

Kupka-Smale theorem for automorphisms of \mathbb{C}^n

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1 Introduction

In the study of dynamical systems, a frequently useful tool has been the Kupka-Smale theorem. Briefly, this theorem says that for a generic diffeomorphism, all periodic points are hyperbolic and the stable and unstable manifolds of any two periodic points are transverse. The goal of this paper is to prove this theorem for the space $\text{Aut}(\mathbb{C}^n)$ of biholomorphic automorphisms of \mathbb{C}^n , where this space is endowed with the topology of local uniform convergence of the map and its inverse.

To state the theorem more precisely, we first recall a few ideas. Writing F^m for m -fold composition of the map F , we say that a periodic point p of minimal period m of a differentiable map F is *hyperbolic* if none of the eigenvalues of $(DF^m)(p)$ has modulus 1. Such a point is called a *saddle point* if $(DF^m)(p)$ has at least one eigenvalue bigger than 1 in modulus, and at least one smaller than 1 in modulus. Moreover, p is *linearizable* if there is a neighborhood of p such that F^m is conjugate to a linear map in this neighborhood.

Given such a saddle periodic point, there is an associated immersed manifold $W^s(p)$ through p which is invariant under F , which is tangent to the subspace $E^s(p)$ spanned by

all eigenvectors whose associated eigenvalue has norm less than 1, and which is defined by

$$W^s(p) := \{q : \lim_{k \rightarrow \infty} (F^m)^k(q) = p\}.$$

This is called the *stable manifold* at p . There is also a local version of this manifold which will be defined later, and corresponding versions of the unstable manifold when F is invertible or locally invertible. We write $W^s(p, F)$ and $W^u(p, F)$ when we need to specify the map involved.

Finally, a set is *residual* in a space X if it contains a dense \mathcal{G}_δ subset of X . See [Sh2] for more background on these and related ideas.

In the case of diffeomorphisms, the Kupka-Smale theorem states that there is a residual set of diffeomorphisms such that all of the periodic points are hyperbolic, and such that the stable and unstable manifolds of any two saddle periodic points are transverse. This theorem is valid in many different settings, including C^k flows, C^k diffeomorphisms, C^k endomorphisms, and real-analytic diffeomorphisms [K, S, Sh1, BT, L]. We prove the following version here.

THEOREM 1.1 *Let KS be the set of $F \in \text{Aut}(\mathbb{C}^n)$ such that every periodic point of F is hyperbolic and linearizable, and such that the global stable and unstable manifolds of any two periodic points of F are transverse. Then KS is a residual subset of $\text{Aut}(\mathbb{C}^n)$.*

The techniques of the real-analytic case as used in [L] and [BT] cannot be used for the space $\text{Aut}(\mathbb{C}^n)$ because in the first case the methods only work on compact manifolds, while in the second case the technique is to approximate a given C^k diffeomorphism by a real-analytic one, which is not possible in general for holomorphic automorphisms.

The paper is organized as follows. After constructing some automorphisms with certain prescribed fixed points and derivatives, we then use these to perturb a given automorphism so that all periodic points are hyperbolic. This is done by first working locally around a nonhyperbolic periodic point and using some theory of analytic sets to show that a positive dimensional variety of periodic points can be perturbed away, then using simple covering arguments to extend this result to any given compact ball. Using this, we can make the set of periodic points of a given fixed period discrete, and it is then simple to perturb them to become hyperbolic and even linearizable.

Following this, we perturb away any nontransverse intersections between the stable and unstable manifolds of two periodic points. This procedure is similar in spirit to that above, but here we use shears to perturb the stable and unstable manifolds.

The key idea is to use shears which are not globally defined, but which have easily understood properties and which can be approximated by global automorphisms. More precisely, if q_0 is a point of nontransverse intersection between stable and unstable manifolds, then we can construct a holomorphic map Φ defined only in a neighborhood of the orbit of q_0 which is the identity except in a neighborhood of q_0 , which is a translation near q_0 , and which can be approximated by global automorphisms. For almost every choice of this translation, Sard's theorem shows that the composition of Φ with the original map will have stable and

unstable manifolds which are transverse near q_0 . We can think of this choice of translation as indexing a family of shears.

Applying some theory of analytic sets, we can obtain the transversality result for an indexed family of globally defined shears which approximate this family of nonglobally defined shears. Again we finish by working on compact sets and using covering arguments to obtain global results.

2 Useful automorphisms

In this section we construct some automorphisms which will be used later. Here I denotes the identity automorphism or the identity matrix as appropriate, π_j denotes projection onto the j th coordinate, e_j is the j th standard basis element, $\#E$ denotes the cardinality of E , and 0 is the origin.

The first lemma allows us to arrange a sequence of points so that any two points differ in each coordinate. This is useful for applying shears to such a sequence.

LEMMA 2.1 *Let $p_1, \dots, p_N \in \mathbb{C}^n$. Then there exists $\Gamma_j \in \text{Aut}(\mathbb{C}^n)$ such that $\Gamma_j \rightarrow I$ as $j \rightarrow \infty$ and such that if $1 \leq m \leq n$, $1 \leq k, l \leq N$ with $k \neq l$, and $j \geq 1$, then $\pi_m \Gamma_j(p_k) \neq \pi_m \Gamma_j(p_l)$. Moreover, each Γ_j is an affine linear automorphism, so that $(D\Gamma_j)$ is constant, and $(D\Gamma_j)$ has n distinct eigenvalues and hence is diagonalizable.*

Proof: Using compositions of shears of the form $\phi(z) = z + \epsilon(z_2 - \pi_k p_l)e_m$, we can inductively reduce the number of nondistinct coordinates. This gives affine linear automorphisms Φ_j approaching the identity with $\pi_m \Phi_j(p_k) \neq \pi_m \Phi_j(p_l)$ for all $m, k \neq l$, and $j \geq 1$.

To get distinct eigenvalues, choose invertible linear matrices A_j so that $A_j(D\Phi_j)A_j^{-1}$ is in Jordan form. Then we can choose a diagonal B_j near I so that $B_j A_j(D\Phi_j)A_j^{-1}$ has distinct eigenvalues. For B_j near enough to I , the sequence $\Gamma_j = A_j^{-1} B_j A_j(D\Phi_j)$ will satisfy all of the conclusions of the lemma. ■

COROLLARY 2.2 *Let $\{p_j\}_{j=1}^\infty \subseteq \mathbb{C}^n$. Then there exists a dense \mathcal{G}_δ subset \mathcal{S} of the set of affine linear automorphisms of \mathbb{C}^n such that for each $\phi \in \mathcal{S}$ we have $\pi_m \phi(p_j) \neq \pi_m \phi(p_k)$ if $j \neq k$, and $(D\phi)$ has distinct eigenvalues.*

Proof: Let \mathcal{S}_N be the corresponding set of affine linear automorphisms for which we require the coordinate condition only on the set $\{p_1, \dots, p_N\}$. Then \mathcal{S}_N is easily seen to be open, and given any affine linear automorphism Ψ , we can use the argument from lemma 2.1 to construct a sequence $\Gamma_j \rightarrow \Psi$ with each $\Gamma_j \in \mathcal{S}_N$. Hence each \mathcal{S}_N is open and dense, and taking the intersection of all of them gives the corollary. ■

The next two lemmas show that we can construct an automorphism of \mathbb{C}^n which fixes a given set of points and which has prescribed derivative at each of these points. We first prove this for the case of \mathbb{C}^2 , then use this result for the general case.

LEMMA 2.3 *Given distinct points $p_1, \dots, p_N \in \mathbb{C}^2$ and invertible 2×2 matrices A_j , there exists $\phi \in \text{Aut}(\mathbb{C}^2)$ such that $\phi(p_j) = p_j$ and $(D\phi)(p_j) = A_j$ for each j .*

Proof: First suppose that $\pi_l p_j \neq \pi_l p_k$ for $l = 1, 2$ whenever $j \neq k$. Write

$$A_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}.$$

For each p_j , let $B_j = A_j$ if $a_j \neq 0$ and $B_j = I$ otherwise. Let $C_j = A_j$ if $a_j = 0$ and $d_j \neq 0$, and $C_j = I$ otherwise. Finally, let $D_j = A_j$ if $a_j = d_j = 0$ and $D_j = I$ otherwise. By taking a composition of three automorphisms, it suffices to treat separately the three cases which are given by replacing the set of A_j 's by each of the three sets of matrices just constructed.

First suppose $a_j \neq 0$ for all j , and let $\alpha_j = a_j$, $\beta_j = b_j$, $\gamma_j = c_j/\alpha_j$, and $\delta_j = d_j - b_j c_j/\alpha_j$. It is standard [R, Thm 15.13] that there exist entire f_1 and g_1 such that $g_1(\pi_2 p_j) = \log \alpha_j$, $g_1'(\pi_2 p_j) = 0$, $f_1(\pi_2 p_j) = (\pi_1 p_j)(1 - \alpha_j)$, and $f_1'(\pi_2 p_j) = \beta_j$ for all j . Define

$$H_1(z, w) = (e^{g_1(w)} z + f_1(w), w).$$

Then $H_1(p_j) = p_j$ and a simple calculation shows that

$$(DH_1)(p_j) = \begin{pmatrix} \alpha_j & \beta_j \\ 0 & 1 \end{pmatrix} \tag{2.1}$$

for all j . A similar construction gives an automorphism H_2 which fixes each p_j and has

$$(DH_2)(p_j) = \begin{pmatrix} 1 & 0 \\ \gamma_j & \delta_j \end{pmatrix} \tag{2.2}$$

for all j . Taking $H = H_2 \circ H_1$ gives $H(p_j) = p_j$ and $(DH)(p_j) = A_j$ for all j as desired. A similar construction applies to the case $d_j \neq 0$ for all j .

In the remaining case, either $A_j = I$, or $a_j = d_j = 0$. From the previous two cases and the fact that $b_j c_j \neq 0$ if $a_j = d_j = 0$, there exist automorphisms H_1 and H_2 so that $H_k(p_j) = p_j$ for all j and $k = 1, 2$, and so that

$$(DH_1)(p_j) = \begin{cases} I & \text{if } A_j = I \\ \begin{pmatrix} 1 & 0 \\ 1 & -b_j \end{pmatrix} & \text{if } a_j = d_j = 0, \end{cases}$$

and

$$(DH_2)(p_j) = \begin{cases} I & \text{if } A_j = I \\ \begin{pmatrix} 0 & b_j \\ -c_j/b_j & 1 \end{pmatrix} & \text{if } a_j = d_j = 0. \end{cases}$$

Let $H = H_1 \circ H_2$. Then $H(p_j) = p_j$ for all j , and $(DH)(p_j) = A_j$ for all j .

For the general case without the restrictions on the coordinates of the p_j , use lemma 2.1 to choose an automorphism ψ for which $\pi_l \psi(p_j) \neq \pi_l \psi(p_k)$ for $l = 1, 2$ whenever $j \neq k$. Since ψ is invertible, we can use the construction just given to construct an automorphism H with $H(\psi(p_j)) = \psi(p_j)$ and

$$(DH)(\psi(p_j)) = [(D\psi)(p_j)] A_j [(D\psi)(p_j)]^{-1},$$

for each j . Let $\phi = \psi^{-1} H \psi$. Then each p_j is a fixed point for ϕ , and the chain rule shows that ϕ has the correct derivative at each p_j . ■

LEMMA 2.4 *Let $p_1, \dots, p_N \in \mathbb{C}^n$ and let A_j be an invertible $n \times n$ matrix for each $j = 1, \dots, N$. Then there exists an automorphism ϕ of \mathbb{C}^n such that $\phi(p_j) = p_j$ and $(D\phi)(p_j) = A_j$ for all j .*

Proof: Note that any invertible $n \times n$ matrix can be obtained as a composition of matrices corresponding to the elementary row operations. Using the techniques of the previous lemma, we can realize each of these elementary operations at any given point, so the lemma follows by taking a composition of the automorphisms thus obtained. ■

The following is a simple corollary of this result.

COROLLARY 2.5 *Let $F \in \text{Aut}(\mathbb{C}^n)$ and let p_1, \dots, p_N be periodic points for F such that p_j has period m_j . Then there exists $\phi \in \text{Aut}(\mathbb{C}^n)$ such that $\phi(p_j) = p_j$ for all j , and so that if $G = \phi^{-1}F\phi$, then for each j , $G^{m_j}(p_j) = p_j$ and $(DG^{m_j})(p_j)$ is in Jordan form.*

Proof: For each j , there is a matrix A_j such that $A_j^{-1}[(DF^{m_j})(p_j)]A_j$ is in Jordan form, and by lemma 2.4 there exists $\phi \in \text{Aut}(\mathbb{C}^n)$ such that $\phi(p_j) = p_j$ and $(D\phi)(p_j) = A_j$ for all j . A simple check shows that ϕ has the desired properties. ■

The next lemma shows that we can construct a family of automorphisms depending holomorphically on several parameters with the following properties: each of these automorphisms fixes a given finite set of points, and the differential at each of these points is a diagonal matrix such that each entry of each matrix can be changed independently of all other entries. Moreover, one member of this family is the identity automorphism.

For notation, each α_j in the lemma is an n -tuple $\alpha_j = (\alpha_{j1}, \dots, \alpha_{jn})$, and $\text{diag}(a_1, \dots, a_n)$ is the $n \times n$ diagonal matrix $\{b_{jk}\}$ with $b_{jj} = a_j$ for all j .

LEMMA 2.6 *Given distinct points $p_1, \dots, p_N \in \mathbb{C}^n$, there exists an $N \times n$ parameter family $\{H_{\alpha_1, \dots, \alpha_N}\} \subseteq \text{Aut}(\mathbb{C}^n)$ which is holomorphic in $\mathbb{C}^{N \times n} \times \mathbb{C}^n$ such that $H_{\alpha_1, \dots, \alpha_N}(p_j) = p_j$ and*

$$(DH_{\alpha_1, \dots, \alpha_N})(p_j) = \text{diag}(e^{\alpha_{j1}}, \dots, e^{\alpha_{jn}})$$

for all $j = 1, \dots, N$ and all $\alpha_1, \dots, \alpha_N$, and such that $H_{0, \dots, 0} \equiv I$.

Proof: By permuting coordinates and taking a composition, it suffices to construct a 1-parameter family $\{H_\alpha\} \subseteq \text{Aut}(\mathbb{C}^n)$ which is holomorphic on $\mathbb{C} \times \mathbb{C}^n$ such that $H_0 \equiv I$ with $H_\alpha(p_j) = p_j$ for all j and α , $(DH_\alpha)(p_j) = I$ for all α if $j \neq 1$, and $(DH_\alpha)(p_1) = \text{diag}(e^\alpha, 1, \dots, 1)$.

To construct such a 1-parameter family, first use lemma 2.1 to choose $\phi \in \text{Aut}(\mathbb{C}^n)$ with $(D\phi)$ constant and diagonalizable such that $\pi_l \phi(p_j) \neq \pi_l \phi(p_k)$ for $l = 1, \dots, n$ whenever $j \neq k$. Write $q_j = \phi(p_j)$, and choose f entire such that $f(\pi_2 q_1) = 1$, $f'(\pi_2 q_1) = 0$, and $f(\pi_2 q_j) = f'(\pi_2 q_j) = 0$ for $j = 2, \dots, N$. Define

$$F_\alpha(z_1, \dots, z_n) = (z_1 e^{\alpha f(z_2)} + (1 - e^\alpha)(\pi_1 q_1) f(z_2), z_2, \dots, z_n).$$

The choice of f shows that $F_\alpha(q_j) = q_j$ for all j and α , and that if $j \neq 1$, then $(DF_\alpha)(q_j) = I$ for all α , while $(DF_\alpha)(q_1) = \text{diag}(e^\alpha, 1, \dots, 1)$.

Next, let $A = (D\phi)$, and choose B such that $B^{-1}AB$ is diagonal. By lemma 2.4, there exists $\psi \in \text{Aut}(\mathbb{C}^n)$ such that $\psi(q_j) = q_j$, $\psi(p_j) = p_j$, and $(D\psi)(q_j) = (D\psi)(p_j) = B$ for all j . Let $\Phi = \psi^{-1}\phi\psi$. Then for all j , $\Phi(p_j) = q_j$, and $(D\Phi)(p_j) = B^{-1}AB$ is diagonal.

Finally, let $H_\alpha = \Phi^{-1}F_\alpha\Phi$. Then $H_\alpha(p_j) = p_j$ for all j , $H_0 \equiv I$, and $(DH_\alpha)(p_j) = \text{diag}(e^\alpha, 1, \dots, 1)$. Thus H_α has the desired properties. ■

3 Perturbation to hyperbolic periodic points

In this section, we show that there is a dense \mathcal{G}_δ subset of automorphisms for which every periodic point is hyperbolic. We do this step by step, working first on compact sets. Thus we first consider the set of automorphisms such that all periodic points of period m which are contained in $\overline{\mathbf{B}^n(0; R)}$ are hyperbolic and show that this is an open dense set in the space of automorphisms. We do this by using some elementary results about analytic sets and the results of the previous section to perturb away any varieties of periodic points, then use the results of the previous section to perturb any isolated periodic points to become hyperbolic. Taking the intersection over all m and R gives a dense \mathcal{G}_δ as claimed.

The first lemma provides control on the number of zeroes of a holomorphic map from a polydisk in \mathbb{C}^n into \mathbb{C}^m , $m \geq n$ whose norm is nonvanishing on the boundary of the polydisk.

LEMMA 3.1 *Suppose $F : \overline{\Delta^n(0; r)} \rightarrow \mathbb{C}^m$ is holomorphic, $m \geq n$, and that $\|F\| > 0$ on $\partial\Delta^n(0; r)$. Then there exists $k \geq 0$ and a neighborhood \mathcal{N} of F in the space of holomorphic maps on $\overline{\Delta^n(0; r)}$ such that for all $G \in \mathcal{N}$, the equation $\|G(p)\| = 0$ has at most k solutions in $\Delta^n(0; r)$ and none on the boundary of this polydisk.*

Proof: The case $m = n$ is a standard application of multidimensional residues [GH, pp. 664-665].

For $m > n$ and $j = 1, \dots, m$, let F_j denote the j th component of F , and let $V_j(F) = \{p \in \Delta^n(0; r) : F_1(p) = \dots = F_j(p) = 0\}$. Then each $V_j(F)$ is an analytic set, $\dim(V_1(F)) = n - 1$ since it is the zero set of one holomorphic function, and $\dim(V_m(F)) = 0$ by hypothesis.

If it were the case that $\dim(V_n(F)) = 0$, then we could apply the $m = n$ case to the first n coordinates of F to obtain the lemma. If this is not the case, then there is $j > 1$ minimal such that $\dim(V_j(F)) = n - j + 1$. Since $\dim(V_m(F)) = 0$, we see by [N, p. 60] that for each of the finitely many irreducible components of $V_{j-1}(F)$ which intersect $\overline{\Delta^n(0; r)}$, there is some $k > j - 1$ such that $F_k \not\equiv 0$ on that component. Replacing F_j by $\Lambda = F_j + b_{j+1}F_{j+1} + \dots + b_m F_m$ for some choice of b_{j+1}, \dots, b_m , we can insure that $\Lambda \not\equiv 0$ on each of these components. This corresponds to replacing F by AF for some invertible $m \times m$ matrix A . Then [N, p. 60] implies that $\dim(V_j(AF)) = n - j$.

By induction, we can choose A invertible such that $\dim(V_n(AF)) = 0$, then use the $m = n$ case applied to the first n coordinates of AF and the invertibility of A to obtain the lemma. ■

The following lemma shows that the dimension of an analytic variety cannot go up under small perturbations of the defining functions, and is a simple consequence of [N, p. 46]. We omit the details.

LEMMA 3.2 *Suppose $\Omega \subset \subset \mathbb{C}^n$ is open, U is a neighborhood of $\overline{\Omega}$, $\Gamma : U \rightarrow \mathbb{C}^m$ is holomorphic, and $V = \{p \in \Omega : \Gamma(p) = 0\}$. Let Γ_j be holomorphic on U such that $\Gamma_j \rightarrow \Gamma$ as $j \rightarrow \infty$, and let $V_j = \{p \in \Omega : \Gamma_j(p) = 0\}$. Then there exists J such that if $j \geq J$, then $\dim V_j \leq \dim V$.*

Recall that F^m indicates the m th iterate of an automorphism F .

DEFINITION 3.3 *Let $F \in \text{Aut}(\mathbb{C}^n)$. Then*

$$\text{Fix}_m(F) := \{p \in \mathbb{C}^n : \|(F^m - I)(p)\| = 0\}$$

is the set of fixed points of F^m .

The next lemma shows that if $p \in \text{Fix}_m(F)$ and $\text{Fix}_m(F)$ has dimension $d < n$ at p , then there is a bound, independent of perturbations, on the number of periodic points of period m which are contained in some $n - d$ dimensional polydisk through p .

LEMMA 3.4 *Let $F \in \text{Aut}(\mathbb{C}^n)$, and suppose $0 \in \text{Fix}_m(F)$ for some m and that $\text{Fix}_m(F)$ has dimension $d \in \{0, \dots, n - 1\}$ at 0 as an analytic set. Then there exist neighborhoods U_0 of 0 and \mathcal{N} of F , an integer $k_0 > 0$, and a linear change of coordinates such that U_0 is a polydisk $\Delta^d(0; r') \times \Delta^{n-d}(0; r'')$ and such that for all $G \in \mathcal{N}$ and $q \in \overline{\Delta^d(0; r')}$, $\|G(q, \cdot)\|$ has at most k_0 zeroes in $\Delta^{n-d}(0; r'')$ and none on the boundary of this polydisk.*

Proof: If $d = 0$, then 0 is an isolated point in $\text{Fix}_m(F)$ by [N, p. 36], so we may take a small polydisk U_0 around 0 such that $\|F^m - I\| > 0$ on ∂U_0 and apply lemma 3.1. Hence we may assume that $1 \leq d \leq n - 1$.

Since $0 \in \text{Fix}_m(F)$, we can use the proof of [N, p. 36] to choose a linear change of coordinates and polydisks $\Delta^d(0; r')$ and $\Delta^{n-d}(0; r'')$ such that $\|F^m - I\| > 0$ on $\overline{\Delta^d(0; r')} \times \partial \Delta^{n-d}(0; r'')$. Let $U_0 = \Delta^d(0; r') \times \Delta^{n-d}(0; r'')$.

The lemma then follows by applying lemma 3.1 to $F(0, \cdot)|_{\Delta^{n-d}(0; r'')}$, then shrinking r'' if necessary to obtain the conclusion of the lemma for F , then again applying lemma 3.1 and a compactness argument to obtain the neighborhood \mathcal{N} as desired. ■

The next proposition is the main tool for inductively reducing the dimension of analytic sets of periodic points. First some notation.

DEFINITION 3.5 *Let $F \in \text{Aut}(\mathbb{C}^n)$ and $p \in \mathbb{C}^n$. Then $\mathcal{O}(p)$ denotes the full orbit of p under F :*

$$\mathcal{O}(p) = \{F^j(p) : j \in \mathbb{Z}\}.$$

PROPOSITION 3.6 *Let $F \in \text{Aut}(\mathbb{C}^n)$ with $0 \in \text{Fix}_m(F)$ for some m , and suppose that $\text{Fix}_m(F)$ has dimension $d \geq 1$ at 0 as an analytic set. Then there exists a neighborhood U of 0 , \mathcal{N} of F , and an open dense subset $\mathcal{S} \subseteq \mathcal{N}$ such that for all $G \in \mathcal{S}$, $\text{Fix}_m(G) \cap U$ has dimension at most $d - 1$.*

Proof: Changing coordinates, we may use the previous lemma to obtain $U_0 = \Delta^d(0; r') \times \Delta^{n-d}(0; r'')$, \mathcal{N} , and k_0 satisfying the conclusions of that lemma. Let $U = \Delta^d(0; r'/2) \times \Delta^{n-d}(0; r'')$, and let

$$\mathcal{S} = \{G \in \mathcal{N} : \dim(\text{Fix}_m(G) \cap U) \leq d - 1\}.$$

Since $\text{Fix}_m(G) \cap U$ is the zero set of $G^m - I$ in U , we can apply lemma 3.2 to conclude that \mathcal{S} is open.

For density, assume first that $d = n$, and let $G \in \mathcal{N}$. If $G \notin \mathcal{S}$, then $G^m \equiv I$, and by lemma 2.6, we can choose an automorphism ϕ arbitrarily close to the identity such that ϕG has an attracting periodic point. Hence $\phi G \in \mathcal{S}$, so \mathcal{S} is dense.

For $d < n$, let $G \in \mathcal{N}$, and let $\mathcal{M} \subseteq \mathcal{N}$ be a neighborhood of G . We will produce $H \in \mathcal{S} \cap \mathcal{M}$. First suppose that S is an irreducible component of $\text{Fix}_m(G) \cap U_0$ of dimension d , and let $\pi : S \rightarrow \Delta^d(0; r')$ be projection onto the first d coordinates. Then π is a proper map, so $\pi(S)$ is an analytic set. It is then a simple consequence of [N, p. 46] that $\pi(S) = \Delta^d(0; r')$. In particular, S must intersect $\{0\} \times \Delta^{n-d}(0; r'')$.

Now, by choice of k_0 , each $H \in \mathcal{N}$ has at most k_0 points in the set $A_H := \text{Fix}_m(H) \cap (\{0\} \times \Delta^{n-d}(0; r''))$. If each one of the points in A_H is isolated in $\text{Fix}_m(H)$, then in particular, there can be no irreducible component of $\text{Fix}_m(H)$ of dimension d through any such point, and hence $\text{Fix}_m(H)$ must have dimension at most $d - 1$ in U_0 by the argument above.

If there are points in A_H which are not isolated in $\text{Fix}_m(H)$, we can use lemma 2.6 to choose an automorphism ψ arbitrarily close to the identity which fixes each point in $\mathcal{O}(p)$ for each $p \in A_H$ and such that no eigenvalue of $(D(\psi H)^m)(p)$ has modulus 1 for any $p \in A_H$. Then $A_H \subseteq A_{\psi H}$, and each point in A_H is isolated in $\text{Fix}_m(\psi H)$.

Applying this procedure repeatedly and using k_0 to insure that the eventually every point in A_H is isolated, we obtain some $H \in \mathcal{N}$ arbitrarily near G such that A_H is isolated in $\text{Fix}_m(H)$, and hence $\text{Fix}_m(H)$ has dimension at most $d - 1$ in U_0 . Thus $H \in \mathcal{S} \cap \mathcal{M}$, so \mathcal{S} is dense in \mathcal{N} . ■

COROLLARY 3.7 *Let $F \in \text{Aut}(\mathbb{C}^n)$, $p \in \text{Fix}_m(F)$, and suppose that $\dim(\text{Fix}_m(F)) = d$ at p . Then there exist neighborhoods U of p and \mathcal{N} of F , and an open dense subset $\mathcal{S} \subseteq \mathcal{N}$ such that for all $G \in \mathcal{S}$, $\text{Fix}_m(G) \cap U$ consists of finitely many isolated points.*

Proof: Induct on d . If $d = 0$, then we can apply lemma 3.4, while if $d = 1$, then proposition 3.6 applies.

For $d \geq 2$, apply the previous lemma to get neighborhoods U_1 of p and \mathcal{N}_1 of F , and an open dense $\mathcal{S}_1 \subseteq \mathcal{N}_1$ such that for all $G \in \mathcal{S}_1$ we have $\dim(\text{Fix}_m(G) \cap U_1) \leq d - 1$. Let U be a neighborhood of p with $\bar{U} \subseteq U_1$, let $\mathcal{N} = \mathcal{N}_1$, and let

$$\mathcal{S} = \{G \in \mathcal{N} : \text{Fix}_m(G) \cap \bar{U} \subseteq U\}.$$

For each $G \in \mathcal{S}$, $\text{Fix}_m(G) \cap \bar{U}$ is a compact analytic set in U , hence consists of finitely many points, and lemma 3.2 shows that this property persists under small perturbations, so that \mathcal{S} is open.

Moreover, for $G \in \mathcal{S}_1$, $\text{Fix}_m(G) \cap \overline{U}$ is a compact set with dimension at most $d-1$ at each point. A simple covering argument and induction shows that \mathcal{S} is dense in \mathcal{S}_1 and hence in \mathcal{N} . ■

Next we use these local results to pass to global information. We will produce a dense \mathcal{G}_δ set of automorphisms such that for each one of these automorphisms, every periodic point is hyperbolic, and in fact, if $p \in \text{Fix}_m(F)$, then F^m is linearizable in a neighborhood of p .

A sufficient condition for this linearization is that the eigenvalues of $(DF^m)(p)$ satisfy a diophantine condition [H], and the set of eigenvalues which satisfy such a condition is a dense \mathcal{G}_δ subset of \mathbb{C}^n . Hence we can write this set as the intersection of open dense sets $A_k \subseteq \mathbb{C}^n$, $k \in \mathbb{Z}^+$, and we may also assume that no component of any element of any A_k has modulus 1. Moreover, since the diophantine condition does not depend on the order of the eigenvalues, we may assume that if a given n -tuple is in A_k , then all permutations of this n -tuple are also in A_k .

For the next two propositions, we need some definitions.

DEFINITION 3.8 *For $R > 0$ and m a positive integer, let $\mathcal{H}_{R,m}$ be the set of $F \in \text{Aut}(\mathbb{C}^n)$ such that if $p \in \text{Fix}_m(F) \cap \overline{\mathbb{B}(0; R)}$, then p is an isolated point in $\text{Fix}_m(F)$.*

DEFINITION 3.9 *For $k \in \mathbb{Z}^+$ and R, m as above, let $\mathcal{H}_{R,m}^k$ be the set of $F \in \text{Aut}(\mathbb{C}^n)$ such that if $p \in \text{Fix}_m(F) \cap \overline{\mathbb{B}(0; R)}$, then the set of eigenvalues of $(DF^m)(p)$ is contained in A_k .*

Note that $\mathcal{H}_{R,m}^k \subseteq \mathcal{H}_{R,m}$ by the inverse function theorem.

PROPOSITION 3.10 *The set $\mathcal{H}_{R,m}$ is open and dense in $\text{Aut}(\mathbb{C}^n)$.*

Proof: Given $F \in \mathcal{H}_{R,m}$, we can surround each $p \in \text{Fix}_m(F) \cap \overline{\mathbb{B}(0; R)}$ by a small ball with $\|F^m - I\| > 0$ on the boundary of this ball. For a small perturbation G of F , $\text{Fix}_m(G) \cap \overline{\mathbb{B}(0; R)}$ will still be contained in the interiors of these balls and hence will consist of a finite number of points, each of which is isolated in $\text{Fix}_m(G)$. Thus $\mathcal{H}_{R,m}$ is open.

For density, choose $F \in \text{Aut}(\mathbb{C}^n)$. Then $\text{Fix}_m(F) \cap \overline{\mathbb{B}(0; R)}$ is compact, so we can cover it with finitely many of the neighborhoods U_j from corollary 3.7. For G near enough to F , $\text{Fix}_m(G) \cap \overline{\mathbb{B}(0; R)}$ will still be contained in the union of these neighborhoods, and the intersection of the corresponding open dense subsets \mathcal{S}_j will be open and dense in a neighborhood of F . Since this intersection is contained in $\mathcal{H}_{R,m}$, the density follows. ■

PROPOSITION 3.11 *The set $\mathcal{H}_{R,m}^k$ is open and dense in $\text{Aut}(\mathbb{C}^n)$.*

Proof: Given $F \in \mathcal{H}_{R,m}^k$, let p_1, \dots, p_N be the points in $\text{Fix}_m(F) \cap \overline{\mathbb{B}(0; R)}$. The implicit function theorem implies that for G near F , G^m will have a unique fixed point $p_j(G)$ near p_j and that the eigenvalues of $(DG^m)(p_j(G))$ depend continuously on G . Hence, for G in a neighborhood of F , the eigenvalues of $(DG^m)(p_j(G))$ will be contained in A_k for all j . Using the fact that $\|F^m - I\| > 0$ outside the p_j together with a compactness argument shows that

we can choose this neighborhood so that there are no other periodic points of period m in $\overline{\mathbf{B}(0; R)}$ for any G in this neighborhood. This neighborhood of F is then contained in $\mathcal{H}_{R,m}^k$, so $\mathcal{H}_{R,m}^k$ is open.

For density, choose $F \in \mathcal{H}_{R,m}$, and let p_1, \dots, p_N be the union of all orbits of the points in $\text{Fix}_m(F) \cap \overline{\mathbf{B}(0; R)}$. Suppose there are M distinct orbits in this union, and choose one point q_j , $j = 1, \dots, M$ from each orbit. By corollary 2.5, there exists $\phi \in \text{Aut}(\mathbb{C}^n)$ which fixes each p_j and so that $(D(\phi^{-1}F\phi)^m)(p_j)$ is in Jordan form for each j . By lemma 2.6, there exists an $M \times n$ parameter family of automorphisms $\psi_{\alpha_1, \dots, \alpha_M}$ with $\psi_{0, \dots, 0} \equiv I$ which fixes each point p_j and has

$$(D\psi_{\alpha_1, \dots, \alpha_M})(p) = \begin{cases} \text{diag}(e^{\alpha_{j,1}}, \dots, e^{\alpha_{j,n}}) & \text{if } p = q_j \\ I & \text{if } p \in \{p_j\}_{j=1}^N - \{q_j\}_{j=1}^M. \end{cases}$$

If p is in the orbit of q_j , then the diagonal entries of $(D(\phi^{-1}F\phi)^m)(p)$ are $\lambda_{j,1}, \dots, \lambda_{j,n}$, where the $\lambda_{j,k}$ are the eigenvalues of $(DF^m)(q_j)$. The chain rule shows that then

$$(D(\psi_{\alpha_1, \dots, \alpha_M} \phi^{-1} F \phi)^m)(p)$$

is upper triangular with diagonal entries $\exp(\alpha_{j,1})\lambda_{j,1}, \dots, \exp(\alpha_{j,1})\lambda_{j,1}$. Since A_k is open and dense, we may choose $\alpha_1, \dots, \alpha_M$ arbitrarily near 0 so that each of these sets of eigenvalues is contained in A_k . Since the set of eigenvalues is preserved under conjugation, we see that the eigenvalue condition is still satisfied for $\phi\psi_{\alpha_1, \dots, \alpha_M}\phi^{-1}F$. Given a neighborhood \mathcal{M} of F , this composition will be in \mathcal{M} whenever all of the α_j 's are sufficiently small. Hence, given $F \in \mathcal{H}_{R,m}$, there exists a sequence $G_j \in \mathcal{H}_{R,m}^k$ converging to F . Since $\mathcal{H}_{R,m}$ is dense, we see that $\mathcal{H}_{R,m}^k$ is also dense. ■

Note that the topology on $\text{Aut}(\mathbb{C}^n)$ is the topology induced by local uniform convergence, where we require convergence of both the map and its inverse. This topology is equivalent to that induced by the metric $d = \sum_m 2^{-m} d_m / (1 + d_m)$, where d_m is the sup-norm metric on the ball of radius m applied to an automorphism and its inverse. With this metric, $\text{Aut}(\mathbb{C}^n)$ is a complete metric space, so Baire's theorem applies.

Hence, taking the intersection of the sets $\mathcal{H}_{R,m}^k$ over all $R, m, k \in \mathbb{Z}^+$ gives the following theorem.

THEOREM 3.12 *Let \mathcal{H}_L be the set of $F \in \text{Aut}(\mathbb{C}^n)$ such that each periodic point of F is hyperbolic, and such that if p is periodic of period m , then F^m is linearizable in a neighborhood of p . Then \mathcal{H}_L is residual in $\text{Aut}(\mathbb{C}^n)$.*

4 Basic results for stable and unstable manifolds

In this section we collect some results about stable and unstable manifolds for holomorphic maps.

We first start with a result which shows that the stable manifold of a periodic saddle point p of $F \in \text{Aut}(\mathbb{C}^n)$ is biholomorphically equivalent to \mathbb{C}^d , where d is the dimension of this manifold. This is proved in [W] with the additional hypothesis that the map is linearizable in a neighborhood of p . Although we won't need this proposition in the sequel, we include it here since it has some independent interest.

DEFINITION 4.1 Let $F \in \text{Aut}(\mathbb{C}^n)$ with saddle periodic point p with period m . Then $E^s(p)$ is the eigenspace of $(DF^m)(p)$ corresponding to the set of eigenvalues with norm less than 1, and $E^u(p)$ is the eigenspace corresponding to all eigenvalues with norm larger than 1.

PROPOSITION 4.2 Let $F \in \text{Aut}(\mathbb{C}^n)$ with $F^m(p) = p$ a periodic saddle point. Let $d = \dim(E^s(p))$. Then there exists a biholomorphic map $\Gamma : \mathbb{C}^d \rightarrow W^s(p)$ and an automorphism $H \in \text{Aut}(\mathbb{C}^n)$ with 0 as a globally attracting fixed point such that $F^m\Gamma = \Gamma H$ on \mathbb{C}^d .

Proof: By the holomorphic version of the stable manifold theorem, we can parametrize some neighborhood $W_{\text{loc}}^s(p)$ of p in $W^s(p)$ by a biholomorphic map $\phi : \mathbb{B}^d(0; r) \rightarrow W^s(p)$ with $\phi(0) = 0$. With this parametrization, F^m induces a map on $\mathbb{B}^d(0; r)$ with an attractive fixed point at 0, and by the proof in [RR, appendix], this induced map is locally conjugate to some $H \in \text{Aut}(\mathbb{C}^n)$ such that 0 is a globally attracting fixed point of H . Combining this conjugacy with ϕ , we obtain Γ defined in a neighborhood of 0 in \mathbb{C}^d , and using $F^{mk}\Gamma = \Gamma H^k$ together with the fact that 0 is globally attracting for H , we can extend this conjugacy to all of \mathbb{C}^d . ■

DEFINITION 4.3 For a periodic saddle point p for the map F , let $W_{\text{loc}}^s(p)$ denote a neighborhood of p in $W^s(p)$, and let $W_{\text{loc}}^u(p)$ denote a neighborhood of p in $W^u(p)$.

As before, we write $W_{\text{loc}}^s(p, F)$ and $W_{\text{loc}}^u(p, F)$ when we need to specify the map F .

We next need some results about how the local stable and unstable manifolds change under small perturbations of a map. Note that if F is a holomorphic map with a periodic saddle point p of period m , then from the implicit function theorem, there is a neighborhood D of p and a neighborhood \mathcal{N} of F , such that for each $G \in \mathcal{N}$, G has a unique periodic point p^G of period m contained in \overline{D} . Shrinking \mathcal{N} , we may assume that each p^G is a saddle point and that the dimensions of $E^s(p^G)$ and $E^u(p^G)$ are independent of G .

DEFINITION 4.4 For a holomorphic map F with a periodic saddle p , let p^G denote the periodic point associated to the map G for $G \in \mathcal{N}$ as just described.

From the stable manifold theorem, we know that given a saddle periodic point p of the map F , then for r sufficiently small, $W_{\text{loc}}^s(p)$ is the graph of a map from the ball of radius r in $E^s(p)$ to $E^u(p)$. That is, changing coordinates such that $p = 0$, $E^s(p) = \mathbb{C}^d \times \{0\}$, and $E^u(p) = \{0\} \times \mathbb{C}^{n-d}$, then $W_{\text{loc}}^s(p) = \{(z, \phi(z)) : z \in \mathbb{B}^d(0; r)\}$. Moreover, we can choose r small enough that for G in some neighborhood of F , we will have $W_{\text{loc}}^s(p^G, G) = \{(z, \phi_G(z)) : z \in \mathbb{B}^d(0; r)\}$, and ϕ_G will vary continuously with G . An analogous statement is true for $W_{\text{loc}}^u(p^G, G)$.

The next two lemmas are slight refinements of these ideas. The first says that given a point q_0 in the stable manifold of a saddle periodic point of a map F , there is a neighborhood U_0 of q_0 such that the intersection of U_0 with the corresponding stable manifold for maps near F depends only on the behavior of F in some fixed neighborhood of the forward orbit

of q_0 . The second says that if there is an indexed family of such holomorphic maps such that the family is also holomorphic in the index, then the corresponding stable manifold varies holomorphically with the index.

In the following lemma, we use maps G which are not globally defined but which are defined in a neighborhood of a saddle point p_1^G . Hence $W_{\text{loc}}^s(p_1^G)$ is well-defined, and we can choose it small enough to be relatively compact in the domain of G . We can apply G^{-1} to obtain a larger subset of the stable manifold, although the resulting manifold may not be connected.

For the following two lemmas, let $F \in \text{Aut}(\mathbb{C}^n)$, let p_1 be a saddle periodic point of F of minimal period m_1 with $d = \dim(E^s(p_1))$, and let $0 = q_0 \in W_{\text{loc}}^s(p_1) - \{p_1\}$. Suppose also that U is a neighborhood of q_0 , $r > 0$ and $\Phi : \Delta^d(0; r) \rightarrow \mathbb{C}^{n-d}$ is holomorphic such that $W_{\text{loc}}^s(p_1) \cap U = \{(z, \Phi(z)) : z \in \Delta^d(0; r)\}$.

Let U_1 be a neighborhood of the set $\mathcal{O}(p_1) \cup \{F^j(q_0) : j \geq 0\}$, and let $D_1 \subseteq U_1$ be a neighborhood of p_1 such that p_1 is the only periodic point of period m_1 in $\overline{D_1}$.

LEMMA 4.5 *Let $\epsilon_0 > 0$. Then there exists a polydisk $U_0 = \Delta^d(0; r') \times \Delta^{n-d}(0; 2r'')$ and a neighborhood \mathcal{N} of $F|_{U_1}$ in the space of holomorphic maps on U_1 such that for each $G \in \mathcal{N}$ there exists $\Phi^G : \Delta^d(0; r') \rightarrow \mathbb{C}^{n-d}$ such that*

$$W_{\text{loc}}^s(p_1^G) \cap U_0 = \{(z, \Phi^G(z)) : z \in \Delta^d(0; r')\}$$

and is contained in $\Delta^d(0; r') \times \Delta^{n-d}(0; r''/2)$, and so that $\|\Phi^G(z) - \Phi(z)\| < \epsilon_0$ for $z \in \Delta^d(0; r')$.

Proof: Since $W_{\text{loc}}^s(p_1)$ is an analytic set in a neighborhood of each of its points, the hypotheses imply that there exist $r' \in (0, r)$ and $r'' \in (0, \epsilon_0)$ such that $W_{\text{loc}}^s(p_1) \cap (\Delta^d(0; r') \times \Delta^{n-d}(0; r''))$ is contained in $\Delta^d(0; r') \times \Delta^{n-d}(0; r''/3)$ and is a graph over $\Delta^d(0; r')$. The lemma is then true for q_0 in some neighborhood B_1 of p_1 by the stable manifold theorem.

For the general case, choose $m \geq 0$ such that $F^m(q_0) \in B_1$. For a small enough neighborhood \mathcal{N} of $F|_{U_1}$, given any $G \in \mathcal{N}$, we can apply m iterations of G^{-1} to $W_{\text{loc}}^s(p_1^G) \cap B_1$ to extend the stable manifold of p_1^G to see that $G^{-m}(W_{\text{loc}}^s(p_1^G)) \cap (\Delta^d(0; r') \times \Delta^{n-d}(0; r''))$ is contained in $\Delta^d(0; r') \times \Delta^{n-d}(0; r''/2)$ and is a graph over $\Delta^d(0; r')$. Since $r'' < \epsilon_0$, this gives the lemma. ■

Remark: Suppose in addition that q_0 is in $W_{\text{loc}}^u(p_2)$ for some saddle point p_2 , that a neighborhood of q_0 in $W_{\text{loc}}^u(p_2)$ is a graph over the first d' coordinates, that U_1 is a neighborhood of $\mathcal{O}(q_0)$, and that D_2 is a neighborhood of p_2 with conditions analogous to those on D_1 . Then we can choose r' and r'' such that the conclusions of the lemma are valid for both $W_{\text{loc}}^s(p_1)$ and $W_{\text{loc}}^u(p_2)$ using this choice of r' and r'' for both manifolds. Moreover, we may assume in addition that $r' < r''$.

LEMMA 4.6 *Assume the hypotheses of the previous lemma, and let \mathcal{N} be the neighborhood of $F|_{U_1}$ obtained there. Let $m \geq 1$, let $G : U_1 \times \Delta^m(0; 1) \rightarrow \mathbb{C}^n$ be holomorphic with $G_\alpha :=$*

$G(\cdot, \alpha) \in \mathcal{N}$ for all $\alpha \in \Delta^m(0; 1)$, and suppose that for each α , there exists a holomorphic $\Phi_\alpha : \Delta^d(0; r; \cdot) \rightarrow \mathbb{C}^{n-d}$ so that

$$W_{\text{loc}}^s(p_1(G_\alpha)) \cap U_0 = \{(z, \Phi_\alpha(z)) : z \in \Delta^d(0; r')\}.$$

Then Φ is holomorphic for $(z, \alpha) \in \Delta^d(0; r) \times \Delta^m(0; 1)$.

Proof: The proof that Φ is holomorphic in a neighborhood of each (z_0, α_0) is as in [B, chapter 11], and piecing these neighborhoods together gives the lemma. ■

5 Transversality of stable and unstable manifolds

In this section we show that it is possible to perturb an automorphism so that the stable and unstable manifolds of any two periodic points are transverse.

Remark: Fix $M > 0$ and let

$$\mathcal{H}^M := \bigcap_{m=1}^M \mathcal{H}_{M,m}^M,$$

where $\mathcal{H}_{M,m}^M$ is as in definition 3.9. Note that \mathcal{H}^M is open and dense in $\text{Aut}(\mathbb{C}^n)$ by proposition 3.11. Using techniques like those in [S, lemma 6.1a], there exists a continuous function $R_0 : \mathcal{H}^M \rightarrow (0, \infty)$ such that if $F \in \mathcal{H}^M$ and $p \in \overline{\mathbb{B}^n(0; M/2)}$ is a periodic saddle of period at most M , then defining $W_{\text{loc}}^s(p; R_0)$ to be the component of $W^s(p) \cap \mathbb{B}^n(p; R_0(F))$ containing p , we have

$$\overline{W_{\text{loc}}^s(p; R_0)} \subseteq F^{-1}(W_{\text{loc}}^s(p; R_0)) \subseteq W^s(p)$$

with an analogous definition and condition for $W_{\text{loc}}^u(p; R_0)$.

DEFINITION 5.1 For $j > 0$ and $F \in \mathcal{H}^M$ with periodic saddle $p \in \overline{\mathbb{B}^n(0; M/2)}$ of period at most M , define

$$\begin{aligned} W_j^s(p) &= W_j^s(p, F) := F^{-j}(W_{\text{loc}}^s(p; R_0)), \\ W_j^u(p) &= W_j^u(p, F) := F^j(W_{\text{loc}}^u(p; R_0)), \end{aligned}$$

DEFINITION 5.2 Let KS_j^M be the set of $F \in \mathcal{H}^M$ such that if $p_1, p_2 \in \overline{\mathbb{B}^n(0; M/2)}$ are periodic saddles of period at most M , then $W^s(p_1)$ and $W^u(p_2)$ are transverse at each point of $\overline{W_j^s(p_1)} \cap \overline{W_j^u(p_2)}$.

We will show that KS_j^M is open and dense in \mathcal{H}^M , hence in $\text{Aut}(\mathbb{C}^n)$. Note that since $\overline{W_j^s(p_1)} \subseteq W_{j+1}^s(p_1)$ and $\overline{W_j^u(p_2)} \subseteq W_{j+1}^u(p_2)$, we see that $W^s(p_1)$ and $W^u(p_2)$ are transverse at each point of $\overline{W_j^s(p_1)} \cap \overline{W_j^u(p_2)}$ if and only if $W_{j+1}^s(p_1)$ and $W_{j+1}^u(p_2)$ are transverse at every such point.

For the moment we fix $F_0 \in \mathcal{H}^M$ with periodic saddles $p_1, p_2 \in \overline{\mathbb{B}^n(0; M/2)}$ of period at most M . For some neighborhood $\mathcal{N}_0 \subseteq \mathcal{H}^M$ of F_0 , each $F \in \mathcal{N}_0$ has periodic saddles p_1^F and p_2^F as in definition 4.4.

DEFINITION 5.3 For $F \in \mathcal{N}_0$ and $j > 0$, let

$$V_j^1(F) := W_j^s(p_1^F, F) \cap W_j^u(p_2^F, F),$$

and let $V_j^0(F)$ be the set of points in $V_j^1(F)$ at which $W_j^s(p_1^F, F)$ and $W_j^u(p_2^F, F)$ are not transverse.

LEMMA 5.4 For each $F \in \mathcal{N}_0$, the set $V_j^0(F)$ is an analytic set in a neighborhood of each of its points.

Proof: Suppose $q_0 \in V_j^0(F)$, let d_s and d_u be the dimensions of $W_j^s(p_1^F)$ and $W_j^u(p_2^F)$ respectively, and assume $d_s \leq d_u$. Changing coordinates, we may assume that $q_0 = 0$, and that near q_0 , $W_j^s(p_1^F)$ projects biholomorphically onto the first d_s coordinates and that $W_j^u(p_2^F)$ projects biholomorphically onto the first d_u coordinates. Let π_s and π_u denote the projections from \mathbb{C}^n onto the first d_s and first d_u coordinates, respectively.

From the remark after lemma 4.5, we get $0 < r'_0 < r''_0$ such that the conclusions of that lemma apply to both manifolds, and hence there exist holomorphic maps $\phi_F^s : \Delta^{d_s}(0; r'_0) \rightarrow \Delta^{n-d_s}(0; r''_0)$ and $\phi_F^u : \Delta^{d_u}(0; r'_0) \rightarrow \Delta^{n-d_u}(0; r''_0)$ whose graphs are neighborhoods of q_0 in the stable and unstable manifolds, respectively.

Let $U_0 = \Delta^{d_u}(0; r'_0) \times \Delta^{n-d_u}(0; r''_0)$, and let $\hat{\pi}_u$ denote projection onto the last $n - d_u$ coordinates. To detect an intersection between $W_j^s(p_1^F)$ and $W_j^u(p_2^F)$ in U_0 , we need only consider points of the form $q = (z, \phi_F^s(z))$ for $z \in \Delta^{d_s}(0; r'_0)$ since this gives a parametrization of $W_j^s(p_1^F)$ in U_0 . Then q is a point of intersection if and only if $q - (\pi_u q, \phi_F^u(\pi_u q)) = 0$. Since the first d_u coordinates of this are always zero, we need only consider the last $n - d_u$ coordinates. Moreover, since this corresponds to a change of coordinates such that $W_j^u(p_2^F)$ is contained in $\mathbb{C}^{d_u} \times \{0\}$, we see that q is a point of transverse intersection exactly when the rank of the last $n - d_u$ coordinate functions is $n - d_u$.

Hence we define $\Phi_F : \Delta^{d_s}(0; r'_0) \rightarrow \mathbb{C}^{n-d_u}$ by

$$\Phi_F(z) = \hat{\pi}_u \phi_F^s(z) - \phi_F^u(\pi_u(z, \phi_F^s(z))). \quad (5.1)$$

Then $q \in V_j^1(F) \cap U_0$ if and only if $\Phi_F(\pi_s q) = 0$, and $q \in V_j^0(F) \cap U_0$ if and only if both $\Phi_F(\pi_s q) = 0$ and $(D\Phi_F)(\pi_s q)$ has rank at most $n - d_u - 1$.

To write $V_j^0(F) \cap U_0$ explicitly as an analytic set, let $e(k) \in \mathbb{C}^{d_1}$ be the k th standard basis vector, and let M be the number of subsets of $\{1, \dots, d_s\}$ with exactly $n - d_u$ elements. Let E_1, \dots, E_M be the collection of all $n - d_u$ dimensional subspaces of \mathbb{C}^{d_s} which are spanned by some collection $\{e(j_1), \dots, e(j_{n-d_u})\}$. For $z \in \mathbb{C}^{d_s}$, let $E_j(z)$ denote the affine subspace $E_j + \{z\}$, and define $\mu_F : \Delta^{d_s}(0; r') \rightarrow \mathbb{C}^M$ by

$$\mu_F(z) = (\det((D\Phi_F|E_1(z))(z)), \dots, \det((D\Phi_F|E_M(z))(z))). \quad (5.2)$$

Then μ_F is holomorphic, and $q \in V_j^0(F) \cap U_0$ if and only if $(\Phi_F(\pi_s q), \mu_F(\pi_s q)) = (0, 0)$. Thus, $V_j^0(F) \cap U_0$ is an analytic set. ■

To perturb to transversality, we define $\Gamma_F : \Delta^{d_s}(0; r'_0) \times \mathbb{C}^{n-d_u} \rightarrow \mathbb{C}^{n-d_u} \times \mathbb{C}^M$ by

$$\Gamma_F(z, \alpha) = (\Phi_F(z), \mu_F(z)) - (\alpha, 0). \quad (5.3)$$

By Sard's theorem, almost every $\alpha \in \mathbb{C}^{n-d_u}$ is a regular value of Φ_F , so that for almost every α , $\Gamma_F(\cdot, \alpha)$ has no zeroes in $\Delta^{d_s}(0; r'_0)$. This implies that the manifolds $W_j^s(p_1^F)$ and $W_j^u(p_2^F) + (0, \alpha)$ are transverse for almost every choice of $\alpha \in \mathbb{C}^{n-d_u}$.

The goal is to be able to produce an indexed family of automorphisms, F_α , such that $F_0 = F$ and such that the resulting map Γ defined using the stable and unstable manifolds for F_α is near Γ_F . We would like to be able to say that if Γ is near enough to Γ_F , then we still have $\Gamma(\cdot, \alpha) \neq 0$ in $\Delta^{d_s}(0; r'_0)$ for almost every α , so that we could take α near 0 for which F_α has only transverse intersections in U_0 .

Unfortunately, this isn't true in general, but we can get perturb to get rid of all zeroes on a lower dimensional slice, then use an inductive argument to get rid of all zeroes on $\Delta^{d_s}(0; r'_0)$.

To carry out this perturbation, fix $F \in \mathcal{N}_0$, then change coordinates as in lemma 5.4, so that $\Gamma_F(0, 0) = 0$, and define

$$A_F(\alpha) := \{z \in \Delta^{d_s}(0; r'_0) : \Gamma_F(z, \alpha) = 0\}.$$

Since the dimension of $A_F(0)$ at 0 is $d \in \{0, \dots, d_s\}$, we can change coordinates in \mathbb{C}^{d_s} to get a neighborhood $U' = \Delta^d(0; r') \times \Delta^{d_s-d}(0; r'')$ of 0 such that

$$A_F(0) \cap \overline{U'} \subseteq \overline{\Delta^d(0; r')} \times \Delta^{d_s-d}(0; r''/2), \quad (5.4)$$

where $\Delta^0(0; r) = \{0\}$ for any $r > 0$. Note that in this case, for each $w \in \overline{\Delta^d(0; r')}$, the intersection of $A_F(0)$ with $\{w\} \times \overline{\Delta^{d_s-d}(0; r'')}$ consists of finitely many points in $\Delta^{d_s-d}(0; r''/2)$.

By the remark after lemma 4.5, there exists a neighborhood $\mathcal{N} \subseteq \mathcal{N}_0$ of F such that if $G \in \mathcal{N}$, then Γ_G is well-defined using the same coordinates just chosen for F , and (5.4) holds with $A_G(0)$ in place of $A_F(0)$. For such G , there exists $\epsilon_1 > 0$ such that if $\|\alpha\| < \epsilon_1$, then (5.4) holds for $A_G(\alpha)$ in place of $A_F(0)$. In this case, define

$$B_G = \{(w, \alpha) \in \Delta^{d_s-d}(0; r'') \times \Delta^{n-d_u}(0; \epsilon_1) : \Gamma_G(0, w, \alpha) = 0\}.$$

LEMMA 5.5 *For all $G \in \mathcal{N}$, the set B_G is an analytic set with dimension at most $n-d_u-1$. Moreover, if $\Gamma_j \rightarrow \Gamma_G$ uniformly on $\overline{U'} \times \overline{\Delta^{n-d_u}(0; \epsilon_1)}$, then there exists J such that for all $j \geq J$ and $\epsilon > 0$, there exists $\|\alpha\| < \epsilon$ such that $\Gamma_j(0, \cdot, \alpha)$ has no zeroes in $\Delta^{d_s-d}(0; r'')$.*

Proof: The assumptions on G imply that the projection $\pi : B_G \rightarrow \Delta^{n-d_u}(0; \epsilon_1)$ is proper with finite fibers, in which case $\pi(B_G)$ is an analytic set with the same dimension as B_G . By Sard's theorem, for almost every choice of $\alpha \in \Delta^{n-d_u}(0; \epsilon_1)$, we have $\Gamma_G(0, \cdot, \alpha) \neq 0$ on $\Delta^{d_s-d}(0; r'')$, hence $\alpha \notin \pi(B_G)$. Thus $\dim \pi(B_G) < n - d_u$.

For j large, the corresponding set B_j defined with Γ_j in place of Γ_G will satisfy $\dim B_j \leq \dim B_G$ by lemma 3.2, and the projection $\pi : B_j \rightarrow \Delta^{n-d_u}(0; \epsilon_1)$ will still be proper with finite fibers. Hence $\pi(B_j)$ is analytic with dimension at most $n - d_u - 1$. Choosing $\alpha \in \Delta^{n-d_u}(0; \epsilon) - \pi(B_j)$ completes the lemma. ■

Remark: Note that if $\Gamma_j(0, \cdot, \alpha)$ has no zeroes in $\Delta^{d_s-d}(0; r'')$, then an argument like that in lemma 3.6 shows that $\{z \in U' : \Gamma_j(z, \alpha) = 0\}$ has dimension at most $d - 1$. We will use this fact inductively to reduce the dimension of nontransversality.

Next, we use shears to reduce the dimension of nontransversality. We use the convention that a set has negative dimension if and only if it is empty.

PROPOSITION 5.6 *Let $F \in \mathcal{N}_0$ and suppose $q_0 \in V_j^0(F)$ with $d = \dim V_j^0$ at q_0 . Then there exist neighborhoods U of q_0 and $\mathcal{N} \subseteq \mathcal{N}_0$ of F , plus an open dense subset $\mathcal{S} \subseteq \mathcal{N}$, such that if $G \in \mathcal{S}$, then $V_j^0(G) \cap U$ has dimension at most $d - 1$.*

Proof: If $d < 0$, then the manifolds are transverse at q_0 , and this will be true for all G near F in some fixed neighborhood of q_0 by the stable manifold theorem.

For $d \geq 0$, we use the coordinates and notation from the previous two lemmas and associated remarks. Note that the set $K := \overline{\mathcal{O}(q_0)}$ is countable since $\mathcal{O}(q_0)$ accumulates only on the finite orbits of p_1 and p_2 . Thus we can conjugate by a map near the identity as in corollary 2.2 and assume in addition that $\pi_1 p \neq \pi_1 q$ whenever p and q are distinct points in K . Since $q_0 = 0$, we can choose $r_0 > 0$ such that $\overline{\Delta(0; r_0)}$ and $\pi_1(K - \{0\})$ are disjoint, then choose a bounded simply connected neighborhood $U_1 \subseteq \mathbb{C}$ of $\pi_1(K - \{0\})$ such that U_1 and $\overline{\Delta(0; r_0)}$ are disjoint.

Let U' and \mathcal{N} be as in the remarks before lemma 5.5. Shrinking U' and \mathcal{N} , we can use the remark after lemma 4.5 to find $r_1, r'_1 > 0$ such that if $U = U' \times \Delta^{n-d_s}(0; r_1)$, then there is a continuous map from $G \in \mathcal{N}$ to the map $\phi_G^s : \pi_s U \rightarrow \Delta^{n-d_s}(0; r_1)$ whose graph is $W_j^s(p_1^G) \cap U$, and a continuous map from $G \in \mathcal{N}$ to the map $\phi_G^u : \pi_u U \rightarrow \Delta^{n-d_u}(0; r'_1)$ whose graph is $W_k^u(p_2^G) \cap U$. Define

$$\mathcal{S} = \{G \in \mathcal{N} : \dim A_G(0) \leq d - 1\}.$$

Lemma 3.2 implies that \mathcal{S} is open, and since $V_j^0(G) \cap U = \{(z, \phi_G^s(z)) : \Gamma_G(z, 0) = 0\}$, we see by [N, p. 46] that $\dim(V_j^0(G) \cap U) \leq d - 1$ for all $G \in \mathcal{S}$.

To show that \mathcal{S} is dense in \mathcal{N} , let $G \in \mathcal{N}$, and let $\mathcal{M} \subseteq \mathcal{N}$ be a neighborhood of G . The choice of U_1 implies that we can find polynomials f_j on \mathbb{C} such that $|f_j| < 1/j$ on U_1 and $|f_j - 1| < 1/j$ on $\Delta(0; r_0)$ for all $j \geq 1$. Define $\Psi_j : \mathbb{C}^n \times \mathbb{C}^{n-d_u} \rightarrow \mathbb{C}^n$ by

$$\Psi_j(p, \alpha) = p + (0, f_j(\pi_1 p)\alpha), \tag{5.5}$$

where here $0 \in \mathbb{C}^{d_u}$. Then for fixed $\alpha \in \mathbb{C}^{n-d_u}$, we have $\Psi_j(\cdot, \alpha) \in \text{Aut}(\mathbb{C}^n)$ and $\Psi_j(\cdot, 0) \equiv I$ for all j . For $j \geq 1$, let $G_{j,\alpha}(p) = \Psi_j(G(p), \alpha)$.

Now, let $f \equiv 0$ on U_1 and $f \equiv 1$ on $\Delta(0; r_0)$, and define $\Psi^\alpha = \Psi(\cdot, \alpha)$ as in (5.5) with f in place of f_j . In this case, $(\Psi^\alpha G)^{-1} = G^{-1}$ on $U_1 \times \mathbb{C}^{n-1}$, which contains a neighborhood of the forward orbit of q_0 under G . In particular, p_1^G is a saddle point for $\Psi^\alpha G$, and by iterating this map, we see that near q_0 , the local stable manifold of p_1^G for this map agrees with $W_j^s(G)$.

The same argument shows that p_2^G is a saddle point for $\Psi^\alpha G$ and that the unstable manifold for this map agrees with that for G near $G^{-1}(q_0)$. But then to this piece of the

unstable manifold, we first apply G and then the map Ψ^α . This has the effect of translating the original unstable manifold for G by $(0, \alpha)$ to obtain the unstable manifold for $\Psi^\alpha G$.

By lemma 4.5, we see that given $\epsilon > 0$, we can choose J large enough that if $j \geq J$ and $\|\alpha\| \leq \epsilon_1$, then $G_{j,\alpha}$ has periodic saddles $p_1(j, \alpha)$ and $p_2(j, \alpha)$, and that the parts of the corresponding stable and unstable manifolds obtained by iteration as in that lemma are graphs of holomorphic functions $\phi_{j,\alpha}^s : \pi_s U \rightarrow \mathbb{C}^{n-d_s}$ and $\phi_{j,\alpha}^u : \pi_u U \rightarrow \mathbb{C}^{n-d_u}$ with $\|\phi_{j,\alpha}^s - \phi_G^s\| < \epsilon$ and $\|\phi_{j,\alpha}^u - (\phi_G^u + \alpha)\| < \epsilon$ on their respective domains. Lemma 4.6 implies that $\phi_{j,\alpha}^s$ and $\phi_{j,\alpha}^u$ are holomorphic in (z, α) .

Using $\phi_{j,\alpha}^s$ and $\phi_{j,\alpha}^u$ in place of ϕ_F^s and ϕ_F^u in (5.1), we define $\Phi_j^\alpha(z) = \Phi_j(z, \alpha)$, and using this in place of Φ_F in (5.2) we define $\mu_j(z, \alpha)$, then use Φ_j^α and μ_j to define $\Gamma_j(z, \alpha)$ as in (5.3). Then each Γ_j is holomorphic, and $\Gamma_j \rightarrow \Gamma_G$ uniformly on $\overline{\pi_s U} \times \overline{\Delta^{n-d_u}(0; \epsilon)}$. Since $\pi_s U = U'$, we can apply lemma 5.5 to obtain J satisfying the conclusions of that lemma. Hence we can fix $j \geq J$ and choose α such that $G_{j,\alpha} \in \mathcal{M}$ and $\Gamma_j(0, \cdot, \alpha)$ has no zeroes in $\Delta^{d_s-d}(0; r'')$.

Finally, an argument like that in lemma 3.6 shows that the set $\{z \in \pi_s U : \Gamma_j(z, \alpha) = 0\}$ has dimension at most $d-1$, and since $V_j^0(G_{j,\alpha}) \cap U$ is the image of this set under the map $z \mapsto (z, \phi_{j,\alpha}^s(z))$, we see by [N, p. 46] that this latter set also has dimension at most $d-1$, as desired. ■

In the next lemma we apply the previous result inductively to perturb the stable and unstable manifolds to be transverse in a small neighborhood.

LEMMA 5.7 *Let $F \in \mathcal{N}_0$ with periodic saddle points p_1 and p_2 , and suppose $q_0 \in V_j^1(F)$. Then there exist neighborhoods $\mathcal{N} \subseteq \mathcal{N}_0$ of F and U of q_0 , plus an open dense subset $\mathcal{S} \subseteq \mathcal{N}$, such that if $G \in \mathcal{S}$, then $V_j^0(G) \cap U = \emptyset$.*

Proof: Let d be the dimension of $V_j^0(F)$ at q_0 . By the previous lemma, there exist neighborhoods U_0 of q_0 and $\mathcal{N} \subseteq \mathcal{N}_0$ of F , plus an open dense subset $\mathcal{T} \subseteq \mathcal{N}$ such that if $G \in \mathcal{T}$, then $\dim(V_j^0(G) \cap U) \leq d-1$. Let U be a neighborhood of q_0 which is relatively compact in U_0 , and let

$$\mathcal{S} = \{G \in \mathcal{N} : V_j^0(G) \cap \overline{U} = \emptyset\}.$$

If $G \in \mathcal{S}$, then $W_j^s(p_1^G)$ and $W_j^u(p_2^G)$ are transverse in a neighborhood of \overline{U} , and this will remain true for maps in a neighborhood of G by the stable manifold theorem. Thus \mathcal{S} is open.

For the density of \mathcal{S} , note that if $d \leq 0$, then the previous lemma applies to give the desired conclusion here. For the general case, we induct on d . Since \mathcal{T} is dense in \mathcal{N} , it suffices to show that \mathcal{S} is dense in \mathcal{T} , so let $G \in \mathcal{T}$.

Since $V_j^1(G) \cap \overline{U}$ is compact, we can cover it with finitely many neighborhoods from the previous lemma. For H in some neighborhood \mathcal{M} of G , $V_j^1(H) \cap \overline{U}$ will still be covered by these neighborhoods. Taking the intersection of \mathcal{M} with the corresponding neighborhoods of G and the open dense subsets from the previous lemma, we obtain a neighborhood of G and an open dense subset contained in \mathcal{S} . Thus \mathcal{S} is also dense. ■

Since the intersection $\overline{W_j^s(p_1^F)} \cap \overline{W_j^u(p_2^F)}$ is compact and contained in $W_{j+1}^s(p_1^F) \cap W_{j+1}^u(p_2^F)$, and since $W_{j+1}^s(p_1^G)$ and $W_{j+1}^u(p_2^G)$ vary continuously as complex manifolds for maps G in some neighborhood of the given map F , we can use the previous result together with a simple covering argument to obtain the following lemma.

LEMMA 5.8 *Let $F \in \mathcal{N}_0$ with periodic saddles p_1 and p_2 , and let $j > 0$. Then there exists a neighborhood \mathcal{N} of F and an open dense subset $\mathcal{S} \subseteq \mathcal{N}$ such that if $G \in \mathcal{S}$, then $W^s(p_1^G)$ and $W^u(p_2^G)$ are transverse at each point of $\overline{W_j^s(p_1^G)} \cap \overline{W_j^u(p_2^G)}$.*

Recall the definition of KS_j^M from definition 5.2.

PROPOSITION 5.9 *KS_j^M is an open dense subset of $\text{Aut}(\mathbb{C}^n)$.*

Proof: The openness is clear from transversality theory and the stable manifold theorem.

For density, it suffices to show that KS_j^M is dense in \mathcal{H}^M , so let $F \in \mathcal{H}^M$ and let $\mathcal{M} \subseteq \mathcal{H}^M$ be a neighborhood of F . Let p_1, \dots, p_N be the periodic saddles of F contained in $\overline{\mathbb{B}^n(0; M/2)}$ with period at most M . Shrinking \mathcal{M} , we may assume that if $G \in \mathcal{M}$, then any periodic saddle of G satisfying these conditions has the form p_k^G for some $k \in \{1, \dots, N\}$.

From the previous lemma, for each pair $1 \leq k, l \leq N$, there exists a neighborhood $\mathcal{N}_{k,l}$ of F and an open dense subset $\mathcal{S}_{k,l} \subseteq \mathcal{N}_{k,l}$ such that if $G \in \mathcal{S}_{k,l}$, then $W^s(p_k^G)$ and $W^u(p_l^G)$ are transverse at each point of $\overline{W_j^s(p_k^G)} \cap \overline{W_j^u(p_l^G)}$.

Taking the intersection of all of these neighborhoods and \mathcal{M} and the intersection of all of the dense subsets and \mathcal{M} , we obtain a neighborhood \mathcal{N} of F and an open dense subset \mathcal{S} of \mathcal{N} which is contained in KS_j^M . Hence KS_j^M is dense. ■

Finally, taking the intersection of all KS_j^M for $M, j \in \mathbb{Z}_+$ gives theorem 1.1.

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