ , our method can be modified so that we approximate to increasingly high order on complements of compact subsets of  $\mathbf{R}$ . We shall not go into details of this. Another possible extension is to approximate a proper smooth embedding by a proper holomorphic embedding on a set of disjoint lines or other real curves in  $\mathbf{C}$ .

A seemingly more difficult question is whether a result of this type holds when  $N$  is a *complex* submanifold of positive dimension in a complex manifold  $M$ . For instance, *does every proper holomorphic embedding  $\lambda: \mathbf{C} \hookrightarrow \mathbf{C}^n$  for  $n \geq 3$  extend to a proper holomorphic embedding  $f: \mathbf{C}^2 \hookrightarrow \mathbf{C}^n$ ?*

The paper is organized as follows. In sect. 2 we introduce the notation and give an outline of the proof of theorem 1.1. In sect. 3 we collect some technical lemmas needed in

the proof. The details of the proof of theorem 1.1 are given in sect. 4.

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## &2. Outline of proof

Since the proof of theorem 1.1 is somewhat intricate, we give in this section an outline of the proof. We also recall a technical result from [10] (proposition 2.1 below) which will be used in the proof.

We begin by explaining the notation.  $\Delta_\rho$  is the closed disk in  $\mathbf{C}$  of radius  $\rho$  and center 0.  $\mathbf{B}$  is the open unit ball in  $\mathbf{C}^n$  with center 0, and  $R\mathbf{B}$  is the ball of radius  $R$ . For a set  $A \subset \mathbf{C}^n$  and  $\rho > 0$ , let  $A + \rho\overline{\mathbf{B}} = \{a + z : a \in A, |z| \leq \rho\}$ . We identify  $\mathbf{C}$  and  $\mathbf{R}$  with their images in  $\mathbf{C}^n$  under the embedding  $\zeta \rightarrow (\zeta, 0, \dots, 0)$ . For  $1 \leq j \leq n$  we denote by  $\pi_j$  the coordinate projection  $\pi_j(z_1, \dots, z_n) = z_j$ .

In the proof we shall use special automorphisms of  $\mathbf{C}^n$  of the form

$$\Psi(z) = z + f(\pi z)v, \quad z \in \mathbf{C}^n,$$

where  $v \in \mathbf{C}^n$ ,  $\pi: \mathbf{C}^n \rightarrow \mathbf{C}^k$  is a linear map for some  $k < n$  with  $\pi v = 0$  (in most cases  $k = 1$ ), and  $f$  is an entire function on  $\mathbf{C}^k$ . An automorphism of this form is called a *shear*; clearly  $\Psi^{-1}(z) = z - f(\pi z)v$ .

One of the main technical tools in our construction is the following result from [10]. The case  $r = 0$  was obtained earlier in [9]. This result can also be obtained by methods in [6].

**2.1 Proposition.** *Let  $K \subset \subset \mathbf{C}^n$  ( $n \geq 2$ ) be a compact, polynomially convex set, and let  $C \subset \mathbf{C}^n$  be a smooth embedded arc of class  $\mathcal{C}^\infty$  which is attached to  $K$  in a single point of  $K$ . Given a  $\mathcal{C}^\infty$  diffeomorphism  $F: K \cup C \rightarrow K \cup C' \subset \mathbf{C}^n$  such that  $F$  is the identity on  $(K \cup C) \cap U$  for some open neighborhood  $U$  of  $K$ , and given numbers  $r \geq 0$ ,  $\epsilon > 0$ , there exist a neighborhood  $W$  of  $K$  and an automorphism  $\Phi \in \text{Aut}\mathbf{C}^n$  satisfying*

$$\|\Phi - Id\|_{\mathcal{C}^r(W)} < \epsilon, \quad \|\Phi - F\|_{\mathcal{C}^r(C)} < \epsilon.$$

(Here  $Id$  denotes the identity map.) Moreover, for each finite subset  $Z \subset K \cup C$  we can choose  $\Phi$  such that it agrees with the identity to order  $r$  at each point  $z \in Z \cap K$  and  $\Phi|_C$  agrees with  $F$  to order  $r$  at each point of  $Z \cap C$ .

The same result holds with any finite number of disjoint arcs attached to  $K$ .

We now give the outline of the proof of theorem 1.1. We wish to approximate a given proper embedding  $\lambda: \mathbf{R} \hookrightarrow \mathbf{C}^n$  by the restriction to  $\mathbf{R}$  of a proper holomorphic embedding  $f: \mathbf{C} \hookrightarrow \mathbf{C}^n$ . By standard results we may assume that  $\lambda$  is  $\mathcal{C}^\infty$  and that any  $\mathcal{C}^r$  map of  $\mathbf{R}$  into  $\mathbf{C}^n$  which has  $\mathcal{C}^r$  distance less than  $\eta(t)$  from  $\lambda$  (as in theorem 1.1) is a proper embedding.

We start with the standard embedding  $\gamma_0(t) = (t, 0, \dots, 0)$  and identify  $\mathbf{R}$  with  $\gamma_0(\mathbf{R})$ . We inductively define automorphisms of  $\mathbf{C}^n$  of the form  $f_k = \Phi_k \circ \dots \circ \Phi_1 \circ \Psi_1 \circ \dots \circ \Psi_k$ ,

where each  $\Phi_j$  and  $\Psi_j$  is an automorphism of  $\mathbf{C}^n$  chosen so that  $f_k|_{\gamma_0(\mathbf{R})}$  approximates  $\lambda$  on larger and larger compact sets. Moreover, we construct the sequence  $f_k$  such that the limit  $f = \lim_{k \rightarrow \infty} f_k$  exists on an open set  $D \subset \mathbf{C}^n$  containing  $\mathbf{C} \times \{0\}$ , and  $f$  is a biholomorphic map of  $D$  onto  $\mathbf{C}^n$ . The restriction of  $f$  to the  $z_1$ -coordinate axis is then a proper holomorphic embedding of  $\mathbf{C} = \mathbf{C} \times \{0\}$  into  $\mathbf{C}^n$  satisfying the required properties.

The inductive correction proceeds as follows. We assume that there is an interval  $I_k \subset \mathbf{R}$  such that  $f_k$  approximates  $\lambda$  on  $I_k$  in the sense of theorem 1.1 and that both  $\lambda(t)$  and  $f_k(t)$  lie outside some closed ball  $B_k$  for  $t \notin I_k$ . We want to produce an interval  $I_{k+1}$  and a ball  $B_{k+1}$ , each of which has radius at least one greater than the corresponding set at the  $k$ th stage, and a map  $f_{k+1}$  which gives the desired approximation on  $I_{k+1}$ .

We do this by applying a version of proposition 2.1 to get an automorphism  $\Phi_{k+1}$  which is close to the identity map on the sets  $f_k(I_k)$ ,  $B_k$ , and  $f_k(\Delta_{k+1})$ , and such that  $\Phi_{k+1} \circ f_k$  approximates  $\lambda$  on  $I_{k+1}$ . In order to apply proposition 2.1, we define a polynomially convex set  $K_k$  which includes  $B_k$ ,  $f_k(\Delta_{k+1})$ , and most of  $f_k(I_k)$ , and we also define a smooth, proper embedding  $\lambda_k: \mathbf{R} \rightarrow \mathbf{C}^n$  which agrees with  $\lambda$  on  $\mathbf{R} \setminus I_k$ , agrees with  $f_k$  on most of  $I_k$ , and is  $\mathcal{C}^r$ -near  $\lambda$  everywhere. We can then apply proposition 2.1 via lemma 3.4 to get  $\Phi_{k+1}$  so that  $\Phi_{k+1} \circ f_k$  approximates  $\lambda$  on some larger interval  $I_{k+1}$  and is near the identity on  $K_k$  (this is required for convergence).

The problem now is that the point  $\Phi_{k+1} f_k(t)$  may come very close to the previous ball  $B_k$  for some  $t \in \mathbf{R} \setminus I_{k+1}$ . Unless we control this distance from below, the limit map may not be a proper embedding. Hence we precompose  $\Phi_{k+1} f_k$  with a shear of the form  $\Psi_{k+1}(z) = z + \mu_{k+1}(z_1)\nu_{k+1}$ , for some  $\mu_{k+1}$  holomorphic in one variable and some vector  $\nu_{k+1}$  with  $\pi_1 \nu_{k+1} = 0$ . By lemma 3.2, we can choose  $\Psi_{k+1}$  near the identity on  $\Delta_{k+1} \cup I_{k+1}$  and such that  $\Phi_{k+1} f_k \Psi_{k+1}(\mathbf{R} \setminus I_{k+1})$  avoids some larger ball  $B_{k+1}$ . Except for technicalities, this finishes the induction.

The proof is completed by showing that  $f = \lim_{k \rightarrow \infty} f_k$  exists and gives a proper holomorphic embedding of  $\mathbf{C} = \mathbf{C} \times \{0\}$  to  $\mathbf{C}^n$ . This is so because the limit  $\Psi = \lim_{k \rightarrow \infty} \Psi_1 \circ \dots \circ \Psi_k$  exists and is an automorphism of  $\mathbf{C}^n$ , while the limit  $\Phi = \lim_{k \rightarrow \infty} \Phi_k \circ \dots \circ \Phi_1$  exists on an open set  $\Omega \subset \mathbf{C}^n$  containing  $\Psi(\mathbf{C} \times \{0\})$ , and  $\Phi: \Omega \rightarrow \mathbf{C}^n$  is a biholomorphic map *onto*  $\mathbf{C}^n$ . Thus  $f = \Phi\Psi$  is a biholomorphic map from  $D = \Psi^{-1}(\Omega)$  onto  $\mathbf{C}^n$  whose restriction to  $\mathbf{C} \times \{0\} \subset D$  provides the desired proper holomorphic embedding into  $\mathbf{C}^n$ . The approximation properties of  $f$  are clear from the inductive step.

### &3. Some lemmas

The following is standard, e.g., [13, prop. 2.15.4].

**3.1 Lemma.** *Let  $\lambda: \mathbf{R} \hookrightarrow \mathbf{C}^n$  be a  $\mathcal{C}^\infty$  proper embedding. Then there exists a continuous  $\eta: \mathbf{R} \rightarrow (0, \infty)$  such that if  $\gamma: \mathbf{R} \rightarrow \mathbf{C}^n$  with  $|\gamma^{(s)}(t) - \lambda^{(s)}(t)| < \eta(t)$  for all  $t \in \mathbf{R}$ ,  $s = 0, 1$ , then  $\gamma$  is a proper embedding.*

Recall that a compact set  $A \subset \subset \mathbf{C}^n$  is polynomially convex if for each  $z \in \mathbf{C}^n \setminus A$  there is a holomorphic polynomial  $P$  on  $\mathbf{C}^n$  such that  $|P(z)| > \max\{|P(w)|: w \in A\}$ . We refer the reader to [12] for properties of such sets.

**3.2 Lemma.** Let  $A \subset \mathbf{C}^n$  be compact and polynomially convex and  $\rho > 0$ . Let  $I \subset \mathbf{R}$  be an interval whose endpoints lie in  $\mathbf{C}^n \setminus (A \cup \Delta_\rho)$ , and let  $r, \epsilon > 0$ . Then there exists an automorphism  $\Psi(z) = z + g(z_1)e_2$  of  $\mathbf{C}^n$  such that

- (i)  $|\Psi(z) - z| < \epsilon$  for  $z \in \Delta_\rho$ ,
- (ii)  $\|\Psi|_{\mathbf{R}}(t) - t\|_{C^r(I)} < \epsilon$ , and
- (iii)  $\Psi(t) \notin A$  for  $t \in \overline{\mathbf{R}I}$ .

If  $Z \subset I$  is finite, we can choose  $\Psi$  as above so that  $g^{(s)}(t) = 0$  for  $t \in Z$ ,  $0 \leq s \leq r$ .

*Proof.* Let  $\mu_1 < \mu_2$  denote the endpoints of  $I$  in  $\mathbf{R}$ , and let  $\Gamma_j = \{(\mu_j, \zeta, 0, \dots, 0) : \zeta \in \mathbf{C}\}$  for  $j = 1, 2$ . Let  $R > \max\{|\mu_1|, |\mu_2|\} + 1$  such that  $A \subset R\mathbf{B}$ . Consider the set  $E_j = A \cap \Gamma_j$ . Since  $A$  is polynomially convex,  $E_j$  is polynomially convex in  $\Gamma_j$  and hence  $\Gamma_j \setminus E_j$  is connected. Since the endpoints of  $I$  lie in  $\Gamma_j \setminus E_j$ , there exists a smooth curve  $\gamma_j: [0, 1] \rightarrow \Gamma_j \setminus E_j$  with  $\gamma_j(0) = (\mu_j, 0, \dots, 0)$  and  $|\pi_2 \gamma_j(1)| > R + 1$  for  $j = 1, 2$ .

Since  $A$  is compact, there exists  $\delta > 0$  such that  $\gamma_j([0, 1]) + 3\delta\overline{\mathbf{B}} \subset \mathbf{C}^n \setminus A$ . Let  $\pi_2(z) = z_2$ . Let  $K = \{x + iy \in \mathbf{C} : \mu_1 - \delta/2 \leq x \leq \mu_2 + \delta/2, |y| \leq \rho + 1\}$ . Define a function  $h: K \cup [-R, R] \rightarrow \mathbf{C}$  by

$$h(t) = \begin{cases} \pi_2 \gamma_1(1) & \text{if } t \in [-R, \mu_1 - 2\delta]; \\ \pi_2 \gamma_1((\mu_1 - \delta - t)/\delta) & \text{if } t \in [\mu_1 - 2\delta, \mu_1 - \delta]; \\ \pi_2 \gamma_2((t - \mu_2 - \delta)/\delta) & \text{if } t \in [\mu_2 + \delta, \mu_2 + 2\delta]; \\ \pi_2 \gamma_2(1) & \text{if } t \in [\mu_2 + 2\delta, R]; \\ 0 & \text{otherwise.} \end{cases}$$

Choose  $\eta$ ,  $0 < \eta < \min\{\epsilon, \delta\}$ . By Mergelyan's theorem [17, p.386] there is an entire function  $g$  on  $\mathbf{C}$  such that  $|h(z) - g(z)| < \eta$  for  $z \in K \cup [-R, R]$ . The shear  $\Psi(z) = z + g(z_1)e_2$  then satisfies (i) and (iii). Since  $I \subset \text{Int}K$ , Cauchy's estimates imply that it also satisfies (ii) provided that  $\eta > 0$  is chosen sufficiently small. The last condition on  $g$  is a trivial addition to Mergelyan's theorem.  $\spadesuit$

**3.3 Lemma.** Let  $\lambda: \mathbf{R} \hookrightarrow \mathbf{C}^n$  be a proper,  $C^\infty$  embedding,  $K \subset \mathbf{C}^n$  compact,  $\epsilon > 0$ , and  $r \in \mathbf{Z}_+$ . Let  $Z \subset \mathbf{R}$  be finite, and suppose  $\lambda(t) \in \mathbf{C} = \mathbf{C} \times \{0\}^{n-1}$  for each  $t \in Z$ . Then there exists a shear  $\Psi(z) = z + h(z_1)v$  for some  $v \in \mathbf{C}^n$  with  $\pi_1 v = 0$  such that

- (i)  $\Psi(\mathbf{C}) \cap \lambda(\mathbf{R}) = \lambda(Z)$ ,
- (ii)  $|\Psi(z) - z| < \epsilon$  for  $z \in K$ , and
- (iii)  $\Psi(z) = z + O(|z - \lambda(t)|^{r+1})$  as  $z \rightarrow \lambda(t)$ , for all  $t \in Z$ .

*Proof.* Let  $Z = \{t_j\}_{j=1}^s$ . Define a polynomial on  $\mathbf{C}$  by  $h(\zeta) = \prod_{1 \leq j \leq s} (\zeta - \pi_1 \lambda(t_j))^{r+1}$ . Consider the map  $\Phi: \mathbf{C} \times \mathbf{C}^{n-1} \rightarrow \mathbf{C}^n$  given by

$$\Phi(z_1, \alpha_2, \dots, \alpha_n) = (z_1, 0, \dots, 0) + h(z_1)(0, \alpha_2, \dots, \alpha_n).$$

Clearly  $\Phi$  is an automorphism of  $(\mathbf{C} \setminus \lambda(Z)) \times \mathbf{C}^{n-1}$ . Let  $\Delta_{R,j}$  denote the closed disk of radius  $R$  in  $\mathbf{C}$  with center  $\pi_1 \lambda(t_j)$  for  $j = 1, 2, \dots, s$ . Choose  $R > 0$  such that the discs

$\Delta_{R,j}$  for  $1 \leq j \leq s$  are pairwise disjoint. Choose a  $\rho$ ,  $0 < \rho < R$ , such that  $\rho^2$  is a regular value of  $\mu_j(t) = |\pi_1 \lambda(t) - \pi_1 \lambda(t_j)|^2$  ( $t \in \mathbf{R}$ ) for each  $j = 1, 2, \dots, s$ .

Let  $M_\rho = \mathbf{C} \setminus \bigcup_{1 \leq j \leq s} \text{int} \Delta_{\rho,j}$ . Let  $\Phi_\rho = \Phi|_{M_\rho \times \mathbf{C}^{n-1}}$  and  $\partial \Phi_\rho = \Phi|_{\partial M_\rho \times \mathbf{C}^{n-1}}$ . A simple check shows that  $\Phi_\rho$  and  $\partial \Phi_\rho$  are transverse to  $\lambda(\mathbf{R})$ . Hence by the transversality theorem, there exists a set  $A_\rho \subset \mathbf{C}^{n-1}$  of full measure such that for each  $\alpha = (\alpha_2, \dots, \alpha_n) \in A_\rho$ ,  $\Phi(M_\rho, \alpha) = \{\Phi(z_1, \alpha) : z_1 \in M_\rho\}$  and  $\lambda(\mathbf{R})$  are transverse, hence disjoint by dimension considerations.

Let  $A = \bigcap_{j=1}^\infty A_{1/j}$ . Then  $A \subset \mathbf{C}^{n-1}$  has full measure, and for each  $\alpha \in A$  we see that  $\Phi(\mathbf{C} \setminus \lambda(Z), \alpha)$  and  $\lambda(\mathbf{R})$  are disjoint. Finally, choose  $\alpha \in A$  such that  $|h(z_1)\alpha| < \epsilon$  for  $z_1 \in \pi_1(K)$ , and let  $\Psi(z) = z + h(z_1)\alpha$ . Then  $\Psi(z_1, 0, \dots, 0) = \Phi(z_1, \alpha_2, \dots, \alpha_n)$ , and  $\Psi$  satisfies the conclusions of the lemma.  $\spadesuit$

**3.4 Lemma.** *Let  $\lambda: \mathbf{R} \hookrightarrow \mathbf{C}^n$  be a  $\mathcal{C}^\infty$  embedding,  $f: \mathbf{C} \hookrightarrow \mathbf{C}^n$  a proper holomorphic embedding, and  $I \subset \mathbf{R}$  a closed interval with  $f|_I = \lambda|_I$ . Let  $K \subset \mathbf{C}^n$  be compact and polynomially convex,  $a, r, \epsilon > 0$ , and  $T \subset \mathbf{R}$  discrete. Suppose that  $\lambda(t), f(t) \notin K$  for  $t \in \mathbf{R} \setminus I$ . Then there exists  $\Phi \in \text{Aut} \mathbf{C}^n$  such that if  $g = \Phi \circ f$ , then*

- (i)  $|g^{(s)}(t) - \lambda^{(s)}(t)| < \epsilon$  for  $t \in [-a, a]$ ,  $0 \leq s \leq r$ ,
- (ii)  $g^{(s)}(t) = \lambda^{(s)}(t)$  for  $t \in T \cap [-a, a]$ ,  $0 \leq s \leq r$ , and
- (iii)  $|\Phi(z) - z| < \epsilon$  for  $z \in K$ .

*Proof.* We may assume that  $I \subset (-a, a)$ . Let  $I_1, I_2$  be the two connected components of  $\{\zeta \in I : f(\zeta) \in \mathbf{C}^n \setminus K\}$  containing the respective endpoints of  $I$ , and let  $I_0 = I \setminus (I_1 \cup I_2)$ . Let  $A$  be the polynomial hull of  $K \cup f(I_0)$ . Then  $A$  is the union of  $K \cup f(I_0)$  and the bounded connected components of  $f(\mathbf{C}) \setminus (K \cup f(I_0))$ . Note that  $f(I_1)$  and  $f(I_2)$  lie in  $f(\mathbf{C}) \setminus A$  since  $f(t) \notin K$  for all  $t \in \mathbf{R} \setminus I$ .

Let  $L = A \cup f([-a, a])$ . Then  $C = \overline{L} \setminus A$  is the union of two embedded arcs, each containing an endpoint of  $f([-a, a])$ . Define  $F$  on  $L$  by  $F(z) = z$  if  $z \in A$ , and  $F(z) = \lambda f^{-1}(z)$  if  $z \in f([-a, a])$ . Then  $F$  is a  $\mathcal{C}^\infty$  diffeomorphism of  $L$  which extends as the identity map on  $(A \cup C) \cap U$  for some neighborhood  $U$  of  $A$ . Apply proposition 2.1 to get  $\Phi \in \text{Aut} \mathbf{C}^n$  such that  $|\Phi(z) - z| < \epsilon$  for  $z \in K$  and such that  $g = \Phi \circ f$  satisfies (i) and (ii).  $\spadesuit$

#### &4. Proof of theorem 1.1

Choose a smooth cutoff function  $\chi$  on  $\mathbf{R}$  such that  $\chi(t) = 1$  for  $|t|$  small and  $\text{supp } \chi \subset (-1, 1)$ . Define the constant  $C = C_r > 1$  such that  $\|\chi h\|_{\mathcal{C}^r} \leq C \|h\|_{\mathcal{C}^r}$  for each  $h \in \mathcal{C}^r(\mathbf{R})$ . We fix such  $C$  for the entire proof.

By approximation we may assume that  $\lambda: \mathbf{R} \hookrightarrow \mathbf{C}^n$  in theorem 1.1 is a proper  $\mathcal{C}^\infty$  embedding. Decreasing  $\eta$  if necessary we may also assume that  $\eta$  satisfies lemma 3.1 for  $\lambda$  and  $\eta(t) < 1/2$  for all  $t \in \mathbf{R}$ .

We use an inductive procedure to obtain a sequence of proper holomorphic embeddings  $f_k: \mathbf{C} \hookrightarrow \mathbf{C}^n$  such that  $f = \lim_{k \rightarrow \infty} f_k$  exists on  $\mathbf{C}$  and satisfies theorem 1.1. Each  $f_k$  will be a restriction to  $\mathbf{C} = \mathbf{C} \times \{0\}^{n-1}$  of a holomorphic automorphism of  $\mathbf{C}^n$ . The next map  $f_{k+1}$  will be of the form  $f_{k+1} = \Phi_{k+1} \circ f_k \circ \Psi_{k+1}$  for suitably chosen  $\Phi_{k+1}, \Psi_{k+1} \in \text{Aut} \mathbf{C}^n$ .

We will describe the case  $k = 1$  after the inductive step is given. Recall that  $\Delta_k$  is the closed disc in  $\mathbf{C} = \mathbf{C} \times \{0\}^{n-1}$  with center 0 and radius  $k$ , and  $\mathbf{B}$  is the unit ball in  $\mathbf{C}^n$ . For the induction at step  $k$ , suppose we have the following:

- (a) closed balls  $B_j = R_j \overline{\mathbf{B}} \subset \mathbf{C}^n$  with  $R_j \geq \max\{j + 1, R_{j-1} + 1\}$ ,  $j = 1, \dots, k$ ,
- (b) automorphisms  $\Phi_1, \dots, \Phi_k$  of  $\mathbf{C}^n$  with  $|\Phi_j(z) - z| < 2^{-j}$  for  $z \in B_{j-1}$ ,  $j = 2, \dots, k$ ,
- (c) numbers  $\epsilon_j > 0$  such that  $\epsilon_1 < 2^{-1}$  and  $\epsilon_j < \epsilon_{j-1}/2 < 2^{-j}$  for  $j = 1, \dots, k$ ,
- (d) automorphisms  $\Psi_1, \dots, \Psi_k$  of  $\mathbf{C}^n$  of the form  $\Psi_j(z) = z + g_j(z_1)e_2 + h_j(z_1)v_j$ , where  $\pi_1(v_j) = 0$  and  $|\Psi_j(z) - z| < \epsilon_j$  for  $|z| \leq j$ ,
- (e) closed intervals  $I_j = [-a_j, a_j]$ ,  $j = 1, \dots, k$ , with  $a_j > \max\{a_{j-1} + 2, j + 2\}$ , and
- (f) numbers  $0 < \delta_j < C^{-1} \inf\{\eta(t) : t \in I_j\}$ ,  $j = 1, \dots, k$ ,

such that the automorphism

$$f_k = \Phi_k \circ \dots \circ \Phi_1 \circ \Psi_1 \circ \dots \circ \Psi_k \in \text{Aut} \mathbf{C}^n$$

(whose restriction to  $\mathbf{C} = \mathbf{C} \times \{0\}^{n-1}$  provides an embedding  $\mathbf{C} \hookrightarrow \mathbf{C}^n$ ) satisfies:

- (1<sub>k</sub>)  $f_k(\Delta_j + \epsilon_k \overline{\mathbf{B}}) \subset \text{Int} B_j$  for  $j = 1, \dots, k$ ,
- (2<sub>k</sub>)  $|f_k^{(s)}(t) - \lambda^{(s)}(t)| < \eta(t)$  for  $t \in I_k$  and  $0 \leq s \leq r$ ,
- (3<sub>k</sub>)  $|f_k^{(s)}(t) - \lambda^{(s)}(t)| < \delta_k$  for  $t \in I_k \setminus (-a_k + 1, a_k - 1)$  and  $0 \leq s \leq r$ ,
- (4<sub>k</sub>)  $f_k^{(s)}(t) = \lambda^{(s)}(t)$  for  $t \in T \cap I_k$ ,  $0 \leq s \leq r$ ,
- (5<sub>k</sub>)  $f_k(\mathbf{C}) \cap \lambda(\mathbf{R}) = \lambda(T \cap I_k)$ ,
- (6<sub>k</sub>)  $|\lambda(t)| > R_k + 1$  for  $|t| \geq a_k - 1$ ,
- (7<sub>k</sub>)  $|f_k(t)| > R_k$  for  $|t| \geq a_k - 1$ .

We will now show how to obtain these hypotheses at step  $k + 1$ . Let  $I_k^1$  and  $I_k^2$  be the two connected components of the set  $\{\zeta \in I_k \setminus \Delta_{k+1} : |f_k(\zeta)| > R_k\}$  containing the respective endpoints of the interval  $I_k$ . Let  $I_k^0 = I_k \setminus (I_k^1 \cup I_k^2)$  be the middle interval. By (7<sub>k</sub>) we have  $I_k^0 \subset (-a_k + 1, a_k - 1)$ .

Let  $K_k$  be the polynomial hull of the set  $B_k \cup f_k(\Delta_{k+1} \cup I_k^0)$ . Since  $f_k(\mathbf{C})$  is a complex submanifold of  $\mathbf{C}^n$  and  $B_k$  is polynomially convex, it is seen easily that  $K_k$  is contained in  $B_k \cup f_k(\mathbf{C})$ , and it is the union of  $B_k \cup f_k(\Delta_{k+1} \cup I_k^0)$  and the bounded connected components of the complement  $f_k(\mathbf{C}) \setminus (B_k \cup f_k(\Delta_{k+1} \cup I_k^0))$  (see Lemma 5.4 in [7]). Note that (7<sub>k</sub>) and (e) imply that  $f_k(\mathbf{R} \setminus (-a_k + 1, a_k - 1)) \subset \mathbf{C}^n \setminus K_k$ .

Choose  $R_{k+1} > R_k + 1$  such that  $K_k \subset (R_{k+1} - 1)\mathbf{B}$ , and let  $B_{k+1} = R_{k+1} \overline{\mathbf{B}}$ . Choose  $a_{k+1} > a_k + 2$  to get (6<sub>k+1</sub>). We now want to approximate  $\lambda$  on the larger interval  $I_{k+1} = [-a_{k+1}, a_{k+1}]$  by the image of the next embedding  $\mathbf{C} \hookrightarrow \mathbf{C}^n$  (to be constructed). In order to apply lemma 3.4 we first approximate  $\lambda$  as follows:

**4.1 Lemma.** *There exists a proper  $\mathcal{C}^\infty$  embedding  $\lambda_k : \mathbf{R} \hookrightarrow \mathbf{C}^n$  satisfying*

- (i)  $\lambda_k = f_k$  on  $[-a_k + 1, a_k - 1]$ ,
- (ii)  $\lambda_k = \lambda$  on  $\mathbf{R} \setminus I_k$ ,
- (iii)  $|\lambda_k^{(s)}(t) - \lambda^{(s)}(t)| < \eta(t)$  for  $t \in I_k \setminus (-a_k + 1, a_k - 1)$ ,  $0 \leq s \leq r$ ,

- (iv)  $\lambda_k^{(s)}(t) = \lambda^{(s)}(t)$  for  $t \in T$ ,  $0 \leq s \leq r$ , and  
(v)  $\lambda_k(t) \notin K_k$  when  $|t| \geq a_k - 1$ .

*Proof.* We define the cutoff function  $\chi_k$  on  $\mathbf{R}$  using  $\chi$ , so that  $\chi_k = 1$  on  $[-a_k + 1, a_k - 1]$ ,  $\chi_k = 0$  on  $\mathbf{R} \setminus I_k$ , and  $\|\chi_k h\|_{C^r} < C\|h\|_{C^r}$  as before. Let

$$\hat{\lambda}_k(t) = f_k(t)\chi_k(t) + \lambda(t)(1 - \chi_k(t)), \quad t \in \mathbf{R}.$$

By lemma 3.1, (3<sub>k</sub>), (4<sub>k</sub>), and choice of  $\eta$  and  $\delta_k$ , we see that (i)–(iv) are satisfied for  $\hat{\lambda}_k$  in place of  $\lambda_k$ .

To obtain (v) we use a transversality argument to perturb  $\hat{\lambda}_k$  on  $I_k \setminus (-a_k + 1, a_k - 1)$ . First note that if  $|t| > a_k$ , then  $|\hat{\lambda}_k(t)| = |\lambda(t)| > R_k + 1$  by (6<sub>k</sub>), so  $\hat{\lambda}_k(t) \notin B_k$ . Also, by (5<sub>k</sub>), we see that  $\hat{\lambda}_k(t) \notin f_k(\mathbf{C})$ , so  $\hat{\lambda}_k(t) \notin K_k$ . Next, if  $t \in T \cap (I_k \setminus (-a_k + 1, a_k - 1))$ , then by (4<sub>k</sub>), (7<sub>k</sub>), and (e) we see that  $\hat{\lambda}_k(t) = f_k(t) \notin K_k$ . Hence there exists a neighborhood  $V$  of  $T \cap (I_k \setminus (-a_k + 1, a_k - 1))$  such that  $\hat{\lambda}_k(\bar{V}) \cap K_k = \emptyset$ .

Thus we need only perturb  $\hat{\lambda}_k$  on  $I_k \setminus (V \cup (-a_k + 1, a_k - 1))$  to get (v). Note that if  $t \in I_k \setminus (-a_k + 1, a_k - 1)$ , then from (6<sub>k</sub>) and (2<sub>k</sub>) we see that  $|\hat{\lambda}_k(t)| > R_k + 1/2$ , so  $\hat{\lambda}_k(t) \notin B_k$ . Finally, a simple transversality argument implies that we can make an arbitrarily small  $C^\infty$  perturbation of  $\hat{\lambda}_k$  to avoid  $f_k(\mathbf{C})$ , and hence we get  $\lambda_k$  with  $\lambda_k = \hat{\lambda}_k$  outside  $I_k \setminus (V \cup (-a_k + 1, a_k - 1))$  and  $\lambda_k$  satisfying (i)–(v). ♠

Now we can use lemma 3.4 to approximate  $\lambda_k$ , hence to approximate  $\lambda$ . Set

$$\begin{aligned} \delta_{k+1} &= \min\{\eta(t) : t \in I_{k+1}\} / (2C), \\ \sigma_{k+1} &= \min\{\eta(t) - |\lambda_k^{(s)}(t) - \lambda^{(s)}(t)| : t \in I_{k+1}, 0 \leq s \leq r\} > 0. \end{aligned}$$

Choose  $\epsilon > 0$  so small that

$$\epsilon < \min\{2^{-(k+1)}, \delta_{k+1}, \sigma_{k+1}\}, \quad f_k(\Delta_j + \epsilon_k \bar{\mathbf{B}}) + \epsilon \bar{\mathbf{B}} \subset \text{Int} B_j, \quad 1 \leq j \leq k.$$

Apply lemma 3.4 with  $\lambda = \lambda_k$ ,  $f = f_k$ ,  $I = [-a_k + 1, a_k - 1]$ ,  $K = K_k$ ,  $a = a_{k+1}$ ,  $r$  and  $T$  unchanged, and  $\epsilon$  as above. This provides  $\Phi_{k+1} \in \text{Aut} \mathbf{C}^n$  and  $G = \Phi_{k+1} \circ f_k \in \text{Aut} \mathbf{C}^n$  satisfying

$$\begin{cases} |\Phi_{k+1}(z) - z| < \epsilon & \text{for } z \in K_k, \text{ hence on } B_k; \\ |G^{(s)}(t) - \lambda_k^{(s)}(t)| < \epsilon & \text{for } t \in I_{k+1}, 0 \leq s \leq r; \\ G^{(s)}(t) = \lambda_k^{(s)}(t) & \text{for } t \in T \cap I_{k+1}, 0 \leq s \leq r. \end{cases}$$

In particular, (2<sub>k+1</sub>)–(4<sub>k+1</sub>) hold with  $G$  in place of  $f_{k+1}$ .

Since  $f_k(\Delta_{k+1}) \subset K_k \subset (R_{k+1} - 1)\mathbf{B}$ , we can choose  $\epsilon'_{k+1} < \epsilon_k$  small enough that (1<sub>k+1</sub>) holds with  $G$  in place of  $f_{k+1}$  and  $\epsilon'_{k+1}$  in place of  $\epsilon_{k+1}$ , and such that if  $\psi \in \text{Aut} \mathbf{C}^n$  with  $\|\psi(t) - t\|_{C^r(I_{k+1})} < \epsilon'_{k+1}$ , then (2<sub>k+1</sub>) and (3<sub>k+1</sub>) hold with  $G \circ \psi$  in place of  $f_{k+1}$ . Let  $\epsilon_{k+1} = \epsilon'_{k+1}/2$ . Then with  $G$  in place of  $f_{k+1}$ , we have (1<sub>k+1</sub>)–(4<sub>k+1</sub>), (6<sub>k+1</sub>), and  $G(-a_{k+1}), G(a_{k+1}) \notin B_{k+1}$  by (6<sub>k+1</sub>) and (2<sub>k+1</sub>).

Next we want to obtain  $(7_{k+1})$ . We do this using lemma 3.2 to change the embedding so that the image of  $\mathbf{R} \setminus I_{k+1}$  misses  $B_{k+1}$  while leaving the embedding essentially unchanged on  $\Delta_{k+1} \cup I_{k+1}$ . Apply lemma 3.2 with  $A = G^{-1}(B_{k+1})$ ,  $\rho = k + 1$ ,  $I = I_{k+1}$ ,  $r$  unchanged,  $Z = T \cap I_{k+1}$  and  $\epsilon = \epsilon_{k+1}/2$ . This gives a shear

$$\psi_{k+1}(z) = z + g_{k+1}(z_1)e_2$$

with

$$\begin{cases} |\psi_{k+1}(z) - z| < \epsilon_{k+1}/2, & z \in \Delta_{k+1}; \\ \|\psi_{k+1}|_{\mathbf{R}}(t) - t\|_{C^r(I_{k+1})} < \epsilon_{k+1}/2; \\ g_{k+1}^{(s)}(t) = 0, & t \in T \cap I_{k+1}, 0 \leq s \leq r; \\ \psi_{k+1}(t) \notin G^{-1}(B_{k+1}), & t \in \overline{\mathbf{R} \setminus I_{k+1}}. \end{cases}$$

Let  $H = G\psi_{k+1}$ . Then with  $H$  in place of  $f_{k+1}$ , we have  $(1_{k+1})$ – $(4_{k+1})$ ,  $(6_{k+1})$ , and  $(7_{k+1})$ .

For the final correction, we use lemma 3.3 to obtain  $(5_{k+1})$  while maintaining the other properties. Let  $R > a_{k+1}$  be such that  $A = G^{-1}(B_{k+1}) \subset R\mathbf{B}$ . Let  $\delta > 0$  be such that

$$\psi_{k+1}([-R, R] \setminus (-a_{k+1}, a_{k+1}) + \delta\overline{\mathbf{B}}) \cap A = \emptyset,$$

and such that if  $\theta \in \text{Aut}\mathbf{C}^n$ , with  $|\theta(z) - z| < \delta$  on  $R\overline{\mathbf{B}}$ , then

$$\|\psi_{k+1}\theta|_{\mathbf{R}}(t) - t\|_{C^r(I_{k+1})} < \epsilon_{k+1}. \quad (1)$$

Apply lemma 3.3 with  $\lambda$  replaced by  $H^{-1}\lambda$ ,  $K = R\overline{\mathbf{B}}$ ,  $r$  unchanged,  $Z = T \cap I_{k+1}$ , and  $\epsilon = \min\{\delta, \epsilon_{k+1}/2\}$ . This gives a shear  $\theta_{k+1}(z) = z + h_{k+1}(z_1)v_{k+1}$  with  $\pi_1 v_{k+1} = 0$  such that

$$\begin{cases} |\theta_{k+1}(z) - z| < \min\{\delta, \epsilon_{k+1}/2\}, & z \in \Delta_{k+1}; \\ \theta_{k+1}(\mathbf{C}) \cap H^{-1}\lambda(\mathbf{R}) = H^{-1}\lambda(T \cap I_{k+1}); \\ h_{k+1}^{(s)}(t) = 0, & t \in T \cap I_{k+1}, 0 \leq s \leq r, \end{cases}$$

and such that (1) holds with  $\theta = \theta_{k+1}$ . Also, by the choice of  $R$  and  $\delta$ ,

$$\psi_{k+1}\theta_{k+1}(\overline{\mathbf{R} \setminus I_{k+1}}) \cap A = \emptyset.$$

Taking  $\Psi_{k+1} = \psi_{k+1}\theta_{k+1}$  and

$$f_{k+1} = H \circ \theta_{k+1} = \Phi_{k+1} \circ f_k \circ \Psi_{k+1}$$

we obtain  $(5_{k+1})$  and preserve the remaining hypotheses. Hence we obtain  $(1_{k+1})$ – $(7_{k+1})$ . Note that  $(k+1)\mathbf{B} \subset B_{k+1}$  so we also obtain (a)–(f), thus finishing the inductive step.

The case  $k = 1$  is similar to the general step. First apply proposition 2.1 with  $K = \emptyset$ ,  $C = [-3, 3] \subset \mathbf{C}$ ,  $F = \lambda$ ,  $\epsilon = C^{-1} \inf\{\eta(t) : t \in [-3, 3]\}$ , and  $Z = T \cap [-3, 3]$  to get  $\phi_1^1 \in \text{Aut}\mathbf{C}^n$  satisfying the conclusions of that proposition. Choose  $R_1 \geq 2$  such that  $\phi_1^1(\Delta_1) \subset (R_1 - 1)\mathbf{B}$ , choose  $a_1 > 4$  to get  $(6_1)$ , and let  $I_1 = [-a_1, a_1]$ . Choose  $\delta_1$  to satisfy (f) for  $j = 1$ .

Define a proper  $\mathcal{C}^\infty$  embedding  $\lambda_0$  as in lemma 4.1 so that (i)–(v) are satisfied with  $\lambda_0$  in place of  $\lambda_k$ ,  $\phi_1^1$  in place of  $f_k$ ,  $3$  in place of  $a_k$ ,  $[-3, 3]$  in place of  $I_k$ , and  $\phi_1^1(\Delta_1)$  in place of  $K_k$ . Apply lemma 3.4 with  $\lambda = \lambda_0$ ,  $f = \phi_1^1$ ,  $I = [-2, 2]$ ,  $K = \phi_1^1(\Delta_1)$ ,  $a = a_1$ ,  $\epsilon = \delta_1$ , and  $T$  and  $r$  unchanged. This gives  $\phi_1^2 \in \text{Aut}\mathbf{C}^n$  such that

$$|\phi_1^2(z) - z| < \delta_1 \leq 1/2, \quad z \in \phi_1^1(\Delta_1),$$

and such that  $\Phi_1 = \phi_1^2\phi_1^1$  satisfies

$$\begin{cases} \|\Phi_1 - \lambda_0\|_{\mathcal{C}^r(I_1)} < \epsilon; \\ \Phi_1^{(s)}(t) = \lambda_0^{(s)}(t), \quad t \in T \cap I_1, \quad 0 \leq s \leq r. \end{cases}$$

As before, we can apply lemmas 3.2 and 3.3 to obtain  $\epsilon_1 > 0$  and a shear  $\Psi_1$  such that the hypotheses (1<sub>1</sub>)–(7<sub>1</sub>) hold for  $f_1 = \Phi_1\Psi_1$ , and (a)–(f) hold for  $k = 1$ . This completes the base case.

To finish the proof of theorem 1.1, note that

$$\Psi_1 \cdots \Psi_k(z) = z + \sum_{j=1}^k (g_j(z_1)e_2 + h_j(z_1)v_j)$$

and that (c) implies

$$|g_j(z_1)e_2 + h_j(z_1)v_j| < 2^{-j}, \quad |z_1| < j.$$

Hence this sum converges uniformly on compacts to a shear  $\Psi(z) = z + G(z_1)$  for some holomorphic map  $G: \mathbf{C} \rightarrow \{0\} \times \mathbf{C}^{n-1}$ .

By proposition 4.2 in [7], the composition  $\Phi_k \circ \cdots \circ \Phi_1$  converges locally uniformly to a biholomorphic map from a domain  $\Omega$  onto  $\mathbf{C}^n$ , and

$$\Omega = \cup_{k=1}^\infty (\Phi_k \cdots \Phi_1)^{-1}(B_{k-1}).$$

We claim that  $\Psi(\mathbf{C} \times \{0\}) \subset \Omega$ . Let  $k > 1$ . By (1<sub>k</sub>) we have

$$\Psi_1 \cdots \Psi_k(\Delta_{k-1} + \epsilon_k \bar{\mathbf{B}}) \subset (\Phi_k \cdots \Phi_1)^{-1}(B_{k-1}).$$

Since  $|\Psi_j(z) - z| < \epsilon_j$  on  $\Delta_{k-1}$  for  $j \geq k$ , and  $\sum_{j=k+1}^\infty \epsilon_j < \epsilon_k$  by (c), we see that

$$\lim_{m \rightarrow \infty} \Psi_{k+1} \cdots \Psi_m(z) \in \Delta_{k-1} + \epsilon_k \bar{\mathbf{B}}$$

for each  $z \in \Delta_{k-1}$ . Hence

$$\Psi(\Delta_{k-1}) \subset (\Phi_k \cdots \Phi_1)^{-1}(B_{k-1}) \subset \Omega, \quad k > 1,$$

so the claim holds. In particular,  $\Phi\Psi: \mathbf{C} \rightarrow \mathbf{C}^n$  is a proper holomorphic embedding.

Finally, using the conditions  $(1_k)$ – $(7_k)$ , we see that  $\Phi\Psi: \mathbf{C} \hookrightarrow \mathbf{C}^n$  is a proper holomorphic embedding with the desired properties. ♠

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