KUPKA-SMALE THEOREM FOR POLYNOMIAL AUTOMORPHISMS OF $\mathbb{C}^2$ AND PERSISTENCE OF HETEROCLINIC INTERSECTIONS

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Abstract. A map is *Kupka-Smale* if all periodic points are hyperbolic and the stable and unstable manifolds of any two saddle points are transverse. Here we prove that Kupka-Smale maps form a residual set of full Lebesgue measure in the space $\mathcal{P}_d$ of polynomial automorphisms of $\mathbb{C}^2$ of fixed dynamical degree $d \geq 2$. We also prove that a heteroclinic point of two saddle periodic orbits may be continued over (almost) the entire parameter space for this set of maps. This is one of the first *persistence theorems* proved in holomorphic dynamics in several variables.

1. Introduction

1.1. **Background and main results.** One way to study a family of dynamical systems is to look for structural features that are shared by a large subset of the family. One of the well-known structural properties of dynamical systems is the Kupka-Smale property, which is defined by requiring that all the periodic orbits of the system are hyperbolic (have no multiplier on the unit circle), and that the stable and unstable manifolds of any two saddle periodic points intersect transversally. The classical Kupka-Smale theorem for diffeomorphisms [Sma63] (see [Kup63] for the vector field case) claims that in the space of all C^r-diffeomorphisms of a real manifold, $r \geq 1$, the set of maps with the Kupka-Smale property contains a *residual subset.* By definition, a residual subset is a set containing a countable intersection of open, dense sets. A property shared by all systems from a residual subset is called *topologically generic.* This concept of genericity has been criticized by many experts because a residual subset may be small in the measure-theoretic sense. Indeed, a residual subset of the line may have Lebesgue measure zero.

An alternative concept of *metrically typical* or, in more modern terminology, *prevalent* properties goes back to Kolmogorov [Kol57] and Arnold, and was studied recently in [HSY92] and [Kala, Kalb]. In a finite dimensional space of maps, this is defined by saying that the set of maps with a given

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property is of full measure. The concept of prevalence in Banach spaces (like vector fields on a manifold) was defined by Hunt, Sauer, and Yorke [HSY92]. The same concept for nonlinear infinite dimensional spaces, (like the space of smooth maps of a compact manifold to itself) was defined by Kaloshin [Kala], who modified the definition of Arnold. In [Kalb] it was shown that the Kupka-Smale property for smooth vector fields and maps is *prevalent*, that is, typical in the metric sense.

In addition to the aforementioned work, the Kupka-Smale theorem has been proven in many settings, including smooth endomorphisms of a compact manifold [Shu69], real-analytic diffeomorphisms [BT86, Les83] and holomorphic automorphisms of $\mathbb{C}^n$ [Buz98]. However, one commonality of this previous work is that all of the underlying spaces are infinite dimensional.

By contrast, in this paper we consider the finite dimensional space of invertible polynomial dynamical systems of given degree in $\mathbb{C}^2$. For smooth diffeomorphisms, the Kupka-Smale theorem is proved by using local perturbation techniques to convert nonhyperbolic periodic points to hyperbolic and to convert tangencies between invariant manifolds into transverse intersections. However, in the space of fixed degree polynomial maps, perturbations must be made by changing the coefficients, and in general it's very difficult to know how stable and unstable manifolds move in response to a change in coefficients. Therefore the genericity of the Kupka-Smale property in this setting is a difficult problem requiring a completely new approach.

Loosely following [FM89], we define the set $\mathcal{P}_d$ as the set of all the "normalized" polynomial automorphisms of dynamical degree $d \geq 2$. We give precise definitions in Section 2.1. An important irreducible component of $\mathcal{P}_d$ is the set $\mathcal{H}_d$ of generalized Henon maps

$$F : (x, y) \mapsto (y, P(y) - ax),$$

where P is a degree d monic polynomial and $a \neq 0$. The parameter space for $\mathcal{H}_d$ is $\mathbb{C}^d \times \mathbb{C}^*$, hence is Stein, while $\mathcal{P}_d$ is formed by taking the composition of generalized Henon maps to obtain total degree d.

In this paper we prove

Theorem 1.1. *Let $d > 1$. Then there exists a set $KS \subset \mathcal{P}_d$ such that KS is residual and of full measure and such that for any F in KS, every periodic point of F is hyperbolic and the stable and unstable manifolds of any two saddle periodic points are transverse.*

Thus for $\mathcal{P}_d$, the set KS is both residual and prevalent in the metric sense. This theorem implies almost immediately the corresponding theorem for real maps. I.e., let $\mathcal{P}_d^{\mathbb{R}}$ be the set of all real maps $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F \in \mathcal{P}_d$.

Theorem 1.2. *Theorem 1.1 holds for $\mathcal{P}_d^{\mathbb{R}}$ instead of $\mathcal{P}_d$; namely, a residual set of full Lebesgue measure of real polynomial automorphisms of $\mathbb{R}^2$ of fixed dynamical degree $d > 1$ has the Kupka-Smale property.*

Theorem 1.1 and its proof provide several additional results. First, since the order of contact between two one-dimensional complex stable and unstable manifolds is invariant under topological conjugacy [Buz99], Theorem 1.1 implies that any $F \in \mathcal{P}_d$ with a tangency between stable and unstable manifolds of saddle periodic points is not globally topologically conjugate to a map in the set KS. Since KS is dense, this means F is not structurally stable. Thus we obtain the following corollary.

Corollary 1.1. *Let $f \in \mathcal{P}_d$ and suppose that f has a tangency between stable and unstable manifolds of saddle periodic points. Then f is not structurally stable in $\mathcal{P}_d$.*

Also, for sufficiently large d, there are open subsets of $\mathcal{P}_d$ in which maps with tangencies are dense [Buz97]. Hence Theorem 1.1 gives an alternative proof of the following theorem, which was proved in [Buz99] with different methods.

Theorem 1.3. *There exists $N > 0$ so that if $d \geq N$, then the set of structurally stable maps in $\mathcal{P}_d$ is not dense in $\mathcal{P}_d$.*

In proving Theorem 1.1, we prove also the following theorem, which says roughly that there are no dependencies between two multipliers (eigenvalues) of periodic points when viewed as functions of parameter space.

Theorem 1.4. *Let p and q be saddle periodic points for $F \in \mathcal{P}_d$ of period m and n, respectively, let λ be the stable multiplier (eigenvalue) for p, and let μ be the unstable multiplier for q. Then λ and μ are independent in the sense that the map from a small neighborhood of F in $\mathcal{P}_d$ to the (λ, μ) plane has rank 2 at generic points.*

The tools we develop to prove Theorem 1.1 allow us also to follow intersections of stable and unstable manifolds as the map varies through parameter space; this is the persistence part of the title. We give a fuller account of these tools and results in Section 2, but here we give a weakened form of this kind of persistence result. Roughly, it says that given a heteroclinic intersection between two saddle points for $F \in \mathcal{P}_d$ and any curve in $\mathcal{P}_d$ starting at F so that the corresponding periodic points remain saddles, there is an arbitrarily nearby curve in $\mathcal{P}_d$ so that the heteroclinic intersection may be continued along this new curve.

Theorem 1.5. *Let $\gamma : [0,1] \to \mathcal{P}_d \times \mathbb{C}^4$ be continuous, and write $\gamma(s) = (F(s), p(s), q(s))$ where $p(s) \in \mathbb{C}^2$, $q(s) \in \mathbb{C}^2$. Suppose $p(s)$ and $q(s)$ are saddle periodic points for $F(s)$ for each $s \in [0,1]$ and that $T \in \mathbb{C}^2$ is a heteroclinic intersection for $p(0)$ and $q(0)$.*

Then for each $\varepsilon > 0$ there exists a curve $\gamma_\varepsilon : [0,1] \to \mathcal{P}_d \times \mathbb{C}^4$ and a continuous map $T_\varepsilon : [0,1] \to \mathbb{C}^2$ with the following properties:

- $\gamma_\varepsilon(0) = \gamma(0)$ *and* $T_\varepsilon(0) = T$
- $|\gamma_\varepsilon(s) - \gamma(s)| < \varepsilon$ *for each* $s \in [0,1]$

- $\gamma_\varepsilon(s) = (F_\varepsilon(s), p_\varepsilon(s), q_\varepsilon(s))$ *where* $p_\varepsilon(s)$ *and* $q_\varepsilon(s)$ *are saddle periodic points of* $F_\varepsilon(s)$ *for each* $s \in [0,1]$
- $T_\varepsilon(s)$ *is a heteroclinic intersection between* $p_\varepsilon(s)$ *and* $q_\varepsilon(s)$ *for each* $s \in [0,1]$.

An important part of the proof of this theorem is Theorem 2.1, which gives a parametrization of generalized stable and unstable manifolds that depends holomorphically on the map throughout a large portion of parameter space.

In the next section, we describe the main ideas in the proof of Theorem 1.1 then in the following section give a general analytic continuation result that lets us follow heteroclinic points.

1.2. **Petrovski-Landis strategy.** In order to check that a heteroclinic tangency cannot persist throughout any open domain of the parameter space, we use an argument that may be called the Petrovski-Landis (PL) strategy. The idea is the following. To prove that some property cannot occur throughout an open domain of the parameter space $\mathcal{P}_d$, a *persistence theorem* is proved for this property: if the property persists in an open domain, then by some form of analytic continuation, the property persists in an open, dense, connected subset of the parameter space. Then another domain in the parameter space is found, perhaps very distant from the first one, where the given property is known not to occur at all. In conjunction with the persistence result, this shows that the property could not be observed throughout the first domain.

This strategy was proposed in [PL55] as an attempt to solve Hilbert's 16th problem. The attempt failed, but the strategy has independent significance. In particular, we apply it below to prove Theorem 1.1.

To prove Theorem 1.1, we proceed roughly as follows. Given a map, F, with an intersection between stable and unstable manifolds for saddle periodic points p and q, we define an analytic set M_{pers} that tracks this point of intersection as the map varies through $\mathcal{P}_d$. To prove that this intersection cannot be a tangency over an open set in parameter space, we assume to the contrary that there is such an open set. Then M_{pers} is a kind of analytic continuation of this tangency. We show that this continuation extends to another domain in parameter space in which the maps have no tangencies at all. This is a contradiction that shows there can be no open set in parameter space throughout which an intersection can be a tangency.

The hard part of this approach is to find an appropriate path through parameter space. Naively, we might try to maintain the saddle property of the two original periodic points. However, it's very difficult to know when a periodic point may lose its saddle property and hence lose its one-dimensional stable and unstable manifolds. Instead, we don't require the periodic points p and q to remain saddles throughout M_{pers}. Rather, one or the other may become attracting or repelling, but we require that each maintain a one-dimensional invariant manifold of the same type, contracting or expanding, as that for the original periodic points and that these invariant manifolds vary holomorphically and continue to intersect throughout M_{pers}. We show that given any curve in $\mathcal{P}_d$ so that the corresponding periodic points have such invariant manifolds, there is a nearby curve that lifts to M_{pers}, and hence the point of intersection persists over this nearby curve. This is our main persistence result for heteroclinic intersections. We then prove a persistence result about tangencies by showing that the subset of M_{pers} for which the intersection is a tangency is an analytic set. On the other hand, given any initial data, there is a curve in M_{pers} from this initial data to a map with no tangencies. Thus the set of maps with tangencies for this pair of periodic points is analytic with codimension at least 1, hence has measure 0. Taking the union over all pairs of periodic points gives a residual full measure set of maps with no tangencies. The full measure result for hyperbolic periodic points is straightforward using the fact that the eigenvalues of a periodic point are nonconstant analytic.

In the next section we describe a general analytic continuation property that allows us to track heteroclinic points through parameter space. This continuation property is reminiscent of Iversen's property in one variable.

1.3. **Algebraic versus transcendental behavior.** The way in which we track a heteroclinic intersection leads us to consider the following type of question. Consider a parameterized equation

$$F(z, \alpha) = 0, \ z \in \mathbb{C}^m, \ \alpha \in \mathbb{C}^n, \tag{1}$$

where $F : \mathbb{C}^{m+n} \to \mathbb{C}^m$ is an entire vector function. Suppose $F(0, 0) = 0$ and $F(z, 0)$ is a local biholomorphism in a neighborhood of $z = 0$ in $\mathbb{C}^m$. Then equation (1) defines a germ of an analytic vector function $z = f(\alpha)$ at zero. The question is: Given a curve $\gamma : I \to \mathbb{C}^n$, can f be continued along this curve? This is equivalent to finding a lift of the curve from $\mathbb{C}^n$ to the analytic set in $\mathbb{C}^{m+n}$ defined by (1). More generally, *what is the maximal domain to which f may be analytically continued?*

If $m = n = 1$, a theorem of Stoilow [Sto56] says that for any such curve, there is a nearby curve so that f may be continued along the nearby curve. In the appendix we give a precise statement and a short proof based on the techniques developed in the proof of Theorem 2.2.

If $m = 2, \ n = 1$, the situation is completely different: there is a function F in equation (1) and a disk D for which the solution f may not be extended beyond D. That is, for any curve extending beyond D, there is no continuation of f along this curve. We present such an example in Section 4.3. We say that equation (1) (or the solution f) has *transcendental behavior* if there is an open set, U, in $\mathbb{C}^n$ so that for any curve from 0 to a point in U, f cannot be continued along this curve.

On the other hand, if F is polynomial in z, then f may be analytically continued outside a nowhere dense subset of $\mathbb{C}^n$. Eventually, this continuation may not be univalent, and perhaps not well defined along a particular curve $\gamma : [0, 1] \to \mathbb{C}^n, \ \gamma(0) = 0$. But any curve γ may be slightly perturbed in such a way that f may be continued along the perturbed curve, or equivalently that the perturbed curve may be lifted to the analytic set defined by equation (1). We say that equation (1) (or the solution f) has *algebraic behavior* if any curve can be perturbed and lifted in this way.

In Section 4.3, we define this lifting property formally and call it the ε-lift property, which is similar to but stronger than the Iversen property of [Sto52]. We also give a condition so that the solution of (1) will have algebraic behavior. In fact, we generalize to the case of an irreducible analytic subset N of $M \times M'$, where M and M' are connected complex manifolds, $\pi : N \to M$ is the natural projection, π is locally proper and $\dim(M) = \dim(N)$. With this setting, we say in Definition 4.7 that N is *tame on disks over* M if for all $\phi : D \to N$ such that the image of $\pi \circ \phi$ is contained in an injective holomorphic embedding of a disk in M, ϕ has radial limits in $M \times M'$ for a.e. point of ∂D. This condition implies the algebraic behavior of N over M:

Theorem 1.6. *Let $\pi : N \to M$ be as above, and suppose that N is tame on disks over M. Then π has the ε-lift property.*

An analytic set which is not tame on disks is presented in Section 4.3, Example 2.

For the remainder of this section, we discuss some open problems, both for persistence and for Kupka-Smale, then give an outline of the paper.

1.4. **Open problems on persistence.** A general persistence problem for polynomial dynamical systems may be formulated at a heuristic level as follows:

Given a locally analytic function related to the geometric features of a polynomial dynamical system, does it have algebraic or transcendental behavior?

Let us give several precise versions of the problem above. The question "what may be said about the domain of some analytic function?" means, in particular "does this function have algebraic or transcendental behavior?"

A. Persistence of complex limit cycles.

Consider the set of polynomial vector fields of degree at most n in $\mathbb{C}^2$. Denote by $\mathcal{A}_n^{\mathbb{R}}$ the subset of real vector fields. Fix one of these, say α_0, and consider a robust limit cycle γ_0 of this field in $\mathbb{R}^2$. Take a line L that intersects γ_0 transversally. Let z be a chart on L. For any real equation $\alpha \in \mathcal{A}_n^{\mathbb{R}}$ close to α_0 there exists a limit cycle γ close to γ_0 that intersects L at a point $z(\alpha)$. The germ of the function $z(\alpha)$, "the initial condition of the limit cycle on the cross-section" is analytic.

Problem 1.1. *What may be said about the domain of analytic continuation of a function "initial condition of a limit cycle" of a planar vector field of fixed degree as a function of the parameter α?*

The complexification of the germ $z(\alpha)$ is in fact the initial condition of a complex limit cycle for the vector field α. This notion goes back to [PL55] and may be found also in [Ily02].

Conjecture 1.1. *The initial condition of a limit cycle of a polynomial vector field as a function of the coefficients has algebraic behavior.*

This conjecture goes back to the seminal work of Petrovski-Landis [PL55]. It is not yet proved, but it gave rise to the persistence problems stated below and to the persistence theorems of the present paper.

B. Persistence of the hypersurface of saddle connections.

For a real polynomial vector field $\alpha \in \mathcal{A}_n^{\mathbb{R}}$, let $R_\theta \alpha$ be the field α rotated by the angle θ; this new vector field is still in $\mathcal{A}_n^{\mathbb{R}}$. Now, fix $\alpha_0 \in \mathcal{A}_n^{\mathbb{R}}$ with a saddle connection. Then, under mild restrictions on α_0, there exists a germ of an analytic function $\theta(\alpha)$ at α_0 such that $R_{\theta(\alpha)}\alpha$ has a saddle connection depending analytically on α.

Problem 1.2. *What may be said about the analytic continuation of the germ $\theta(\alpha)$ related to polynomial vector fields with a saddle connection?*

C. Domain of the Poincaré map

Let us consider a simpler construction with no parameter involved. Consider a real polynomial vector field, a limit cycle of this field and the corresponding Poincaré map.

Problem 1.3. *What may be said about the domain of the analytic continuation of the Poincaré map of a limit cycle of a fixed polynomial vector field?*

Theorem 1.5 above may be considered as a solution of a persistence problem of the same type: i.e., a heteroclinic intersection point has algebraic behavior as a function of the parameter.

1.5. **Open problems on Kupka-Smale.**

Problem 1.4. *Is the Kupka-Smale theorem true for polynomial automorphisms of $\mathbb{C}^n$, $n > 2$?*

Problem 1.5. *Is the Kupka-Smale property generic for 2-dimensional polynomial vector fields?*

The positive answer to problem 1.5 in the real case is trivial: saddle connections may be destroyed by a rotation of a vector field. In the complex plane the absence of common separatrices (complex saddle connections) of two singular points for a generic field is not yet proved. Note that the leaves of generic complex foliations are dense. It is natural to suggest that the set of the complex vector fields with a complex saddle connection is dense in the parameter space.

Problem 1.6. *Is the Kupka-Smale property typical for volume preserving polynomial automorphisms of the plane?*

The first two problems were suggested by Sheldon Newhouse and the last one by John Franks at the discussion after the talk given by the third author at the Matherfest at Princeton, October 2002.

1.6. **Outline of paper.** The remainder of the paper is organized as follows. In Section 2, we give a more detailed outline of the proof as well as precise definitions and formulations for the persistence results. In Section 3, we define special contracting and expanding holomorphic invariant manifolds and show that these can be simultaneously parametrized by $\mathbb{C}$ with the parametrization varying holomorphically with the map over a large subset of $\mathcal{P}_d$; this is the content of Theorem 2.1. In Section 4, we recall some results of one-variable complex analysis, then discuss the ε-lift property, including examples that do and do not have the ε-lift property. We then prove Theorem 1.6 and use this to prove Theorem 2.2, which lets us track points of heteroclinic intersection based on information about the multipliers. In Section 5, we prove Theorem 1.4, then prove Theorem 2.3 on the existence of a path to a hyperbolic map. In Section 6, we prove Theorem 2.4 to the effect that the set of maps with tangencies has codimension one in the set M_{pers}, as well as complete the proofs of Theorems 1.1 and 1.2. In the appendix, we give a proof of Stoilow's theorem using the ideas of the ε-lift property.

More precisely, the first part of Theorem 1.1 (generic hyperbolicity of periodic points) is proved in Section 6.1, while the remainder of the proof is in Section 6.3. Theorem 1.2 is proved in Section 6.4. Theorem 1.4 is proved in Section 5.2. Theorem 1.5 is a special case of Theorem 2.2, which is proved in Section 4.4. Theorem 1.6 is proved in Section 4.3.

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2. Framework for persistence and Kupka-Smale

In this section we give the precise definitions needed to state our persistence results and show how these results fit together to prove the Kupka-Smale theorem.

2.1. **The parameter space for polynomial automorphisms.** Friedland and Milnor [FM89] have classified polynomial automorphisms of $\mathbb{C}^2$ based on their dynamical behavior. *Elementary* automorphisms have simple dynamics, and are polynomially conjugate to an automorphism of the form $(x, y) \to (ax + p(y), cy + d)$ (p polynomial, $a, c \neq 0$). A *nonelementary* automorphism is conjugate to a composition of *generalized Hénon mappings*

(Hénon maps for short), which are of the form $F(x, y) = (y, p(y) - ax)$, where $p(y)$ is a monic polynomial of degree $d > 1$ and $a \neq 0$. Let $\mathcal{H}_d$ denote the space of Hénon maps of degree d, and let $\mathcal{P}_d$ be the space of compositions of Hénon maps so that the composition has total degree d. This total degree is the product of the degrees of each of the component Hénon maps. Since each Hénon map depends on finitely many coefficients, we see that $\mathcal{P}_d$ is the union of finitely many smooth complex manifolds, each biholomorphic to $\mathbb{C}^j \times (\mathbb{C} \setminus \{0\})^k$ for some j and k, hence each of which is Stein [FM89, lemma 2.4]. (As an example of these different components, let $F_d(x, y) := (y, y^d - x)$. Then the component containing F_{2d} is different from the component containing $F_d \circ F_2$.) We use the affine metric on $\mathbb{C}^{j+k}$ as the metric on the corresponding component of $\mathcal{P}_d$.

Note that if F is any polynomial automorphism of $\mathbb{C}^2$ and is not an elementary automorphism, then by [FM89], F is conjugate to a map $G \in \mathcal{P}_d$ for some $d \geq 2$ via a polynomial automorphism. I.e., there is a polynomial automorphism ϕ so that $\phi F \phi^{-1} = G$. As mentioned above, G is contained in a component, S, of $\mathcal{P}_d$, and the Kupka-Smale theorem holds on S. Then F is contained in a set S' obtained from S by taking each $H \in S$ to $\phi^{-1} H \phi$. Hence S' is biholomorphic to S, and the Kupka-Smale theorem (including the full measure result) holds also on S'. For this reason, Theorem 1.1 is stated for $\mathcal{P}_d$, but with the proper interpretation it applies also to the set of all nonelementary polynomial automorphisms of $\mathbb{C}^2$.

2.2. **Invariant manifolds and strictly heteroclinic points.** Recall that a periodic point of a map $F \in \mathcal{P}_d$ is a fixed point of some iterate of the map:

$$F^m(p) = p. \tag{2}$$

The minimal m with this property is called the *period* of p. The eigenvalues of $D_p F^m$ are called the *multipliers* of p. The periodic point is *hyperbolic* if neither of the multipliers belongs to the unit circle. It is of *saddle* type provided that the multipliers lie on different sides of the unit circle and is of *nodal* type if both are inside, or both are outside the unit circle. In the first case the orbit is an *attracting* node and in the second case a *repelling* node.

A periodic node is called *resonant* if its multipliers, λ and ν satisfy the equation $\lambda^n = \nu$ or $\nu^n = \lambda$ for some integer $n > 1$.

Recall that given a hyperbolic periodic point p of period m of a map F, the *stable manifold* of p, $W^s(p)$, is the set of points q so that $F^{km}(q)$ converges to p as k tends to ∞. The *unstable manifold* $W^u(p)$ is defined similarly with k tending to $-\infty$.

For $F \in \mathcal{P}_d$ and a saddle periodic point p, both $W^s(p)$ and $W^u(p)$ are holomorphic immersed manifolds parametrized biholomorphically by $\mathbb{C}$ (this result has been known for many years, but see [Buz98, prop. 4.3] for a proof of a generalization of this statement in the n-dimensional case). On the other hand, for an attracting node, the stable manifold is biholomorphic to $\mathbb{C}^2$. This provides a famous example (due to Fatou) of a biholomorphic embedding of $\mathbb{C}^2$ into itself as a proper subset.

An intersection point of a stable manifold of one periodic orbit and an unstable manifold of another is called a *heteroclinic point* of these two orbits; if these orbits coincide, the intersection point above is called *homoclinic.* For conciseness, we allow heteroclinic to stand for heteroclinic or homoclinic.

For maps in $\mathcal{P}_d$ with an attracting periodic point, p, the unstable manifold of any saddle periodic point may intersect the basin of attraction of p. Since the basin of attraction is open, this leads to open subsets of the unstable manifold being heteroclinic points. As mentioned in the outline of proof, we'd like to be able to follow heteroclinic points uniquely even between a node and a saddle, so we restrict to periodic points which satisfy a certain nonresonance condition, then define the notion of strictly heteroclinic points.

Definition 2.1. Let p be a periodic point for F with an ordered pair of multipliers (λ, ν), with $\nu \neq 1$.

- (λ, ν) is *stably nonresonant* means $|\lambda| < 1$ and $\lambda^n \neq \nu$ for all $n \in \mathbb{Z}_+$.
- (λ, ν) is *unstably nonresonant* means $|\lambda| > 1$ and $\lambda^n \neq \nu$ for all $n \in \mathbb{Z}_+$.

When the context is clear, we will say that λ is (un)stably nonresonant rather than identifying the ordered pair (λ, ν).

As a special case, note that the multipliers for a saddle point are automatically stably/unstably nonresonant. We show in Section 3 that if λ is (un)stably nonresonant for p, then there is a 1-dimensional holomorphic immersed manifold tangent to the eigenspace for λ and invariant under F^m. We label this manifold $W^c(p)$ (c for contracting) in the stable case and $W^e(p)$ (e for expanding) in the unstable case. Theorem 2.1 below implies that in both cases, the manifold is biholomorphic to $\mathbb{C}$ and varies holomorphically with the parameter. When p is saddle, these manifolds coincide with $W^s(p)$ and $W^u(p)$, while if p is an attracting node, then $W^c(p)$ is a proper subset of $W^s(p)$. Also, if p is an attracting node, there may be two choices of $W^c(p)$: one for each multiplier. We use the notation $W^c_\lambda(p)$ to specify the multiplier when needed. In the following definition, we use $W^{c/e}$ to refine the notion of heteroclinic points.

Definition 2.2. Let p and q be periodic for F.

- p and q are *potentially heteroclinic* if p has a stably nonresonant multiplier, λ, and q has an unstably nonresonant multiplier, μ.
- A point T is a *strictly heteroclinic point of p and q related to λ and μ*, provided that p and q are potentially heteroclinic with associated multipliers λ and μ, and $T \in W^c_\lambda(p) \cap W^e_\mu(q)$.

2.3. **Domain of potential heteroclinic intersection.** In this section, we define the *domain of potential heteroclinic intersection* of two potentially heteroclinic periodic points p_0 and q_0 for F_0. Roughly, this is a complex manifold, M_{pot}, spread over a subset of $\mathcal{P}_d$ such that along any path in this manifold starting at the point corresponding to F_0, there are uniquely defined and continuously varying potentially heteroclinic periodic points with distinguished multipliers extending p_0, q_0, λ_0 and μ_0.

First we define a manifold of periodic points.

Definition 2.3. Let $F_0 \in \mathcal{P}_d$ with a periodic point, p_0, of period m with distinct multipliers, neither equal to 1. Then $\mathrm{Per}(F_0, p_0)$ is the maximal irreducible analytic subset of $\mathcal{P}_d \times \mathbb{C}^2$ such that

- $(F_0, p_0) \in \mathrm{Per}(F_0, p_0)$;
- For each $(F, p) \in \mathrm{Per}(F_0, p_0)$, $F^m(p) = p$ and D_pF^m has multipliers $\nu \neq \lambda$ and $\nu \neq 1$, $\lambda \neq 1$.

Remark. Note that $\mathrm{Per}(F_0, p_0)$ is a quasi affine algebraic manifold. Indeed, it is the component through (F_0, p_0) of the set of all (F, p) satisfying the following polynomial equation and inequality:

$$F^m(p) = p, \;\; G(F, p) \neq 0, \tag{3}$$

where $G = (t^2 - 4\Delta)(1 - t + \Delta)$, $t(F, p) = tr(D_pF^m)$, $\Delta = \det(D_pF^m)$. Note also that by the implicit function theorem, $\mathrm{Per}(F_0, p_0)$ is parametrized by F in an open subset of the component of $\mathcal{P}_d$ containing F_0.

Given this manifold of periodic points, we want to identify the regions in which we have (un)stably nonresonant multipliers. First we restrict to multipliers not on the unit circle, then to the nonresonant conditions, using the superscripts c/e to indicate the nonresonant conditions, just as was done for the invariant manifolds $W^{c/e}$. Here, EV is a mnemonic for eigenvalue.

Definition 2.4. Let F_0, p_0 and $\mathrm{Per}(F_0, p_0)$ be as above and assume that p_0 has a multiplier λ_0. Let $\mathrm{EV}(F_0, p_0, \lambda_0)$ be the maximal irreducible analytic subset of $\mathrm{Per}(F_0, p_0) \times \mathbb{C}$ such that

- $(F_0, p_0, \lambda_0) \in \mathrm{EV}(F_0, p_0, \lambda_0)$;
- For $(F, p, \lambda) \in \mathrm{EV}(F_0, p_0, \lambda_0)$, $F^m(p) = p$, and λ is a multiplier of p under F.

Moreover, specify the following cases:

- $|\lambda_0| < 1$: $\mathrm{EV}^s(F_0, p_0, \lambda_0) = \mathrm{EV}(F_0, p_0, \lambda_0) \cap \{|\lambda| < 1\}$.
- $|\lambda_0| > 1$: $\mathrm{EV}^u(F_0, p_0, \lambda_0) = \mathrm{EV}(F_0, p_0, \lambda_0) \cap \{|\lambda| > 1\}$.

Then finally,

- $|\lambda_0| < 1$: $\mathrm{EV}^c(F_0, p_0, \lambda_0)$ is the subset of $\mathrm{EV}^s(F_0, p_0, \lambda_0)$ such that λ is stably nonresonant.
- $|\lambda_0| > 1$: $\mathrm{EV}^e(F_0, p_0, \lambda_0)$ is the subset of $\mathrm{EV}^u(F_0, p_0, \lambda_0)$ such that λ is unstably nonresonant.

Remark. $\mathrm{EV}(F_0, p_0, \lambda_0)$ is the component through (F_0, p_0, λ_0) of the quasi affine algebraic manifold given by (3) and $J = 0$, where $J = \det(D_pF^m - \lambda I)$. Since p and λ are locally determined by F, we see that $\mathrm{EV}(F_0, p_0, \lambda_0)$ is locally biholomorphic to $\mathrm{Per}(F_0, p_0)$.

Now we are ready to define the domain of potential heteroclinic intersection mentioned above.

Definition 2.5. Given a tuple

$$\mathfrak{a}_0 = (F_0, p_0, q_0, \lambda_0, \mu_0)$$

such that p_0, q_0 are periodic points of $F_0 \in \mathcal{P}_d$, λ_0 is a stably nonresonant multiplier of p_0, and μ_0 is an unstably nonresonant multiplier of q_0, the *domain of potential (heteroclinic) intersection* is defined as

$$M_{\mathrm{pot}}(\mathfrak{a}_0) = \{(F, p, q, \lambda, \mu) \mid (F, p, \lambda) \in \mathrm{EV}^c(F_0, p_0, \lambda_0), \\ (F, q, \mu) \in \mathrm{EV}^e(F_0, q_0, \mu_0)\}.$$

Note that the natural projections

$$\pi^c : M_{\mathrm{pot}}(\mathfrak{a}_0) \to \mathrm{EV}^c(F_0, p_0, \lambda_0), \quad \pi^e : M_{\mathrm{pot}}(\mathfrak{a}_0) \to \mathrm{EV}^e(F_0, q_0, \mu_0)$$

are well defined and locally biholomorphic.

2.4. **Simultaneous uniformization and heteroclinic persistence.** In this section we define the set of heteroclinic persistence for a strictly heteroclinic point, T_0, of p_0 and q_0. This is a subset, M_{pers}, of $M_{\mathrm{pot}} \times \mathbb{C}^2$: roughly, M_{pers} is the set on which the original heteroclinic intersection between p_0 and q_0 can be followed continuously. First we need to discuss the uniformization of invariant manifolds.

Given a noncompact analytic curve, it is possible to *uniformize* it, that is, to map the universal cover of the curve biholomorphically onto $\mathbb{C}$ or onto a (topological) disc in $\mathbb{C}$. Such a map is called a *uniformizing map*. The *simultaneous uniformization problem* concerns families of analytic curves: *given a family of such curves depending analytically on a parameter, is it possible to choose the uniformizing map for the curves of the family so that the uniformizing maps also depends analytically on the same parameter?*

The answer depends strongly on the family. The following theorem shows that for the families we consider it is possible to give such a simultaneous uniformization. In general, it may be not be possible even if the total space of the family is a Stein manifold [Glu01], [Glu02]. The key point in our case is that the curves W^c, W^e under consideration are parabolic (biholomorphic to $\mathbb{C}$) and the base space is Stein.

Theorem 2.1. *Let $EV^c = EV^c(F_0, p_0, \lambda_0)$ be as in Definition 2.4. Then for any $\alpha = (F, p, \lambda) \in EV^c$ there exists a uniformization by $\mathbb{C}$ of $W^c_\lambda(p, F)$, and this uniformization may be chosen to depend globally analytically on α. In other words, there exists a holomorphic map*

$$\psi^c : EV^c \times \mathbb{C} \to \mathbb{C}^2$$

such that $\psi^c_\alpha = \psi^c|_{\{\alpha\}\times\mathbb{C}}$ maps $\mathbb{C}$ biholomorphically onto $W^c_\lambda(p)$. Moreover, ψ^c_α linearizes the map F on the invariant manifold in the following sense:

$$F \circ \psi^c_\alpha(\tau) = \psi^c_\alpha(\lambda\tau) \tag{4}$$

An analogous statement holds for EV^c replaced by EV^e.

We prove this theorem in Section 3, including the existence of the manifolds $W^{c/e}(p)$. This simultaneous uniformization property is one of the key tools in the proof of the persistence theorems. In [BV01, Theorem 5.6], there is a similar uniformization result for stable and unstable manifolds of points in the invariant set J of a hyperbolic map in $\mathcal{P}_d$.

This simultaneous uniformization gives us a means to follow a heteroclinic intersection through parameter space. That is, given $\mathfrak{a}_0 = (F_0, p_0, \lambda_0, q_0, \mu_0)$, we can parametrize $W^c_\lambda(p)$ and $W^e_\mu(q)$ by ψ^c and ψ^e as above, where each of these varies holomorphically with $\mathfrak{a} \in M_{\text{pot}}(\mathfrak{a}_0)$. Then a point of strict heteroclinic intersection between $W^c_\lambda(p)$ and $W^e_\mu(q)$ corresponds to a point in common between the image of $\psi^c_\mathfrak{a}$ and $\psi^e_\mathfrak{a}$, and hence gives rise to an analytic set. We make this precise in the following definition.

Note: when working with $\mathfrak{a} \in M_{\text{pot}}$ we will also use $\psi^c_\mathfrak{a}$ as opposed to $\psi^c_{\pi^c(\mathfrak{a})}$, and likewise for ψ^e.

Definition 2.6. Let $\mathfrak{a}_0 = (F_0, p_0, q_0, \lambda_0, \mu_0)$ be as in Definition 2.5, and let $\psi^{c/e}$ be the uniformizations of $W^{c/e}$ over $\text{EV}^{c/e}$ as in the previous definition. Suppose there is a strict heteroclinic intersection

$$T_0 \in W^c_\lambda(p_0) \cap W^e_\mu(q_0),$$

and hence there are $\tau^c_0, \tau^e_0 \in \mathbb{C}$ so that $\psi^c_{\mathfrak{a}_0}(\tau^c_0) = \psi^e_{\mathfrak{a}_0}(\tau^e_0) = T_0$. Let $\tau_0 = (\tau^c_0, \tau^e_0)$. Then the *set of heteroclinic persistence*, denoted

$$M_{\text{pers}} = M_{\text{pers}}(\mathfrak{a}_0, \tau_0),$$

is defined to be the maximal irreducible component through $(\mathfrak{a}_0, \tau^c_0, \tau^e_0)$ of the analytic set

$$\{(\mathfrak{a}, \tau^c, \tau^e) \in M_{\text{pot}}(\mathfrak{a}_0) \times \mathbb{C} \times \mathbb{C} : \psi^c_\mathfrak{a}(\tau^c) = \psi^e_\mathfrak{a}(\tau^e)\}. \tag{5}$$

Remark. Note that there is a natural identification between M_{pers} and a parametrized set of strict heteroclinic intersections in dynamical space. I.e., given $(\mathfrak{a}, \tau^c, \tau^e) \in M_{\text{pers}}$, we have that $\psi^c_\mathfrak{a}(\tau^c) = \psi^e_\mathfrak{a}(\tau^e)$ gives the dynamical space coordinates for a strict heteroclinic intersection, so $(\mathfrak{a}, \tau^c, \tau^e) \mapsto (\mathfrak{a}, \psi^c_\mathfrak{a}(\tau^c))$ is this identification. The following definition notates the inverse of this identification.

Definition 2.7. Let $\mathfrak{a} = (F, p, q, \lambda, \mu) \in M_{\text{pot}}(\mathfrak{a}_0)$ with uniformizations $\psi^{c/e}$ as above, and let $T \in \mathbb{C}^2$ be a point of strict heteroclinic intersection between $W^c_\lambda(p)$ and $W^e_\mu(q)$. Then define

$$\psi^{-1}_\mathfrak{a}(T) = (\tau^c, \tau^e)$$

where τ^c, τ^e are chosen so that $\psi^c_\mathfrak{a}(\tau^c) = \psi^e_\mathfrak{a}(\tau^e) = T$.

Note also that there is a natural projection

$$\pi : M_{\text{pers}}(\mathfrak{a}_0, \tau_0) \to M_{\text{pot}}(\mathfrak{a}_0),$$

where π is defined as just forgetting the τ-component.

Remark. Since both M_{pot} and M_{pers} may be viewed as subsets of complex Euclidean space, we use the Euclidean metric to measure distances in both spaces.

In the next theorem, we formulate our main persistence result in full generality. In it, we describe the structure of the set of heteroclinic persistence in terms of the domain of potential intersection in two ways. First, the natural projection from M_{pers} to M_{pot} covers an open dense subset of M_{pot}. Second, for any curve in M_{pot}, there is a nearby curve that lifts to a curve in M_{pers}.

We make this lifting property precise in the following definitions. First, let M, M' be connected complex manifolds and let N be an irreducible, complex analytic subset of $M \times M'$. Let $\pi : N \to M$ be the natural projections, and suppose that π is locally proper at each point of N and $\dim(N) = \dim(M)$. In this case, π is a local biholomorphism outside a nowhere dense closed set (e.g. [Chi89, sec. 3.7]).

Definition 2.8. Let $\pi : N \to M$ be as above. Let $\gamma : I \to M$, let $\mathfrak{b}_0 \in N$ with $\pi(\mathfrak{b}_0) = \gamma(0)$, and let $\varepsilon > 0$. An *ε-lift of γ to N beginning at* $\mathfrak{b}_0$ is a curve $\hat{\gamma} : I \to N$ with $\hat{\gamma}(0) = \mathfrak{b}_0$ so that $\pi \circ \hat{\gamma}$ has C^0-distance at most ε from γ.

Definition 2.9. Let $\pi : N \to M$ be as above. The projection π has *the ε-lift property* if for every curve $\gamma : I \to M$, for every point $\mathfrak{b}_0 \in N$ so that $\pi(\mathfrak{b}_0) = \gamma(0)$, and for all $\varepsilon > 0$, there exists an ε-lift of γ to $\hat{\gamma}$ beginning at $\mathfrak{b}_0$.

The ε-lift property is similar to but stronger than that of the Iversen property of Stoilow's theorem [Sto52]; see also Proposition 4.2 below. In the language of Section 1.3, the ε-lift property implies that the map π^{-1} has algebraic behavior.

Theorem 2.2. *Let $F_0 \in \mathcal{P}_d$ be a map with a strictly heteroclinic intersection T_0 between invariant manifolds of periodic points p_0 and q_0 corresponding to multipliers λ_0 and μ_0 respectively, $|\lambda_0| < 1 < |\mu_0|$. Let*

$$\mathfrak{a}_0 = (F_0, p_0, q_0, \lambda_0, \mu_0), \;\; \tau_0 = \psi_{\mathfrak{a}_0}^{-1}(T_0),$$
$$M_{pot} = M_{pot}(\mathfrak{a}_0), \;\; M_{pers} = M_{pers}(\mathfrak{a}_0, \tau_0),$$

and let $\pi : M_{pers} \to M_{pot}$ be the natural projection defined above. Then π has the ϵ-lift property. Moreover, the image $\pi(M_{pers}) \subset M_{pot}$ is open and dense.

We prove this theorem in Section 4.

Remark. The conclusion that $\pi(M_{\text{pers}})$ is open and dense in M_{pot} follows immediately from the conclusion about the ε-lift property plus the fact that the conditions on π imply that it is an open map [Chi89, sec. 5.8]. In Section 4.3, we show that any curve in M_{pot} with initial point in $\pi(M_{\text{pers}})$ has an ε-lift to M_{pers}.

2.5. **Saddle hyperbolicity and heteroclinic tangencies.** The previous theorem allows us to navigate through parameter space quite freely in that it essentially reduces the problem of making sure that two invariant manifolds intersect to the problem of satisfying very mild restrictions on the multipliers of the periodic points. In order to use this theorem in the framework of the PL strategy to show that the set of polynomial automorphisms of $\mathbb{C}^2$ with a strict heteroclinic tangency (that is, strict heteroclinic intersection with a tangency between the corresponding invariant manifolds) is small in some sense, we first need to show that we can navigate in M_{pot} to a map that is known not to have any tangencies. This is made precise in Theorem 2.3. Following this we discuss the set of strict heteroclinic tangencies.

Definition 2.10. A map $F \in \mathcal{P}_d$ is *saddle hyperbolic* means that it is hyperbolic on its nonwandering set and that all its periodic points are of saddle type.

In particular, for a saddle hyperbolic map, there can be no tangencies between $W^c(p)$ and $W^e(q)$ for any periodic p and q; indeed, the assumption that all periodic points are saddles implies that $W^c(p)$ and $W^e(q)$ are actually the stable and unstable manifolds, respectively, and since the map is hyperbolic on its nonwandering set, there can be no tangencies between stable and unstable manifolds. The following theorem, proved in Section 5, gives us a path to a saddle hyperbolic map.

Theorem 2.3. *Let $F_0 \in \mathcal{P}_d$ be a map with potentially heteroclinic points p_0, q_0 with corresponding multipliers λ_0, μ_0, and let $\mathfrak{a}_0 = (F_0, p_0, q_0, \lambda_0, \mu_0)$. Then there is a curve $\gamma : [0,1] \to M_{pot}(\mathfrak{a}_0)$ so that $\gamma(0) = \mathfrak{a}_0$ and so that the natural projection of $\gamma(1)$ to $\mathcal{P}_d$ is a saddle hyperbolic map.*

Now we are ready to discuss heteroclinic tangencies as a subset of M_{pers} and use Theorems 2.2 and 2.3 to conclude that maps with tangencies are rare.

Definition 2.11. Let $F_0 \in \mathcal{P}_d$ and $\mathfrak{a}_0 = (F_0, p_0, q_0, \lambda_0, \mu_0)$ be as in the previous theorem, and suppose also that $W^c_{\lambda_0}(p_0)$ and $W^e_{\mu_0}(q_0)$ are tangent at a point T_0. Let $\tau_0 = \psi_{\mathfrak{a}_0}^{-1}(T_0)$. Then

$$M_{\text{tang}} = M_{\text{tang}}(\mathfrak{a}_0, \tau_0)$$

is the set of all $(a, \tau^c, \tau^e) \in M_{\text{pers}}(\mathfrak{a}_0, \tau_0)$ so that $W^c_\lambda(p)$ and $W^e_\mu(q)$ are tangent at the intersection point $\psi^c_{\mathfrak{a}}(\tau^c) = \psi^e_{\mathfrak{a}}(\tau^e)$.

In Section 6.2, we show that M_{tang} is obtained from M_{pers} by adding the condition

$$\det\left(\frac{\partial \psi^c_{\mathfrak{a}}}{\partial \tau^c}, \frac{\partial \psi^e_{\mathfrak{a}}}{\partial \tau^e}\right) = 0.$$

Thus M_{tang} is an analytic subset of M_{pers}.

Theorems 2.2 and 2.3 allow us to follow any strictly heteroclinic intersection through parameter space to a saddle hyperbolic map. Since M_{tang}, the set of heteroclinic tangencies, defines an analytic subset of M_{pers}, and since a tangency cannot exist in the domain of saddle hyperbolicity, we deduce the following theorem from Theorems 2.2 and 2.3.

Theorem 2.4. *Let $\mathfrak{a}_0$, τ_0 be as in the previous definition. Then $M_{tang}(\mathfrak{a}_0, \tau_0)$ is an analytic subset of $M_{pers}(\mathfrak{a}_0, \tau_0)$ with complex codimension 1.*

In particular, Theorem 2.4 implies that the set of maps having heteroclinic tangency is small, both topologically and metrically. This is the main step in the proof of Theorem 1.1.

In addition to controlling the tangencies of the map, we need to be sure that the set of maps having all periodic points hyperbolic is also residual and of full measure. This follows relatively easily from the fact that for a given periodic point, the set of maps for which this periodic point is not hyperbolic is a real-analytic subset of $\mathcal{P}_d$: it suffices to show that each such multiplier is nonconstant and take the union of all such real-analytic subsets. The only difficulty is that, as mentioned in Section 2.1, $\mathcal{P}_d$ is the union of finitely many smooth manifolds, and we need to show that the multipliers are nonconstant on each of these slices. Buzzard [Buz99, proposition 2.1 and the remark following] claims to prove this. However, the proof of that proposition contains a gap in that the derivatives of the map F_μ depend more delicately on μ than indicated there. We provide a complete proof of this step in Section 5 to give the following theorem.

Theorem 2.5. *There is a residual set, $\mathcal{E}$, of full measure in $\mathcal{P}_d$ such that if $F \in \mathcal{E}$, then each periodic point of F is hyperbolic.*

Theorem 2.4 applies immediately to real maps. As in the introduction, let $\mathcal{P}_d^{\mathbb{R}}$ be the set of all real maps $F : \mathbb{R}^2 \to \mathbb{R}^2,\ F \in \mathcal{P}_d$. A point $T \in \mathbb{R}^2$ is a real point of strict heteroclinic intersection for F if T is a point of strict heteroclinic intersection for the natural complexification. Likewise, we use the obvious definitions for real points of strictly heteroclinic tangency and the sets $M_{\mathrm{pers}}^{\mathbb{R}}, M_{\mathrm{tang}}^{\mathbb{R}}$. This gives the following immediate corollary of Theorem 2.4, and this corollary allows us to prove Theorem 1.2.

Theorem 2.6. *Theorem 2.4 holds with $\mathcal{P}_d$, M_{pers}, M_{tang} and heteroclinic tangency replaced by $\mathcal{P}_d^{\mathbb{R}}$, $M_{pers}^{\mathbb{R}}$, $M_{tang}^{\mathbb{R}}$ and real heteroclinic tangency, respectively.*

3. Simultaneous uniformization for stable and unstable manifolds

In this section we prove Theorem 2.1, which says that the invariant manifolds $W^c(p)$ may be uniformized by $\mathbb{C}$ and that this uniformization is holomorphic over EV^c. That is, there is a holomorphic $\psi^c : \mathrm{EV}^c \times \mathbb{C} \to \mathbb{C}^2$ so that for fixed $\alpha \in \mathrm{EV}^c$, ψ^c_α is biholomorphic from $\mathbb{C}$ onto $W^c(p)$.

For saddle periodic points, the simultaneous uniformization is an easy consequence of the Hadamard-Perron theorem for maps depending on a parameter. On the other hand, for our purposes, we need to move through parameter space with more freedom on the multipliers: the only requirement is that λ is stably nonresonant for all parameters that we consider. Hence $|\lambda| < 1$, and the remaining multiplier, ν, satisfies $\nu \neq 1$, $\lambda^n \neq \nu$ for all $n \geq 1$ (with analogous restrictions in the unstable case).

For nodes, we can use either the classical Poincaré theorem for nonresonant nodes or an invariant manifold theorem for resonant nodes, but that still leaves the case $|\nu| = 1$. We did not succeed to find in the literature the statements that we need in the case of a resonant node or the nonhyperbolic case, so we have provided proofs even for the saddle case because they are uniform and simple. In all these cases we need to find only a local invariant manifold holomorphically depending on the parameter. The global uniformization is then provided by a classical principal: the local linearization may be globalized by the dynamics.

The proof of Theorem 2.1 proceeds in two steps. In the next subsection we show that EV^c is a complex manifold and prove a local version of Theorem 2.1 that gives a uniformization that is holomorphic throughout a small neighborhood of EV^c. In the final subsection, we show that EV^c is Stein, and a linear bundle over a Stein manifold may be trivialized, so the local result may be globalized to give Theorem 2.1. Finally, Theorem 2.1 for EV^e is reduced to that for EV^c by using inverse maps.

Throughout this section, $\mathcal{O}(M)$ is, as usual, the space of all holomorphic functions on an analytic manifold M, D is the open unit disc, and D_r is the open disk $|z| < r$.

3.1. **A local simultaneous uniformization theorem.** The following theorem is a local simultaneous uniformization theorem for the invariant manifolds $W^c_\lambda(p)$ corresponding to points $(F, p, \lambda) \in \mathrm{EV}^c(F_0, p_0, \lambda_0)$.

Theorem 3.1. *Let* $(F, p, \lambda) \in EV^c$ *as in Theorem 2.1. Then there exists a neighborhood* U *of* (F, p, λ) *in* EV^c*, and a holomorphic map*

$$\psi^c_U : U \times \mathbb{C} \to \mathbb{C}^2$$

such that for any $\alpha \in U$, $\psi^c_\alpha = \psi^c_U(\alpha, \cdot)$ *maps* $\mathbb{C}$ *biholomorphically onto* $W^c_\lambda(p)$ *and linearizes the map* F *on the invariant manifold as in Theorem 2.1.*

The statement of this theorem and the theorem that EV^c is a Stein manifold uses the fact that EV^c (and EV^e) are complex manifolds. We prove this next.

Proposition 3.1. *Let* $F_0 \in \mathcal{P}_d$*, and suppose* F_0 *has a periodic point* p_0 *with a stably nonresonant multiplier* λ_0*. Let* $\mathcal{P}_d(F_0)$ *be the irreducible component of* $\mathcal{P}_d$ *containing* F_0*. Then* $EV^c(F_0, p_0, \lambda_0)$ *is a complex manifold spread over an open subset of* $\mathcal{P}_d(F_0)$*. I.e., the projection* $\pi(F, p, \lambda) = F$ *is locally biholomorphic. An analogous statement holds for* EV^e*.*

Proof. Let $\mathrm{EV} = \mathrm{EV}(F_0, p_0, \lambda_0)$ as in Definition 2.4 and let m be the period of p_0. We show first that the projection $\pi : \mathrm{EV} \to \mathcal{P}_d(F_0)$ is locally biholomorphic, then show that $\mathrm{EV}^c(F_0, p_0, \lambda_0)$ is open in EV, which suffices to prove the proposition.

Fix a point

$$(F_1, p_1, \lambda_1) \in EV.$$

By definition of EV, p_1 has no multiplier equal to 1, so by the implicit function theorem, $F^m(p) = p$ defines a vector-valued function $F \mapsto p \in \mathbb{C}^2$ that is holomorphic near F_1. Moreover, the multiplier λ_1 of p_1 under F_1 is different from the other multiplier of p_1. Hence, for F near F_1, the multiplier λ depends holomorphically on F. Thus $F \mapsto (F, p, \lambda)$ is locally biholomorphic from a neighborhood of F_1 in $\mathcal{P}_d(F_0)$ to a neighborhood of (F_1, p_1, λ_1) in EV. Therefore, the natural projection $\pi : \mathrm{EV} \to \mathcal{P}_d(F_0)$ given by $\pi(F, p, \lambda) = F$ is locally biholomorphic.

It remains to prove that EV^c is open in EV. As noted above, λ is holomorphic on EV, so the set $\mathrm{EV}^s = \mathrm{EV} \cap \{|\lambda| < 1\}$ is open, so it suffices to prove that EV^c is open in EV^s.

For any $\alpha = (F, p, \lambda) \in \mathrm{EV}^s$, denote by ν the second multiplier of p under F. We use $\lambda(\alpha)$ and $\nu(\alpha)$ when needed to stress the dependence on α. In what follows, we use the fact that if $\alpha \in \mathrm{EV}^s \setminus \mathrm{EV}^c$, then $|\lambda| < 1$ but λ is not stably nonresonant, and hence there exists an integer $n \geq 1$ such that

$$\lambda^n(\alpha) = \nu(\alpha). \tag{6}$$

Suppose now that $\alpha_0 \in \mathrm{EV}^c$. Then $|\lambda(\alpha_0)| < 1$, so there exists N large enough that if $n > N$, then $|\lambda(\alpha_0)|^n < |\nu(\alpha_0)|$, and by continuity, this inequality persists for all $\alpha \in U_0$, U_0 a neighborhood of α_0 in EV^s, and for all $n > N$. On the other hand, since $\alpha_0 \in \mathrm{EV}^c$, equation (6) cannot hold for any value of $n > 1$, so in particular cannot hold for $1 \leq n \leq N$. Again by continuity, these N inequalities will persist for all $\alpha \in U_1$, U_1 a neighborhood of α_0 in EV^s. Then for $\alpha \in U_0 \cap U_1$, equation (6) will fail for all $n \geq 1$, so $U_0 \cap U_1$ is contained in EV^c, and hence EV^c is open in EV^s. □

We now begin the proof of Theorem 3.1. By the definition of EV^c, the multipliers λ and ν for a point p with $(F, p, \lambda) \in \mathrm{EV}^c$ satisfy the conditions that neither is equal to 1, $|\lambda| < 1$, and $\lambda^n \neq \nu$ for all integers $n \geq 1$. We subdivide these conditions into two cases: the first case is $|\nu| \leq |\lambda|$, and the second is $|\lambda| < |\nu|$. Note that in the first case, it is impossible to have $\nu^n = \lambda$ for any $n > 1$ since $|\nu| < 1$. Since all other resonances are prohibited by the conditions coming from EV^c, we see that p is a nonresonant node in the first case. For this case we state the linearization result in terms of nonresonant nodes and cite the Poincaré theorem to give a proof. The second case includes the possibilities that p is a saddle, or p is nonhyperbolic, or a resonant node. In the saddle case Theorem 3.1 is classical. In the case of resonant node the result is very close to results of [Bru70], which unfortunately are stated for vector fields only. On the other hand, in this third case, the invariant manifold may be identified as the strong stable manifold of p, so we give a single proof to cover all three subcases.

Lemma 3.1. *Theorem 3.1 holds in the case when p is of type "nonresonant node": $|\lambda| < 1$, $|\nu| < 1$, and $\lambda^n \neq \nu$, $\nu^n \neq \lambda$ for all integers $n \geq 1$. In particular, it holds when $|\nu| \leq |\lambda|$ and λ is stably nonresonant.*

Proof. Let $(F_1, p_1, \lambda_1) \in \mathrm{EV}^c$, and let m be the period of p_1. Since p_1 is a nonresonant node for F_1, we see as in the proof of the previous proposition that the same is true for all $(F, p, \lambda) \in U$, U a neighborhood of (F_1, p_1, λ_1) in EV^c. By Proposition 3.1, F parametrizes U. By the Poincaré theorem (see e.g. [RR88]), there exists a holomorphic coordinate change defined near p that conjugates F^m with its linear part at p, and this coordinate change depends holomorphically on F. Thus we obtain a holomorphically varying parametrization of a neighborhood of p in $W^c_\lambda(p)$ so that equation (4) is valid. Iterating equation (4) allows us to extend this parametrization to give a biholomorphic map from $\mathbb{C}$ to all of $W^c_\lambda(p)$. □

Proposition 3.2. *Theorem 3.1 holds in the case when $|\lambda| < |\nu|$.*

Proof. Suppose that $p = (0, 0)$ and near p,

$$F^m(z, w) = (\lambda z + f(z, w),\ \nu w + g(z, w)),\ j^1_0 f = j^1_0 g = 0,$$

where $j^1_0 f$ denotes the 1-jet of f at 0. This may be achieved by an affine coordinate change depending holomorphically on F. Note that it is sufficient to find a local invariant curve W^c_{loc} defined as the image of $\psi^c(D_\varepsilon)$ for which equation (4) holds: as before, if we have such a local curve, then iterating (4) provides a global continuation of ψ^c to the whole of $\mathbb{C}$.

First we find W^c_{loc} as a graph of a function $\varphi : D_\varepsilon \to \mathbb{C}$, $\varphi(0) = 0$, such that $F^m(W^c_{loc}) \subset W^c_{loc}$. After that the linearizing parameter t that satisfies (4) can be found by the Schröder theorem. See e.g., [Buz98, prop. 4.3] for a similar normalization.

The invariance equation for W^c_{loc} takes the form $F^m(z, \varphi(z)) = (z', \varphi(z'))$. Using the form given for F^m above, we can equate the first coordinates to get $z' = \lambda z + f(z, \varphi(z))$. Then equating the second coordinates gives

$$\nu\varphi(z) + g(z, \varphi(z)) = \varphi(\lambda z + f(z, \varphi(z))).$$

Consider the operator $\varphi \mapsto H(\varphi)$ (H of Hadamard) defined by solving this equation for $\varphi(z)$:

$$H(\varphi)(z) = \nu^{-1}[\varphi(\lambda z + f(z, \varphi(z))) - g(z, \varphi(z))].$$

Let $\mathcal{A}(D_\varepsilon)$ be the space of all functions holomorphic in D_ε and continuous on the boundary with the norm $||\varphi||_\varepsilon = \max_{D_\varepsilon} |\varphi|$. Denote by B_ε the following closed subset of $\mathcal{A}(D_\varepsilon)$:

$$B_\varepsilon = \{\varphi \in \mathcal{A}(D_\varepsilon) \mid ||\varphi'||_\varepsilon \leq 1,\ \varphi(0) = 0\}.$$

Lemma 3.2. *For ε small, $H(B_\varepsilon) \subset B_\varepsilon$, and H is a contraction on B_ε.*

Proof. To prove the first assertion, let $f_\varphi(z) = f(z, \varphi(z))$. Note that for $z \in D_\varepsilon$, $|\lambda z + f_\varphi(z)| = (|\lambda| + O(\varepsilon))|z|$. Since $|\lambda| < 1$, we have for sufficiently small ε and for $\varphi \in B_\varepsilon$, $z \in D_\varepsilon$, that the composition $\varphi(\lambda z + f_\varphi(z))$ is well defined.

On the other hand, for $(z, w) \in D_\varepsilon \times D_\varepsilon$ we have:

$$|f(z,w)| = O(\varepsilon^2), \ |g(z,w)| = O(\varepsilon^2),$$

$$|\nabla f(z,w)| = O(\varepsilon), \ |\nabla g(z,w)| = O(\varepsilon).$$

Hence,

$$||f'_\varphi||_\varepsilon = O(\varepsilon), \ ||g'_\varphi||_\varepsilon = O(\varepsilon).$$

Therefore, for ε sufficiently small, we have

$$||(H\varphi)'||_\varepsilon \leq |\nu|^{-1}(|\lambda| + ||f'_\varphi|| + ||g'_\varphi||) = \frac{|\lambda| + O(\varepsilon)}{|\nu|} < 1.$$

This proves the relation

$$HB_\varepsilon \subset B_\varepsilon.$$

Let us now prove that H is contracting on B_ε. Indeed, for $\varphi, \psi \in B_\varepsilon$:

$$\begin{aligned} H(\varphi)(z) - H(\psi)(z) &= \nu^{-1}\{[(\varphi(\lambda z + f_\varphi(z)) - g_\varphi(z)) \\ &\qquad - (\psi(\lambda z + f_\psi(z)) - g_\psi(z))]\} \\ &= \nu^{-1}\{[\varphi(\lambda z + f_\varphi(z)) - \psi(\lambda z + f_\varphi(z))] \\ &\qquad + [\psi(\lambda z + f_\varphi(z)) - \psi(\lambda z + f_\psi(z))] \\ &\qquad + [g_\psi(z) - g_\varphi(z)]\}. \end{aligned}$$

The second and third expressions in square brackets are majorized by $O(\varepsilon)||\varphi - \psi||_\varepsilon$. The first one is more delicate. Let $\varphi - \psi = \zeta$. By the Schwarz lemma, we have $|\zeta(z)| \leq |z| \|\zeta\|_\varepsilon / \varepsilon$ for $z \in D_\varepsilon$. Also, $|\lambda z + f_\varphi(z))| \leq |\lambda|\varepsilon + o(\varepsilon)$ in the same disk. Since $|\lambda| < 1$ we see that for ε small and $z \in D_\varepsilon$,

$$|\nu|^{-1}|\zeta(\lambda z + f_\varphi(z))| \leq |\nu|^{-1}||\zeta||_\varepsilon \frac{|\lambda\varepsilon + o(\varepsilon)|}{\varepsilon} = \frac{|\lambda| + o(1)}{|\nu|}||\zeta||_\varepsilon.$$

The inequality $|\lambda| < |\nu|$ implies that H is contracting. □

Proposition 3.2 follows immediately from Lemma 3.2 as explained above. □

Obvious modifications prove that if F depends analytically on a parameter, then φ depends analytically on the same parameter.

3.2. Stein property of $\mathbf{EV}^c$ and $\mathbf{EV}^e$.

Proposition 3.3. *Let $F_0 \in \mathcal{P}_d$, p_0 periodic for F_0, and λ_0 a stably nonresonant multiplier for p_0. Then the manifold $EV^c = EV^c(F_0, p_0, \lambda_0)$ is Stein. An analogous statement holds for EV^e.*

Proof. The manifold EV^s is Stein. This follows from Definition 2.4. Indeed, $P = \mathrm{Per}(F_0, p_0, \lambda_0)$ is a quasi affine algebraic manifold by Remark 2.3, hence is Stein. Also, EV^s is an open subset of P defined by $P \cap \{|\lambda| < 1\}$, where $\lambda \in \mathcal{O}(P)$. Thus EV^s is a holomorphically convex subset of a Stein manifold, hence is Stein.

For EV^u, note that $\lambda \neq 0$ on P. Hence, $\lambda^{-1} \in \mathcal{O}(P)$. Then EV^u is determined by the inequality $|\lambda^{-1}| < 1$, hence is also Stein.

Let us turn back to EV^c, which is obtained from EV^s by removing points $\alpha \in \mathrm{EV}^s$ where $\lambda^n(\alpha) = \nu(\alpha)$ for some $n \geq 1$. By Proposition 3.1, EV^c is open in the Stein manifold EV^s. Hence it suffices to show that EV^c is holomorphically convex. So, let K be a compact subset of EV^c, let K^c denote the holomorphic hull of K with respect to EV^c, and let K^s denote the hull of K with respect to EV^s. Note that K^s is compact in EV^s since EV^s is Stein. Also, $K^c \subset K^s$ since $\mathrm{EV}^c \subset \mathrm{EV}^s$.

Since K^s is compact in EV^s, the value of the multiplier ν is bounded away from 0 and $|\lambda|$ is bounded away from 1, hence K^s intersects at most a finite number of surfaces $S_n = \{\alpha \in \mathrm{EV}^s : \lambda^n(\alpha) = \nu(\alpha)\}$. Fix one such n. Then the function $h(\alpha) = (\lambda^n(\alpha) - \nu(\alpha))^{-1}$ is holomorphic on EV^c and is meromorphic on EV^s with a pole set exactly on S_n. Since K is compact and contained in the complement of S_n, we see that the closure of K^c in EV^s cannot intersect S_n. Since this is true for all S_n intersecting K^s, we have that the closure of K^c in EV^s is contained in EV^c. Thus EV^c is holomorphically convex, hence Stein.

An analogous argument applies to show that EV^e is Stein. □

Proof of Theorem 2.1. Theorem 3.1 implies that for each point $\alpha \in \mathrm{EV}^c$, there is a neighborhood U of α so that $W^c_\lambda(p)$ is a trivial line bundle over U. Proposition 3.3 implies that EV^c is Stein, and a complex line bundle over a Stein manifold may be globally trivialized by solving a Cousin problem. This global trivialization implies Theorem 2.1 for EV^c. The result for EV^e follows by replacing f with f^{-1}. □

4. Following strictly heteroclinic intersections

In this section we first recall some results from complex analysis concerning the boundary values of functions holomorphic in the disk, then give a condition for the ε-lift property to hold. We apply this to M_{pers} and M_{pot} to prove Theorem 2.2. Theorem 1.5 is an immediate corollary of Theorem 2.2, and we do not discuss it below.

4.1. Boundary values of holomorphic functions.

Below is a summary of the standard results used in Section 4.

Definition 4.1. Let f be holomorphic on D.

- A complex number w is a *nontangential limit value* of f at a point $\zeta \in \partial D$ if there exists a Stolz angle $S \subset D$ with vertex ζ and a sequence $\{\zeta_n\} \subset S$ such that $\zeta_n \to \zeta$ and $f(\zeta_n) \to w$ as $n \to \infty$.

- Given a Stolz angle S in D with endpoint ζ, the *cluster set of f in S* is the set of all limits of $f(\zeta_n)$ obtained along sequences $\{\zeta_n\} \subset S$ with $\zeta_n \to \zeta$.
- The *cluster set* of f at ζ is the union over all Stolz angles, S, with endpoint ζ, of the cluster set of f in S.

The following definition highlights the two extreme possibilities for the cluster set of f.

Definition 4.2. Let f be holomorphic in D and let ζ be a point in ∂D.

- ζ is a *Fatou point* for f if the cluster set of f at ζ is a single point (possibly ∞).
- ζ is a *Plessner point* for f if for any Stolz angle S centered at ζ, the cluster set of f in S is all of $\mathbb{C}$.

The two following theorems describe the cluster sets of a.e. point in the unit circle. Both theorems may be found in [CL66]: the first is theorem 2.5, while the second is theorem 8.2.

Theorem 4.1 (Fatou)**.** *If f is bounded and holomorphic on D, then a.e. point in ∂D is a Fatou point for f.*

Theorem 4.2 (Plessner)**.** *If f is meromorphic on D, then a.e. point of ∂D is either Fatou or Plessner for f.*

We need also the following results on the uniqueness of functions with certain boundary behavior. The first is an immediate consequence of the Poisson formula for harmonic functions in the disk. The second is a corollary of a theorem of Privalov and is found in [CL66, cor. 8.1].

Theorem 4.3. *If u is a bounded harmonic function in D and u has radial limit 0 along a set of radii whose endpoints form a set of full measure on ∂D, then $u \equiv 0$.*

Theorem 4.4. *Let $f(z)$ be meromorphic in D and suppose there is a positive measure subset $E \subset \partial D$ such that each $\zeta \in E$ is a Fatou point for f and f has nontangential limit 0 at ζ. Then f is identically 0 in D.*

4.2. **Interior ends for holomorphic maps.** In this section we consider the image of a curve under a holomorphic map. This will play an important role in proving the existence of the ε-lift property.

Definition 4.3. Let U be a bounded domain in $\mathbb{C}$, let $\Phi : D \to U$ be a holomorphic function, and let $\gamma : [0,1] \to \overline{D}$ be a curve such that $\gamma([0,1)) \subset D$ and $\gamma(1) \in \partial D$. The curve γ is called an *interior end* (respectively, *boundary end*) for Φ with respect to U provided that the limit

$$z = \lim_{r \to 1} \Phi \circ \gamma(r),$$

exists and $z \in U$ (respectively, $z \in \partial U$).

The following proposition gives a condition for the existence of interior ends for a self-map of the disk.

Proposition 4.1. *Let $\Phi : D \to D$ be holomorphic. Then either $\overline{\Phi(D)} = \overline{D}$ or there exists a positive measure set of radii that are interior ends for Φ with respect to D (or both).*

Proof. Suppose $\overline{\Phi(D)} \neq \overline{D}$. Then there exists $z_0 \in D \setminus \overline{\Phi(D)}$. Composing with a Möbius transformation, we may assume that $z_0 = 0$. Also, there exists $r > 0$ so that $D_r \subset D \setminus \overline{\Phi(D)}$. Then $f(z) = \log \Phi(z)$ is holomorphic on D with image contained in the strip $S_r = \{z : \log r < \Re e\, z < 0\}$. Since S_r is simply connected, there is a biholomorphic map $\phi_+ : S_r \to D_+$, where D_+ is the half-disk $D \cap \{z : \Im m\, z > 0\}$. Moreover, ϕ_+ extends continuously to the two lines bounding S_r, and ϕ_+ can be chosen so that $\phi_+(\{\Re e\, z = 0\})$ is contained in the diameter $[-1, 1]$. Then $(\phi_+ \circ f)$ is a bounded holomorphic function; hence, the harmonic function $v = \Im m\,(\phi_+ \circ f)$ has radial limits along a.e. radius.

If Φ is constant, then every radius is an interior end, so we may assume that Φ is nonconstant. Hence f and v are also nonconstant. Since Φ is bounded, it has radial limits along almost every radius, hence a.e. radius is either a boundary end or an interior end for Φ with respect to D. If ζ is the endpoint of a radius that is a boundary end for Φ with respect to D, then $f(\zeta t)$ converges to a point in $\{\Re e\, z = 0\}$ as $t \to 1$. Hence $v(\zeta t)$ converges to 0 for such ζ.

Thus if $v(\zeta t)$ does not converge to 0 as $t \to 1$, then the corresponding radius is not a boundary end for Φ. By Theorem 4.3 there is a positive measure set of radii along which v has nonzero radial limit. Since none of these radii can be boundary ends and since a.e. radius is either a boundary end or interior end, this gives a positive measure set of radii that are interior ends for Φ with respect to D. □

Applying this proposition to a domain that is conformally equivalent to the disk via a conformal map that extends homeomorphically to the boundary, we obtain the following immediate corollary.

Corollary 4.1. *Let $U \subset \mathbb{C}$ be bounded, open, simply connected, and having a simple Jordan curve as boundary, and let $\Phi : D \to U$ be holomorphic. Then either $\overline{\Phi(D)} = \overline{U}$ or there is a positive measure set of radii that are interior ends for Φ with respect to U.*

4.3. **The ε-lift property.** Let M, M' be connected complex manifolds and let N be an irreducible, complex analytic subset of $M \times M'$; i.e., for each $\mathfrak{b} \in M \times M'$, there exists a neighborhood U of $\mathfrak{b}$ and a vector-valued holomorphic function $H = H_{\mathfrak{b}}$ on U so that $N \cap U$ is the set of $(z, w) \in U$ with $H(z, w) = 0$. Let $\pi : N \to M$, $\pi' : N \to M'$ be the natural projections, and suppose that π is locally proper at each point of N and $\dim(N) = \dim(M)$. In this case, π is a local biholomorphism outside a nowhere dense closed set (e.g. [Chi89, sec. 3.7]).

For the convenience of the reader, we recall below Definitions 2.8 and 2.9.

Definition 4.4. Let $\pi : N \to M$ be as above. Let $\gamma : I \to M$, let $\mathfrak{b}_0 \in N$ with $\pi(\mathfrak{b}_0) = \gamma(0)$, and let $\varepsilon > 0$. An ε-lift of γ to N beginning at $\mathfrak{b}_0$ is a curve $\hat{\gamma} : I \to N$ with $\hat{\gamma}(0) = \mathfrak{b}_0$ so that $\pi \circ \hat{\gamma}$ has C^0-distance at most ε from γ.

Definition 4.5. Let $\pi : N \to M$ be as above. The projection π has the ε-lift property if for every curve $\gamma : I \to M$, for every point $\mathfrak{b}_0 \in N$ so that $\pi(\mathfrak{b}_0) = \gamma(0)$, and for all $\varepsilon > 0$, there exists an ε-lift of γ to $\hat{\gamma}$ beginning at $\mathfrak{b}_0$.

Remark. The conditions on π imply that it is an open map [Chi89, sec. 5.8], while the ε-lift property implies that $\pi(N)$ is dense in M and that $\pi(N)$ is connected since M is connected.

To illustrate the ε-lift property, we give two examples, one which has the ε-lift property and one which does not.

Example 1. Let $M = \mathbb{C}$, $N = \{(z, w) \in \mathbb{C}^2 : zw = 1\}$. Here $\pi(z, w) = z$ is a biholomorphism of N to $\mathbb{C} \setminus \{0\}$ and has the ε-lift property.

Example 2. First choose a proper holomorphic embedding $h : D \to \mathbb{C}^2$. Then let $M = \mathbb{C}$ and let N be the graph of h in $\mathbb{C}^3$: $N = \{(z, w) \in \mathbb{C} \times \mathbb{C}^2 : w = h(z)\}$. Since N is a Stein manifold in $\mathbb{C}^3$, we can represent N as the zero set of a holomorphic map $H : \mathbb{C}^3 \to \mathbb{C}^2$. Then $\pi(N) = D$. Also, any curve $\gamma : I \to D$ has an ε-lift to N (even an exact lift to N), but if γ is not contained in D, then it does not have an ε-lift to N.

Next we consider the relationship between an ε-lift of a curve and analytic continuation along the curve. Let $\gamma : I \to M$, and suppose $\gamma(0) = \pi(\mathfrak{b}_0)$ for some $\mathfrak{b}_0 \in N$. If π is a local biholomorphism at $\mathfrak{b}_0$, then $h = \pi' \circ \pi^{-1}$ defines a germ of a holomorphic map at $\pi(\mathfrak{b}_0)$. Moreover, a lift of γ to N defines a continuous extension of h to γ. However, as seen in the examples, there may be no such lift of all of γ; h may become unbounded along γ. We examine this possibility more in the next few results.

Definition 4.6. Let $\mathfrak{b}_0 \in N$ with π a local biholomorphism at $\mathfrak{b}_0$ and let $\mathfrak{a}_0 = \pi(\mathfrak{b}_0)$. Let $\gamma : I \to M$ with $\gamma(0) = \mathfrak{a}_0$, and let A be either $[0, 1]$ or $[0, 1)$. Then $h = \pi' \circ \pi^{-1}$ *can be continued along* $\gamma|_A$ means that there exists a continuous $\gamma_N : A \to N$ so that $\gamma_N(0) = \mathfrak{b}_0$ and $\pi\gamma_N(t) = \gamma(t)$ for all $t \in A$. In this case we define $h(\gamma(t)) = \pi'(\gamma_N(t))$ for all $t \in A$.

Since a point in N is specified by a point in M plus a choice for π', we see that given the values of π' along a curve in M, these values determine a lift of the curve to N. The following lemma states that the existence of finite limits for π' along an open portion of a curve is sufficient for continuation along the closed curve. This could be strengthened to give continuation along a neighborhood of the limiting point on the curve, but we do not need this result here.

Lemma 4.1. *Let* $\gamma : I \to M$ *and let* $A = [0, 1)$. *Suppose* $\mathfrak{b}_0 \in N$ *so that* π *is a local biholomorphism at* $\mathfrak{b}_0$ *and* $\gamma(0) = \pi(\mathfrak{b}_0)$. *Suppose also that* $h = \pi' \circ \pi^{-1}$ *can be continued along* $\gamma|_A$ *and that* $\lim_{t\to 1} h(\gamma(t))$ *exists as an element of* M'. *Then* h *can be continued along* $\gamma|_I$. *In particular,* h *defines a lift of* γ *to a curve in* N *with initial point* $\mathfrak{b}_0$.

Proof. The assumptions on γ imply that $(\gamma(t), h(\gamma(t)))$ converges as $t \to 1$ to a point $\mathfrak{b} \in M \times M'$. Then the corresponding defining function $H_{\mathfrak{b}}$ is well-defined in a neighborhood of $\mathfrak{b}$. Also, $H_{\mathfrak{b}}(\gamma(t), h(\gamma(t))) = 0$ for all t sufficiently near 1, hence for $t = 1$ by continuity.

Hence we can define $\gamma_N(t) = (\gamma(t), h(\gamma(t)))$ for $t \in [0, 1]$, in which case the preceding paragraph implies $\gamma_N(t) \in N$ for all $t \in I$. Thus $\gamma_N : I \to N$ with $\gamma_N(0) = \mathfrak{b}_0$ and $\pi\gamma_N(t) = \gamma(t)$. Hence h can be continued along γ and γ lifts to γ_N. □

The following condition gives the finite limits needed by the previous lemma in order to show that π has the ε-lift property.

Definition 4.7. Let $\pi : N \to M$ be as above. Then N is *tame on disks over* M means that for all $\phi : D \to N$ such that $\pi \circ \phi : D \to L$, where $\overline{L}$ is an injective holomorphic embedding of a closed disk into M, the map ϕ has radial limits in $M \times M'$ for a.e. point of ∂D.

Remark. To say that $\overline{L}$ is an injective holomorphic embedding of a closed disk into M means that there exists $\varepsilon > 0$ and an injective holomorphic map $h : D_{1+\varepsilon} \to M$ so that $h(D) = L$.

First a lemma on the existence of a weaker form of lifting in which the approximating curve is required only to lie in an ε-neighborhood of the image of the original curve and have endpoint near the endpoint of the original curve. This form of the lifting property is known as the Iversen property. Here $U^\varepsilon(E)$ represents an ε-neighborhood of a set E.

Proposition 4.2 (Iversen property for analytic curves)**.** *Let* $\pi : N \to M$ *be as above, and suppose that* N *is tame on disks over* M. *Let* $\gamma_0 : I \to M$ *be injective and real-analytic with* $\gamma_0(0) = \pi(\mathfrak{b}_0)$ *for some* $\mathfrak{b}_0 \in N$ *and with* $\gamma_0'(t) \neq 0$ *for all* $t \in I$. *Let* $\varepsilon > 0$. *Then there exists* $\gamma_\varepsilon : I \to M$ *such that*

- $\gamma_\varepsilon(0) = \gamma_0(0)$
- $|\gamma_\varepsilon(1) - \gamma_0(1)| < \varepsilon$
- $\gamma_\varepsilon(I) \subset U^\varepsilon(\gamma_0(I))$
- γ_ε *lifts to* $\hat{\gamma}_\varepsilon : I \to N$ *with* $\hat{\gamma}_\varepsilon(0) = \mathfrak{b}_0$ *and* $\pi\hat{\gamma}_\varepsilon(t) = \gamma_\varepsilon(t)$ *for all* $t \in I$.

Proof. First we show that we may assume that π is a local biholomorphism at $\mathfrak{b}_0$. By [Chi89, sec. 3.7], regular points of π are open and dense in N, so there are regular points of π arbitrarily near $\mathfrak{b}_0$ in N. Since N is irreducible, for such regular points near enough to $\mathfrak{b}_0$, there is a short curve, σ, in N from the regular point to $\mathfrak{b}_0$. We can project this short curve to M,

$$
\begin{array}{ccccc}
(D,0) & \xrightarrow{\hat{\Phi}} & (\hat{S}, \hat{\mathfrak{b}}_0) & & \\
 & & \downarrow \hat{\pi} & & \\
 & \overset{\Phi}{\searrow} & (S, \mathfrak{b}_0) & \xrightarrow{\pi'} & M' \\
 & & \downarrow \pi & & \\
(I,0) & \xrightarrow{\gamma_0} & (L, \gamma_0(0)) & &
\end{array}
$$

FIGURE 1. Maps used in Proposition 4.2

then follow this with γ_0, then approximate this composite curve with a real-analytic curve, then approximate this real-analytic curve with a curve as in the conclusion of the lemma. The curve γ_ε is then obtained by traversing $\pi \circ \sigma$ in reverse, then traversing the approximate curve. All of this can be done subject to the constraints of the lemma. Thus we may assume π is a local biholomorphism at $\mathfrak{b}_0$.

The assumptions on γ_0 imply that there exists $\delta > 0$ so that γ_0 extends to a holomorphic embedding of $U^{2\delta}(I)$ and so that the image of this set is contained in an ε-neighborhood of $\gamma_0(I)$ in M. Let $L = \gamma_0(U^\delta(I))$, and let S be the irreducible component of $\pi^{-1}(L)$ containing $\mathfrak{b}_0$. Since L is one-dimensional and π is a local biholomorphism at regular points, S is also one-dimensional. Thus S has a desingularization $\hat{\pi} : \hat{S} \to S$. Also, $\hat{S}$ is a hyperbolic Riemann surface since $\gamma_0^{-1} \circ \pi \circ \hat{\pi} : \hat{S} \to U^\delta(I)$ is a nonconstant, bounded holomorphic function. Hence $\hat{S}$ has universal covering map $\hat{\Phi} : D \to \hat{S}$. Define $\Phi = \pi \circ \hat{\pi} \circ \hat{\Phi} : D \to L$. After a Möbius transformation, we may assume that $\hat{\pi} \circ \hat{\Phi}(0) = \mathfrak{b}_0$, hence $\Phi(0) = \gamma_0(0)$. See figure 1.

Note that $\gamma_0 : U^\delta(I) \to L$ is biholomorphic and extends to be injective holomorphic on $U^{2\delta}(I)$, hence is a homeomorphism of $\overline{U^\delta(I)}$ to $\overline{L}$. Hence we may regard L as a subset of $\mathbb{C}$ satisfying the conditions of Corollary 4.1, so either $\overline{\Phi(D)} = \overline{L}$ or there is a positive measure set of radii that are interior ends for Φ with respect to L. If $\overline{\Phi(D)} = \overline{L}$, then there exists a point $z_1 \in D$ so that $|\Phi(z_1) - \gamma_0(1)| < \varepsilon$. Define $\gamma_\varepsilon(t) = \Phi(z_1 t)$ and $\hat{\gamma}_\varepsilon(t) = \hat{\pi}\hat{\Phi}(z_1 t)$ for $t \in I$. Then γ_ε and $\hat{\gamma}_\varepsilon$ satisfy the conclusions of the proposition.

In the alternative case, there is a positive measure set $E \subset \partial D$ of end-points of radii that are interior ends of Φ with respect to L. We will show that this leads to a contradiction. By the assumption that N is tame on disks over M and the fact that Φ maps D to L, we see that $\hat{\pi}\hat{\Phi}$ has radial limits in $M \times M'$ along a.e. radius. In particular, there exists $\zeta \in E$ which is the endpoint for such a radius. Thus, at this ζ, Φ has nontangential limit in L and $\hat{\pi}\hat{\Phi}$ has radial limit in $M \times M'$.

Since π is a local biholomorphism at $\mathfrak{b}_0$, we have also that $\hat{\pi}$ and $\hat{\Phi}$ are local biholomorphisms at 0 and $\hat{\mathfrak{b}}_0$, respectively, hence Φ is a local biholomorphism at 0. Thus we can define the germ of a holomorphic map $h(z) = \pi'\hat{\pi}\hat{\Phi}\Phi^{-1}(z)$ at $\gamma_0(0)$, and this germ agrees with the germ defined by $h(z) = \pi'(\pi^{-1}(z))$.

In particular, with $\zeta \in E$ as above, we can define the curve $\gamma(t) = \Phi(\zeta t)$, $t \in [0,1]$, then continue h along $\gamma|_{[0,1)}$ by defining $h(\gamma(t)) = \pi' \hat{\pi} \hat{\Phi}(\zeta t)$.

The choice of ζ implies that $\gamma(1) \in L$ and that $\lim_{t\to 1} \hat{\pi}\hat{\Phi}(\zeta t)$ exists in $M \times M'$, hence $\lim_{t\to 1} h(\gamma(t))$ exists as an element of M'. Thus by Lemma 4.1, h continues along $[0,1]$, so γ lifts to a curve in N with initial point $\mathfrak{b}_0$. This lift of γ to N is given by $\gamma_N(t) = \hat{\pi}\hat{\Phi}(\zeta t)$ for $t \in [0,1)$ and by extending to the limit at $t = 1$. By continuity, $\pi(\gamma_N(1)) = \gamma(1)$, and this point lies in L, so $\gamma_N(1) \in S$. Hence h defines a lift of $\gamma : I \to L$ to a curve in S with initial point $\mathfrak{b}_0$, hence to a curve in D with initial point 0. But this lift of $\gamma|_{[0,1)}$ to D is exactly the radius from 0 to ζ, and this lift cannot be extended to a lift of $\gamma(I)$ to D since ζ lies outside D, a contradiction.

Hence $\overline{\Phi(D)} = \overline{L}$, so γ_ε and $\hat{\gamma}_\varepsilon$ exist as claimed. □

Remark. The alternative case in Proposition 4.2 can also be shown to be contradictory as follows. First define the maximal Riemann surface spread over L on which h can be analytically continued as a multi-valued function. Then show that this surface is equivalent to S, then construct the lifts to $\hat{S}$ and D as above. Then γ is defined in the same way, and the lift of γ to S contradicts the maximality of S.

We prove Theorem 1.6 by subdividing a given curve into many small pieces, then applying Proposition 4.2 to each of these pieces in turn, connecting the endpoints along the way to fill in any gaps.

Proof of Theorem 1.6. Let $\gamma : I \to M$ with $\gamma(0) = \pi(\mathfrak{b}_0)$, let $\varepsilon > 0$, and cover $\gamma(I)$ by finitely many open subsets of M, $B_1, \dots, B_k$, each biholomorphic to a ball of the same dimension as M and each with diameter less than ε in M. Then there are points $0 = t_0 < t_1 < \cdots < t_{l+1} = 1$ and indices $k(j)$ so that $\gamma([t_j, t_{j+1}])$ is contained in $B_{k(j)}$ for $j = 0, \dots, l$.

Since $B_{k(0)}$ is biholomorphic to a ball, there is a curve in $B_{k(0)}$ from $\gamma(t_0)$ to $\gamma(t_1)$ satisfying the conditions of Proposition 4.2. By that same proposition, we may approximate this curve, then reparametrize to get $\gamma_\varepsilon : [t_0, t_1] \to B_{k(0)}$ so that $\gamma_\varepsilon(t_0) = \gamma(t_0)$ and $\gamma_\varepsilon(t_1) \in B_{k(0)} \cap B_{k(1)}$ and so that γ_ε lifts to N with initial point $\mathfrak{b}_0$ and some end point $\mathfrak{b}_1 \in N$.

Likewise, since $\gamma_\varepsilon(t_1)$ and $\gamma(t_2)$ are contained in $B_{k(1)}$, there is a curve in $B_{k(1)}$ from $\gamma_\varepsilon(t_1)$ to $\gamma(t_2)$ satisfying the conditions of Proposition 4.2. Again we may approximate and reparametrize to get $\gamma_\varepsilon : [t_1, t_2] \to B_{k(1)}$ so that $\gamma_\varepsilon(t_1)$ is consistent with the previous definition of this point and $\gamma_\varepsilon(t_2) \in B_{k(1)} \cap B_{k(2)}$ and so that this portion of γ_ε lifts to N with initial point $\mathfrak{b}_1$ and some end point $\mathfrak{b}_2$.

Continuing in this way we get γ_ε defined on $[0,1]$. Then $\gamma_\varepsilon : I \to M$ has C^0-distance at most ε from γ, and γ_ε lifts to $\hat{\gamma}_\varepsilon : I \to N$ with $\hat{\gamma}_\varepsilon(0) = \mathfrak{b}_0$ and $\pi\hat{\gamma}_\varepsilon = \gamma_\varepsilon$. Thus γ has an ε lift, so π has the ε-lift property. □

4.4. **Continuation of heteroclinic points and the ε-lift property.** In this section we show that the dimensions of M_{pers} and M_{pot} are equal and $\pi : M_{\text{pers}} \to M_{\text{pot}}$ is proper, then show that π has the ε-lift property.

Let $F_0 \in \mathcal{P}_d$ be a map with a strictly heteroclinic point T_0 between periodic points p_0 and q_0. Let $|\lambda_0| < 1$, $|\mu_0| > 1$ be the corresponding multipliers of p_0, q_0. Let $\mathfrak{a}_0 = (F_0, p_0, q_0, \lambda_0, \mu_0)$ and let $\tau_0 = \psi_{\mathfrak{a}_0}^{-1}(T_0)$ as in Definition 2.7. Let $M_{\text{pers}} = M_{\text{pers}}(\mathfrak{a}_0, \tau_0)$ and $M_{\text{pot}} = M_{\text{pot}}(\mathfrak{a}_0)$.

Proposition 4.3. *Let $\pi : M_{pers} \to M_{pot}$ be the natural projection as defined after Definition 2.7. Then the fibers of π are analytic subsets of $\mathbb{C}^2$ of dimension zero, hence discrete, and $\pi : M_{pers} \to M_{pot}$ is locally proper. Also, $\dim(M_{pers}) = \dim(M_{pot})$, and this dimension is equal to the dimension of the component of $\mathcal{P}_d$ containing F_0.*

Proof. Suppose that for some $\mathfrak{a} \in M_{\text{pot}}$ the fiber $\{(\tau^c, \tau^e) : \psi_{\mathfrak{a}}^c(\tau^c) = \psi_{\mathfrak{a}}^e(\tau^e)\}$ has dimension $d > 0$. Note that for $\mathfrak{a} = (F, p, q, \lambda, \mu)$ fixed, $\psi_{\mathfrak{a}}^c$ parametrizes $W_\lambda^c(p)$ biholomorphically and $\psi_{\mathfrak{a}}^e$ parametrizes $W_\mu^e(q)$ biholomorphically.

If $d = 2$, then $\psi_{\mathfrak{a}}^c(\tau^c) = \psi_{\mathfrak{a}}^e(\tau^e)$ for all τ^c and τ^e, which is impossible since ψ^c and ψ^e are injective.

If $d = 1$ then since both ψ^c and ψ^e are injective holomorphic, we see that $W_\lambda^c(p)$ and $W_\mu^e(q)$ intersect in a Riemann surface that is open in both of them. Hence the union $W = W_\lambda^c(p) \cup W_\mu^e(q)$ is a single connected Riemann surface. Moreover, W contains an embedded copy of $\mathbb{C}$, namely $W_\lambda^c(p)$, and also contains the point $q \notin W_\lambda^c(p)$. Hence W must be biholomorphic to the Riemann sphere, which is impossible since W is a Riemann surface contained in $\mathbb{C}^2$.

Thus $d = 0$ as desired, so the fibers of π are discrete. Hence for any $\mathfrak{b} = (\mathfrak{a}, \tau^c, \tau^e) \in M_{\text{pers}}$ there is a small neighborhood of $\mathfrak{b}$ in M_{pers} so that the restriction of π to this neighborhood is proper. Also, M_{pers} is obtained from $M_{\text{pot}} \times \mathbb{C}^2$ by the addition of two equations, so the fact that π is locally proper implies that $\dim(M_{\text{pers}}) = \dim(M_{\text{pot}})$.

Finally, since $\text{Per}(F_0, p_0)$ is parametrized by an open subset of the component of $\mathcal{P}_d$ containing F_0 and since EV^c and EV^e are locally biholomorphic to open subsets of $\text{Per}(F_0, p_0)$, we see from the definition of M_{pot} that $\dim(M_{\text{pot}}) = \dim(\text{Per}(F_0, p_0))$, which is equal to the dimension of the component of $\mathcal{P}_d$ containing F_0. □

Now we show that M_{pers} is tame on disks over M_{pot}, which implies the ε-lift property.

Proof of Theorem 2.2. By Theorem 1.6, it suffices to show that M_{pers} is tame on disks over M_{pot}. So, let $\phi : D \to M_{\text{pers}}$ so that the projection $\pi\phi : D \to M_{\text{pot}}$ maps D into the image of an injective holomorphic embedding of a closed disk. In particular, the composition of $\pi\phi$ with the inverse of this embedding has nontangential limits in $\mathbb{C}$ for a.e. point of ∂D, hence $\pi\phi$ has nontangential limits in M_{pot} at a.e. point of ∂D. Let E be the subset of ∂D for which $\pi\phi$ has nontangential limits in M_{pot}. Note that on M_{pers},

τ^e and τ^c are simply the coordinate functions in $\mathbb{C}^2$ corresponding to the projection $\pi' : M_{\text{pot}} \times \mathbb{C}^2 \to \mathbb{C}^2$. Thus we have the maps

$$\begin{array}{ccccc} D & \xrightarrow{\phi} & M_{\text{pers}} & \xrightarrow{\tau=\pi'} & \mathbb{C}^2 \\ & & \pi \downarrow & & \\ & & M_{\text{pot}} & & \end{array}$$

and we can define holomorphic functions $\hat{\tau}^{e/c} = \tau^{e/c}\phi : D \to \mathbb{C}$. Then it suffices to verify that $\pi'\phi$ has radial limits in $\mathbb{C}^2$ along a.e. radius, or equivalently that $\hat{\tau}^{e/c}$ have finite radial limits along a.e. radius.

By Plessner's Theorem (Theorem 4.2), a.e. point in E is either a Fatou point or a Plessner point for $\hat{\tau}^e$, and likewise for $\hat{\tau}^c$. We will show that no point of E is a Plessner point for either map, hence a.e. point is Fatou for both. This proves immediately the existence of (possibly infinite) nontangential limits at a.e. point of ∂D. To show that these limits are finite for a.e. boundary point, note that $1/\hat{\tau}^e$ is meromorphic (even holomorphic) and not identically 0 in D. Hence by [CL66, cor. 8.1], the set of points in E for which $1/\hat{\tau}^e$ has nontangential limit 0 has measure 0. Thus $\hat{\tau}^e$ has finite nontangential limits at almost every boundary point, hence finite radial limits along a.e. radius, and likewise for $\hat{\tau}^c$.

To show that no point of E is a Plessner point for $\hat{\tau}^e$, let $\zeta \in E$. Then $\pi\phi$ has a radial limit $\mathfrak{a}_1 \in M_{\text{pot}}$ along the radius ending at ζ. Write $\mathfrak{a}_1 = (F_1, p_1, q_1, \lambda_1, \mu_1)$ with $|\lambda_1| < 1$ and $|\mu_1| > 1$. From Theorem 2.1, for each $\mathfrak{a}$ near $\mathfrak{a}_1$ in M_{pot}, there are corresponding periodic points p and q (depending on $\mathfrak{a}$) and uniformizing maps $\psi^c_{\mathfrak{a}} : \mathbb{C} \to W^c_{\mathfrak{a}}(p)$ and $\psi^e_{\mathfrak{a}} : \mathbb{C} \to W^e_{\mathfrak{a}}(q)$ depending holomorphically on $\mathfrak{a}$.

By [BS91], there is a neighborhood $\mathcal{Q}$ of F_1 in $\mathcal{P}_d$ and a compact set $B \subset \mathbb{C}^2$ such that for each $F \in \mathcal{Q}$, the set of all heteroclinic orbits (indeed all orbits bounded in both forward and backwards time) is contained in B.

On the other hand, the set $W^e_{\mathfrak{a}}(q)$ is an injective immersion of $\mathbb{C}$ into $\mathbb{C}^2$, hence is unbounded. Thus, there exists a disk $D_0 \subset \mathbb{C}$ such that $\psi^e_{\mathfrak{a}_1}(\overline{D}_0) \cap B = \emptyset$. For $\mathfrak{a}$ in a small neighborhood of $\mathfrak{a}_1$ in M_{pot}, the projection of $\mathfrak{a}$ to $\mathcal{P}_d$ will be in $\mathcal{Q}$ and $\psi^e_{\mathfrak{a}}(\overline{D}_0)$ will still be disjoint from B. In particular, for such $\mathfrak{a}$, any corresponding value of τ^e (or $\hat{\tau}^e$) must lie outside D_0.

Finally, since $\pi\phi$ has nontangential limit $\mathfrak{a}_1$ at ζ, we see that in a fixed nontangential approach region for ζ, $\pi\phi(z)$ will be near $\mathfrak{a}_1$ when z is near ζ. Hence the corresponding value $\hat{\tau}^e(\mathfrak{a})$ will lie outside D_0. Thus the cluster set of $\hat{\tau}^e$ at ζ is disjoint from D_0, so ζ is not a Plessner point for $\hat{\tau}^e$, as claimed. A similar proof applies to $\hat{\tau}^c$. As noted, this implies that M_{pers} is tame on disks over M_{pot}, hence $\pi : M_{\text{pers}} \to M_{\text{pot}}$ has the ε-lift property. □

5. A path to hyperbolicity

This section is devoted to proving Theorem 2.3. In other words, we prove that given a map F having saddle periodic points p and q, there is a path to a saddle hyperbolic map so that the corresponding periodic points are potentially heteroclinic throughout the path. We'll see in the next section that in conjunction with the ε-lift property, this path through M_{pot} implies that the set of maps with heteroclinic tangencies has codimension at least 1 in M_{pers}.

5.1. **A two-parameter family of maps and saddle hyperbolicity.** We first set up some notation. From Friedland-Milnor [FM89], we can write any element in $\mathcal{P}_d$ in the form $F(x,y) = F_1 \circ \cdots \circ F_k(x,y)$, where $F_j(x,y) = (y, p_j(y) - a_j x)$, $p_j(y) = y^{d_j} + O(y^{d_j - 1})$. Given such an F, we make a two-parameter family corresponding loosely to the quadratic Hénon family as follows: For $a, c \in \mathbb{C}$, let

$$F_{j,a,c}(x,y) = (y, p_j(y) + c^{d_j} - a a_j x),$$

and let

$$F_{a,c}(x,y) = F_{1,a,c} \circ \cdots \circ F_{k,a,c}(x,y).$$

We interpret $a = 0$ as giving a degenerate 1-dimensional map: $F_{0,c}(x,y) = (P_c(y), Q_c(P_c(y)))$ for some polynomials P_c and Q_c. Hence $F_{0,c}$ collapses horizontal lines so that all of $\mathbb{C}^2$ is collapsed onto the 1-dimensional curve $\{(y, Q_c(y)) : y \in \mathbb{C}\}$, which is biholomorphic to $\mathbb{C}$. In this case, we may view $F_{0,c}$ in two ways. First, the map $H(x,y) = y$ gives a semiconjugation from $F_{0,c}$ to the map $Q_c \circ P_c(y)$. That is, $Q_c \circ P_c \circ H = H \circ F_{0,c}$. On the other hand, if we consider the restriction of $F_{0,c}$ to its image curve, then the parametrizing map $h(y) = (y, Q_c(y))$ gives a conjugation to the map $P_c Q_c(y)$. That is, h^{-1} is well-defined as a map on the image curve, and $P_c Q_c(y) = h^{-1} \circ F_{0,c} \circ h(y)$ for each y in $\mathbb{C}$. In any case, regarding $F_{0,c}$ as a map of $\mathbb{C}^2$ to itself, each periodic point still has 2 multipliers, one of which is 0, corresponding to the collapsing direction, the other of which may be any value in $\mathbb{C}$.

Proposition 5.1. *Given $A > 0$ and $\varepsilon > 0$, there exists $C > 0$ such that if $|a| < A$ and $|c| > C$, then $F_{a,c}$ is hyperbolic and such that if λ is a multiplier of a periodic point of $F_{a,c}$, then $|\lambda|$ is not contained in the interval $[\varepsilon, 1/\varepsilon]$. Moreover, all periodic points are of saddle type.*

Proof. For $c \in \mathbb{C}$, let $\Delta_c = \{(x,y) \mid |x| \le 2|c|,\ |\frac{c}{2}| \le |y| \le 2|c|\}$. We will find $C > 0$ so that if $|a| < A$ and $|c| > C$, then the nonwandering set of $F_{a,c}$ is contained in Δ_c. That is,

$$\Omega(F_{a,c}) \subset \Delta_c, \tag{7}$$

where Ω indicates the nonwandering set. To do this, we use filtration ideas like those in [BS91]. Define

$$V^+ = V_c^+ = \{(x,y) \mid |x| \le |y|,\ |y| \ge |2c|\},$$

$$V^0 = V_c^0 = \{(x,y) \mid |x| \le |2c|,\ |y| \le \frac{|c|}{2}\},$$

$$V^- = V_c^- = \{(x,y) \mid |x| \ge |2c|,\ |y| \le |x|\}.$$

Then the closure of $\mathbb{C}^2 \setminus \Delta_c$ is exactly $V^+ \cup V^0 \cup V^-$.

Let π_j denote projection to the jth coordinate, $j = 1, 2$. To prove (7), we will prove that for some $C > 0$ and for $|c| > C$, the following relations hold:

$$F_{a,c}(V^+) \subset V^+ \text{ and } |\pi_2 F_{a,c}(x,y)| > |y| + 1 \text{ for } (x,y) \in V^+, \tag{8}$$

$$F_{a,c}(V^0) \subset V^+, \tag{9}$$

$$F_{a,c}^{-1}(V^-) \subset V^- \text{ and } |\pi_1 F_{a,c}^{-1}(x,y)| > |x| + 1 \text{ for } (x,y) \in V^-. \tag{10}$$

Hence points in $V^+ \cup V^0$ tend to infinity under forward iterates, while points in V^- tend to infinity under backwards iterates, so these three formulas together imply that there is no nonwandering point of $F_{a,c}$ outside Δ_c.

To formulate the assumptions on C, write $p_j(y) = y^{d_j} + q_j(y)$. Choose $B > 0$ so that if $|y| \geq 1$, then $B|y|^{d_j - 1} \geq |q_j(y)|$ for all j, and let $A_1 = \max_{j=1,\dots,k} |a_j|$. Let A be as in the statement of Proposition 5.1, and suppose that

$$C > 4(1 + AA_1), \tag{11}$$

$$C > 2(B + 2AA_1 + 1), \tag{12}$$

$$C \geq 8. \tag{13}$$

We will prove (8)-(10) for $F_{j,a,c}$ instead of $F_{a,c}$. From this, a simple induction on j implies the result for $F_{a,c}$. For simplicity of notation, we drop the subscript j on d and q and let

$$F : (x,y) \mapsto (y, y^d + q(y) + c^d - a a_j x).$$

Also, let $(x^+, y^+) = F(x,y)$ and $(x^-, y^-) = F^{-1}(x,y)$.

To prove (8) we need: $|y^+| - |y| \geq 1$ in V^+. Recall that in V^+ we have $|y| \geq 2|c|$ and $|x| \leq |y|$. Since $d \geq 2$, we have

$$\begin{aligned} |y^+| - |y| &\geq |y|^d - B|y|^{d-1} - |c|^d - AA_1|x| - |y| \\ &\geq |y|^{d-1}(|y| - (B + AA_1 + 1)) - |c|^d \\ &\geq \left(\frac{1}{2} - \frac{1}{2^d}\right)|y|^d \geq |c|^d > 1. \end{aligned}$$

The third inequality uses (12) plus $C < |c| \leq |y|/2$.

A similar string of inequalities shows that

$$|y^+| \geq \frac{1}{4}|y|^d \geq |y| = |x^+|.$$

This proves (8). To prove (9), note that for $(x, y) \in V^0$, we have $|x^+| = |y| \leq \frac{|c|}{2}$. So we have to check that $|y^+| \geq 2|c|$. Indeed,

$$\begin{aligned}|y^+| &\geq |c|^d - |y|^d - B|y|^{d-1} - AA_1|x| \\ &\geq |c|^d \left(1 - \frac{1}{2^d}\right) - B|c|^{d-1} - 2AA_1|c| \\ &\geq |c|^{d-1}\left(\frac{3}{4}|c| - B - 2AA_1\right) \\ &\geq \frac{|c|^d}{4} \geq 2|c|.\end{aligned}$$

The fourth inequality uses (12), and the last one uses (13) and $d \geq 2$. This proves (9).

To prove (10) note that

$$F^{-1}(x, y) = (x^-, y^-) = \left(\frac{x^d + q(x) + c^d - y}{aa_j}, x\right).$$

In V^- we have $|y| \leq |x|$ and $|x| \geq 2|c|$, hence

$$\begin{aligned}AA_1(|x^-| - |x|) &\geq |x|^d - B|x|^{d-1} - |c|^d - |x| - AA_1|x| \\ &\geq |x|^{d-1}(|x| - B - \frac{|x|}{2^d} - (1 + AA_1)) \\ &\geq \frac{1}{4}|x|^d \geq |c|^d \geq AA_1.\end{aligned}$$

Dividing by AA_1 gives $|x^-| \geq |x| + 1$. Also, this string of inequalities shows that $|x^-| \geq |x|^d/4AA_1$, and together with (11) and $d \geq 2$, this gives $|x^-| \geq |x| = |y^-|$. This proves (10), thus completing the proof of (7).

Finally, note that

$$D_{(x,y)}F_{j,a,c} = \begin{pmatrix} 0 & 1 \\ -aa_j & p_j'(y) \end{pmatrix}.$$

Choosing C sufficiently large depending on A, A_1, and B, each such matrix with $|y| > C/2$ and $|a| < A$ will preserve the vertical cone field $|x| < |y|$, and if $a \neq 0$, the inverse will preserve the horizontal cone field $|y| < |x|$. Moreover, as C tends to ∞, the rates of expansion and contraction tend to ∞ and 0, respectively. Hence for C sufficiently large, $|a| < A$ and $|c| > C$, the map $F_{a,c}$ will be hyperbolic on its nonwandering set, which is contained in Δ_c, and the eigenvalues of any periodic point will lie outside the interval $[\varepsilon, 1/\varepsilon]$, and any such periodic point will be saddle. □

Note that the family of maps $\{F_{1,c}\}$ can be used in place of $\{F_\mu\}$ in [Buz99] to give a complete proof of the first half of the Kupka-Smale theorem: that is, that there is a full measure, dense G_δ subset of $\mathcal{P}_d$ such that all periodic points are hyperbolic. We provide details of this proof with an indication of the full measure result in the next section.

We will need also to guarantee the hyperbolicity of the inverse of a map such as $F_{a,c}$ in the case that a tends to ∞. For this we use the following lemma.

Lemma 5.1. *Consider a family G_t of polynomial automorphisms of $\mathbb{C}^2$ depending on a real parameter $t \in [0,1]$, such that $G_t = G_{k,t} \circ \cdots \circ G_{1,t}$ with $G_{j,t}(x,y) = (y, y^{d_j} + q_{j,t}(y) - a_{j,t}x + c_{j,t}^{d_j})$. Moreover, $a_{j,t}$ and each coefficient of $q_{j,t}$ tends to zero as $t \to 1$. Suppose also that all the functions c_j are algebraic in t near $t = 1$, and at least one of them tends to infinity as $t \to 1$. Then G_t is saddle hyperbolic for all t sufficiently close to* 1.

Proof. Quotients of algebraic functions are algebraic as well. Hence, each quotient $c_j/c_k = c_{j,t}/c_{k,t}$ has a limit as $t \to 1$, either finite or infinite. We say that c_j is larger than c_k if this limit is larger than 1 in modulus. Hence, amidst the functions c_j there exists a largest one, that is a function whose quotient with any other c_k has a limit no smaller than 1 in modulus.

Also, note that saddle hyperbolicity is invariant under smooth conjugation. Hence by conjugating G_t with $G_{j,t}$ as in [FM89], we can cyclically permute the compositional factors of G_t. Thus we may assume the largest $c_{j,t}$ is $j = 1$ and that this one tends to ∞ as $t \to 1$.

We show that for t near 1 and $c(t) = c_{1,t}$, G_t satisfies

$$\Omega(G_t) \subset \Delta_{c(t)} \tag{14}$$

To do this, we need to prove counterparts of (8)-(10):

$$G_t(V^+_{c(t)}) \subset V^+_{c(t)}$$
$$G_t(V^0_{c(t)}) \subset V^+_{c(t)}$$
$$G_t^{-1}(V^-_{c(t)}) \subset V^-_{c(t)}$$

plus the corresponding bounds on growth of iterates in V^+ and V^-. Note that the bound involving $|y|$ in (8) immediately implies that if $c' > c$, then $F(V^+_{c'}) \subset V^+_{c'}$. In particular, the current assumptions on $q_{j,t}$ and $a_{j,t}$ imply that for t near 1, (8) with c replaced with $c(t)$ applies directly to $G_{1,t}$ and applies to each $G_{j,t}$ since $c_{1,t}$ is the largest of the $c_{j,t}$. Likewise, in (10), c may be replaced with $c' > c$, so again for t near 1, (10) applies to each $G_{j,t}$, $j = 1, \ldots, k$. By induction on j, we see that $G_t(V^+_{c(t)}) \subset V^+_{c(t)}$ and $G_t^{-1}(V^-_{c(t)}) \subset V^-_{c(t)}$ and that the bounds in (8) and (10) apply to G_t also.

Finally, (9) applies to $G_{1,t}$ directly to show that $G_{1,t}(V^0_{c(t)}) \subset V^0_{c(t)}$ for t near 1. Since $G_{j,t}(V^+_{c(t)}) \subset V^+_{c(t)}$ for each j, we have $G_t(V^0_{c(t)}) \subset V^0_{c(t)}$ for t near 1. This proves (14).

Calculating the Jacobian derivative $D_{(x,y)}G_t$ for $(x,y) \in \Delta_{c(t)}$ as in Proposition 5.1 shows that for t close to 1, G_t is saddle hyperbolic. □

5.2. **Independence of multipliers.** In this section we define an analytic set S that we use in this section and the next, then we prove Theorem 1.4.

Let $F_0 \in \mathcal{P}_d$ and let p_0, q_0, be potentially heteroclinic periodic points for F_0 with corresponding multipliers λ_0 and μ_0. Let m and n be the periods of p_0 and q_0, respectively, and define a two-parameter family $F_{a,c}$ through F as at the beginning of Section 5.1. Define an algebraic set in $\mathbb{C}^8$ by

$$\{(a, c, p, q, \lambda, \mu) \in \mathbb{C}^8 \mid F^m_{a,c}(p) = p,\ F^n_{a,c}(q) = q,\\ \det(D_p F^m_{a,c} - \mu I) = 0,\ \det(D_q F^n_{a,c} - \lambda I) = 0\}.$$

Since these are all polynomial equations, this extends to a projective variety in $\mathbb{P}_a \times \mathbb{P}_c \times \mathbb{P}^2_p \times \mathbb{P}^2_q \times \mathbb{P}_\lambda \times \mathbb{P}_\mu$. Let $a_0 = 1$ and $c_0 = 0$. Then there is an irreducible component of this variety through the point $\mathfrak{a}_0 = (a_0, c_0, p_0, q_0, \lambda_0, \mu_0)$, and this component projects to an irreducible projective variety, $S = S(\mathfrak{a}_0)$, in $\mathbb{P}_a \times \mathbb{P}_c \times \mathbb{P}_\lambda \times \mathbb{P}_\mu$. Since the periodic points and multipliers are determined by a and c, both the original component and S are 2-dimensional.

The projection, $\pi_{a,c}$, of S to $\mathbb{P}_a \times \mathbb{P}_c$ is surjective by the implicit function theorem since the periodic points p_0 and q_0 can be followed for any perturbation of a_0 and c_0. However, the projection, $\pi_{\lambda,\mu}$, of S to $\mathbb{P}_\lambda \times \mathbb{P}_\mu$ may conceivably not be onto. Hence we need the following lemma, which almost immediately implies Theorem 1.4.

Lemma 5.2. *The projection $\pi_{\lambda,\mu} : S \to \mathbb{P}_\lambda \times \mathbb{P}_\mu$ is surjective, hence $\pi_{\lambda,\mu}$ is a local biholomorphism outside a closed, nowhere dense subset of S.*

Proof. The image of S under $\pi_{\lambda,\mu}$ is a projective variety, and if $\pi_{\lambda,\mu}$ is not onto, then this image has dimension either 0 or 1. If the dimension is 0, then the multipliers are constant independent of a and c, which is not possible by Proposition 5.1.

Suppose then that the image has dimension 1. Then for any $(\lambda_1, \mu_1) \in \pi_{\lambda,\mu}(S)$, the preimage $\pi^{-1}_{\lambda,\mu}(\lambda_1, \mu_1)$ is a variety of dimension at least 1 (otherwise $\pi_{\lambda,\mu}$ is locally proper onto a 1-dimensional image, and hence S has dimension 1, a contradiction). If the dimension of the preimage is 2 for some point, then the preimage is all of S, in which case the image of S has dimension 0, which is impossible, as already noted. Hence each such preimage has dimension 1.

Recall that λ_0 and μ_0 are the original multipliers for p_0 and q_0, hence both λ_0 and μ_0 are nonzero. Let $S_0 = \pi^{-1}_{\lambda,\mu}(\lambda_0, \mu_0)$. Then S_0 is 1-dimensional as just noted. Also, S_0 projects to give varieties in $\mathbb{P}_a$ and in $\mathbb{P}_c$. Thus on S_0, each of a and c is either constant or attains all values in $\mathbb{P}$. Moreover, a must be nonconstant on S_0 by Proposition 5.1: otherwise, a is constant and c is nonconstant, hence unbounded, in which case the multipliers cannot be constant. Thus the projection of S_0 to $\mathbb{P}_a$ is onto.

Hence we can find a path in S_0 along which $a \to 0$ (and $\lambda = \lambda_0$, $\mu = \mu_0$ are constant by definition of S_0). Since a is a meromorphic function on S_0, we may bypass the poles and hence assume that a is bounded along this path. Consider c along this path. One possibility is that c is unbounded. In this case, Proposition 5.1 implies that λ_0 and μ_0 cannot be constant along this path, a contradiction.

Hence c must be bounded along this path. In this case $a = 0$ corresponds to a degenerate map collapsing all of $\mathbb{C}^2$ to an invariant curve as described just before Proposition 5.1. Also, any periodic point for such a map has a multiplier equal to 0 for the collapsing direction. Since $\lambda_0 \neq 0$ and $\mu_0 \neq 0$, we see that one of the periodic points we are tracking has multipliers 0 and λ_0, while the other has multipliers 0 and μ_0. Moreover, the fact that $|\lambda_0| < 1$ means that the corresponding periodic point is attracting for the one-dimensional map given by $a = 0$.

Now, starting from this map in S_0 with $a = 0$, we continue along a path in S. Since the projection of S to $\mathbb{P}_a \times \mathbb{P}_c$ is onto, we can keep $a = 0$ and find a path in S by adjusting c to get $\lambda = 0$ while keeping $|\lambda| < 1$ throughout this path but placing no constraint on μ. Note that for the maps $F_{a,c}$ along this path, the condition $|\lambda| < 1$ implies that each map on the path has at least one attracting periodic orbit with multipliers $(\lambda, 0)$. Therefore, by Proposition 5.1, c is bounded along this path. Hence, the multiplier μ has a finite limit, say μ_1, along this path. Likewise, c has a finite limit, say c_1, and the end of the path in S has coordinates $(0, c_1, 0, \mu_1)$.

Now let S_1 be the irreducible component of $\pi^{-1}_{\lambda,\mu}(0, \mu_1)$ through the point $(0, c_1, 0, \mu_1)$. Since we are still assuming that the image of S is 1-dimensional, the second paragraph of this proof implies that S_1 is 1-dimensional. Moreover, on S_1, a is identically zero since each map in S_1 has a periodic orbit with multiplier $\lambda = 0$ and since at least one map in S_1 has $a = 0$ (this is needed to exclude the possibility $a = \infty$).

Now, if c is constant on S_1, then $a = 0$, $c = c_1$, $\lambda = 0$ and $\mu = \mu_1$ are all fixed on S_1 and hence S_1 has dimension 0, which is impossible. Thus c must be nonconstant on S_1, hence unbounded, while $a = 0$. But then by Proposition 5.1, μ_1 cannot stay constant throughout S_1, which contradicts the definition of S_1. Thus S_1 must be 0-dimensional, but as noted, this is a contradiction to the assumption that the image of S is 1-dimensional.

Thus, $\pi_{\lambda,\mu}$ must be onto. The fact that S has dimension 2 implies that $\pi_{\lambda,\mu}$ is a local biholomorphism outside of $\pi^{-1}_{\lambda,\mu}(\sigma)$, where σ is an analytic subset of $\mathbb{P}_\lambda \times \mathbb{P}_\mu$ and $\pi^{-1}_{\lambda,\mu}(\sigma)$ is nowhere dense [Chi89, sec. 3.7]. □

Now we are ready to prove Theorem 1.4. Let $\mathcal{P}_d(F)$ denote the component of $\mathcal{P}_d$ containing the map $F \in \mathcal{P}_d$.

Proof of Theorem 1.4. By the previous lemma, the projection $\pi_{\lambda,\mu} : S \to \mathbb{P}_\lambda \times \mathbb{P}_\mu$ is onto and is a local biholomorphism outside a closed, nowhere dense subset of S. In particular, π has generic rank 2 in S. As noted above, S is parametrized by a and c, so the map from S to $\mathcal{P}_d(F_0)$ given by $(a, c, \lambda, \mu) \to F_{a,c}$ defines a germ of an analytic subset of $\mathcal{P}_d(F_0)$ through F_0. Since the map from a small neighborhood of F_0 in this analytic subset to the (λ, μ) plane has generic rank 2, we see that the map from a small neighborhood of F_0 in $\mathcal{P}_d$ to the (λ, μ) plane also has generic rank 2. □

5.3. **Path to a saddle hyperbolic map.** In this section we prove Theorem 2.3, which says roughly that given any initial point in M_{pot}, there is a curve in M_{pot} from this initial point to a point corresponding to a saddle hyperbolic map.

Proof of Theorem 2.3. Let F_0, p_0, q_0, λ_0, μ_0 and $\mathfrak{a}_0$ be as in the statement of Theorem 2.3, and let S be as in the previous section with this initial. By Lemma 5.2 the projection $\pi_{\lambda,\mu} : S \to \mathbb{P}_\lambda \times \mathbb{P}_\mu$ is onto. We construct the desired path in M_{pot} by first moving λ to 0 while leaving μ fixed, then if needed moving μ to ∞ while leaving λ fixed. In what follows, we will construct a path that may use degenerate maps not in $\mathcal{P}_d$, but such a path can be approximated with a curve in $\mathcal{P}_d$ satisfying the conditions of the theorem.

Let $\pi_\mu : S \to \mathbb{P}_\mu$ be projection, which must also be onto. Lemma 5.2 implies that π_μ is a nonconstant meromorphic function from S onto $\mathbb{P}_\mu$. Since S is an irreducible complex surface, we see that for fixed μ_1, the equation $\pi_\mu = \mu_1$ defines a pure 1-dimensional subvariety of S. Hence the set $S_{\mu_1} = \pi_\mu^{-1}(\mu_1)$ is a pure 1-dimensional variety in S. Likewise, the set $\pi_\lambda^{-1}(\lambda_1)$ is pure 1-dimensional for each fixed λ_1.

In particular, S_{μ_0} is 1-dimensional. In fact, without loss of generality, we may assume that $\pi_\lambda : S_{\mu_0} \to \mathbb{P}_\lambda$ is onto. If not, then the image $\pi_\lambda(S_{\mu_0})$ consists of a finite number of points. Hence, S_{μ_0} is an exceptional divisor of the projection $\pi_{\lambda,\mu} : S \to \mathbb{P}_\lambda \times \mathbb{P}_\mu$. But Lemma 5.2 implies that this projection has at most a finite number of exceptional divisors. Hence a generic choice of μ_0 will avoid the μ coordinate of the projection of any of these divisors under $\pi_{\lambda,\mu}$. Since the projection of S to the (λ, μ) plane has generic rank two, we may make an initial perturbation within S so that this condition is satisfied. For simplicity of notation, we assume μ_0 itself satisfies this condition.

Hence, we can take $\lambda \to 0$ within S_{μ_0}. Since a is an algebraic function of λ, we may choose a path for λ so that a is neither 0 nor ∞ before $\lambda = 0$. Also, since the initial periodic points are (un)stably nonresonant and the multipliers depend analytically on λ, we may choose the path so that the corresponding periodic points are (un)stably nonresonant throughout this path. Thus the path may be lifted to $M_{\text{pot}}(\mathfrak{a}_0)$. In fact, a generic line segment will satisfy these conditions, so we may assume that the path consists of two line segments, hence is piecewise algebraic. Note that if c reaches ∞ along this path before its end, then a nearby map with a finite and c sufficiently large is already saddle hyperbolic by Proposition 5.1, so we are done. Thus we may assume that there is a path to $\lambda = 0$ so that prior to reaching $\lambda = 0$, c is never ∞ and a is never 0 or ∞. For the reference to Lemma 5.1 in case 3, we parametrize this path by $t \in [0, 1]$ so that $\lambda = 0$ at $t = 1$, and we note that both a and c are algebraic functions of λ near $t = 1$.

There are 3 possibilities for a. Either it tends to 0, or to some finite nonzero value, or to ∞ as $\lambda \to 0$. We show in each case that either the path already contains a saddle hyperbolic map or that it can be continued to contain such a map.

Case 1: $a \to 0$. If $|c|$ is sufficiently large near the end of the path, then by Proposition 5.1, the corresponding map is already saddle hyperbolic, so we are done.

Otherwise, c tends to a finite limiting value, say c_1, and the limiting map is F_{0,c_1}, which collapses all of $\mathbb{C}^2$ onto a complex curve. Moreover, this map has one periodic point with multiplier μ_0 and another periodic point (possibly the same one) with multiplier $\lambda = 0$.

We claim that $\lambda = 0$ corresponds to the collapsing direction for F_{0,c_1}, as described before Proposition 5.1. If not, then the periodic point associated with λ is superattracting for F_{0,c_1}. In this case, any perturbation to $a \neq 0$ will create a diffeomorphism with no multiplier equal to 0. On the other hand, if $a = 0$ is fixed, then λ is a nonconstant algebraic function of c by Proposition 5.1, hence any perturbation of c with $a = 0$ will also produce a map with no multiplier equal to 0. Thus, a superattracting periodic point can exist only for isolated values in $\mathbb{P}_a \times \mathbb{P}_c$. Hence $\pi_\lambda^{-1}(0)$ is 0-dimensional in this case. However, as in the second paragraph of this proof, for each fixed λ_1 the set $\pi_\lambda^{-1}(\lambda_1)$ is pure 1-dimensional, a contradiction. Hence $\lambda = 0$ corresponds to the collapsing direction and $\pi_\lambda^{-1}(0)$ is pure 1-dimensional.

Let S_1 be the irreducible component of $\pi_\lambda^{-1}(0)$ through the point $a = 0$, $c = c_1$, $\lambda = 0$, $\mu = \mu_0$. As noted at the beginning of this proof, $\pi_\lambda^{-1}(0)$ is pure 1-dimensional, hence S_1 is 1-dimensional. Also, as at the end of the proof of Lemma 5.2, a is identically 0 throughout S_1 since each map in S_1 has a periodic point with multiplier 0 and since at least one map in S_1 has $a = 0$. As in that proof, c must be nonconstant on S_1 since otherwise S_1 would be 0-dimensional. Thus there is a path in S_1 along which $a = 0$ and c is unbounded. By Proposition 5.1, this path in S_1 contains a saddle hyperbolic map.

Case 2: $a \to a_1 \neq 0$. If c is bounded along the path, then the set of maps along the path lies in a compact subset of $\mathcal{P}_d$, so the multipliers also lie in a compact set not containing 0. This is impossible since $\lambda \to 0$. Thus $c \to \infty$, so again there is a path to a saddle hyperbolic map by Proposition 5.1 .

Case 3: $a \to \infty$. In this case, we consider the inverse of $F_{a,c}$ and apply Lemma 5.1. After conjugating each factor in the inverse with the involution $(z, w) \mapsto (w, z)$, we see that $F_{a,c}^{-1}$ is conjugate to $\hat{G}_{a,c} = \hat{G}_{k,a,c} \circ \cdots \circ \hat{G}_{1,a,c}$, where

$$\hat{G}_{j,a,c}(z, w) = \left(w, \frac{p_j(w) - z + c^{d_j}}{aa_j} \right).$$

As in [FM89], we can conjugate $\hat{G}_{a,c}$ to the normal form $G_{a,c}$ given in Lemma 5.1 by replacing each $\hat{G}_{j,a,c}$ by $G_{j,a,c} = H_{j+1,a}^{-1} \circ \hat{G}_{j,a,c} \circ H_{j,a}$. Here

$H_{j,a}(z,w) = (\alpha_{j-1}z, \alpha_j w)$, and the indices on α are taken modulo k so that $\alpha_k = \alpha_0$ and $\alpha_{k+1} = \alpha_1$ and hence $H_{k+1,a} = H_{1,a}$. Moreover, $\alpha_j^{d_j} = aa_j\alpha_{j+1}$, for $j = 1, \dots, k$. This gives

$$G_{j,a,c}(z,w) = \left(w, \frac{p_j(\alpha_j w) - \alpha_{j-1}z + c^{d_j}}{\alpha_j^{d_j}} \right),$$

and $G_{a,c} = G_{k,a,c} \circ \cdots \circ G_{1,a,c}$. Again as in [FM89], the conditions on α_j imply that $\alpha_j = \kappa_j a^{r_j}$ for some nonzero constants κ_j, independent of a, and rational numbers r_j with $0 < r_j \leq 1$, also independent of a (assuming the correct choice of roots).

In particular, as a tends to ∞, $|\alpha_j|$ tends to ∞ for each j. The polynomial in $G_{j,a,c}$ is $p_j(\alpha_j w)/\alpha_j^{d_j}$, which is monic. Also, as a tends to ∞, the remaining terms tend to 0. Moreover, the coefficient of z in $G_{j,a,c}$ is $-\alpha_{j-1}/aa_j\alpha_{j+1}$. Since $r_{j-1} - r_{j+1} - 1 < 0$, this coefficient tends to 0 as a tends to ∞. Moreover, at least one of the constant terms $c^{d_j}/aa_j\alpha_{j+1}$ must tend to ∞ since otherwise all of the coefficients in $G_{a,c}$ are bounded, hence there can be no multiplier tending to ∞, but $1/\lambda$ is a multiplier tending to ∞.

To show that $G_{a,c} = G_{a_t,c_t,t}$ satisfies all the conditions of Lemma 5.1 along the path to $\lambda = 0$ in S_{μ_0}, it remains to check only that the constant terms $c_{j,t}$ in the formula for $G_{j,t}$ is algebraic in t near $t = 1$. Indeed, $c_{j,t} = (c_t/\alpha_{j,t})^{d_j} = (c_t/\kappa_j a_t^{r_j})^{d_j}$, with a_t and c_t algebraic in t near $t = 1$. Hence by Lemma 5.1, for t near 1, $G_t = G_{a_t,c_t}$ is saddle hyperbolic. Therefore its inverse, F_{a_t,c_t} is also saddle hyperbolic.

Thus, in each case there is a path to a saddle hyperbolic map keeping the designated periodic points potentially heteroclinic. □

6. Proofs of main theorems

6.1. Prevalence of maps with all periodic points hyperbolic. In this section we prove part I of the Kupka-Smale theorem on the prevalence of maps having all periodic points hyperbolic.

As before, let $\mathcal{P}_d(F)$ denote the component of $\mathcal{P}_d$ containing the map $F \in \mathcal{P}_d$.

Proof of Theorem 1.1, part I. We make a slight variation on Definition 2.4 in that we now allow multipliers equal to 1 and we now follow both multipliers: Let $F_0 \in \mathcal{P}_d$ and consider the analytic subset of $\mathcal{P}_d(F_0) \times \mathbb{C}^2 \times \mathbb{C}$ consisting of all (F, p, λ) so that $F^m(p) = p$ and λ is a multiplier for p under F. Then given a periodic point p_0 for F_0 of period m and having multipliers λ_0 and ν_0, let $\overline{\mathrm{EV}}^2 = \overline{\mathrm{EV}}^2(F_0, p_0)$ be the union of the irreducible components through (F_0, p_0, λ_0) and (F_0, p_0, ν_0) of this analytic set.

For fixed m, [FM89] implies that the number of periodic points of period m is at most d^m. Also, for a compact subset, $\mathcal{Q}$ of $\mathcal{P}_d(F_0)$, [BS91] implies that there is a compact subset, B, of $\mathbb{C}^2$ so that all periodic points for any map in $\mathcal{Q}$ are contained in B. Since the multipliers for such periodic points are bounded on such a compact set, we see that the natural projection $\pi : \overline{\mathrm{EV}}^2(F_0, p_0) \to \mathcal{P}_d(F_0)$ is proper. Since $\overline{\mathrm{EV}}^2$ is an analytic subset of $\mathcal{P}_d(F_0) \times \mathbb{C}^3$ given by 3 independent equations, we see that $\dim(\overline{\mathrm{EV}}^2) = \dim(\mathcal{P}_d(F_0))$. By [Chi89, sec 3.7], there is an analytic subset, A, of $\mathcal{P}_d(F_0)$ of codimension at least 1 so that $\pi^{-1}(A)$ is nowhere dense and π is a local biholomorphism outside $\pi^{-1}(A)$.

By Theorem 1.4, λ cannot be constant on $\overline{\mathrm{EV}}^2$, so the condition $|\lambda| = 1$ defines a real-analytic subset of $\overline{\mathrm{EV}}^2$ of real codimension at least 1, as does $|\nu| = 1$. Taking the union of these analytic subsets and projecting to $\mathcal{P}_d(F_0)$ gives a nowhere dense F_σ subset of measure zero in $\mathcal{P}_d(F_0)$. Hence the complement is a dense G_δ set of full measure, and on this complement the corresponding periodic point is hyperbolic. Let $\mathcal{H}(F_0, p_0)$ denote this dense G_δ subset.

Given a point in $\overline{\mathrm{EV}}^2$ at which π is a local biholomorphism, we can perturb the corresponding map F to have all coefficients rational, then take π^{-1} of the perturbed map to get a point in $\overline{\mathrm{EV}}^2$. Thus any $\overline{\mathrm{EV}}^2$ as above can be obtained from a map in $\mathcal{P}_d(F_0)$ having all coefficients rational. Note that the set of such maps is countable, and that for a fixed map in $\mathcal{P}_d$ the set of all periodic points is countable. Hence we can take the intersection of $\mathcal{H}(F, p)$ over all such F and p to get a dense, full measure, G_δ subset of $\mathcal{P}_d(F_0)$ so that all periodic points are hyperbolic. □

Remark. Since the resonance conditions also define a countable set of analytic subsets, a similar proof shows that there is a dense, full measure, G_δ subset of $\mathcal{P}_d(F_0)$ so that all periodic points are hyperbolic and nonresonant.

6.2. **Tangencies have codimension 1.** In this section we prove Theorem 2.4 (M_{tang} has codimension 1 in M_{pers}) by using Theorem 2.3 to get a saddle hyperbolic map in M_{pot} and using Theorem 2.2 to get a nearby saddle hyperbolic map that lifts to M_{pers}.

Proof of Theorem 2.4. Let $F_0 \in \mathcal{P}_d$, $\mathfrak{a}_0 = (F_0, p_0, q_0, \lambda_0, \mu_0)$, and $\tau_0 = (\tau_0^c, \tau_0^e)$ be as in the statement of Theorem 2.4. As usual, we assume $|\lambda_0| < 1$ and $|\mu_0| > 1$. Also, let $M_{\mathrm{pot}} = M_{\mathrm{pot}}(\mathfrak{a}_0)$ and $M_{\mathrm{pers}} = M_{\mathrm{pers}}(\mathfrak{a}_0, \tau_0)$.

Let ψ^e, ψ^c be the uniformizing maps as in Theorem 2.1. From Definition 2.11 we know that M_{tang} is the set of $(\mathfrak{a}, \tau^c, \tau^e) \in M_{\mathrm{pers}}$ for which $T = \psi^e_{\mathfrak{a}}(\tau^e) = \psi^c_a(\tau^c)$ is a point of tangency between the curves $W^c_\lambda(p)$ and $W^e_\mu(q)$. The tangent vectors to these curves at T are obtained by taking the derivative of $\psi^{e/c}_{\mathfrak{a}}$ with respect to $\tau^{e/c}$, and these curves are tangent precisely when the determinant of the 2×2 matrix formed from these vectors is 0:

$$\det\left(\frac{\partial \psi^c_{\mathfrak{a}}}{\partial \tau^c}(\tau_e),\ \frac{\partial \psi^e}{\partial \tau^e}(\tau^e)\right) = 0.$$

Thus M_{tang} is an analytic subset of M_{pers} defined by a single equation. Since M_{pers} is irreducible, we see that either M_{tang} is a codimension 1 subset of M_{pers} or else it equals M_{pers}. We will prove that the latter case is impossible.

To do that, note that by Theorem 2.3, there exists $\mathfrak{a} \in M_{\text{pot}}$ so that the projection of $\mathfrak{a}$ to $\mathcal{P}_d$ is saddle hyperbolic. Hence there is a neighborhood U of $\mathfrak{a}$ in M_{pot} so that each element of U projects to a saddle hyperbolic map in $\mathcal{P}_d$.

By Theorem 2.2, $\pi(M_{\text{pers}})$ is open and dense in M_{pot}, hence there is a point $\mathfrak{b} \in M_{\text{pers}}$ so that $\pi(\mathfrak{b}) \in U$. Since a saddle hyperbolic map has no tangencies between $W^c(p)$ and $W^e(q)$ for any periodic points p and q, we see that $\mathfrak{b}$ cannot be in the set M_{tang}. Hence M_{tang} is not equal to M_{pers}, hence must have codimension 1. □

6.3. **Kupka-Smale property for complex automorphisms.** Here we complete the proof of Theorem 1.1. We show that the maps from $\mathcal{P}_d$ that *do not* have the Kupka-Smale property form a countable union of nowhere dense closed sets having measure zero.

Proof of Theorem 1.1, part II. First fix a map $F_0 \in \mathcal{P}_d$ and consider the component $\mathcal{P}_d(F_0)$. We enumerate the sets M_{pers} as follows. First enumerate the points $\mathfrak{a}_n = (F_n, p_n, q_n, \lambda_n, \mu_n)$ and $T_n \in \mathbb{C}^2$, $n \geq 1$ satisfying the conditions: $F_n \in \mathcal{P}_d(F_0)$ is a polynomial map with rational coefficients, p_n and q_n are saddle periodic points for F_n, λ_n with $|\lambda_n| < 1$ is a multiplier for p_n, μ_n with $|\mu_n| > 1$ is a multiplier for q_n, and T_n is a heteroclinic point for p_n and q_n corresponding to λ_n and μ_n.

For a given map, the set of periodic points is countable by [FM89], and the set of heteroclinic points is countable by Proposition 4.3. Hence the set $\mathcal{A}$ of all such pairs $(\mathfrak{a}_n, T_n)$ is countable. Moreover, since p_n and q_n are saddles, their multipliers are automatically nonresonant, so we can apply Theorem 2.1 to get global parametrizations ψ^c for $W^c(p_n)$ and ψ^e for $W^e(q_n)$. Let $\tau_n = \psi^{-1}(T_n)$ as in Definition 2.7.

Let $\mathfrak{b}_n = (\mathfrak{a}_n, \tau_n)$ and let $M^n_{\text{pers}} = M_{\text{pers}}(\mathfrak{a}_n, \tau_n)$, which is a subset of $\mathcal{P}_d(F_0) \times \mathbb{C}^8$. Then any intersection between stable and unstable manifolds of saddle points of a map in $\mathcal{P}_d(F_0)$ is identified in one of these sets M_{pers}. Indeed, saddle periodic points persist under small perturbations of the map, as do intersections of stable and unstable manifolds. Thus we can perturb a given map to a nearby map with rational coefficients and still follow the corresponding intersection (possibly obtaining more than one intersection point for the new map).

Therefore for any point T of heteroclinic tangency for a map $F \in \mathcal{P}_d(F_0)$, there exists n so that M^n_{pers} contains a point $(\mathfrak{a}, \tau)$ that encodes this tangency. Let M^n_{tang} be the set of points in M^n_{pers} for which the corresponding manifolds are tangent, as in Definition 2.11. By Theorem 2.4, M^n_{tang} is either empty or a codimension 1 analytic subset of M^n_{pers}.

Let $\mathcal{K}_m$ be a sequence of compact domains that exhaust $\mathcal{P}_d(F_0)$, and let B_m be a ball of radius m and center 0 in $\mathbb{C}^8$. Let $\mathcal{Q}_m = \mathcal{K}_m \times \overline{B}_m$. Denote by $T_{n,m}$ the intersection

$$T_{n,m} = M^n_{\text{tang}} \cap \mathcal{Q}_m.$$

The set $T_{n,m}$ is compact and is the restriction to $\mathcal{Q}_m$ of an analytic set of dimension at most $\dim(\mathcal{P}_d(F_0))-1$. Hence, the projection of $T_{n,m}$ to $\mathcal{P}_d(F_0)$ is a nowhere dense closed subset of $\mathcal{K}_m \subset \mathcal{P}_d(F_0)$ having Lebesgue measure zero. Call this projection $R_{m,n}$. The set of all maps $F \in \mathcal{P}_d(F_0)$ that have heteroclinic tangencies is the union of all the $R_{m,n}$. The complement of this union is a residual closed set of full Lebesgue measure.

Taking the intersection of this residual set with the residual set of full Lebesgue measure of part I of the proof, we obtain Theorem 1.1. □

6.4. **Kupka-Smale property for real automorphisms.** Theorem 2.6 is an immediate corollary of Theorem 2.4. Indeed, let $M^{\mathbb{R}}_{\text{tang}}$ and $M^{\mathbb{R}}_{\text{pers}}$ be real analogs of M_{tang} and M_{pers}. Then the complexification of the first pair of sets gives the second one. Hence, $M^{\mathbb{R}}_{\text{tang}}$ is a real analytic subset of $M^{\mathbb{R}}_{\text{pers}}$ of codimension no less than 1, since otherwise M_{tang} would coincide with M_{pers}, a contradiction. A similar argument implies the density of real maps having all periodic points hyperbolic. Finally, Theorem 1.2 is deduced from Theorem 2.6 using the same enumeration technique as in the proof of Theorem 1.1.

Appendix A. Stoilow's theorem

As mentioned in Section 1.3 the solutions of equation (1) have algebraic behavior for $m = n = 1$. We make this precise in the following theorem, which is a slight reformulation of a result of Stoilow [Sto52].

Theorem A.1. *Let S be a nonempty irreducible analytic subset of $\mathbb{C}^2$ determined by an entire, nonconstant function $F : \mathbb{C}^2 \to \mathbb{C}$. That is, $S = \{(z,\alpha) : F(z,\alpha) = 0\}$. Let $\pi : S \to \mathbb{C}$ given by $\pi(z,\alpha) = \alpha$ be the natural projection, and suppose that $\pi(S)$ is not a single point. Then π has the ε-lift property.*

Proof. By Theorem 1.6, it suffices to show that S is tame on disks over $\mathbb{C}$. So, let $\phi : D \to S$ be holomorphic so that $\pi\phi$ is contained in the image of an injective holomorphic image of a closed disk. In this case, the base space is simply $\mathbb{C}$, so the injective image of a closed disk is the closure of a bounded domain. In particular, $\pi\phi$ has nontangential limits at almost every boundary point of ∂D.

Let $\pi'(z,\alpha) = z$. By Plessner's theorem, almost every point of ∂D is either Fatou or Plessner for $\pi'\phi$. Suppose $\zeta \in \partial D$ is a Plessner point of $\pi'\phi$ and that $\pi\phi$ has nontangential limit α_0 at ζ. Then in any Stolz angle ending at ζ, the cluster set of $\pi'\phi$ is all of $\mathbb{C}$, while $\pi\phi(w)$ tends to α_0 as w tends to ζ within this angle. Since the image of ϕ is contained in S, we see that for any $\varepsilon > 0$, if we intersect S with $\mathbb{C} \times D(\alpha_0, \varepsilon)$, then take the closure, this closure contains the set $\mathbb{C} \times \{\alpha_0\}$.

In particular, $\overline{S}$ contains the line $\mathbb{C} \times \{\alpha_0\}$, and hence S contains this line since S is closed. But since S is irreducible, it must equal this line, which is a contradiction to the assumption that $\pi(S)$ is not a single point.

Thus at each point $\zeta \in \partial D$ so that $\pi\phi$ has nontangential limit at ζ, we must have that ζ is a Fatou point for $\pi'\phi$. Hence for a.e. point $\zeta \in \partial D$, $\pi'\phi$ has radial limit at ζ. Thus S is tame on disks over $\mathbb{C}$, hence π has the ε-lift property, as claimed. □

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