

Extensions of Hénon maps to the closed 4-ball

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Abstract

Any polynomial automorphism of \mathbb{C}^2 with nontrivial dynamics is conjugate to a diffeomorphism of the 4-ball such that this diffeomorphism extends to a diffeomorphism of the closed 4-ball. Moreover, the conjugating map is a smooth bijection of \mathbb{C}^2 to itself. On the sphere at infinity, the extension has an attracting and a repelling solenoid, and the dynamics near these invariant solenoids are described by conjugation to a model solenoidal map.

1 Introduction

An outgrowth of the great successes in the study of complex dynamics in one variable is the recent attention given to the dynamical behavior of polynomial automorphisms of \mathbb{C}^2 ; e.g., by Friedland and Milnor, Bedford and Smillie, Fornæss and Sibony, Hubbard, Oberste-Vorth, Papadopol, Veselov, and others. In particular, the potential theoretic techniques first developed in one variable have proven to be invaluable in higher dimensions. An introduction to this subject, with further references, can be found in [5]. On the other hand, there is a large body of work on the dynamics of diffeomorphisms of compact manifolds, and some of these techniques have also proven to be of great value in exploring the dynamics of holomorphic maps in several variables. [10] is one introduction to this branch of real dynamics, and [3] and [4] give some applications to holomorphic dynamics. In the hope of providing further links between real dynamics on compact manifolds and polynomial automorphisms of \mathbb{C}^2 , it is natural to search for ways to extend a polynomial automorphism of \mathbb{C}^2 to a diffeomorphism of a compact manifold. Since there is no obvious way of extending a general polynomial automorphism to a holomorphic (or even real-analytic) diffeomorphism of a compact manifold, we look instead for a smooth extension.

This paper presents one method for constructing such a smooth extension to a compact manifold with boundary: Given a polynomial automorphism with nontrivial dynamics, it is conjugate to a diffeomorphism of the 4-ball such that this diffeomorphism extends to a diffeomorphism of the closed 4-ball. Moreover, the conjugating map is a smooth bijection of \mathbb{C}^2 to itself. On the sphere at infinity, the extension has an attracting and a repelling solenoid, and the dynamics near these invariant solenoids are described by conjugation to a model solenoidal map introduced by [8]. The methods used here are analytic in character: cut-off functions are used to piece together part of the original automorphism and part of a map which extends naturally to the 3-sphere at infinity. This piecing together is done in a parametrized way, and the resulting one-parameter family of maps is used to construct an isotopy which forms the basis of the conjugating map.

Some of the results in this paper are similar to portions of the work by Hubbard and Oberste-Vorth [8] and Hubbard, Papadopol, and Veselov [9]. In particular, those papers also describe the extension of polynomial automorphisms to homeomorphisms of the closed

4-ball. However, the techniques of those papers owe more to algebraic geometry than to analysis. This algebraic-geometric approach has the advantage of preserving as much of the complex analytic structure as possible, but it leads to obstructions which prevent the resulting homeomorphism from being a global diffeomorphism.

For clarity of presentation, we first consider the case of a single Hénon map. In section 2, we construct a homeomorphism of \mathbb{C}^2 which equals a given Hénon map on a large compact set, which extends to a homeomorphism of the closed 4-ball obtained by adding a 3-sphere at infinity, and which is a diffeomorphism outside of a small set. The extension to the 3-sphere at infinity given by this homeomorphism is interesting in itself and is perhaps the simplest possible extension (see equation (2.1)). This construction is done in a parametrized way to give a one-parameter family containing the original Hénon map. In section 3, we use this one-parameter family together with some cut-off functions to construct an isotopy which maps a fundamental domain for the Hénon map to a fundamental domain for the map constructed in section 2 and which conjugates the two maps on their fundamental domains. This isotopy then extends easily to give a global conjugacy of the Hénon map to the other members of the one-parameter family. In section 4, we show that on the 3-sphere at infinity, the extended map has two invariant solenoids, one attracting and one repelling, and that the dynamics on large parts of this 3-sphere can be described by conjugation to a model solenoidal map. In section 5 we modify the proofs in the previous sections to obtain the same extension and conjugation results, but now using C^∞ smooth maps. Finally, in section 6, we discuss the corresponding results for compositions of several Hénon maps.

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2 A family of homeomorphisms

In this section we construct a homeomorphism of \mathbb{C}^2 which equals a given Hénon map on a large bidisk and which extends to the closed 4-ball obtained by adding a 3-sphere at infinity.

First we fix some notation which will be used throughout the paper. Let $F(x, y) = (y, p(y) - ax)$ for $x, y \in \mathbb{C}$, with $a \neq 0$ and $p(y) = y^d + \hat{p}(y)$ a polynomial with $d \geq 2$, $\hat{p}(y) = by^{d-k} + O(y^{d-k-1})$ a polynomial, $k \geq 2$, $b \neq 0$. A map of this form is called a Hénon map and is a polynomial automorphism of \mathbb{C}^2 . By [6], any polynomial automorphism of \mathbb{C}^2 with nontrivial dynamics is conjugate to a composition of Hénon maps.

Choose $r_0 \geq \max\{8|a| + 8, 16|b|\}$ large enough that if $|y| \geq r_0/2$, then $|p(y) - y^d| = |\hat{p}(y)| \leq 2|b||y|^{d-2}$ (hence $\leq |y|^{d-1}/4$) and $|\hat{p}'(y)| \leq |y|^{d-2}$.

For $r \geq r_0$, let

$$H_r(x, y) = \left(y, R \frac{y^d}{|y|^{d-1}} - x \right), \quad (2.1)$$

where $R = R(r)$ is a smooth positive function to be specified in lemma 2.2, and where we make the standing assumption that $y^d/|y|^{d-1} = 0$ if $y = 0$. Then each H_r is a homeomorphism

of \mathbb{C}^2 to itself which is real-analytic on $\mathbb{C}^2 - \{(x, 0) : x \in \mathbb{C}\}$. Also, H_r is \mathbb{R}^+ homogeneous of degree 1, hence defines a map of S^3 to itself. Moreover, H_r is conjugate to its inverse by the involution $\sigma(x, y) = (y, x)$. I.e., $H_r^{-1} = \sigma^{-1}H_r\sigma$.

Notation: Let $S_r(y) = R(r)y^d/|y|^{d-1}$. Also, let $\|(x, y)\|_\infty = \max\{|x|, |y|\}$.

PROPOSITION 2.1 *There exists a one-parameter family of homeomorphisms F_r , $r \geq r_0$ of \mathbb{C}^2 and a C^∞ function $R_0(r) > r$ such that $F_r(x, y) = F(x, y)$ if $\|(x, y)\|_\infty < r$ and $F_r(x, y) = H_r(x, y)$ if $\|(x, y)\|_\infty > R_0(r)$. Moreover, $F_r(x, y)$ is C^∞ in (r, x, y) except on the set $\{(r, x, 0) : r \geq r_0, |x| \geq r\}$.*

Before proving this proposition, we need some preliminaries and a lemma.

Choose a C^∞ function $\phi : \mathbb{R} \rightarrow [0, 1]$ such that $\phi(t) = 1$ if $t \leq 1/8$, $\phi(t) = 0$ if $t \geq 7/8$, and $-2 \leq \phi'(t) \leq 0$. For $y \in \mathbb{C}$, let $\phi_r(y) = \phi(|y| - r)$.

Notation: For a smooth function $f : \mathbb{C} \rightarrow \mathbb{C}$, we write $D_z f$ for the 2×2 matrix representing the Jacobian of f at z . If f is holomorphic, then $D_z f = f'(z)$ with the usual identification. For a matrix A , we use the notation $\|A\|_m$ for the minimum $\min\{\|Av\| : \|v\| = 1\}$ and $\|A\|$ for the usual matrix norm. Note that if f is holomorphic, then the matrix norm $\|f'(z)\|$ equals $|f'(z)|$.

LEMMA 2.2 *There exists a function $p_r : \mathbb{C} \rightarrow \mathbb{C}$ which is C^∞ for $(r, y) \in [r_0, \infty) \times \mathbb{C}$ and a C^∞ function $R : [r_0, \infty) \rightarrow \mathbb{R}$ with $R = R(r) \geq (r + 1)^{d-1}$ such that*

1. $p_r(y) = p(y)$ if $|y| \leq r$.
2. $p_r(y) = S_r(y)$ if $|y| \geq r + 2$.
3. $\operatorname{Re} \frac{p_r(y)}{y^d} \geq \frac{r^{d-1}}{2|y|^{d-1}}$ if $|y| \geq r$.
4. $\|D_y p_r\|_m \geq \begin{cases} |p'(y)| & \text{if } |y| \leq r \\ \frac{d}{2}|y|^{d-1} & \text{if } r \leq |y| \leq r + 1 \\ (r + 1)^{d-1} & \text{if } |y| \geq r + 1. \end{cases}$

Proof of Lemma 2.2: For $|y| \leq r + 1$, let

$$p_r(y) = y^d + \phi_r(y)\hat{p}(y).$$

Then $p_r(y) = p(y)$ if $|y| \leq r + 1/8$, and $p_r(y) = y^d$ if $r + 7/8 \leq |y| \leq r + 1$. Moreover, if $|y| \geq r_0/2$, then by the choice of r_0 ,

$$\begin{aligned} \|D_y(\phi_r(y)\hat{p}(y))\| &\leq |\phi'_r(|y|)| |\hat{p}(y)| + \phi_r(y) |\hat{p}'(y)| \\ &\leq 4|b||y|^{d-2} + |y|^{d-2}. \end{aligned}$$

Hence for $r + 1 \geq |y| \geq r \geq r_0$,

$$\begin{aligned}\|D_y p_r\|_m &\geq \|D_y y^d\|_m - (4|b| + 1)|y|^{d-2} \\ &\geq d|y|^{d-1} - \frac{4|b| + 1}{r_0}|y|^{d-1} \\ &\geq \frac{d}{2}|y|^{d-1}.\end{aligned}$$

Thus condition (1) is satisfied, as are the first two parts of condition (4).

Next we need to pass from y^d to $S_r(y) = Ry^d/|y|^{d-1}$. For $R, t \geq 0$, define

$$A_r(t, R) = \int_0^t (ds^{d-1} \phi_{r+1}(s) + (1 - \phi_{r+1}(s))R) ds.$$

Then $A_r(t, R) = t^d$ if $t \leq r + 9/8$ and $\frac{\partial}{\partial t} A_r(t, R) = R$ if $t \geq r + 2$. We will define $R = R(r)$ implicitly by $A_r(r + 2, R(r)) = R(r)(r + 2)$. To see that R can be so defined, let $B(r, R) = A_r(r + 2, R) - (r + 2)R$. Then using the fact that $0 \leq \phi_{r+1}(t) \leq 1$, we see that $(\partial/\partial R)B(r, R) < 0$, and integrating first from 0 to $r + 1$, then from $r + 1$ to $r + 2$ shows that $B(r, (r + 1)^{d-1}) > 0$ while $B(r, d(r + 2)^{d-1}) < 0$.

Hence $R(r)$ is well-defined, is unique, and by the implicit function theorem is C^∞ . Moreover, $R(r) \geq (r + 1)^{d-1}$. For $t \geq 0$, let $A_r(t) = A_r(t, R(r))$. Then $A_r(t) = t^d$ for $t \leq r + 9/8$ and $A_r(t) = R(r)t$ for $t \geq r + 2$. Also, $A'_r(t) \geq (r + 1)^{d-1}$ if $r + 1 \leq t \leq r + 2$.

For $|y| \geq r + 1$, let

$$p_r(y) = A_r(|y|) \frac{y^d}{|y|^d}.$$

Then condition (2) is satisfied, and $p_r(y)$ is C^∞ in (r, y) . To check the remaining part of condition (4), we calculate $\|D_y p_r\|_m$ for $|y| \geq r + 1$. By the rotational symmetry of $p_r(y)$ for $|y| \geq r + 1$, it suffices to find $\|D_t p_r\|_m$ for $t > r + 1$. Note that for ϵ real and near 0,

$$p_r(t + i\epsilon) = \frac{t^d + idt^{d-1}\epsilon}{|t + i\epsilon|^d} A_r(|t + i\epsilon|) + O(\epsilon^2),$$

so using the fact that $(\frac{\partial}{\partial \epsilon}|t + i\epsilon|)|_{\epsilon=0} = 0$ and $t^d = |t|^d$, we obtain

$$D_t p_r = \begin{pmatrix} A'_r(t) & 0 \\ 0 & \frac{d}{t} A_r(t) \end{pmatrix}. \quad (2.2)$$

Since $A'_r(t) \geq (r + 1)^{d-1}$ and $dA_r(t)/t \geq d(r + 1)^{d-1}$ for $t \geq r + 1$, condition (4) follows.

To verify condition (3), note that if $r \leq |y| \leq r + 1$, then $|p_r(y) - y^d| = \phi_r(y)|\hat{p}(y)| \leq |y|^{d-1}/2$, so dividing both sides by $|y|^d$ and taking real parts gives $\operatorname{Re}(p_r(y)/y^d) \geq 1 - 1/2|y| \geq r^{d-1}/2|y|^{d-1}$, where the last inequality follows from the fact that $|y| \geq r \geq r_0 > 1$.

If $|y| \geq r + 1$, then integrating from 0 to $r + 1$ and $r + 1$ to $|y|$ and using $R(r) \geq (r + 1)^{d-1}$ gives

$$\begin{aligned}A_r(|y|) &\geq (r + 1)^d + (r + 1)^{d-1}(|y| - (r + 1)) \\ &= |y|(r + 1)^{d-1},\end{aligned}$$

and hence $\operatorname{Re}(p_r(y)/y^d) = A_r(|y|)/|y|^d \geq r^{d-1}/|y|^{d-1}$. Hence condition (3) holds. \square

Proof of Proposition 2.1: Let $p_r(y)$ be as in the previous lemma. Note that if $|y| \leq r$, then $p_r(y) = p(y)$, and if $|y| \geq r + 2$, then $p_r(y) = S_r(y)$. Also, the map $(x, y) \mapsto (y, p_r(y) - ax)$ is a homeomorphism of \mathbb{C}^2 for each fixed r .

We next modify this homeomorphism by making a diffeomorphism of the plane for each y which equals the map $x \mapsto p_r(y) - ax$ if $|y| \geq r + 1$ or if $\|(x, y)\|_\infty \leq r$, and which equals $x \mapsto S_r(y) - ax$ if $|x|$ is sufficiently large (depending on r).

Define $\alpha_r(y) = |p_r(y) - S_r(y)|$. Then $\alpha_r(y)$ is continuous in (r, y) and uniformly continuous and bounded on $[r_0, R] \times \mathbb{C}$ for any $R > r_0$ since $p_r(y) = S_r(y)$ for $|y| \geq r + 2$. Hence, we can choose a positive C^∞ function $L(r)$ with $L(r) \geq \sup\{\alpha_r(y) : y \in \mathbb{C}\}/|a|$.

Next, define

$$\psi_r(x) = \phi\left(\frac{|x| - r - 2}{3L(r)}\right).$$

Then $\psi_r(x) = 1$ when $|x| \leq r + 2$, and $\psi_r(x) = 0$ if $|x| \geq r + 2 + 3L(r)$. Also, using the bounds on ϕ_r and ϕ' , we see that for $x_1, x_2, y \in \mathbb{C}$ and $t \in \mathbb{R}$,

$$\left|\frac{\partial}{\partial t}\psi_r(x_1 + x_2 t)\right| \leq \frac{2}{3L(r)}|x_2|. \quad (2.3)$$

Let

$$q_r(x, y) = \psi_r(x)p_r(y) + (1 - \psi_r(x))S_r(y),$$

and

$$G_r(x, y) = (y, q_r(x, y) - ax).$$

Note that $G_r(x, y)$ is a continuous map of \mathbb{C}^2 into itself. To show that G_r is a homeomorphism, we see from the form of G_r that it suffices to show that the function $g_{r,y}(x) = q_r(x, y) - ax$ is a homeomorphism of the plane for each fixed y . Given any $x_1, x_2 \in \mathbb{C}$, let $g(t) = g_{r,y}(x_1 + (x_2 - x_1)t)$ for $t \in \mathbb{R}$. Then g is C^∞ , and using (2.3) and the definition of $\alpha_r(y)$ we get

$$|g'(t) + a(x_1 - x_2)| \leq \frac{2|x_2 - x_1|}{3L(r)}\alpha_r(y).$$

Integrating g' from 0 to 1 and using $\alpha_r(y)/L(r) \leq |a|$ gives $|g_{r,y}(x_2) - g_{r,y}(x_1)| \geq |x_2 - x_1||a|/3$. Hence $g_{r,y}$ is injective. Moreover, $g_{r,y}$ is continuous, and $g_{r,y}(x) = S_r(y) - ax$ for $|x| \geq r + 2 + 3L(r)$, hence $g_{r,y}(x)$ extends to \mathbb{CP}^1 by fixing ∞ . By compactness, this extension is a homeomorphism of \mathbb{CP}^1 , hence $g_{r,y}$ is a homeomorphism of \mathbb{C} . As noted, this implies that G_r is a homeomorphism of \mathbb{C}^2 .

For the last modification, first note that $q_r(x, y) = S_r(y)$ if $|x| \geq r + 2 + 3L(r)$ or $|y| \geq r + 2$. Hence for such (x, y) we have $G_r(x, y) = (y, S_r(y) - ax)$. Define $M(r) = r + 2 + 3L(r)$.

We claim that there exist C^∞ functions $a(r, x, y)$ for $(r, x, y) \in [r_0, \infty) \times \mathbb{C}^2$, and $R_0(r)$ for $r \geq r_0$, such that

1. $R_0(r) \geq M(r) + 1$.
2. $a(r, x, y) = ax$ if $\|(x, y)\|_\infty \leq M(r)$.
3. $a(r, x, y) = x$ if $\|(x, y)\|_\infty \geq R_0(r)$.
4. For fixed r and y , and $c > 0$, there exists $c' \in \mathbb{C}$, $c' \neq 0$ such that $a(r, ce^{i\theta}, y) = c'e^{i\theta}$ for all $\theta \in \mathbb{R}$.
5. For all (r, x, y) , $|a(r, x, y)| \leq \max\{|ax|, |x|\}$.
6. For $t > 0$, $\frac{\partial}{\partial t}|a(r, t, y)| > 1 - \frac{|1-a|}{1+|a|}$.
7. For any $|x| < |y|$, $\|D_x a(r, x, y)\| + \|D_y a(r, x, y)\| < 7 \max\{|a|, 1\}$.

To construct such an $a(r, x, y)$, let

$$a_1(r, y) = \exp \left(\phi \left(\frac{|y| - M(r)}{N(r)} \right) \log |a| \right),$$

where $N(r) > M(r)$ is a positive function to be specified later. Note first that $a_1(r, y) = |a|$ if $|y| \leq M(r)$, and $a_1(r, y) = 1$ if $|y| \geq M(r) + N(r)$. Also, $a_1(r, y) \leq \max\{|a|, 1\}$, and given $\epsilon_0 > 0$, we can choose $N(r) > 0$ sufficiently large to obtain $\|D_y a_1(r, y)\| < \epsilon_0/(M(r) + 1)$. For $t > 0$ define

$$a_2(r, t, y) = t + \int_0^t \phi_{M(r)}(s)(a_1(r, y) - 1)ds.$$

Then $a_2(r, t, y) = a_1(r, y)t$ for $t \leq M(r)$, and $a_2(r, t, y) - t$ is constant for $t \geq M(r) + 1$.

Let $\epsilon(r, t, y) = a_2(r, t, y) - t$. Then $\epsilon(r, t, y) = \epsilon(r, M(r) + 1, y)$ whenever $t \geq M(r) + 1$, and for fixed r ,

$$|\epsilon(r, M(r) + 1, y)| \leq |1 - |a|| (M(r) + 1).$$

Let $\epsilon_r = |1 - |a|| (M(r) + 1)$, let $R_0(r) = (1 + 2N(r)(1 + |a|))(M(r) + 1)$, and let

$$a_3(r, t, y) = a_2(r, t, y) - \left(1 - \phi \left(\frac{t - M(r) - 1}{R_0(r) - M(r) - 1} \right) \right) \epsilon(r, M(r) + 1, y).$$

Then $a_3(r, t, y) = a_2(r, t, y)$ for $t \leq M(r) + 1$, and $a_3(r, t, y) = t$ for $t \geq R_0(r)$. Moreover, using $|\phi'| < 2$, we see that for $t \geq M(r) + 1$,

$$\left| 1 - \frac{\partial}{\partial t} a_3(r, t, y) \right| \leq 2 \frac{\epsilon_r}{R_0(r) - M(r) - 1} \leq \frac{|1 - |a||}{N(r)(1 + |a|)}. \quad (2.4)$$

Finally, let

$$a(r, x, y) = a_3(r, |x|, y) \exp \left(i\phi \left(\frac{|x| - M(r)}{N(r)} \right) \phi \left(\frac{|y| - M(r)}{N(r)} \right) \operatorname{Im}(\log a) \right) \frac{x}{|x|},$$

where $\log a$ is any branch of the logarithm. Then $a(r, x, y)$ satisfies properties 1 through 6 immediately.

Moreover, for $t > 0$, we see from (2.4) and $|\phi'| < 2$ that

$$\left| \frac{\partial}{\partial t} a(r, t, y) \right| \leq \max \left\{ |a|, 1 + \frac{1}{N(r)} \right\} + |a_3(r, M(r) + N(r), y)| \frac{2}{N(r)},$$

which by property 5 is bounded by $(3 + \epsilon_0) \max\{|a|, 1\}$ for $N(r)$ large. Also, for $t > 0$,

$$\left| \frac{\partial}{\partial \epsilon} a(r, t + i\epsilon, y)|_{\epsilon=0} \right| = |a_3(r, t, y)| \frac{1}{t},$$

which again by property 5 is bounded by $\max\{|a|, 1\}$. By rotational symmetry, we see that $\|D_x a(r, x, y)\| \leq (4 + \epsilon_0) \max\{|a|, 1\}$.

Next, note that $\|D_y a_2(r, |x|, y)\| \leq (M(r) + 1) \|D_y a_1(r, y)\| \leq \epsilon_0$, hence $\|D_y a_3(r, |x|, y)\| \leq 2\epsilon_0$. Thus

$$\|D_y a(r, x, y)\| \leq \|D_y a_3(r, |x|, y)\| + |a_3(r, |x|, y)| \left\| D_y \phi \left(\frac{|y| - M(r)}{N(r)} \right) \right\|,$$

and since this derivative of ϕ is 0 for $|y| \geq M(r) + N(r)$, property 5 implies that if $|x| < |y|$, then the right hand side is bounded by $2\epsilon_0 + 2(M(r) + N(r))/N(r)$. Combining this with the bound for $\|D_x a\|$, we see that for $\epsilon_0 < 1/4$ and $N(r)$ sufficiently large, property 7 is satisfied also. Thus, $a(r, x, y)$ exists as claimed.

In particular, for each fixed r and y , the map $x \mapsto S_r(y) - a(r, x, y)$ is a smooth injective map of \mathbb{C} to itself which equals $x \mapsto S_r(y) - x$ for $x \geq R_0(r)$, and hence this map is a diffeomorphism of \mathbb{C} .

Finally, let

$$F_r(x, y) = \begin{cases} G_r(x, y) = (y, q_r(x, y) - ax) & \text{if } \|(x, y)\|_\infty \leq M(r), \\ (y, S_r(y) - a(r, x, y)) & \text{if } \|(x, y)\|_\infty \geq M(r). \end{cases} \quad (2.5)$$

Note that the two definitions agree in a neighborhood of $\{\|(x, y)\|_\infty = M(r)\}$. To show that F_r is a homeomorphism of \mathbb{C}^2 to itself, it suffices to check that for each fixed y , $F_r(\cdot, y)$ is a homeomorphism of \mathbb{C} to itself, which follows from the remarks about $S_r(y) - a(r, x, y)$. \square

3 Conjugation

In this section we show that the map F_r constructed in the previous section is conjugate to the Hénon map F . First we develop some of the basic properties of the maps F_r , then we use the family of maps F_r to define an isotopy which can be used to conjugate F to F_r .

Notation: Let $F_\infty = F$.

LEMMA 3.1 *Let $r \in [r_0, \infty]$. If $|y| \geq r_0/2$ and $|x| \leq 2|y|$, then $(z, w) = F_r(x, y)$ satisfies $|w| \geq 3|z| \geq 3r_0/2$.*

Proof: Since $F_\infty = F_r$ on $\Delta^2(0; r)$ for any $r \geq r_0$, we may assume without loss that $r < \infty$. First suppose $\|(x, y)\|_\infty \leq M(r)$. Choose $\theta \in \mathbb{R}$ such that $e^{i\theta}y^d = |y|^d$. Since $|y| \geq r_0/2$, we have $|e^{i\theta}p(y) - e^{i\theta}y^d| \leq |y|^d/4$ by choice of r_0 . If $|y| \leq r$, then $\operatorname{Re}(e^{i\theta}p_r(y)) = \operatorname{Re}(e^{i\theta}p(y)) \geq 3|y|^d/4$. If $|y| \geq r$, then $\operatorname{Re}(e^{i\theta}p_r(y)) = \operatorname{Re}(p_r(y)/y^d)|y|^d \geq (r^{d-1}/2)|y|$. In either case, $\operatorname{Re}(e^{i\theta}p_r(y)) \geq 3r_0|y|/8$. Hence

$$\begin{aligned} |q_r(x, y)| &\geq \operatorname{Re}(e^{i\theta}q_r(x, y)) \\ &\geq \psi_r(x, y)r_0\frac{3|y|}{8} + (1 - \psi_r(x, y))R(r)|y| \\ &\geq \frac{3r_0|y|}{8} \end{aligned}$$

Since $r_0 \geq 8|a| + 8$ and $|x| \leq 2|y|$, we obtain $|w| = |q_r(x, y) - ax| \geq 3r_0|y|/8 - 2|a||y| \geq 3|y| = 3|z|$. Hence $(z, w) = F_r(x, y)$ satisfies $|w| \geq 3|z| \geq 3r_0/2$.

To finish the proof, suppose $\|(x, y)\|_\infty \geq M(r)$, in which case $(x, y) = (y, S_r(y) - a(r, x, y))$. Then $|w| = |S_r(y) - a(r, x, y)| \geq r_0|y| - \max\{|ax|, |x|\} \geq 3|y| = 3|z|$. Hence the lemma holds. \square

LEMMA 3.2 *Let $r \in [r_0, \infty]$. If $|x| \geq r_0/2$ and $|y| \leq 2|x|$, then $(z, w) = F_r^{-1}(x, y)$ satisfies $|z| \geq 3|w| \geq 3r_0/2$.*

Proof: Again we may assume $r < \infty$. Suppose that $\|(z, w)\|_\infty \leq M(r)$. Then $(x, y) = F_r(z, w) = (w, q_r(z, w) - az)$. Since $|w| = |x| \geq r_0/2$, we have as in the previous lemma that $|q_r(z, w)| \geq 3r_0|w|/8$. Since $|y| \leq 2|w|$, we have $|az| \geq |q_r(z, w)| - |y| \geq 3r_0|w|/8 - 2|w| \geq 3|a||w|$. Hence $|z| \geq 3|w| \geq 3r_0/2$.

If $\|(z, w)\|_\infty \geq M(r)$, then $(x, y) = (w, S_r(w) - a(r, z, w))$, so examining the second coordinate and using property 5 of $a(r, x, y)$ we obtain

$$\begin{aligned} \max\{|az|, |z|\} &\geq |a(r, z, w)| \\ &\geq R(r)|w| - |y| \\ &\geq ((r+1)^{d-1} - 2)|w| \\ &\geq (8|a| + 6)|w|. \end{aligned}$$

Hence $|z| \geq 3|w| \geq 3r_0/2$, as desired. \square

Notation: Let $\Delta_0^2 = \{(x, y) : |x| \leq r_0/2, |y| \leq r_0\}$.

Notation: For a map f , let $\Omega(f)$ denote the nonwandering set of f . I.e., $\Omega(f)$ consists of all points p such that for any neighborhood V of p , there exists $n > 0$ so that $V \cap f^n(V) \neq \emptyset$.

LEMMA 3.3 *For each $r \geq r_0$, $\Omega(F_r) = \Omega(F)$.*

Proof: By [2], $\Omega(F) \subseteq \Delta_0^2$. By the two previous lemmas, if $\|(x, y)\|_\infty \geq r_0/2$, then (x, y) is wandering for F_r . Since $F = F_r$ on Δ_0^2 , the lemma follows. \square

Notation: Let $V = \{|x| > r_0/2, |x| > |y|/2\}$, and for $r \in [r_0, \infty]$, let $D_r = (F_r(\overline{V}) \setminus V) \cup \overline{\Delta_0^2}$.

LEMMA 3.4 *Let $r \in [r_0, \infty]$ and $(x, y) \in \mathbb{C}^2$. Then there exists $n \in \mathbb{Z}$ such that $F_r^n(x, y) \in D_r$*

Proof: If there exists n so that $F_r^n(x, y) \in \overline{\Delta_0^2}$, then the proof is done. Otherwise, lemma 3.2 implies that $\cap_{n \leq 0} F_r^n(V) = \emptyset$, so $\cup_{n \leq 0} F_r^n(\mathbb{C}^2 \setminus V) = \mathbb{C}^2$. Hence there exists $n \geq 0$ such that $F_r^n(x, y) \in \mathbb{C}^2 \setminus V$. Similarly, lemma 3.1 implies that there exists $n \leq 0$ so that $F_r^n(x, y) \in V$. Since $F_r^{-1}(V) \subseteq V$ and $F_r(\mathbb{C}^2 \setminus V) \subseteq \mathbb{C}^2 \setminus V$, there exists a unique integer n such that $F_r^{n-1}(x, y) \in \overline{V}$ and $F_r^n(x, y) \in \mathbb{C}^2 \setminus V$, hence $F_r^n(x, y) \in F_r(\overline{V}) \setminus V$ as desired. \square

Notation: Let $U_0 = \{|x| < r_0, |y| \leq r_0\}$, $U_1 = \{|y| \geq r_0, |x| < |y|\}$, and $U = U_0 \cup U_1$. Note that U is a neighborhood of ∂V .

LEMMA 3.5 *There exists a neighborhood V_0 of \overline{V} such that $F_r(\overline{U}) \cap \overline{V}_0 = \emptyset$ for all $r \in [r_0, \infty]$.*

Proof: Lemma 3.2 implies that $F^{-1}(\overline{V}) \cap \overline{U}_0 = \emptyset$, so applying F gives $F(\overline{U}_0) \cap \overline{V} = \emptyset$. Note also that $F_r(\overline{U}_0) = F(\overline{U}_0)$ for all $r \geq r_0$. Moreover, if $(x, y) \in \overline{U}_1$, then by lemma 3.1, $(z, w) = F_r(x, y)$ has $|w| \geq 3|z| \geq 3r_0/2$, which is bounded away from \overline{V} . Hence for all $r \in [r_0, \infty]$, $F_r(\overline{U})$ is contained in the closed set $F(\overline{U}_0) \cup \{(x, y) : |y| \geq 3|x| \geq 3r_0/2\}$, and this union does not intersect \overline{V} . Hence there exists V_0 as claimed. \square

Notation: For $c \in \mathbb{C}$, let $X_c = \{(cy, y) : |y| > r_0/2\}$. For $w \in \mathbb{C}$, let $Y_w = \{(z, w) : z \in \mathbb{C}\}$.

LEMMA 3.6 *Let $|c| < 1$, and $w \in \mathbb{C}$. Then for all $r \in [r_0, \infty]$, $F_r(X_c)$ and Y_w are transverse. In particular, given $(z_0, w_0) \in F_r(X_c) \cap Y_{w_0}$, there exists a neighborhood W of r in $[r_0, \infty]$ and a unique C^∞ function $z : W \rightarrow \mathbb{C}$ such that $z(r) = z_0$ and for all $s \in W$, $(z(s), w_0) \in F_s(X_c) \cap Y_{w_0}$.*

Proof: Let $(z, w) \in F_r(X_c) \cap Y_w$, where $|c| < 1$ and $r \in [r_0, \infty]$. By lemma 3.1, $|w| \geq 3r_0/2$. Moreover, the tangent space of Y_w at (z, w) is $\mathbb{C}(1, 0)$. Hence to check transversality, it suffices to check that

$$\|D_y(\pi_2 F_r(cy, y))\|_m > 0.$$

In fact, we will prove the stronger result, which will be useful later, that if $|x| < |y|$, $|y| \geq r_0/2$, and $|c| < 1$, then for each real θ ,

$$\left| (D_{(x,y)} \pi_2 F_r) \begin{pmatrix} c \\ e^{i\theta} \end{pmatrix} \right| > \left| (D_{(x,y)} \pi_1 F_r) \begin{pmatrix} c \\ e^{i\theta} \end{pmatrix} \right|, \quad (3.1)$$

where $D_{(x,y)} \pi_j F_r$ represents the 2×4 matrix of partial derivatives of $\pi_j F_r$ at the point (x, y) .

From the definition of F_r in equation (2.5), we see that for any x and y ,

$$(D_{(x,y)} \pi_1 F_r) \begin{pmatrix} c \\ e^{i\theta} \end{pmatrix} = e^{i\theta},$$

and hence in (3.1) it suffices to show that the left hand side is bigger than 1. The definition of F_r also implies that if $\|(x, y)\|_\infty \leq M(r)$, then

$$\pi_2 F_r(x, y) = \psi_r(x)p_r(y) + (1 - \psi_r(x))S_r(y) - ax.$$

If also $|y| \geq r + 2$, then $p_r(y) = S_r(y)$, so

$$\pi_2 F_r(x, y) = R(r)|y| \frac{y^d}{|y|^d} - ax,$$

and a calculation like that leading to (2.2) together with $R(r) > 8|a| + 8$ and $|c| < 1$ shows that the left hand side of (3.1) is at least $R(r) - |ac| > 1$ in this case.

If $|y| < r + 2$, then $|x| < r + 2$ by assumption, so $\psi_r(x) = 1$ and hence $\pi_2 F_r(x, y) = p_r(y) - ax$. Since $|y| > r_0/2$, part 4 of lemma 2.2 and the choice of r_0 shows that

$$\|D_y p_r(y)\|_m \geq (|y| - 1)^{d-1} > 4|a| + 3,$$

so the left hand side of (3.1) is at least $3|a| + 3 > 1$. Hence (3.1) holds whenever $|x| < |y|$, $|c| < 1$, and $r_0/2 < |y| \leq M(r)$.

If $|y| \geq M(r)$, then

$$\pi_2 F_r(cy, y) = R(r)|y| \frac{y^d}{|y|^d} - a(r, x, y).$$

By property 7 of $a(r, x, y)$ and a calculation like that for equation (2.2), we see that the left hand side of (3.1) is at least

$$R(r) - \|D_x a(r, x, y)\||c| - \|D_y a(r, x, y)\| > 1.$$

Hence the transversality part of the current lemma is seen to be true, and the rest of the lemma follows from the implicit function theorem. \square

LEMMA 3.7 *There exists a C^∞ map $g : [r_0, \infty] \times \mathbb{C}^2 \rightarrow \mathbb{C}^2$ such that $g_r = g(r, \cdot)$ satisfies*

1. $g_\infty = Id$.
2. g_r is a diffeomorphism of \mathbb{C}^2 and a diffeomorphism of D_∞ onto D_r .
3. $\pi_2 g_r(x, y) = y$ for all $(x, y) \in \mathbb{C}^2$ and all $r \in [r_0, \infty]$.
4. $g_r = Id$ on a neighborhood (independent of r) of $\partial V \cup \Delta_0^2$.
5. $g_r(F(X_c)) = F_r(X_c)$ for all $r \in [R_0, \infty]$ and all $|c| \leq 3/4$.

Proof: We will define g_r as the time- t map ($t = r_0/r$) of a time dependent vector field which preserves the sets Y_w and which is compactly supported within each Y_w . This approach avoids questions of the completeness of vector fields defined on noncompact sets. First we define an isotopy, h_r , of $F(U)$ to $F_r(U)$ which preserves the sets Y_w , then use the ideas of the isotopy extension theorem to obtain g_r .

Fix $(x_0, y_0) \in U$. If $|y_0| \leq r_0$, then also $|x_0| \leq r_0$, so $F_r(x_0, y_0) = F(x_0, y_0)$ for all $r \in [r_0, \infty]$. Define $h_r(F(x_0, y_0)) = F(x_0, y_0)$ for all $r \in [r_0, \infty]$.

Next, let $(x_0, y_0) \in U$ with $|y_0| > r_0$, and let $|c| < 1$ such that $x_0 = cy_0$. Since $F_r(x_0, y_0) = F(x_0, y_0)$ for $r \geq \|(x_0, y_0)\|_\infty$, we define $h_r(F(x_0, y_0)) = F(x_0, y_0)$ for such r . To complete the definition of h_r , let $w = \pi_2 F(x_0, y_0)$, and suppose that $h_r(F(x_0, y_0))$ is defined and C^∞ for $r \in (r_1, \infty]$ with $h_r(F(x_0, y_0)) \in F_r(X_c) \cap Y_w$ for all $r \in (r_1, \infty]$.

In order to extend $h_r(F(x_0, y_0))$ to $[r_0, \infty]$, we first extend the definition to $r = r_1$, then use transversality to extend to a neighborhood of r_1 . By compactness, this is sufficient to extend to all of $[r_0, \infty]$.

The definition of F_r shows that $\pi_2 F_r(cy, y)$ behaves like a polynomial in the sense that $\pi_2 F_r(cy, y) = w$ has exactly d zeros counted with multiplicity (note that lemma 3.6 implies that the multiplicity of each zero is 1 except possibly at points where $F_r(cy, y) = F(cy, y)$ is holomorphic). In particular, $h_{r_1}(F(x_0, y_0)) = \lim_{r \rightarrow r_1} h_r(F(x_0, y_0))$ must exist and equal one of the images of the d zeros corresponding to $r = r_1$. Simple estimates imply that $h_{r_1}(F(x_0, y_0))$ is contained in $F_{r_1}(X_c) \cap Y_w$, so lemma 3.6 implies that $h_r(F(x_0, y_0))$ extends uniquely to a neighborhood of r_1 with $h_r(F(x_0, y_0)) \in F_r(X_c) \cap Y_w$, hence h_r extends to $r \in [r_0, \infty]$, as desired.

The implicit function theorem implies that h_r is smooth on $[r_0, \infty] \times F(U)$. Note also that $h_r = Id$ on Δ_0^2 for all r . As in the isotopy extension theorem, e.g., [7, theorem 8.1.4], the trajectory of a point under h_r defines a smooth vector field by

$$Q_t(p) = \begin{cases} \left(\frac{d}{dr} h_r(h_{r_0/t}^{-1}(p)) \right) \Big|_{r=r_0/t} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0, \end{cases}$$

where $p \in h_{r_0/t}(F(U)) = F_{r_0/t}(U)$ and $t \in [0, 1]$. Note that $\pi_2 Q_t \equiv 0$, so that Q_t restricts to a vector field on each Y_w .

Let $\mu : \mathbb{C}^2 \rightarrow [0, 1]$ be a C^∞ function which is 0 on $\mathbb{C}^2 \setminus U$ and 1 in a neighborhood of $\mathbb{C}^2 \setminus V$ and such that $\mu = 1$ on each set X_c with $|c| \leq 3/4$. Then $Z_t(p) = \mu(F_{r_0/t}^{-1}(p))Q_t(p)$ is a C^∞ vector field for $p \in F_{r_0/t}(U)$, $t \in [0, 1]$, with $\pi_2 Z_t \equiv 0$, $Z_t \equiv Q_t$ on $F_{r_0/t}(\mathbb{C}^2 \setminus V)$ and on $F_{r_0/t}(X_c)$ for each $|c| \leq 3/4$, and $Z_t \equiv 0$ on a neighborhood of $F_{r_0/t}(\partial U)$ and on $\Delta_0^2 \cap F_{r_0/t}(U)$ since $h_r \equiv Id$ there for any r . Hence Z_t extends to a smooth vector field on \mathbb{C}^2 by setting $Z_t \equiv 0$ on the complement of $F_{r_0/t}(U)$. In particular, Z_t preserves each Y_w , and $Z_t \equiv 0$ on \overline{V} by lemma 3.5, so Z_t is complete.

For $r \in [r_0, \infty]$, let g_r be the time- (r_0/r) map of Z_t . Then properties 1-4 are immediate, and property 5 is true since $h_r(F(X_c)) = F_r(X_c)$ and $g_r = h_r$ on $F(X_c)$ for $|c| \leq 3/4$. \square

THEOREM 3.8 *For each $r \geq r_0$, the maps F_r and F are conjugate via a homeomorphism of \mathbb{C}^2 . I.e., there exists $\Phi_r : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ a homeomorphism such that $\Phi_r F \Phi_r^{-1} = F_r$.*

Proof: We will construct a homeomorphism $\Phi_r : D_\infty \rightarrow D_r$ such that Φ_r is the identity on $\partial V \cup \overline{\Delta_0^2}$, and $\Phi_r = F_r \circ F^{-1}$ on $F(\partial V)$.

Before constructing Φ_r , we show how this proves the theorem. Suppose $p \in D_\infty$ and $F(p) \in D_\infty$. By lemma 3.1, either $p \in \partial V$ or $p \in \overline{\Delta_0^2}$. If $p \in \partial V$, then $\Phi_r(p) = p$, so $F_r \Phi_r(p) = F_r(p)$. Also, $F(p) \in F(\partial V)$, so $\Phi_r F(p) = (F_r F^{-1})F(p) = F_r(p) = F_r \Phi_r(p)$. On

the other hand, if $p \in \overline{\Delta_0^2}$ (and $F(p) \in D_\infty$), then $F(p) \in \overline{\Delta_0^2}$, so $\Phi_r(p) = p$, $\Phi_r F(p) = F(p)$, and $F(p) = F_r(p)$, so $F_r \Phi_r(p) = \Phi_r F(p)$. Hence $F_r \Phi_r = \Phi_r F$ on $D_\infty \cap F^{-1}(D_\infty)$.

Moreover, given Φ_r defined on D_∞ , we can extend Φ_r to \mathbb{C}^2 by iteration. For $p \in \mathbb{C}^2$, lemma 3.4 implies that there exists $n \in \mathbb{Z}$ so that $F^n(p) \in D_\infty$. Define $\Phi_r(p) = F_r^{-n} \Phi_r F^n(p)$. Since $\Phi_r(p) = F_r \Phi_r F(p)$ for $p \in D_\infty \cap F^{-1}(D_\infty)$ and $F_r = F$ on $\overline{\Delta_0^2}$, we see that Φ_r is well-defined and continuous. Moreover, Φ_r^{-1} is defined by $\Phi_r^{-1}(p) = F^n \Phi_r^{-1} F^n(p)$, where $n \in \mathbb{Z}$ is chosen so that $F^n(p) \in D_r$, and as before, Φ_r^{-1} is well-defined and continuous. Hence Φ_r is a homeomorphism of \mathbb{C}^2 with $\Phi_r F \Phi_r^{-1} = F_r$ on \mathbb{C}^2 .

To construct Φ_r , let $\mu : \mathbb{C}^2 \rightarrow [0, 1]$ be C^∞ with $\mu \equiv 0$ in a neighborhood of $F(\partial V) \setminus F(\overline{\Delta_0^2})$ and $\mu \equiv 1$ in a neighborhood of $\partial V \cup \overline{\Delta_0^2}$. We also require that μ is constant on each set $F(X_c)$ for $|c| \leq 1/2$ and $\mu(p) = 1$ for each $p = F(cy, y)$ with $|c| \geq 1/2$. Define $\Phi_r : D_\infty \rightarrow \mathbb{C}^2$ by

$$\Phi_r(p) = F_r F_r^{-1} g_{r/\mu(p)}(p).$$

Note that if $|c| \leq 3/4$, then part 5 of the previous lemma implies that $\Phi_r(F(X_c)) = F_r(X_c)$. Moreover, parts 2 and 5 of that lemma imply that g_r is a diffeomorphism of $D_\infty \setminus \{F(X_c) : |c| \leq 1/2\}$ to $D_r \setminus \{F_r(X_c) : |c| \leq 1/2\}$. Since μ is constant on each $F(X_c)$, this implies that Φ_r is a diffeomorphism of D_∞ to D_r . In fact, setting $\gamma(p) = \mu(F(F_r^{-1}(p)))$, we have

$$\Phi_r^{-1}(p) = g_{r/\gamma(p)}^{-1} F_{r/\gamma(p)} F_r^{-1}(p).$$

Moreover, if p is in the neighborhood of $\partial V \cup \overline{\Delta_0^2}$ where $\mu = 1$ and $g_r = Id$, then $\Phi_r(p) = p$. Similarly, if p is in the neighborhood of $F(\partial V) \setminus F(\overline{\Delta_0^2})$ where $\mu = 0$, then $\Phi_r(p) = F_r F^{-1}(p)$. Finally, if $p \in F(\partial V) \cap F(\overline{\Delta_0^2})$, then $F_s = F$ at $F^{-1}(p)$ for any $s \geq r_0$, and hence $g_s(p) = p$ and thus $\Phi_r(p) = p$. Thus Φ_r satisfies all of the needed requirements. \square

4 Behavior on the sphere at infinity

In this section we describe the behavior of the map F_r on the 3-sphere at infinity. Much of this section is similar to and inspired by [8], where the theory of solenoidal mappings is developed more fully.

Let $\mathbb{T} = D \times S^1$, where D is the disk of radius 2 in \mathbb{C} , and let (z, ζ) be the coordinates in \mathbb{T} . Let C_+ and C_- be the families of cones

$$\begin{aligned} C_+(z, \zeta) &= \{(u, \xi) \in T_{(z, \zeta)} \mathbb{T} : |\xi| \geq |u|\}, \\ C_-(z, \zeta) &= \{(u, \xi) \in T_{(z, \zeta)} \mathbb{T} : |\xi| \leq |u|\} \end{aligned}$$

in the tangent bundle of \mathbb{T} . Following [8], we say that a map $\tau : \mathbb{T} \rightarrow \mathbb{T}$ is *solenoidal* of degree d if τ is a C^1 injective immersion of degree d such that $D\tau(C_+) \subseteq C_+$, and if there exists a constant $K > 1$ such that for all $(z, \zeta) \in \mathbb{T}$ we have that $(u, \xi) \in C_+(z, \zeta)$ and $D_{(z, \zeta)} \tau(u, \xi) = (u_1, \xi_1)$ imply $|\xi_1| > K|\xi|$ and that $(u, \xi) \in C_-(z, \zeta)$ and $D_{(z, \zeta)} \tau(u, \xi) = (u_1, \xi_1)$ imply $|u_1| < 1/K|u|$.

A standard model of a solenoidal mapping is given by defining

$$\tau_d(z, \zeta) = (\zeta + \epsilon z \zeta^{1-d}, \zeta^d)$$

for $(z, \zeta) \in \mathbb{C}^2$, $|z| \leq 2$, $|\zeta| = 1$. In [8] and [9], two different methods are given for extending a Hénon map to a compactification of \mathbb{C}^2 so that the compactification is homeomorphic to a closed 4-ball and so that the Hénon map extends to a homeomorphism of this closed 4-ball to itself. Moreover, for both methods, the authors show that on the sphere at infinity, this homeomorphism has an attracting solenoid and a repelling solenoid, and that in a neighborhood of the attracting solenoid the homeomorphism is conjugate to τ_d , where d is the degree of the Hénon map. In a neighborhood of the repelling solenoid, the homeomorphism is conjugate to τ_d^{-1} .

These results are also true for the present extension. Let \hat{H}_r denote the map induced by H_r (or equivalently F_r) on the 3-sphere. Let $V^+ = \{(x, y) : |x| < 2|y|, |x|^2 + |y|^2 = 1\}$. Note that V^+ is a solid torus and that by lemma 3.1, \hat{H}_r maps V^+ into the interior of V^+ . Also, let $\sigma(x, y) = (y, x)$.

THEOREM 4.1 *For r sufficiently large, $\hat{H}_r|V^+$ (resp. $\hat{H}_r^{-1}|\sigma(V^+)$) is solenoidal and conjugate to τ_d (resp. τ_d^{-1}).*

Proof: We can write \hat{H}_r in coordinates (z, ζ) , $|z| \leq 2$, $|\zeta| = 1$ by

$$\hat{H}_r(z, \zeta) = \left(\frac{\zeta}{|R\zeta^d - z|}, \frac{R\zeta^d - z}{|R\zeta^d - z|} \right),$$

where $R = R(r)$. Conjugating ζ by the map $\theta \mapsto e^{i\theta}$, which induces an isometry on the tangent bundle, we can write \hat{H}_r as

$$\begin{aligned} \hat{H}_r(z, \theta) &= \left(\frac{e^{i\theta}}{|Re^{i\theta} - z|}, \operatorname{Im} \log(Re^{i\theta} - z) \right) \\ &= \left(\frac{e^{i\theta}}{R} \left| 1 - \frac{ze^{-i\theta}}{R} \right|^{-1}, d\theta + \operatorname{Im} \log \left(1 - \frac{ze^{-i\theta}}{R} \right) \right). \end{aligned}$$

From this, we see that as $r \rightarrow \infty$ (and hence $R \rightarrow \infty$), the partial derivatives of all terms except $d\theta$ tend to 0. In particular, for r sufficiently large, \hat{H}_r satisfies the conditions for being solenoidal.

The fact that $\hat{H}_r : V^+ \rightarrow V^+$ is conjugate to τ_d follows as in corollary X.2 of [9]: \hat{H}_r restricted to V^+ is conjugate to one of the maps $\tau_{d,k}$ as in theorem 3.11 of [8], and proposition 4.6 of [8] implies that $\tau_{d,0} = \tau_d$ is the only one of these maps which can be extended to an orientation preserving homeomorphism of the 3-sphere. Since \hat{H}_r extends, it must be conjugate to τ_d . Since \hat{H}_r^{-1} is conjugate to \hat{H}_r by the involution $\sigma(x, y) = (y, x)$, we see that \hat{H}_r^{-1} is conjugate to τ_d^{-1} on $\sigma(V^+)$. \square

In general, \hat{H}_r need not be conjugate to \hat{H}_s on all of the 3-sphere for $r \neq s$. John Hubbard is planning a detailed study of the conjugacy classes of maps similar to \hat{H}_r .

5 Smooth results

In this section we modify the proofs of the previous sections to obtain the same results except that here the maps F_r and the one parameter family are all C^∞ smooth.

Let F , r_0 , r , and $R = R(r)$ be as before, and define

$$J_r(x, y) = \left(y, R \frac{y^d}{|y|^{d-1}} \phi \left(\frac{|x|}{CR|y|} \right) - x \right),$$

where $C = C(r)$ is a positive function to be specified. Then J_r is \mathbb{R}^+ homogeneous of degree 1, and J_r is C^∞ on $\mathbb{C}^2 \setminus \{(0, 0)\}$.

PROPOSITION 5.1 *There exists a one-parameter family of C^∞ diffeomorphisms F_r , $r \geq r_0$ of \mathbb{C}^2 and a C^∞ function $R_0(r) > r$ such that $F_r(x, y) = F(x, y)$ if $\|(x, y)\|_\infty < r$ and $F_r(x, y) = J_r(x, y)$ if $\|(x, y)\|_\infty > R_0(r)$. Moreover, $F_r(x, y)$ is C^∞ in (r, x, y) .*

Proof: The proof is nearly the same as that for proposition 2.1. Referring to that proof, define $\alpha_r(y)$ as before, define $\beta_r = \sup\{|p_r(y)| + |S_r(y)| : |y| \leq 1\}$, then choose $L(r)$ a positive C^∞ function so that $L(r) \geq \max\{\beta_r, \sup\{\alpha_r(y) : y \in \mathbb{C}\}\}/|a|$.

Define ψ_r as before with this new $L(r)$, let $M(r) = r + 2 + 3L(r)$, and let $C = C(r)$ be C^∞ with $C(r) \geq \max\{24/|a|, M(r), 2 + 1/|a|\}$. Let

$$q_r(x, y) = \psi_r(x)p_r(y) + (1 - \psi_r(x))S_r(y)\phi \left(\frac{|x|}{CR|y|} \right)$$

and

$$G_r(x, y) = (y, q_r(x, y) - ax).$$

Since $|\phi'| \leq 2$, we have for any $x_1, x_2 \in \mathbb{C}$ that

$$\begin{aligned} \left| \frac{\partial}{\partial t} q_r(x_1 + (x_2 - x_1)t, y) \right| &\leq |S_r(y)| \frac{2|x_2 - x_1|}{CR|y|} (1 + \psi_r(x_1 + (x_2 - x_1)t)) \\ &+ \left| \frac{\partial}{\partial t} \psi_r(x_1 + (x_2 - x_1)t) \right| \left| p_r(y) - S_r(y)\phi \left(\frac{x_1 + (x_2 - x_1)t}{CR|y|} \right) \right|. \end{aligned} \quad (5.1)$$

Note that in this expression both ψ_r and $(\partial/\partial t)\psi_r$ are 0 if $|x_1 + (x_2 - x_1)t| \geq M(r)$, in which case the right hand side is bounded by $|a||x_2 - x_1|/12$. On the other hand, if $|x_1 + (x_2 - x_1)t| \leq M(r)$, then $\phi(|x|/CR|y|) = 1$ whenever $|y| \geq 8M(r)/CR$. Since $8M(r)/CR \leq 1$, we see that if $|x_1 + (x_2 - x_1)t| \leq M(r)$, then

$$\left| p_r(y) - S_r(y)\phi \left(\frac{x_1 + (x_2 - x_1)t}{CR|y|} \right) \right| \leq \max\{\beta_r, \alpha_r(y)\} \leq L(r)|a|.$$

This plus equation (2.3) and $C \geq 24/|a|$, implies that the right side of (5.1) is bounded by $5|a||x_1 - x_2|/6$, so continuing as in the proof of proposition 2.1, we see that G_r is a diffeomorphism of \mathbb{C}^2 .

Finally, with $a(r, x, y)$ as in the proof of proposition 2.1, we can define

$$F_r(x, y) = \begin{cases} G_r(x, y) = (y, q_r(x, y) - ax) & \text{if } \|(x, y)\|_\infty \leq M(r), \\ \left(y, S_r(y)\phi \left(\frac{|x|}{C(r)R(r)|y|} \right) - a(r, x, y) \right) & \text{if } \|(x, y)\|_\infty \geq M(r). \end{cases} \quad (5.2)$$

Note that $\phi(t) = 0$ for $t > 1$, so F_r is smooth even along $|y| = 0$. Moreover, properties 4 and 6 of $a(r, x, y)$ together with the fact that

$$\left| \frac{\partial}{\partial t} S_r(y) \phi \left(\frac{t}{CR|y|} \right) \right| \leq \frac{2}{C}$$

and the fact that $C > 2 + 1/|a|$ implies that $F_r(\cdot, y)$ is a diffeomorphism of \mathbb{C} for each fixed y , hence that F_r is a diffeomorphism of \mathbb{C}^2 . By property 3 of $a(r, x, y)$, $F_r(x, y) = J_r(x, y)$ for $\|(x, y)\|_\infty \geq R_0(r)$, as desired. \square

THEOREM 5.2 *The maps F and F_r are conjugate via a C^∞ diffeomorphism of \mathbb{C}^2 . I.e., there exists $\Phi_r : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ a diffeomorphism such that $\Phi_r^{-1} F \Phi_r = F_r$.*

Proof: On the set U , the F_r constructed in this section equals the F_r constructed in proposition 2.1. Hence the conjugation in the current theorem is obtained exactly as in theorem 3.8.

The fact that Φ_r is C^∞ follows from the facts that Φ_r is C^∞ on D_∞ , that F and F_r are C^∞ , that $\Phi_r = Id$ on a neighborhood of $\partial V \cup \overline{\Delta_0^2}$ and $\Phi_r = F_r \circ F^{-1}$ on a neighborhood of $F(\partial V)$, and that $\Phi_r(p) = F_r^{-n} \Phi_r F^n(p)$ for $p \in F^{-n}(D_\infty)$. \square

Note that $J_r = H_r$ except at points (x, y) where $|x|/CR|y| \geq 1/8$, which implies the equality of \hat{J}_r and \hat{H}_r except on the analogous set in the 3-sphere. In particular, $\hat{J}_r|V^+ = \hat{H}_r|V^+$ is still conjugate to τ_d . Moreover, if $|x|/CR|y| \geq 1/8$ and $(z, w) = J_r(x, y)$, then $C \geq 24/|a|$ implies that $|ax|/2 \geq R|y|$, hence $|w| \geq |ax|/2 > 2|z|$. Thus, for such (x, y) , the point on the 3-sphere corresponding to $J_r(x, y)$ is not contained in $\sigma(V^+)$. In particular, if $(z, w) \in \sigma(V^+)$, then $\hat{J}_r^{-1}(z, w) = \hat{H}_r^{-1}(z, w)$, so \hat{J}_r^{-1} is conjugate to τ_d^{-1} on $\sigma(V^+)$ by theorem 4.1.

6 Compositions of Hénon maps

In this section we obtain results similar to those in earlier sections, but here the focus is on compositions of Hénon maps.

Let $F_j(x, y) = (y, p_j(y) - a_j x)$ for $j = 1, \dots, n$, where p_j is a polynomial of degree $d_j \geq 2$ having the form of the polynomial p of section 2 and $a_j \geq 0$. As noted earlier, [6] implies that any polynomial automorphism of \mathbb{C}^2 with nontrivial dynamics is conjugate to the map $F = F_n \circ \dots \circ F_1$ for some choice of n , the p_j 's and the a_j 's.

For each j , we can construct $F_{j,r}$ as in proposition 2.1 (or proposition 5.1). Let r_0 be the maximum of the r_0 's corresponding to each individual $F_{j,r}$. Let $\hat{r}_0 > 0$ be large enough that $F(\overline{\Delta_0^2(0; \hat{r}_0)})$ contains $\Delta^2(0; r_0)$.

THEOREM 6.1 *Let $\mathbf{r} = (r_1, \dots, r_n)$ with each $r_j \geq \hat{r}_0$. Then $F = F_n \circ \dots \circ F_1$ is conjugate on \mathbb{C}^2 to $F_{\mathbf{r}} = F_{n,r_n} \circ \dots \circ F_{1,r_1}$. If each F_{j,r_j} is constructed as in proposition 5.1, then the conjugating map is smooth.*

Proof: The proof of lemma 3.6 implies that the inequality in (3.1) is valid for each $F_{j,r}$ for all $r \geq r_0$, $|x| < |y|$, $|y| \geq r_0/2$, $|c| < 1$, and θ real. By lemma 3.1, we can apply each F_{j,r_j} in turn to see that (3.1) is valid for $F_{\mathbf{r}}$. Hence lemma 3.6 is valid for $F_{\mathbf{r}}$.

For each $r \geq 0$, let $F_{\mathbf{r},r} = F_{n,r_n+r} \circ \cdots \circ F_{1,r_1+r}$, let $F_{\mathbf{r},\infty} = F$, and for $r \in [0, \infty]$, let $D_r = (F_{\mathbf{r},r}(\overline{V}) \setminus V) \cap \overline{\Delta}_0^2$.

Since lemma 3.6 is valid in the current setting and by the choice of \hat{r}_0 , we see that lemma 3.7 is valid with $F_{\mathbf{r},r}$ in place of F_r using the same proof (up to some notational changes). Hence the proof of theorem 3.8 is also valid in the current setting. Thus $F_{\mathbf{r}}$ and F are conjugate as claimed, and the smoothness follows as in theorem 5.2. \square

Finally, for r_j sufficiently large, the map induced on the 3-sphere at infinity by F_{j,r_j} will be solenoidal on V^+ as in theorem 4.1, so the composition of these maps, $\hat{F}_{\mathbf{r}}$ will also be solenoidal on V^+ , as can be seen by checking the definition of solenoidal. Moreover, $\hat{F}_{\mathbf{r}}$ is conjugate to $\tau_{d_n} \circ \cdots \circ \tau_{d_1}$ on V^+ . This can be seen by applying proposition 3.8 of [8] inductively to obtain the conjugacy on a fundamental domain for $\hat{F}_{\mathbf{r}}$, then completing the conjugacy as in theorem 3.11 of [8]. Analogous results obtain for the inverses of the relevant maps. For further results on such compositions, the reader is urged to consult section XI of [9].

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