Hyperbolic automorphisms and holomorphic motions in $\mathbb{C}^2$

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1 Introduction

Holomorphic motions have been an important tool in the study of complex dynamics in one variable. In this paper we provide one approach to using holomorphic motions in the study of complex dynamics in two variables. To introduce these ideas more fully, let Δ_r be the disk of radius r and center 0 in the plane, let $\mathbb{P}^1$ be the Riemann sphere, and recall that a holomorphic motion of a set $E \subset \mathbb{P}^1$ is a function $\alpha : \Delta_r \times E \to \mathbb{P}^1$ such that $\alpha(0, z) = z$ for each $z \in E$, $\alpha(\lambda, \cdot) : E \to \mathbb{P}^1$ is injective for each fixed $\lambda \in \Delta_r$, and $\alpha(\cdot, z) : \Delta_r \to \mathbb{P}^1$ is holomorphic for each fixed $z \in E$. For future reference, we note that this definition (as well as most results about holomorphic motions) applies equally well when the parameter λ is allowed to vary in the complex polydisk: $\lambda \in \Delta_r^n$.

One of the first uses of holomorphic motions in the study of complex dynamics was in the paper of Mañé-Sad-Sullivan [MSS], in which they use holomorphic motions to prove the density of structurally stable maps within the family of polynomial maps of $\mathbb{C}$ of degree d. In general, a map $f : M \to M$, M a manifold, is structurally stable within a family of maps, $\mathcal{F}$, if there is some neighborhood of f, say $\mathcal{U} \subset \mathcal{F}$, such that any map in $\mathcal{U}$ is conjugate to f via a homeomorphism of M. Mañé-Sad-Sullivan obtain structural stability for polynomial maps by showing that (subject to certain restrictions) the holomorphic motion defined naturally on the Julia set of a polynomial map extends to give a conjugacy on all of $\mathbb{C}$ to nearby polynomial maps. More precisely, they do this by starting with the canonical holomorphic motion defined on hyperbolic periodic points and on periodic points satsifying a critical orbit relation. By the λ-lemma of [MSS], this holomorphic motion extends uniquely to a holomorphic motion of the closure of the periodic points. They then construct by hand certain holomorphic motions which give partial conjugacies and which extend by iteration to give a holomorphic motion of a dense set of the plane, which again extends uniquely to give a topological conjugacy on the whole sphere.

Shortly after this work, Bers and Royden [BR] used the notion of a harmonic Beltrami coefficient (defined in section 6) to show that given any holomorphic motion of a set E, there is a canonical extension of this motion to a holomorphic motion of the sphere, although with

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a restriction to $\lambda \in \Delta_{r/3}$. The characterization of this extension is that in any component, S, of the complement of $\overline{E}$, the Beltrami coefficient $(\partial\alpha/\partial\bar{z})/(\partial\alpha/\partial z)$ is a harmonic Beltrami coefficient. Using this result, McMullen and Sullivan [MS] prove the density of structurally stable maps within the family of rational maps of $\mathbb{P}^1$ of degree d as follows: As before, given a family, f_λ, $\lambda \in \Delta$, with certain regularity properties, there is a canonical holomorphic motion on the closure of the set consisting of periodic points and orbits of critical points. By the Bers-Royden result, this motion extends canonically to a motion, α_λ, of the entire sphere. Then $f_\lambda^{-1} \circ \alpha_\lambda \circ f_0(z)$ defines a second holomorphic motion which agrees with the original motion on the periodic points and critical orbits, and which also has a harmonic Beltrami coefficient. By the uniqueness of the Bers-Royden extension, this second holomorphic motion agrees with the first, and hence α_λ is a global topological conjugacy.

Turning to higher dimensions, one natural family of maps with interesting dynamics in $\mathbb{C}^2$ is the family of (generalized) Hénon maps: compositions of holomorphic diffeomorphisms of the form $f(z,w) = (w, p(w) - az)$, where p is a polynomial of degree $d \geq 2$ and $a \neq 0$. We note here that for questions of structural stability, we will restrict ourselves to families of maps all having the same degree. This corresponds for example to considering structural stability of quadratic polynomials in one variable. With this restriction, the topology on Hénon maps can be specified either in terms of the coefficients of the defining maps or in terms of the compact-open topology, applied to the map and its inverse. Section 2 provides a more detailed account of Hénon maps and hyperbolicity. For further references, see the bibliography in [BuS].

There is an immediate generalization of holomorphic motions to two dimensions, simply allowing each point $z \in E$ to vary holomorphically within $\mathbb{C}^2$. In fact, by work of Mattias Jonsson [J], given a family, f_λ of hyperbolic Hénon maps, the set J_λ, which is the closure of the set of saddle periodic points of f_λ, varies as a holomorphic motion in this sense. However, this generalization fails to have many of the important properties of one variable holomorphic motions; in particular, given this kind of holomorphic motion on a set E, there is in general no unique extension to $\overline{E}$ and no canonical extension in the sense of Bers and Royden.

Our approach in this paper is to use the technique of McMullen and Sullivan to construct holomorphic motions on dynamically defined one-dimensional subsets of $\mathbb{C}^2$, then show that these maps define homeomorphisms on the union of these one-dimensional subsets. To be more precise, let f be a hyperbolic Hénon map, let J^+ (resp. J^-) be the boundary of the set of points with bounded forward (resp. backward) orbit, and let $J = J^+ \cap J^-$. Then J^+ and J^- are laminated by Riemann surfaces; each of these Riemann surfaces is conformally equivalent to the plane and is the stable or unstable manifold of a point in J. Given a one-parameter family, f_λ, of such maps, the points of intersection between J_λ^- and J_λ^+ define a holomorphic motion in each leaf, which extends canonically to the entire leaf by the Bers-Royden theorem. As in McMullen-Sullivan, this defines a conjugacy between f_0 on a leaf of J_0^+ and f_λ on a leaf of J_λ^+. However, since each leaf of J_0^+ is dense in J_0^+, it is not clear that the resulting conjugacy gives a homeomorphism of J_0^+ to J_λ^+. To establish that this map is a homeomorphism, we use the notion of an affine structure (see [G1], [G2] and [BS7]) to provide a coherent framework for discussing holomorphic motions on the leaves of the lamination. We show that the affine structure of J_λ^+ varies holomorphically with λ and that, suitably normalized, the global parametrizing functions for the leaves of J_λ^+ converge locally uniformly when approaching a limit leaf. With this, the continuity of the conjugacy follows essentially from the uniqueness of the Bers-Royden extension.

The first main result of this paper is the following theorem, which is an analog of the results of [MSS] and [MS], and which states that a hyperbolic Hénon map restricted to $J^+ \cup J^-$ is conjugate to nearby Hénon maps via a holomorphic motion of each leaf of $J^+ \cup J^-$.

THEOREM 1.1 *Let f_λ be a one-parameter family of hyperbolic Hénon maps depending holomorphically on $\lambda \in \Delta^n$. Then there exists $r > 0$ and a map*

$$\Psi : \Delta_r^n \times (J_0^+ \cup J_0^-) \to J_\lambda^+ \cup J_\lambda^-$$

such that defining $\Psi_\lambda(p) = \Psi(\lambda, p)$, we have

1. *$\Psi_0(p) = p$.*
2. *Ψ_λ is a homeomorphism for each fixed λ.*
3. *$\Psi_\lambda(p)$ is holomorphic in λ for each fixed $p \in J_0^+ \cup J_0^-$.*
4. *Ψ_λ maps each leaf of J_0^- (J_0^+) to a leaf of J_λ^- (J_λ^+).*
5. *$\Psi_\lambda f_0 = f_\lambda \Psi_\lambda$ on $J_0^+ \cup J_0^-$.*

The first three of the above properties are direct analogs of holomorphic motions in one variable, while the fourth property shows that the map respects the dynamically defined stable and unstable laminations.

In the study of the dynamics of polynomials in the plane, the polynomials with connected Julia set play a special role. In [BS6], Bedford and Smillie define the notion of an unstably connected Hénon map, which is an analog of a polynomial with a connected Julia set in one variable. They also show that given a hyperbolic Hénon map which is unstably connected, the lamination of J^+ extends to a lamination of $J^+ \cup U^+$, where U^+ is the set of points with unbounded forward orbits. With this additional structure, we obtain a conjugacy as above on $J^+ \cup U^+$.

THEOREM 1.2 *In addition to the hypotheses of theorem 1.1, assume that f_0 is unstably connected. Then the conclusions of that theorem remain valid when J_0^+ and J_λ^+ are replaced by $J_0^+ \cup U_0^+$ and $J_\lambda^+ \cup U_\lambda^+$, respectively.*

In particular, when f_0 is hyperbolic and unstably connected, this gives a canonical conjugacy between f_0 and f_λ on all of $\mathbb{C}^2$ except for the basins of any attracting periodic points.

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2 Preliminaries

We recall some standard terminology and some known results, which are discussed more fully in [BS1], [BS2] and [BS7]. Friedland and Milnor [FM] divide the polynomial automorphisms of $\mathbb{C}^2$ into two classes: elementary, which have relatively simple dynamics, and nonelementary. For brevity, we will use the term Hénon map to describe a nonelementary polynomial automorphism of $\mathbb{C}^2$. Such maps can be characterized by having dynamical degree $d \geq 2$, where the dynamical degree of a polynomial automorphism of $\mathbb{C}^2$ is defined as in [BS2] by

$$d = \lim_{n\to\infty} (\deg f^n)^{1/n},$$

where $\deg f^n$ denotes the maximum of the degrees of the two (polynomial) components of f^n.

Given a Hénon map, f, we let K^+/K^- denote the set of points in $\mathbb{C}^2$ with bounded forward/backward orbits under f, and let $J^\pm = \partial K^\pm$ and $J = J^+ \cap J^-$. Since $\det DF$ is constant on $\mathbb{C}^2$, we may replace f by f^{-1} if necessary to obtain $|\det Df| \leq 1$. From [BS1] and [BS2] it follows that if f is hyperbolic when restricted to J, then f is Axiom A, and in this case the nonwandering set consists of the basic set J plus a finite set of periodic sinks, S. The stable set of J, $W^s(J)$, is $J^+ = \partial K^+$, and the interior of K^+ consists of the basins of the sinks. The unstable set of J, $W^u(J)$ is $J^- \setminus S$, and the interior of K^- is empty. The sets $W^{s/u}(J)$ have dynamically defined Riemann surface laminations $\mathcal{W}^{s/u}$, whose leaves consist of stable/unstable manifolds of points in J. Each leaf of either lamination is conformally equivalent to $\mathbb{C}$. Also, J has local product structure, which means that there exist positive δ and ϵ such that if $x, y \in J$ with $\|x - y\| < \delta$, then $W^s_\epsilon(x)$ and $W^u_\epsilon(y)$ intersect in a unique point which is contained in J. Here $W^s_\epsilon(x)$ is the local stable manifold of x, defined as $\{p : \|f^n(x) - f^n(p)\| < \epsilon, \forall n \geq 0\}$, with an analogous definition for the local unstable manifold. As usual, we will use $W^s(p)$ and $W^u(p)$ for the stable and unstable manifolds of a point p.

Note that if f_λ is a one-parameter family of Hénon maps depending holomorphically on $\lambda \in \Delta$, and if f_0 is hyperbolic, then f_λ is also hyperbolic for all λ in some neighborhood of 0. Also, by [BS1], f_0 is Ω-stable, meaning that there is a one-parameter family of homeomorphisms $\psi_\lambda : J_0 \to J_\lambda$ conjugating $f_0|J_0$ to $f_\lambda|J_\lambda$. In fact, by work of Mattias Jonsson [J], for each $p \in J_0$, the map $\lambda \mapsto \psi_\lambda(p)$ is holomorphic in λ. Hence there is a natural holomorphic motion defined on J_0. Moreover, by restricting the domain of λ and possibly shrinking δ and ϵ, we may assume that the δ and ϵ chosen for the local product structure on J_0 apply equally to J_λ for each λ. For the remainder of the paper, we let δ_0 and ϵ_0 represent such a choice of δ and ϵ.

3 Unstable connectivity and critical points

For theorem 1.2, we need also the notion of an unstably connected Hénon map. Let $U^+ = \mathbb{C}^2 \setminus K^+$ be the set of points with unbounded forward orbit. Bedford and Smillie [BS6] define a Hénon map to be unstably connected with respect to a saddle point p if some component of $W^u(p) \cap U^+$ is simply connected. By theorem 0.1 of that paper, this is equivalent to the condition that for *any* saddle periodic point p, *each* component of $W^u(p) \cap U^+$ is simply connected, and in this case they say that f is *unstably connected.* By theorem 0.2 of the same paper, the assumption $|\det Df| \leq 1$ implies that f is unstably connected if and only if J is connected. As mentioned earlier, if f is hyperbolic, then f is Ω-stable, so if f is hyperbolic with connected J, then all nearby Hénon maps are also hyperbolic with connected J. Summarizing this argument, we have the following.

PROPOSITION 3.1 *Let f be a Hénon map of dynamical degree d, with $|\det Df| \leq 1$, and suppose that f is hyperbolic and unstably connected. Then there is a neighborhood $\mathcal{U}$ of f in the space of Hénon maps of degree d such that each $g \in \mathcal{U}$ is hyperbolic and unstably connected.*

As observed in [H] (see also [HO] and [BS1]), there is a plurisubharmonic function G^+ on $\mathbb{C}^2$ defined by

$$G^+(p) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ \|f^n(p)\|,$$

and this function is pluriharmonic on U^+ and satisfies $G^+ \circ f(p) = d \cdot G^+(p)$ and $G^+(x,y) = \log^+|y| + O(1)$ for $(x,y) \in V_R^+ = \{|y| > R, |x| < |y|\}$, R large. There is an analogous definition of G^- with f^{-n} in place of f^n. Since G^+ is pluriharmonic on U^+, it is locally the real part of a holomorphic function. In fact, in [HO, prop 5.4], it is shown that $G^+ = \mathrm{Re}\ \log(\phi^+)$ in V^+, where $\phi^+(x,y) = y + O(1)$. Hence the level sets of ϕ^+ define a nondegenerate holomorphic foliation $\mathcal{G}^+$ defined in V^+. Since U^+ is the union of all backward images of V^+ under f, and since f is a diffeomorphism, this foliation pulls back to give a holomorphic foliation $\mathcal{G}^+$ on U^+.

Finally, let $\mathcal{W}^s$ denote the lamination of J^+ by stable manifolds of J. We restate here a propostion due to Bedford and Smillie to the effect that if f is hyperbolic and unstably connected, then the foliation $\mathcal{G}^+$ and the lamination $\mathcal{W}^s$ fit together to form a lamination of $J^+ \cup U^+$.

PROPOSITION 3.2 *[BS6, Prop. 2.7] If f is hyperbolic and unstably connected, then there is a locally trivial lamination of $J^+ \cup U^+$ whose leaves are the leaves of $\mathcal{W}^s$ and $\mathcal{G}^+$.*

For polynomials of one complex variable, there is a close connection between connectivity of the Julia set and the behavior of critical points. In two variables, Bedford and Smillie [BS5] define the set of *unstable critical points* of a Hénon map to be the union over points $p \in J$ of the set of critical points of the Green function G^+ restricted to $W^u(p)$ (actually the union over all p for which the unstable manifold exists, which is a set of full μ-measure, where μ is the unique measure of maximal entropy). They show also that such a critical point is exactly a point of tangency between an unstable manifold of a point in J and a leaf of the foliation $\mathcal{G}^+$.

In case f is hyperbolic and unstably connected, there are no tangencies between the leaves of the unstable set $W^u(J)$ and the foliation $\mathcal{G}^+$, or equivalently, for each $p \in J$, the set $W^u(p) \cap U^+$ contains no unstable critical points. This fact is used in the proof of corollary A2 of [BS7], but is not stated explicity. Rather, Bedford and Smillie show in [BS6, Theorem 7.3] that f is unstably connected if and only if for μ almost every point p, $W^u(p) \cap U^+$ contains no unstable critical points. For completeness, we provide here a proof of the stronger result when f is hyperbolic and unstably connected.

PROPOSITION 3.3 *Let f be hyperbolic. Then f is unstably connected if and only if for each point $p \in J$, $W^u(p) \cap U^+$ has no unstable critical points, if and only if for each point $p \in J$, $W^u(p)$ is nowhere tangent to the leaves of the foliation $\mathcal{G}^+$.*

Proof: From [BS6, Theorem 7.3], f is unstably connected if and only if for μ almost every point p, $W^u(p) \cap U^+$ contains no unstable critical points, and by [BS5, Proposition B.1], an unstable critical point in $W^u(p) \cap U^+$ is exactly a tangency between $W^u(p)$ and a leaf of the foliation $\mathcal{G}^+$. Thus, we need prove only that if f is unstably connected, then for each point $p \in J$, $W^u(p) \cap U^+$ has no unstable critical points.

Now, the fact that f is hyperbolic implies that $W^u(p)$ exists for each $p \in J$ and that the unstable set $W^u(J)$ is a locally trivial lamination of J^-. Suppose there exists $p \in J$ such that $W^u(p)$ is tangent to a leaf of $\mathcal{G}^+$. Making a local biholomorphic change of coordinates in a neighborhood of the point of tangency, we may assume that the point of tangency is the origin in (z, w) coordinates, that $\mathcal{G}^+$ has leaves which are complex lines parallel to the z-axis, and that $W^u(p)$ is locally the graph of a holomorphic function $z \mapsto z^k h(z)$, $h(0) \neq 0$, $k \geq 2$. For any piece of a leaf of $W^u(q)$ sufficiently near this graph, the derivative of the corresponding graph for $W^u(q)$ will have a zero near the origin, hence there will be a tangency between between $W^u(q)$ and $\mathcal{G}^+$. Since each leaf of $W^u(J)$ is dense in J^- [BS2] and since these leaves form a locally trivial lamination, we see that for each $p \in J$, there is a tangency between $W^u(p)$ and $\mathcal{G}^+$.

Thus, if f is hyperbolic, then a tangency between $W^u(p)$ and $\mathcal{G}^+$ for one $p \in J$ implies a tangency between $W^u(q)$ and $\mathcal{G}^+$ for all $q \in J$, hence for a set of full μ measure, hence f is not unstably connected, as noted above. Taking the contrapositive, if f is unstably connected, then for each $p \in J$ there is no tangency between $W^u(p)$ and $\mathcal{G}^+$, hence no unstable critical points on $W^u(p) \cap U^+$. As noted above, this completes the proof. ■

4 Holomorphic families of laminations

In this section we discuss some uniformization properties of Riemann surface laminations and of holomorphic families of such laminations. Roughly, the main result is that given a holomorphic family of Riemann surface laminations in which each leaf is conformally equivalent to the complex plane, and given two holomorphic transversals to these laminations, there is a natural way of parametrizing a given leaf by the plane so that the parametrization of this leaf varies holomorphically with the family, and so that the points of intersection of this leaf with the two transversals are the images of 0 and 1 under the parametrization. Moreover, locally, this parametrization can be done in such a way that the parametrization converges locally uniformly when approaching a limit leaf. Precise definitions and results are given below.

We first recall the definition of a Riemann surface lamination of a topological space X, following [BS6] (see also [C], [G1], and [G2]). A chart consists of an open set $U_j \subset X$, a topological space Y_j, and a map $\rho_j : U_j \to \mathbb{C} \times Y_j$ which is a homeomorphism onto its image. An atlas consists of a collection of charts which covers X. For fixed $y \in Y_j$, the set of points $\rho_j^{-1}(\mathbb{C} \times \{y\})$ is called a plaque. For coordinate charts (ρ_i, U_i, Y_i) and (ρ_j, U_j, Y_j) with $U_i \cap U_j \neq \emptyset$, the transition function is the homeomorphism from $\rho_j(U_i \cap U_j)$ to $\rho_i(U_i \cap U_j)$

defined by $\rho_{ij} = \rho_i \circ \rho_j^{-1}$. A Riemann surface lamination, $\mathcal{L}$, of a topological space X is determined by an atlas of charts which satisfy the following consistency condition: the transition functions may be written in the form $\rho_{ij} = (g(z,y), h(y))$, where for fixed $y \in Y_j$, the function $z \mapsto g(z,y)$ is holomorphic. The condition on the transition functions gives a consistency between the plaques defined in U_j and those in U_i. thus plaques fit together to make global manifolds called leaves of the lamination, and each leaf has the structure of a Riemann surface.

In the current setting, we are interested in the Riemann surface laminations of J^+ and J^- given by stable and unstable manifolds and in the lamination of U^+ given by the foliation $\mathcal{G}^+$. Since these leaves have a natural holomorphic structure induced from $\mathbb{C}^2$, we will require additionally that each map ρ_j is holomorphic on each plaque. With this additional requirement, we can view a lamination of X as locally the "graph" of a holomorphic motion: At a point $p \in X$, let $\mathbf{v}$ be a vector in $\mathbb{C}^2$ such that $T = \mathbb{C}\mathbf{v}$ is a complex line transverse to the plaque through p. After a biholomorphic change of coordinates, we may assume that p is the origin and that $\mathbf{v} = (0,1)$. Let V be a small neighborhood of p, and let E be the set of points in V which lie on T. Then the plaques in $\mathcal{L}$ near the origin define a holomorphic motion with parameter z. I.e., there is a function $\alpha(z,w)$ defined for $(z,w) \in \Delta_\epsilon \times E$ which is holomorphic in z for each fixed $w \in E$, such that $\alpha(0,w) = w$, $\alpha(z,\cdot)$ is injective for each z, and such that a plaque of $\mathcal{L}$ through the point $(0,w)$ is given by the set of points $(z, \alpha(z,w))$, $z \in \Delta_\epsilon$. Moreover, there is a coherence property corresponding to the consistency requirement on the transition functions given above. In the current setting, the map $H(z,w) = (z, \alpha(z,w))$ is a homeomorphism from $\Delta_\epsilon \times E$ to an open set $U \subset X$ which is holomorphic for each fixed $w \in E$. Given a second point $\hat{p}$ and $\hat{H} : \Delta_\epsilon \times \hat{E} \to \hat{U}$ with $U \cap \hat{U} \neq \emptyset$, we have a transition function $H^{-1} \circ \hat{H}$, which can be written in the form $H^{-1} \circ \hat{H}(z,w) = (g(z,w), h(w))$, where for fixed w, the map $z \mapsto g(z,w)$ is holomorphic.

A holomorphic family of laminations is a generalization in which each plaque varies holomorphically with some parameter $\lambda \in \Delta_r^n$. For this purpose, we will restrict ourselves to families of laminations of sets in $\mathbb{C}^2$, and we will adopt the holomorphic motion view of laminations. So we say that $\mathcal{L}_\lambda$ is a holomorphic family of laminations depending on the parameter $\lambda \in \Delta_r^n$ if each for each fixed λ, $\mathcal{L}_\lambda$ is a lamination of a set X_λ in $\mathbb{C}^2$ such that each plaque is a Riemann surface as above and such that each plaque depends holomorphically on λ in the following sense. As above, for each point $p \in X_{\lambda_0}$ there is a local biholomorphic change of coordinates so that the image of p is the origin and $\mathbf{v} = (0,1)$ is transverse to the plaque of $\mathcal{L}_{\lambda_0}$ through the origin. Let E be the intersection of $T = \mathbb{C}\mathbf{v}$ and a small neighborhood of p in X_{λ_0}. Then we require $\epsilon > 0$ and the existence of a function $\alpha(z,w,\lambda)$ defined on $\Delta_\epsilon \times E \times \Delta_\epsilon^n(\lambda_0)$ which is holomorphic in (z,λ) for each fixed w, such that $\alpha(0,w,\lambda_0) = w$, $\alpha(z,\cdot,\lambda)$ is injective for each fixed (z,λ), such that the point $(0,\alpha(0,w,\lambda))$ is contained in X_λ for each $\lambda \in \Delta_\epsilon^n$, and such that for each $\lambda \in \Delta_\epsilon^n$, the plaque of $\mathcal{L}_\lambda$ through $(0,\alpha(0,w,\lambda))$ is given by the set of points $(z,\alpha(z,w,\lambda))$, $z \in \Delta_\epsilon$. I.e., α is a holomorphic motion of points $w \in E$ with parameters $(z,\lambda) \in \Delta_\epsilon \times \Delta_\epsilon^n(\lambda_0)$.

We will need a coherence condition on families of laminations also. We can view the family $\mathcal{L}_\lambda$ as sitting in $\mathbb{C}^2 \times \Delta_r^n$. Given a point $p \in X_{\lambda_0}$ and local change of coordinates as above, we require that the map $H(z,w,\lambda) = (z,\alpha(z,w,\lambda),\lambda)$ is a homeomorphism from $\Delta_\epsilon \times E \times \Delta_\epsilon^n(\lambda_0)$ to an open set U in $\cup_\lambda(X_\lambda \times \{\lambda\})$. Moreover, given a second point $\hat{p} \in X_{\hat{\lambda}_0}$

with $\hat{H} : \Delta_\epsilon \times \hat{E} \times \Delta^n_\epsilon(\hat{\lambda}_0) \to \hat{U}$, we require that the transition function $H^{-1} \circ \hat{H}$ can be written in the form $H^{-1} \circ \hat{H}(z, w, \lambda) = (g(z, w, \lambda), h(w), \lambda)$, where for fixed w, the map $(z, \lambda) \mapsto g(z, w, \lambda)$ is holomorphic in z and λ.

Note that the set $\{(z, \alpha(z, w, \lambda), \lambda) : z \in \Delta_\epsilon, \lambda \in \Delta^n_\epsilon(\lambda_0)\}$ is an $(n+1)$-dimensional holomorphic submanifold of $\mathbb{C}^{n+2}$. Hence the plaque of $\mathcal{L}_{\lambda_0}$ through p can be said to vary holomorphically with λ by viewing it as a slice of this submanifold. We call this submanifold a family of plaques associated with p. Each plaque in this family is associated with a unique leaf in the corresponding lamination $\mathcal{L}_\lambda$, so we may speak also of the family of leaves associated with p. We will see shortly that in the cases of interest for Hénon maps, the family of leaves through p is biholomorphic to $\mathbb{C} \times \Delta^n_r$.

The following is an immediate consequence of the implicit function theorem and the definitions given above. It says essentially that a point of transverse intersection between a holomorphic family of curves and a holomorphic family of plaques associated with a point varies holomorphically with the parameter.

LEMMA 4.1 *Let $\mathcal{L}_\lambda$ be a holomorphic family of laminations, let P_λ be the family of plaques associated with a point $p \in L_0$, and let $F : \Delta \times \Delta^n \to \mathbb{C}^2$ be holomorphic such that $F(0,0) = p$ and such that for each fixed λ, $F(\cdot, \lambda)$ is an injective immersion which is transverse to P_λ. Then there exists $\epsilon > 0$ and a holomorphic function $p : \Delta^n_\epsilon \to \mathbb{C}^2$ such that $p(0) = p_0$ and $p(\lambda) \in P_\lambda \cap F(\Delta, \lambda)$ for all $\lambda \in \Delta^n_\epsilon$.*

Note that if the point $p(\lambda)$ does not escape out the boundary of the image of F or the boundary of a plaque P_λ, then by the monodromy theorem $p(\lambda)$ may be analytically continued to all of Δ^n.

5 Stable manifolds and affine structures

Let f_λ be a one-parameter family of hyperbolic Hénon maps and recall from section 2 that there is a homeomorphism ψ_λ from J_0 to J_λ which is holomorphic in λ and which conjugates $f_0|J_0$ to $f_\lambda|J_\lambda$. Given a point p_0 in J_0, let p_λ be its image under ψ_λ, and let $W^{s/u}(p_\lambda)$ be the corresponding stable and unstable manifolds. In this section we show that the stable (and unstable) manifolds of p_λ can be parametrized by $\mathbb{C}$ in a way which depends holomorphically on λ and so that the parametrizations of nearby leaves converge locally uniformly to the parametrization of the family of leaves through p_λ.

Let S_λ denote the set of sink orbits for f_λ, and let $\mathcal{W}^u_\lambda$ denote the lamination of $J^-_\lambda \setminus S_\lambda$. Given $p \in J^-_\lambda$, write $L_\lambda(p)$ for the leaf of the lamination $\mathcal{W}^u_\lambda$ containing p.

As in [G1], [G2], and [BS7], we define an affine structure on a holomorphic curve L to be an atlas consisting of holomorphic diffeomorphisms χ_j from open sets U_j of L to open sets of $\mathbb{C}$ such that the U_j cover L and the $\chi_j \circ \chi_k^{-1}$ are restrictions of affine diffeomorphisms of $\mathbb{C}$ to their domains of definition. For three distinct points x, y, z in $\mathbb{C}$, the ratio $(x-y)/(x-z)$ is invariant under the group of affine diffeomorphisms of $\mathbb{C}$. If x, y, z are distinct nearby points of U_j, then the ratio $(\chi_j(x) - \chi_j(y))/(\chi_j(x) - \chi_j(z))$ depends only on the points x, y, and z not on the particular coordinate chart χ_j whose domain contains x, y, and z. Hence we may denote this function by $(x-y)/(x-z)$, which is holomorphic in x, y, and z, and which in fact is holomorphic as a map into $\mathbb{P}^1$ whenever x, y, and z are not all equal. An affine structure on a simply connected Riemann surface is said to be complete if it is isomorphic to $\mathbb{C}$ with its canonical affine structure.

If f_0 is hyperbolic, then for each $p_0 \in J_0$ there is an injective holomorphic map from $\mathbb{C}$ to the unstable manifold of p_0, and this map defines a complete affine structure on this unstable manifold. Moreover, the iterates of f_0 respect this affine structure in the sense that the pullback or pushforward of the affine structure from one leaf to another agrees with the original affine structure on the new leaf.

Let $\epsilon = \epsilon_0$ as chosen for the local product structure in section 2, fix $x_0 \in J_0^-$, and choose disjoint transversals T_1, T_2 to the local unstable manifold $W^u_\epsilon(x_0)$, and let T_3 be any other transversal to this local unstable manifold. For $x \in J_0^-$ near x_0, there are three points $p_j(x) = T_j \cap W^u_\epsilon(x)$, $j = 1, 2, 3$, and p_1, p_2 are distinct. The ratio $(p_1 - p_3)/(p_1 - p_2)$ is well-defined, independently of any particular choice of complex affine coordinate on $W^u(x)$. To say that the affine structure is continuous is to say that this ratio varies continuously with x, and proposition 5.1 of [BS7] implies that the affine structure on $\mathcal{W}^u$ is continuous. In fact, the following theorem of Ghys implies a stronger continuity property.

THEOREM 5.1 *[G2] Let $\mathcal{L}$ be a Riemann surface lamination of a subset, X, of a complex manifold such that each leaf of $\mathcal{L}$ is parabolic (conformally equivalent to the plane). Then the affine structure on leaves is continuous in the following sense: Let U be a chart of $\mathcal{L}$, and for each $i \geq 0$, let x_i, y_i, z_i be a triple of distinct points in U which for each fixed i are all three contained in the same plaque of $\mathcal{L}$. Suppose also that (x_i, y_i, z_i) converges to distinct points $(x_\infty, y_\infty, z_\infty)$ in U. Then the ratio $(x_i - y_i)/(x_i - z_i)$ converges to $(x_\infty - y_\infty)/(x_\infty - z_\infty)$.*

Note that in [G2], the laminated space is assumed to be compact. However, the compactness is used only to deduce that the conformal type of each leaf is independent of the Riemannian metric on the space. In the current setting, each leaf is parabolic using the standard metric on $\mathbb{C}^2$, so we may dispense with compactness.

We use the continuity of the affine structure to construct holomorphic parametrizations of leaves which converge locally uniformly when approaching a limit leaf. The essential idea is to choose a limit leaf along with two transversals to this leaf. Nearby leaves will also intersect these transversals, and we can choose the parametrization of leaves by the plane so that the images of 0 and 1 lie on these transversals. The continuity of the affine structure gives the local uniform convergence almost immediately. Note that we take a very myopic view when parametrizing leaves. In practice, one leaf will come back and accumulate on itself everywhere. For purposes of the parametrization, we work locally and regard each plaque as part of a separate leaf with its own parametrization. Thus one leaf may have many different parametrizations, any two of which differ by an affine transformation.

For the following proposition, let $\mathcal{L}$ be a lamination of a closed subset X of $\mathbb{C}^2$ such that each leaf, L, of $\mathcal{L}$ is parabolic. Also, let U be a chart of $\mathcal{L}$, and let $I = \mathbb{Z}^+ \cup \{\infty\}$.

PROPOSITION 5.2 *Let $x_i, y_i \in U$ for $i \in I$ with $x_i \to x_\infty$ and $y_i \to y_\infty$, $x_\infty \neq y_\infty$ and such that for each $i \in I$, x_i and y_i are contained in the same leaf, L_i, of $\mathcal{L}$ and in the same plaque within U. Let $\phi_i : \mathbb{C} \to L_i$ be injective holomorphic for $i \in I$ with $\phi_i^{-1}(x_i) \to \phi_\infty^{-1}(x_\infty)$ and $\phi_i^{-1}(y_i) \to \phi_\infty^{-1}(y_\infty)$. Then $\phi_i \to \phi_\infty$ uniformly on each compact subset of $\mathbb{C}$.*

Proof: Let P_i be the plaque of U containing x_i, y_i. We will show first that ϕ_i converges to ϕ_∞ uniformly on each compact subset of $\phi_\infty^{-1}(P_\infty) \subset \mathbb{C}$. By assumption on U, there exists a biholomorphic change of coordinates such that P_∞ is an open set in the z-axis $(\mathbb{C} \times \{0\})$. By restricting to sufficiently large i, we may assume that the projection $\pi_i : P_i \to (\mathbb{C} \times \{0\})$ is injective holomorphic for each i (and $\pi_\infty = \mathrm{Id}$). Moreover, $\pi_i^{-1}\pi_\infty$ converges to the identity uniformly on compact subsets of P_∞ as $i \to \infty$ (e.g. by the λ-lemma of [MSS]).

Let $\gamma \in \pi_\infty(P_\infty)$ be a simple closed curve with $x_\infty, y_\infty \notin \pi_\infty^{-1}(\gamma)$, and let $N_\gamma = U_{i \in I}\, \pi_i^{-1}(\gamma)$. Then N_γ is compact and $x_i, y_i \notin N_\gamma$ for i large. Define $R_i(p)$ on P_i, $i \in I$, by

$$R_i(p) = \frac{\phi_i^{-1}(x_i) - \phi_i^{-1}(p)}{\phi_i^{-1}(x_i) - \phi_i^{-1}(y_i)}.$$

Since $x_\infty \neq y_\infty$ and the preimages of x_i and y_i converge to the preimages of x_∞ and y_∞, repectively, we see that for large i, R_i is well-defined and holomorphic on P_i. Moreover, $R_i(p)$ is precisely the ratio function applied to the triple (x_i, y_i, p). Viewing $R_i(p) = R(i,p)$ as a function on the compact set N_γ, the theorem of Ghys implies that R is continuous on N_γ, hence uniformly continuous. In particular, $\phi_i^{-1} \circ \pi_i^{-1} \to \phi_\infty^{-1} \circ \pi_\infty^{-1}$ uniformly on γ, hence on the interior of γ by Cauchy's formula, hence on each compact subset of P_∞.

Thus $(\pi_i \circ \phi_i)^{-1} \to (\pi_\infty \circ \phi_\infty)^{-1}$ uniformly on compact subsets of $\pi_\infty(P_\infty)$. Since $\pi_\infty \phi_\infty$ is injective holomorphic, this implies that $\pi_i \circ \phi_i$ converges to $\pi_\infty \circ \phi_\infty$ uniformly on compact subsets of $\phi_\infty^{-1}(P_\infty)$ (e.g. by the integral formula for the inverse of a holomorphic map). Since $\pi_i^{-1} \circ \pi_\infty$ converges to the identity uniformly on compact subsets of P_∞, this implies that ϕ_i converges to ϕ_∞ uniformly on compact subsets of $\phi_\infty^{-1}(P_\infty)$.

To complete the proof, let $K \subset \mathbb{C}$ be compact, and cover $\phi_\infty(K)$ by finitely many plaques $P_{\infty,1}, \ldots, P_{\infty,m}$ with $P_{\infty,j} \cap P_{\infty,j+1} \neq \emptyset$ for $j = 1, \ldots, m-1$ and $P_{\infty,1} = P_\infty$. The preceding construction implies that ϕ_i converges to ϕ_∞ uniformly on compact subsets of $\phi_\infty^{-1}(P_{\infty,1})$. Since $P_{\infty,1}$ and $P_{\infty,2}$ are open and have nonempty intersection, we can apply the same argument to two new sequences of points with limits in their intersection to conclude that ϕ_i converges to ϕ_∞ uniformly on compact subsets of $\phi_\infty^{-1}(P_{\infty,2})$. By induction, we obtain uniform convergence on all of K. ■

In dealing with families of Hénon maps, we will need a parametrized version of the above result. First a definition.

DEFINITION 5.3 *Let $\mathcal{L}_\lambda$, $\lambda \in \Delta^n$, be a holomorphic family of laminations. We say that $\mathcal{L}_\lambda$ is leafwise trivial if for each leaf L_{λ_0}, there exists $\epsilon > 0$ such that the set $Z := \{(\lambda, p) : \lambda \in \Delta^n_\epsilon(\lambda_0), p \in L_\lambda\}$ is biholomorphic to $\Delta^n_\epsilon \times \mathbb{C}$.*

As an example of how a holomorphic family of leaves could fail to be trivial in this sense, consider a $\mathbb{P}^1$ bundle over Δ^n, then remove a section over Δ^n which is not holomorphic. Then each leaf is biholomorphic to $\mathbb{C}$, but the bundle is not biholomorphic to $\Delta^n \times \mathbb{C}$.

In the following theorem, $I = \mathbb{Z}^+ \cup \{\infty\}$, as before.

THEOREM 5.4 *Let $\mathcal{L}_\lambda$, $\lambda \in \Delta^n$, be a leafwise trivial holomorphic family of laminations. Let $x_i(\lambda)$, $y_i(\lambda)$, $i \in I$, be holomorphic in λ with $x_i(\lambda) \neq y_i(\lambda)$ for each i and λ, and such that for all λ, $y_i(\lambda)$ is contained in the plaque through $x_i(\lambda)$. Suppose also that $x_i(\lambda)$ converges to $x_\infty(\lambda)$ and $y_i(\lambda)$ converges to $y_\infty(\lambda)$ uniformly on compact subsets of Δ^n as $i \to \infty$. Let $L_{i,\lambda}$*

be the leaf through $x_i(\lambda)$, and let $\phi_{i,\lambda} : \mathbb{C} \to L_{i,\lambda}$ be injective holomorphic with $\phi_{i,\lambda}(0) = x_i(\lambda)$ and $\phi_{i,\lambda}(1) = y_i(\lambda)$.

Then $\phi_i(\lambda, z) = \phi_{i,\lambda}(z)$ is holomorphic in (λ, z), and ϕ_i converges to ϕ_∞ uniformly on compact subsets of $(z, \lambda) \in \mathbb{C} \times \Delta^n$.

Proof: Since $\mathcal{L}_\lambda$ is leafwise trivial, it is a locally trivial fibration over Δ^n, hence is biholomorphic to $\Delta^n \times \mathbb{C}$ by [W, lemma 4.4]. Hence there exist injective holomorphic maps $\Phi_{i,\lambda} : \mathbb{C} \to L_{i,\lambda}$ such that $\Phi_{i,\lambda}(z)$ is holomorphic in $(\lambda, z) \in \Delta^n \times \mathbb{C}$.

Since $x_i(\lambda)$ and $y_i(\lambda)$ are holomorphic in λ, we see that $X_i(\lambda) := \Phi_{i,\lambda}^{-1}(x_i(\lambda))$ and $Y_i(\lambda) := \Phi_{i,\lambda}^{-1}(y_i(\lambda))$ are holomorphic from Δ^n to $\mathbb{C}$, and by the injectivity of $\Phi_{i,\lambda}$, we have $X_i(\lambda) \neq Y_i(\lambda)$. Since injective maps from the plane to itself are unique up to affine map, we see that $\phi_{i,\lambda}(z) = \Phi_{i,\lambda}(X_i(\lambda) + z(Y_i(\lambda) - X_i(\lambda)))$ is holomorphic in (λ, z) as desired.

Finally, the uniform convergence of ϕ_i to ϕ_∞ follows almost exactly as in the proof of proposition 5.2, using the function $R_{i,\lambda}$ given by the formula for R_i with $\phi_{i,\lambda}^{-1}$ in place of ϕ_i^{-1}. ■

Next, we show that the leaves of the dynamical laminations generated by a hyperbolic Hénon map are leafwise trivial holomorphic families of laminations.

THEOREM 5.5 *Let f_λ be a family of hyperbolic Hénon maps depending holomorphically on $\lambda \in \Delta^n$, and let $\mathcal{W}^u_\lambda$ be the lamination of J^-_λ whose leaves are the unstable manifolds of J_λ. Then $\mathcal{W}^u_\lambda$ is a leafwise trivial holomorphic family of laminations. Likewise $\mathcal{W}^s_\lambda$ is a leafwise trivial holomorphic family of laminations.*

Moreover, if each f_λ is unstably connected and $\mathcal{L}_\lambda = \mathcal{W}^s_\lambda \cup \mathcal{G}^+_\lambda$, then again $\mathcal{L}_\lambda$ is a leafwise trivial holomorphic family of laminations.

Proof: The proof of the (un)stable manifold theorem for hyperbolic sets as in [S, Chap. 6] relies on a contraction mapping argument applied to a Banach space of bounded sections over J_λ. Starting with initial approximations to the unstable manifolds which vary holomorphically with λ, the uniform convergence obtained from the contraction implies that the unstable manifolds for J_λ will vary holomorphically with λ in the sense that the family of leaves associated with a point varies holomorphically with λ. Thus $\mathcal{L}_\lambda$ is a holomorphic family of laminations.

For the leafwise triviality, [BS1, theorem 5.4] implies that for $x_{\lambda_0} \in J_{\lambda_0}$, we can exhaust $W^u(x_{\lambda_0})$ by an increasing union of disks. Since the family of leaves L_λ associated with x_{λ_0} varies holomorphically with λ, the same argument implies that there exists $\epsilon > 0$ and injective holomorphic maps $H_j : \Delta \times \Delta^n_\epsilon(\lambda_0) \to Z$, where Z is the manifold of leaves associated with x_{λ_0} as in definition 5.3, such that the image of H_j is contained in the image of H_{j+1} and such that the union of their images is all of Z. Since each leaf is conformally equivalent to $\mathbb{C}$, [FS] implies that Z is biholomorphic to $\mathbb{C} \times \Delta^n_\epsilon$, so $\mathcal{L}_\lambda$ is leafwise trivial.

Finally, suppose f_λ is unstably connected for all λ. The function $G^+_\lambda(p)$ is pluriharmonic in (λ, p) by [BS1, proposition 3.3], hence is locally the real part of a function Ψ which is holomorphic in (λ, p). Then the plaques of $\mathcal{G}^+_\lambda$ are precisely the level sets of $\Psi(\lambda, \cdot)$, hence these plaques vary holomorphically in λ, so $\mathcal{L}_\lambda$ is a holomorphic family of laminations. The fact that $\mathcal{L}_\lambda$ is leafwise trivial in this case follows as above, using the ideas in the proof of theorem 7.2 in [HO] to produce the increasing sequence of biholomorphic images of bidisks. ■

Collecting the results of this section, we obtain the following result, which allows us to parametrize leaves of $\mathcal{W}^u_\lambda$ and $\mathcal{W}^s_\lambda$ holomorphically in λ so that the parametrizations converge locally uniformly when approaching a limit leaf. For this proposition, let $\epsilon = \epsilon_0$ be as chosen for local product structure. Moreover, if necessary we may shrink this ϵ so that at each point of J_λ, the bidisk of size 2ϵ with axes parallel to the stable and unstable directions at this point defines a chart for the stable and unstable laminations.

THEOREM 5.6 *Let f_λ be a family of hyperbolic Hénon maps depending holomorphically on $\lambda \in \overline{\Delta^n}$. Let $p \in J_0$, $q \in J_0 \cap W^s_\epsilon(p)$ with $q \neq p$, and let $p_\lambda = \psi_\lambda(p)$, $q_\lambda = \psi_\lambda(q)$. Then there exists $\phi_\lambda : \mathbb{C} \to \mathbb{C}^2$ injective for each fixed λ and holomorphic in $(z, \lambda) \in \mathbb{C} \times \Delta^n$ such that $\phi_\lambda(\mathbb{C}) = W^s(p_\lambda)$, $\phi_\lambda(0) = p_\lambda$, and $\phi_\lambda(1) = q_\lambda$. Moreover, if $p^j \in J_0$ with $p^j \to p$ and $q^j \in J_0 \cap W^s_\epsilon(p^j)$ with $q^j \to q$ and ϕ^j is the corresponding parametrization for each j, then ϕ^j_λ converges to ϕ_λ uniformly on compact subsets of $\mathbb{C} \times \overline{\Delta^n}$. There is an analogous result for $W^u(p_\lambda)$.*

Proof: By theorem 5.5, $\mathcal{W}^u_\lambda$ is a leafwise trivial family of laminations. Hence theorem 5.4 applies to give ϕ_λ with the stated properties and shows that if p^j_λ and q^j_λ converge uniformly on compacts to p_λ and q_λ respectively, then ϕ^j_λ converges uniformly on compacts to ϕ_λ. Hence it suffices to show the uniform convergence of p^j_λ and q^j_λ to p_λ and q_λ.

To do this, define holomorphic maps $h_j(\lambda) = p^j_\lambda$ and $h(\lambda) = p_\lambda$, where $p^j_\lambda = \psi_\lambda(p^j)$. Note that since we have restricted to λ in the closed polydisk $\overline{\Delta^n}$, the filtration argument in [BS1] implies that there exists some $R > 0$ so that J_λ is contained in Δ^2_R independently of λ. In particular, h_j is uniformly bounded by R, independently of λ and j. Note also that for each fixed λ, ψ_λ is a homeomorphism, and since $p^j \to p$, we have $\psi_\lambda(p^j) \to \psi_\lambda(p)$ for each fixed λ. Hence $\{h_j\}_j$ is a uniformly bounded sequence of holomorphic maps which converges pointwise to h. Since the sequence is uniformly bounded, it is equicontinuous, and this plus pointwise convergence implies uniform convergence. Thus p^j_λ converges uniformly on compacts to p_λ, and likewise for q^j_λ, which as noted above implies the convergence of ϕ^j_λ to ϕ_λ. ■

We need an analogous parametrization for leaves of $\mathcal{G}^+_\lambda$ in the unstably connected case. Since ψ_λ is not defined outside J_0 we will have to work a bit harder. First a theorem which will allow us to extend ψ_λ to $U^+_0 \cap J^-_0$.

THEOREM 5.7 *Let f_λ be a family of unstably connected hyperbolic Hénon maps depending holomorphically on $\lambda \in \overline{\Delta^n}$. Let $p \in (J^+_0 \cup U^+_0) \cap J^-_0$. Let L^+_λ be the family of leaves of $\mathcal{G}^+_\lambda \cup \mathcal{W}^s_\lambda$ through p, and let L^-_λ be the family of leaves of $\mathcal{W}^u_\lambda$ through p. Then there exists a unique map $\lambda \to p_\lambda \in \mathbb{C}^2$ bounded and holomorphic in $\lambda \in \overline{\Delta^n}$ such that $p_0 = p$ and $p_\lambda \in L^-_\lambda \cap L^+_\lambda$ for each λ.*

Moreover, if $p^j \in (J^+_0 \cup U^+_0) \cap J^-_0$ and $p^j \to p$, then p^j_λ converges to p_λ uniformly on $\overline{\Delta^n}$.

Proof: We first construct p_λ. For this purpose, if $p \in J_0$, then $p_\lambda = \psi_\lambda(p)$ satisfies the conclusions, hence we assume $p \in U^+_0$. Choose a chart containing p for the family of laminations $\mathcal{G}^+_\lambda$, $\lambda \in \Delta^n_\epsilon$ and let P^+_λ be the family of plaques through p. Likewise, let P^-_λ be the family of plaques of $\mathcal{W}^u_\lambda$ through p. Since f_0 is hyperbolic and unstably connected, lemma 4.1 implies that p_λ is defined uniquely for λ near 0 as the intersection of P^+_λ and P^-_λ.

Note that by definition of the lamination $\mathcal{G}^+_\lambda$, the function $G^+_\lambda(p_\lambda)$ is constant. Note also that since λ is restricted to the closed polydisk in the hypothesis of the lemma, it follows from [BS1] that there exists $R > 0$ independent of λ so that J^-_λ is contained in $\Delta^2_R \cap V^+_R$ and that for a given constant C, the intersection of $\Delta^2_R \cap V^+_R$ with the level set $\{G^+_\lambda(x, y) = C\}$ is contained in $\{|y| < R'\}$ for some $R' > 0$ independent of λ. Hence replacing R by the max of R and R', we have that p_λ is contained in Δ^2_R, and this will remain true if we continue p_λ within the intersection of J^-_λ and the same level set of G^+_λ.

We now continue p_λ throughout $\overline{\Delta^n}$. Suppose that γ is any closed curve from $[0, 1]$ to $\overline{\Delta^n}$ and suppose that p_λ is defined and holomorphic at each point $\lambda \in \gamma([0, 1))$. Since p_λ is uniformly bounded, we can take a sequence $t_j \in [0, 1)$, t_j increasing to 1 such that for $\lambda_j = \gamma(t_j)$, the points p_{λ_j} converge to some point q. Let $\lambda_0 = \gamma(1)$. Since $p_\lambda \in J^-_\lambda$ for all λ and since the union over $\lambda \in \overline{\Delta^n}$ of $J^-_\lambda \times \{\lambda\}$ is closed as a subset of $\mathbb{C}^2 \times \overline{\Delta^n}$, we have $q \in J^-_{\lambda_0}$. Also, since $G^+_\lambda(p_\lambda)$ is a constant $C > 0$ we have $G^+_\lambda(q) = C$ and hence $q \in U^+_{\lambda_0}$. In particular, q is the point of intersection of plaques of the corresponding laminations, hence has an extension q_λ as above for λ in some neighborhood of λ_0.

Note that if $q_\lambda = p_\lambda$ at some point λ in their set of common definition, then the local unique extension in terms of intersecting plaques implies that they agree on an open set, hence everywhere they are both defined. Thus q_λ will be a continuation of p_λ once we show that they agree at one point.

In a neighborhood of q_{λ_0} let $\Psi_\lambda(x, y)$ be holomorphic in (λ, x, y) with Re $\Psi_\lambda(x, y) = G^+_\lambda(x, y)$. Then the level sets of Ψ_λ define the lamination $\mathcal{G}^+_\lambda$, hence $\Psi_{\lambda_j}(p_{\lambda_j})$ is a constant C independent of j, hence equal to $\Psi_{\lambda_0}(q)$. In a neighborhood of q, and for λ near λ_0, there is a fixed complex line independent of λ through q such that the projection of the level set $\{\Psi_\lambda = C\}$ to this line is injective holomorphic. Moreover, the points of intersection of J^-_λ with this level set define a holomorphic motion via projection to this complex line. Because J^-_λ intersects the set $\{\Psi_\lambda = C\}$ transversally for all λ near λ, we can choose a small neighborhood, Y, of q, then restrict λ to a sufficiently small neighborhood of λ_0 such that each point in Y which is a point of intersection between $\{\Psi_\lambda = C\}$ and J^-_λ has a continuation as such a point of intersection for all λ in this small neighborhood.

For j sufficiently large, p_{λ_j} is such a point of intersection, and the continuation of p_{λ_j} must agree with the extension of p_{λ_k} since p_λ is defined as a point of intersection. Hence p_λ has an extension to λ in a neighborhood of λ_0. Then p_λ and q_λ both project to the complex line chosen above, and their images are points in the holomorphic motion. Corollary 2 of [BR], implies that given $r > 0$ small these points of the holomorphic motion are constrained to lie in a small neighborhood of q for $\|\lambda - \lambda_0\| \leq r$ From the injectivity of a holomorphic motion and the compactness of this parameter range, these two points must be either identical for all such λ or distinct with a positive lower bound on their closest approach. Since p_{λ_j} converges to q by hypothesis, the two images must be identical.

Hence q_λ agrees with p_λ for some λ where both are defined. As noted above, this implies that they agree on an open set, hence q_λ is a continuation of p_λ. By the monodromy theorem, p_λ extends to all of $\overline{\Delta^n}$.

Suppose now that p^j converges to p as in the statement of the theorem. We wish to show that p^j_λ converges uniformly on $\overline{\Delta^n}$ to p_λ. However, since the p^j_λ are uniformly bounded, the argument in the proof of theorem 5.6 implies that we need show only that p^j_λ converges to p_λ for each fixed λ.

Let P^+_λ and P^-_λ be the family of plaques through p for λ in some small neighborhood of 0. Since the family of plaques $\{P^+_\lambda\}_\lambda$ form a holomorphic manifold, M^+, of dimension $n+1$ in $\mathbb{C}^2 \times \Delta^n_\epsilon$, there is an open set in this ambient space and a bounded holomorphic function H^+ defined on this open set such that M^+ is the precisely the zero set of H^+. Likewise, for H^- and M^-.

For j sufficiently large and λ in some small polydisk, D^n, independent of j, the point p^j_λ is contained in the set where $H^\pm$ are defined, and since p^j_λ is defined as the point of intersection of two leaves of the stable and unstable laminations, we see that for fixed j, $H^\pm(p^j_\lambda)$ is either 0 for all λ near 0 or never 0. Moreover, since $H^\pm$ is bounded, the set of functions $h^\pm_j(\lambda) = H^\pm(p^j_\lambda)$ is a normal family. Now, given any subsequence of $h^\pm_j$, we can extract a locally uniformly convergent subsequence, and since $p^j = p^j_0$ converges to $p = p^0$, the limit function must have a zero at $\lambda = 0$, hence the limit function must be identically 0 by Hurwitz' theorem. Since this is true for any initial subsequence, it follows that $h^\pm_j$ converges to 0 pointwise as $j \to \infty$ for each $\lambda \in D^n$. Since the $h^\pm_j$ are uniformly bounded, we have as before that the convergence to 0 is uniform on compact sets. From the definition of $h^\pm_j$ in terms of $H^\pm$, this implies that p^j_λ converges to p_λ uniformly for λ in compact subsets of D^n.

Finally, recall that the points p^j_λ are uniformly bounded, hence form a normal family. Given any subsequence, and any further locally uniformly convergent subsequence, the argument above implies that the limit function agrees with p_λ on some neighborhood of 0, hence everywhere. Since this is true for any initial subsequence, the functions p^j_λ must converge pointwise to p_λ on all of $\overline{\Delta^n}$, and since they are uniformly bounded, we see that the convergence is uniform on this compact set. ■

COROLLARY 5.8 *Let f_λ be as in the previous theorem. Then the map $\psi_\lambda : J_0 \to J_\lambda$ extends to a a map $\psi_\lambda : (J^+_0 \cup U^+_0) \cap J^-_0$ such that ψ_0 is the identity, ψ_λ is a homeomorphism for each fixed λ, and $\psi_\lambda(p)$ is holomorphic in λ for each fixed p.*

Proof: The theorem implies that given $p \in (J^+_0 \cup U^+_0) \cap J^-_0$, we can define $\psi_\lambda(p) = p_\lambda$, and that this extension is continuous and holomorphic in λ. Moreover, for any fixed λ_0, we can apply the theorem to obtain $\psi_{\lambda_0,\lambda}$ taking $J^+_{\lambda_0} \cup U^+_{\lambda_0}) \cap J^-_{\lambda_0}$ to $J^+_\lambda \cup U^+_\lambda) \cap J^-_\lambda$. The uniqueness part of the theorem implies that $\psi^{-1}_\lambda = \psi_{\lambda,0}$, hence ψ_λ is injective with continuous inverse, as desired. ■

We are now ready to give a version of theorem 5.6 in the unstably connected case. The proof is the same as the proof of theorem 5.6, using the corollary to obtain the homeomorphism ψ_λ.

THEOREM 5.9 *Let f_λ be a family of hyperbolic, unstably connected Hénon maps depending holomorphically on $\lambda \in \overline{\Delta^n}$. Let $A_0 = (J^+_0 \cup U^+_0) \cap J^-_0$, let $p \in A_0$, and let $q \in A_0$ be in the same plaque of $\mathcal{W}^s \cup \mathcal{G}^+$ as p with $p \neq q$. Let p_λ and q_λ be the points defined in the* *previous theorem. Then there exists $\phi_\lambda : \mathbb{C} \to \mathbb{C}^2$ injective for each fixed λ and holomorphic in $(z, \lambda) \in \mathbb{C} \times \Delta^n$ such that $\phi_\lambda(\mathbb{C})$ equals the leaf of $\mathcal{W}^s_\lambda \cup \mathcal{G}^+_\lambda$ through p_λ, with $\phi_\lambda(0) = p_\lambda$, and $\phi_\lambda(1) = q_\lambda$. Moreover, if $p^j \in A_0$ with $p^j \to p$ and $q^j \in A_0$ in the same plaque as p^j with $q^j \to q$ and ϕ^j is the corresponding parametrization for each j, then ϕ^j_λ converges to ϕ_λ uniformly on compact subsets of $\mathbb{C} \times \Delta^n$. There is an analogous result for leaves of $\mathcal{W}^u_\lambda$.*

6 Holomorphic motions

We recall the following theorem, due to Bers and Royden [BR], on the canonical extension of a holomorphic motion of a set $E \subset \mathbb{P}^1$ to a holomorphic motion on $\mathbb{P}^1$. For more background, see [BR].

THEOREM 6.1 *[BR] Let $\tau : \Delta \times E \to \mathbb{P}^1$ be a holomorphic motion. Then τ restricted to $\Delta_{1/3} \times E$ has a canonical extension to a holomorphic motion $\tau : \Delta_{1/3} \times \mathbb{P}^1 \to \mathbb{P}^1$. This extension is characterized by the following property: Let $\mu(\lambda, z)$ be the Beltrami coefficient of $z \mapsto \tau(\lambda, z)$ and let S be any component of $\mathbb{P}^1 \setminus \hat{E}$, where $\hat{E}$ is the closure of E in $\mathbb{P}^1$. Then*

$$\mu(\lambda, z) = \rho_S(z)^{-2}\overline{\psi(\lambda, z)} \tag{6.1}$$

for $z \in S$, $\lambda \in \Delta_{1/3}$, where $\rho_S(z)|dz|$ is the hyperbolic metric in S and the function $\psi(\lambda, z)$ is holomorphic in $z \in S$, antiholomorphic in $\lambda \in \Delta_{1/3}$.

This theorem is true also if the disk is replaced by the ball in $\mathbb{C}^n$. See [Su] or [Mi].

A Beltrami coefficient of the form in (6.1) is said to be a harmonic Beltrami coefficient. The hyperbolic metric is also known as the Poincaré metric and the infinitesimal Kobayashi metric.

The parametrization of leaves given in the previous section gives us a way to speak of a holomorphic motion on leaves.

DEFINITION 6.2 *Let $\phi : \Delta^n \times \mathbb{C} \to \mathbb{C}^2$ be holomorphic and suppose that $\phi_\lambda = \phi(\lambda, \cdot)$ is injective for each fixed $\lambda \in \Delta^n$. Let $E_0 \subset \phi(0, \mathbb{C})$. Then $\tau : \Delta^n \times E_0 \to \mathbb{C}^2$ is a holomorphic motion of E_0 on the family of leaves defined by ϕ means that $\tau_\lambda(E_0) = \tau(\lambda, E_0)$ is contained in the leaf $\phi(\lambda, \mathbb{C})$ for each λ, and $\phi_\lambda^{-1}\tau_\lambda\phi_0$ is a standard holomorphic motion in $\mathbb{C}$ of the set $\phi_0^{-1}(E_0)$.*

In particular, given a holomorphic motion on leaves, we can pull it back to a holomorphic motion in the plane, then apply the Bers-Royden extension and push forward to obtain an extended holomorphic motion on leaves. We will call this extension the Bers-Royden extension also.

We record here also a notion for the convergence of holomorphic motions on leaves when approaching a limit leaf. Let $I = \mathbb{Z}^+ \cup \{\infty\}$. In the following definition, the Hausdorff metric on sets in the plane is defined with respect to the spherical metric, denoted here by d_s, on the Riemann sphere.

Notation: With ϕ and τ as in the previous definition, let $\phi_*[\tau_\lambda]$ denote the map $\phi_\lambda^{-1}\tau_\lambda\phi_0$ defined on $\phi_0^{-1}(E_0)$.

DEFINITION 6.3 *For each $i \in I$, let $\phi^i : \Delta^n \times \mathbb{C} \to \mathbb{C}^2$ be holomorphic with $\phi^i_\lambda = \phi^i(\lambda, \cdot)$ injective for each fixed λ, and suppose that ϕ^i converges to ϕ^∞ uniformly on compact sets. Let $E^i \subset \phi^i(0, \mathbb{C})$ for each $i \in I$, and let $\tau^i : \Delta^n \times E^i$ be a holomorphic motion on the leaves defined by ϕ^i. Then τ^i converges uniformly to τ^∞ means that the sets $A^i = (\phi^i_0)^{-1}(E^i)$ converge to A^∞ in the Hausdorff metric and that the corresponding holomorphic motions in the plane converge uniformly on compacts: For each $\epsilon > 0$, there exist $\delta > 0$ and $N > 0$ such that if $i > N$ and $\|\lambda_1 - \lambda_2\| + d_s(z_1, z_2) < \delta$, $z_1 \in A^i$, $z_2 \in A^\infty$, then*

$$d_s(\phi^i_*[\tau^i_{\lambda_1}](z_1), \phi^\infty_*[\tau^\infty_{\lambda_2}](z_2)) < \epsilon.$$

The uniqueness of the Bers-Royden extension allows us to conclude that given a sequence of holomorphic motions on leaves converging as above, then the extensions also converge in this sense.

PROPOSITION 6.4 *Let ϕ^i and τ^i be as in the previous definition, and let $\hat{\tau}^i$ denote the Bers-Royden extension of τ^i. Then $\hat{\tau}^i$ converges uniformly to $\hat{\tau}^\infty$.*

Proof: The fact that A^0 converges to A^∞ in the Hausdorff metric implies that for a given compact $K \subset \mathbb{C} \setminus A^\infty$, K is also contained in the complement of A^i for large i, and that the hyperbolic metric of the component of the complement of A^i containing K converges uniformly on K to the hyperbolic metric of the complement of A^∞. Moerover, since each $\phi^i_*[\hat{\tau}^i_\lambda]$ has a harmonic Beltrami coefficient, say $\mu_i(\lambda, z) = \rho_i(z)^{-2}\overline{\psi_i(\lambda, z)}$, and since $\|\mu_i(\lambda, z)\|\rho_i(z)^2$ is uniformly bounded for $\lambda \in \overline{\Delta^n}$, $z \in K$, we see that the family $\{\psi_i\}$ is a normal family.

Hence there exists a subsequence of ψ^i converging uniformly on each compact subset of $\Delta^n \times (\mathbb{C} \setminus A^\infty)$ to $\psi(\lambda, z)$. Moreover, from theorem 1 of [BR], we have for each i that

$$\|\mu_i(\lambda, z)\|_\infty < \|\lambda\|.$$

Hence this estimate holds also for $\mu(\lambda, z) = \rho_\infty(z)^{-2}\overline{\psi(\lambda, z)}$, and the subsequence of holomorhic motions corresponding to the chosen subsequence of ψ^i converges uniformly to a holomorphic motion with the harmonic Beltrami coefficient μ. But this limit motion must agree with $\phi^\infty_*[\hat{\tau}^\infty_\lambda]$ on A^∞, and since this latter motion also has a harmonic Beltrami coefficient, the uniqueness of the Bers-Rodyden extension implies that the limit motion must equal $\phi^\infty_*[\hat{\tau}^\infty_\lambda]$. Since any subsequence must have the same limit, we obtain pointwise convergence, and corollary 2 of [BR] implies equicontinuity of the sequence, hence uniform convergence as in the preceding definition. ■

We prove next that the natural motion of J_0 given by ψ_λ is a holomorphic motion on leaves and that the motions on a sequence of leaves approaching a limit leaf converges to the motion on the limit leaf.

THEOREM 6.5 *Let f_λ be a family of hyperbolic Hénon maps depending holomorphically on $\lambda \in \overline{\Delta^n}$. Let $\mathcal{L}_\lambda$ be either of the laminations $\mathcal{W}^u_\lambda$ or $\mathcal{W}^s_\lambda$. Let $p \in J_0$, $p_\lambda = \psi_\lambda(p)$, and let $L_\lambda = L_\lambda(p)$ be the leaf of $\mathcal{L}_\lambda$ through p_λ. Let $E_0 = L_0 \cap J_0$. Then $\psi(\lambda, \cdot) = \psi_\lambda(\cdot)$ is a holomorphic motion of E_0 on the family of leaves $\{L_\lambda\}$.*

Moreover, if $p^j \in J_0$ converges to $p \in J_0$ and $L^j_\lambda = L_\lambda(p^j)$ is the leaf through p^j_λ, then the holomorphic motion of $E^j_0 = L_0(p^j) \cap J_0$ on the family of leaves $\{L^j_\lambda\}$ converges uniformly to the holomorphic motion of E_0 on the family of leaves $\{L_\lambda\}$.

Finally, the Bers-Royden extensions of the motions of E^j_0 converge uniformly to the Bers-Royden extensions of the motion of E_0.

Proof: Since ψ_λ is a homeomorphism of J_0 to J_λ which conjugates f_0 to f_λ, it follows that ψ_λ maps $L_0 \cap J_0$ onto $L_\lambda \cap J_\lambda$. Hence $\psi_\lambda(E_0)$ is contained in L_λ. Moreover, theorem 5.6 implies that there exist holomorphically varying parametrizations $\phi_\lambda : \mathbb{C} \to L_\lambda$. Since $\psi_\lambda(q)$ is holomorphic in λ for each fixed $q \in J_0$, we see that $\phi_*[\psi_\lambda]$ is a holomorphic motion in the plane, hence ψ_λ is a holomorphic motion on the family of leaves through p.

For the convergence result, assume without loss of generality that $\mathcal{L}_\lambda$ is the unstable lamination. For the remainder of this proof, let $\delta = \delta_0$ and $\epsilon = \epsilon_0$ be the constants chosen earlier from the definition of local product structure: if $a_\lambda, b_\lambda \in J_\lambda$ with $\|a - b\| < \delta$, then $W^s_\epsilon(a_\lambda)$ and $W^u_\epsilon(b_\lambda)$ intersect in a unique point contained in J_λ.

Theorem 5.6 implies that there exist functions $\phi^j_\lambda : \mathbb{C} \to W^u(p^j_\lambda)$ which are holomorphic in $z \in \mathbb{C}$ and in $\lambda \in \overline{\Delta^n}$, and bijective for each fixed λ, and which converge locally uniformly to the map ϕ^∞_λ parametrizing $W^u(p_\lambda)$.

With these parametrizations, the first part of this proof implies that the holomorphic motion on the family of leaves through p^j is defined on the set $A^j = (\phi^j_0)^{-1}(J_0 \cap W^u(p^j))$ and is given by the pullback $\tau^j_\lambda = \phi^j_*[\psi_\lambda]$. The set A^∞ and τ^∞_λ are defined similarly using p and ϕ.

Choose $R > 0$ and let $K = \overline{\Delta_R} \subset \mathbb{C}$. Since we are using the spherical metric to define the Hausdorff metric, the proposition will be established once we show that $A^j \cap K$ converges to $A^\infty \cap K$ in the Hausdorff metric and that $\tau^j_\lambda = \phi^j_*[\psi_\lambda]$ converges uniformly on $(z, \lambda) \in (K \cap E^j) \times \overline{\Delta^n}$ to τ^∞_λ.

Since $\phi^\infty_\lambda(K)$ is contained in $W^u(p_\lambda)$, it follows that for large n, $f^{-n}_\lambda(\phi^\infty_\lambda(K))$ is contained in $W^u_\delta(f^{-n}_\lambda(p_\lambda))$. Hence for large j, $f^{-n}_\lambda(\phi^j_\lambda(K))$ is also within δ of $f^{-n}_\lambda(p_\lambda)$. It suffices to prove the convergence result near $f^{-n}_\lambda(p_\lambda)$, then apply f^n_λ; for clarity, we drop the f^{-n}_λ for the remainder of the proof.

Choose distinct points a and b in $W^u_\epsilon(p) \cap J_0$ so that each of a_λ and b_λ is of distance no more than $\delta/2$ from p_λ for any λ. Then for large j, $W^u_\epsilon(p^j_\lambda)$ and $W^s_\epsilon(a_\lambda)$ intersect in a unique point of J_λ, and likewise for b_λ.

Using a local biholomorphic change of variables from a neighborhood of $W^u_\delta(p_\lambda)$ to the unit bidisk $\{|u| < 1, |v| < 1\}$ (with the change of variables depending holomorphically on λ), we may assume that $W^u_\delta(p_\lambda)$ is $\Delta \times \{0\}$ and that $W^s_\delta(a_\lambda)$ and $W^s_\delta(b_\lambda)$ are $\{0\} \times \Delta$ and $\{1/2\} \times \Delta$, respectively. Then for each $q \in J_0 \cap W^u_\delta(p_0)$ and for given values of $\lambda \in \Delta^n$ and $v \in \Delta$, we associate the point given by taking the intersection of $W^s_\epsilon(q_\lambda)$ with $\Delta \times \{v\}$, then projecting to the u-coordinate. This defines a holomorphic motion of the point q with parameters λ and v.

We can view this holomorphic motion as a lamination with leaves defined by $\{q_{\lambda,v} : v \in \Delta\}$ as in figure 1, and the holonomy map associated with the leaves of this lamination gives a projection H^j_λ from $W^u_\delta(p_\lambda) \cap J_\lambda$ to $W^u_\delta(p^j_\lambda) \cap J_\lambda$. As j tends to ∞, the v coordinate of $W^u_\delta(p^j_\lambda)$ converges uniformly to 0. Hence the estimate in corollary 2 of [BR] implies that H^j_λ (and $(H^j_\lambda)^{-1}$) converges to the identity uniformly in q and λ. In particular, this establishes the

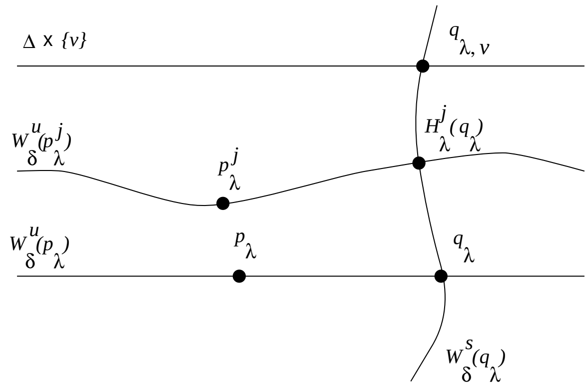

FIGURE 1. The holomorphic motion $q(\lambda, v)$ and the projection H^j_λ

convergence of $E^j \cap K$ to $E^\infty \cap K$ in the Hausdorff metric. Moreover, given $q \in J_0 \cap W^u(p^j_0)$, we have $\psi_\lambda(q) = H^j_\lambda \psi_\lambda (H^j_0)^{-1}(q)$. Hence

$$\tau^j_\lambda = (\phi^j_\lambda)^{-1} H^j_\lambda \psi_\lambda [(\phi^j_0)^{-1} H^j_0]^{-1},$$

where ψ_λ is restricted to $W^u_\delta(p_\lambda)$. The right hand side converges uniformly to

$$(\phi^\infty_\lambda)^{-1} \psi_\lambda (\phi^\infty_0)^{-1} = \tau^\infty_\lambda,$$

as desired.

Finally, the convergence of the Bers-Royden extensions follows from proposition 6.4. ■

We prove an analogous result in the unstably connected case.

PROPOSITION 6.6 *Let f_λ be as in the previous proposition and assume also that each f_λ is unstably connected. Let $\mathcal{L}_\lambda$ be the lamination $\mathcal{W}^s_\lambda \cup \mathcal{G}^+_\lambda$. Let $p \in (J^+_0 \cup U^+_0) \cap J^-_0$, $p_\lambda = \psi_\lambda(p)$, and let $L_\lambda = L_\lambda(p)$ be the leaf of $\mathcal{L}_\lambda$ through p_λ. Let $E_0 = L_0 \cap J^-_0$. Then $\psi(\lambda, \cdot) = \psi_\lambda(\cdot)$ is a holomorphic motion of E_0 on the family of leaves $\{L_\lambda\}$.*

Moreover, if $p^j \in (J^+_0 \cup U^+_0) \cap J^-_0$ converges to p in the same set, then the holomorphic motion of $E^j_0 = L_0(p^j) \cap J^-_0$ on the family of leaves $\{L_\lambda(p^j)\}$ converges uniformly to the holomorphic motion of E_0 on the family of leaves $\{L_\lambda\}$, and the Bers-Royden extensions of the motions of E^j_0 converge uniformly to the Bers-Royden extension of the motion of E_0.

Proof: Since f_λ is unstably connected, we can use corollary 5.8 to obtain the homeomorphism ψ_λ and use theorem 5.9 in place of theorem 5.6 in the proof of the previous theorem to obtain the holomorphic motion of E_0.

For the convergence result, if $p \in J_0$, the proof is the same as that of the previous theorem, so we assume that $p \in U^+_0 \cap J^-_0$. In this case, proof of the previous theorem still applies except for the existence of δ and ϵ. However, instead of applying f^{-n} for some large n, we now apply f^n. Since leaves of the lamination of U^+_0 are super-stable manifolds as shown in [BS5], it follows that for large n and j, $f^n_\lambda(\phi^j_\lambda(K))$ is again contained in a small neighborhood of $f^n_\lambda(p_\lambda)$, and the discussion of G^+ after proposition 3.1 implies that these images of K will be nearly horizontal disks. A simple calcuation implies that the local unstable manifolds of points in J^-_λ near p_λ are nearly vertical disks. Hence again there are unique points of intersection between local stable and unstable leaves, so the remainder of the proof of the previous theorem applies without change. ■

7 Proof of main theorems

Proof of theorem 1.1: Choose $p_0 \in J_0$ and let $p_\lambda = \psi_\lambda(p_0)$. We will first construct the map Ψ_λ on the set $W^u(p_0)$. To this end, let $\phi_\lambda : \mathbb{C} \to W^u(p_\lambda)$ be a parametrization obtained by theorem 5.6. I.e., ϕ_λ is holomorphic in (λ, z), $\phi_\lambda(0) = p_\lambda$, and $\phi_\lambda(1) = \psi_\lambda(q_0)$ for some $q_0 \in W^u(p_0) \setminus \{p_0\}$. Let $E_0 = J_0 \cap W^u(p_0)$, and define a holomorphic motion of $A_0 = \phi_0^{-1}(E_0)$ by

$$\alpha_\lambda = \phi_\lambda^{-1}\psi_\lambda\phi_0 = \phi_*[\psi_\lambda].$$

By the theorem of Bers and Royden, α extends canonically to a holomorphic motion $\hat{\alpha}_\lambda$ of $\mathbb{C}$ with a harmonic Beltrami coefficient.

We define $\Psi_\lambda : W^u(p_0) \to W^u(p_\lambda)$ by $\Psi_\lambda = \phi_\lambda\hat{\alpha}_\lambda\phi_0^{-1}$. Note that on E_0, $\Psi_\lambda = \psi_\lambda$. Moreover, Ψ_λ is independent of the choice of ϕ_λ. To see this, suppose that $\gamma : \Delta^n \times \mathbb{C} \to W^u(p_\lambda)$ is holomorphic in (λ, z), and let B_0 and β_λ be the analogs of A_0 and α_λ with γ in place of ϕ. Then $\phi_\lambda^{-1}\gamma_\lambda : \mathbb{C} \to \mathbb{C}$ is affine linear and holomorphic in λ, say $\phi_\lambda^{-1}\gamma_\lambda(z) = Q_\lambda(z)$, or $\gamma_\lambda(z) = \phi_\lambda Q_\lambda(z)$. Hence

$$\beta_\lambda(z) = Q_\lambda^{-1}\phi_\lambda^{-1}\psi_\lambda\phi_0 Q_0(z) = Q_\lambda^{-1}\alpha_\lambda Q_0^{-1}(z).$$

Since Q_λ is affine linear, the canonical extension of β_λ is $\hat{\beta}_\lambda = Q_\lambda^{-1}\hat{\alpha}_\lambda Q_0(z)$. Using this with the expression for γ_λ given above and canceling terms, we obtain $\gamma_\lambda\hat{\beta}_\lambda\gamma_0^{-1}(p) = \Psi_\lambda(p)$ for each $p \in W^u(p_0)$. I.e., Ψ_λ is independent of the choice of parametrization.

Hence we may apply the construction given above to each $p_0 \in J_0$ to obtain $\Psi_\lambda : J_0^- \setminus S_0 \to J_\lambda^- \setminus S_\lambda$ satisfying properties 1, 3, and 4 of the theorem, where S_λ is the set of sink orbits for f_λ. The same construction applies to give Ψ_λ on J_0^+, and we can define Ψ_λ on S_0 by using the implicit function theorem to follow the sink orbits.

As in McMullen- Sullivan [MS], we can use the uniqueness of the Bers-Royden extension to show that Ψ_λ conjugates f_0 on $W^u(p_0)$ to f_λ on $W^u(p_\lambda)$. To do this, let $\hat{\alpha}_\lambda$ be the holomorphic motion of $\mathbb{C}$ induced as above by ψ_λ on $W^u(p_0)$, and let $\hat{\beta}_\lambda$ be the motion induced by ψ_λ on $W^u(f_0(p_0))$, where $W^u(f_\lambda(p_\lambda))$ is parametrized by γ_λ. (Note that β and γ are different from the maps of the same name in the preceding section.) We obtain the following diagram, with the left and right portions commuting as indicated.

$$\begin{array}{ccccccc}
\mathbb{C} & \xrightarrow{\phi_0} & W^u(p_0) & \xrightarrow{f_0} & W^u(f_0(p_0)) & \xleftarrow{\gamma_0} & \mathbb{C} \\
\downarrow \hat{\alpha}_\lambda & /// & \downarrow \psi_\lambda \rightsquigarrow \Psi_\lambda & & \downarrow \psi_\lambda \rightsquigarrow \Psi_\lambda & /// & \downarrow \hat{\beta}_\lambda \\
\mathbb{C} & \xrightarrow{\phi_\lambda} & W^u(p_\lambda) & \xrightarrow{f_\lambda} & W^u(f_\lambda(p_\lambda)) & \xleftarrow{\gamma_\lambda} & \mathbb{C}
\end{array}$$

Note that $\gamma_\lambda^{-1}f_\lambda\phi_\lambda$ is a biholomorphic map of $\mathbb{C}$ to itself, hence equal to some affine linear map Q_λ depending holomorphically on λ. Hence $Q_\lambda\hat{\alpha}_\lambda Q_0^{-1}$ is a holomorphic motion of $\mathbb{C}$. Moreover, since Q_λ is an affine linear map, the Beltrami coefficient of this new holomorphic motion is simply a constant times the Beltrami coefficient of $\hat{\alpha}_\lambda$, hence the new holomorphic motion has a harmonic Beltrami coefficient.

Moreover, the fact that $\psi_\lambda = f_\lambda\psi_\lambda f_0^{-1}$ on J_0 implies that $\psi_\lambda = f_\lambda\phi_\lambda\alpha_\lambda\phi_0^{-1}f_0^{-1}$ on $W^u(f_0(p_0)) \cap J_0$, and hence $Q_\lambda\alpha_\lambda Q_0^{-1} = \gamma_\lambda^{-1}\psi_\lambda\gamma_0$ on the same set. But also $\beta_\lambda = \gamma_\lambda^{-1}\psi_\lambda\gamma_0$ by construction, so by the uniqueness of the extension of this motion to a motion with harmonic Beltrami coefficient, we see that $\hat{\beta}_\lambda = Q_\lambda\hat{\alpha}_\lambda Q_0^{-1}$.

Since $\hat{\alpha}_\lambda = \phi_\lambda^{-1}\Psi_\lambda\phi_0$ and $\hat{\beta}_\lambda = \gamma_\lambda^{-1}\Psi_\lambda\gamma_0$, we have

$$\gamma_\lambda^{-1}\Psi_\lambda\gamma_0 = Q_\lambda\phi_\lambda^{-1}\Psi_\lambda\phi_0 Q_0,$$

and using $Q_\lambda = \gamma_\lambda^{-1} f_\lambda \phi_\lambda$ and cancelling common factors, we obtain $\Psi_\lambda = f_\lambda \Psi_\lambda f_0^{-1}$.

The argument just given applies to any $p_0 \in J_0$, hence $f_\lambda = \Psi_\lambda f_0 \Psi_\lambda^{-1}$ on $J_0^- \setminus S_0$. Finally, the extension of Ψ_λ to S_0 using the implicit function theorem respects the dynamics on the sink orbits, hence Ψ_λ conjugates f_0 to f_λ on all of J_0^-. Applying this to f_λ^{-1} gives Ψ_λ on $J_0^- \cup J_0^+$ satisfying properties 1, 3, 4, and 5.

Note that Ψ_λ is bijective since it is bijective on each leaf and since it is a 1-to-1 one correspondence between leaves. We need to check that Ψ_λ is continuous with continuous inverse, but it suffices to show that it is continuous and proper (as a map from a subset of $\mathbb{C}^2$ into $\mathbb{C}^2$) since then we can use a one-point compactification to get a continuous 1-to-1 map on a compact set, which automatically has a continuous inverse.

To show continuity, let q^j be a sequence of points in J_0^- converging to a point q^∞ in J_0^-, and suppose first that q^∞ is not a sink. We want to show that $\Psi_\lambda(q^j)$ converges to $\Psi_\lambda(q^\infty)$. Let $p^\infty \in J_0$ so that q^∞ is in the unstable manifold of p^∞ for f_0, and likewise let $p^j \in J_0$ so that q^j is in the unstable manifold of p^j. Dropping to a subsequence if necessary, we may assume that p^j converges to p^∞.

Theorem 6.5 implies that the holomorphic motion of $W^u(p^j) \cap J_0$ converges uniformly to the holomorphic motion of $W^u(p^\infty) \cap J_0$, and that the Bers-Royden extensions of the former motions converge to the Bers-Royden extension of the latter. Since Ψ_λ is precisely the Bers-Royden extension of these motions, it follows at once that $\Psi_\lambda(q^j)$ converges to $\Psi_\lambda(q^\infty)$.

We claim next that $p_0 \in J_0^-$ is in the basin of attraction of a sink orbit if and only if $\Psi_\lambda(p_0)$ is in the basin of attraction of a sink orbit for each λ. First, $p_0 \in J_0^-$ is in J_0^+ precisely when $\Psi_\lambda(p_0) \in J_\lambda^+$ since Ψ is injective and is a homeomorphism of J_0 to J_λ. Since J_0^+ is the boundary of all basins of attraction of sink orbits of f_0, we may assume either that p_0 is in the basin of a sink or that p_0 is in the set of points with unbounded forward orbit. We can then write Δ^n as the disjoint union of the set A of λ such that $\Psi_\lambda(p_0)$ is in the basin of some sink and the set B of λ such that $\Psi_\lambda(p_0)$ has unbounded forward orbit. Note that if $\Psi_\lambda(p_0)$ is attracted to a sink of f_λ, then some small closed neighborhood is attracted to this sink, and for all sufficiently small perturbations of f_λ, this closed neighborhood will still be in the basin of some sink. Since $\Psi_\lambda(p_0)$ is holomorphic in λ, it follows that the set A is open, and likewise, the set B is open. Since Δ^n is connected, only one of these two sets can be nonempty, and since the point 0 is in one of them, the claim follows. This argument can be refined by further decomposing the set A into disjoint sets A_i of λ such that $\Psi_\lambda(p_0)$ is contained in the basin of attraction of $\Psi_\lambda(q^j)$ for each sink q^j of f_0. The conclusion in this case is that p_0 is in the basin of attraction of q_0^j if and only if for all λ, $\Psi_\lambda(p_0)$ is in the basin of attraction of q_λ^j.

To continue the proof of continuity, if q^∞ is a sink, then without loss of generality we may assume that each q^j is contained in the basin of attraction of q^∞ but is not equal to q^∞. Let U be a small neighborhood of J_0 in J_0^-, and let $N = \overline{f_0(U) \setminus U}$. Then N is compact, disjoint from J_0, and for each j there exists n_j such that $f_0^{-n_j}(q^j) \in N$. Moreover, since q^∞

is a sink, it follows that $n_j \to \infty$. Let

$$K = \overline{\{f_0^{-n_j}(q^j) : j \geq 1\}}.$$

Then K is a compact set contained in the intersection of N and the basin of q^∞. The previous paragraph implies that $\Psi_\lambda(K)$ is contained in the basin of attraction of q_λ^∞ for all λ. Hence, for fixed λ, $\Psi_\lambda(K)$ is a compact set in the basin of q_λ^∞, and since $n_j \to \infty$, we see that $f_\lambda^{n_j}\Psi_\lambda(K)$ converges uniformly to q_λ^∞. Since $\Psi_\lambda(q^j) \in f_\lambda^{n_j}\Psi_\lambda(K)$, we have $\Psi_\lambda(q^j)$ converging to $q_\lambda^\infty = \Psi_\lambda(q^\infty)$. Thus Ψ_λ is continuous on all of J_0^-.

For properness, suppose $p_0^j \in J_0^-$ with $\|p_0^j\| \to \infty$ but $\|\Psi_\lambda(p_0^j)\| \leq C$ for some large constant C and fixed λ. Since J_0 is a bounded set, and since any p_0^j which is in K_0^+ must be in J_0, we may assume without loss of generality that each p_0^j is in the complement of K_0^+. By the claim made previously, $p_\lambda^j = \Psi_\lambda(p_\lambda^j)$ is in the complement of K_λ^+. After dropping to a subsequence, we can find a sequence n_j increasing to ∞ such that $q_0^j = f_0^{-n_j}(p_0^j)$ converges to a point q_0^∞ in $J_0^- \setminus K_0^+$. Let $q_\lambda^j = \Psi_\lambda(q_0^j)$. The conjugacy property of Ψ implies that $q_\lambda^j = f_\lambda^{-n_j}(p_\lambda^j)$. Let $A_j = f_\lambda^{-n_j}((J_\lambda^- \setminus (\text{int } K_\lambda^+)) \cap \overline{\Delta_C^2})$. Then each A_j is compact, $A_{j+1} \subset A_j$, and $q_\lambda^j \in A_{n_j}$ for each j. Moreover, the continuity of Ψ_λ implies that q_λ^∞ is the limit of the sequence q_λ^j, so

$$q_\lambda^\infty \in \cap_{m>0}\overline{\cup_{j\geq m}\{q_\lambda^j\}} \subset \cap_{m>0}A_{n_m}.$$

But the intersection of all A_j is precisely J_λ, so q_λ^∞ must be in J_λ. But this is a contradiction since Ψ_λ is injective, is a homeomorphism from J_0 to J_λ and since q_0^∞ is not in J_0. Hence Ψ_λ must be proper, hence a homeomorphism of J_0^- to J_λ^-.

Applying the above proof to f_λ^{-1}, we get a conjugacy of $f_0|J_0^+$ to $f_\lambda|J_\lambda^+$ which agrees with the previously constructed map on J_0, so we get the map Ψ_λ defined on $J_0^+ \cup J_0^-$, as desired. This completes the proof of theorem 1.1. ■

Proof of theorem 1.2: In the case when f_0 is unstably connected and hyperbolic, then proposition 3.1 implies that f_λ is also unstably connected for λ near 0. Moreover, the previous construction applies to give Ψ on $J_0^+ \cup J_0^-$, and by replacing theorem 5.6 with theorem 5.9 and theorem 6.5 with theorem 6.6, the previous proof applies to show that Ψ is continuous.

For the properness, the previous proof does not apply directly, although it still implies that Ψ is proper on $J_0^+ \cup J_0^-$. To finish the proof, suppose that p_0^j is a sequence of points in U_0^+ with $\|p_0^j\| \to \infty$. In this case, either $G_0^+(p_0^j) \to \infty$ or $G_0^-(p_0^j) \to \infty$, and we suppose for now that the former applies. Note that the leaf of the lamination through p_0^j, which is a level set of G_0^+, is biholomorphic to the plane, and G_0^- is subharmonic on this leaf, is nonnegative, nonconstant, and harmonic outside of the zero set, hence must equal 0 somewhere. By definition of G_0^-, a zero of this function is precisely a point of J_0^-. Hence there exists a point q_0^j in J_0^- on the leaf through p_0^j. Then $G_0^+(q_0^j) = G_0^+(p_0^j)$, and since Ψ_λ is a homeomorphism on J_0^-, we must have $G_\lambda^+(\Psi_\lambda(q_0^j)) \to \infty$. Since Ψ_λ takes level sets of G_0^+ to level sets of G_λ^+, we have $G_\lambda^+(\Psi_\lambda(p_0^j)) \to \infty$, hence $\|\Psi_\lambda(p_0^j)\| \to \infty$.

Next, suppose $G_0^-(p_0^j) \to \infty$ but $G_0^+(p_0^j) < C$ and $\|\Psi_\lambda(p_0^j)\| < C$ for some constant C. Again we can choose q_0^j in J_0^- on the leaf through p_0^j. Since the set of points in J_0^- with $G_0^+ < C$ is bounded, we can drop to a convergent subsequence to obtain q_0^j converging

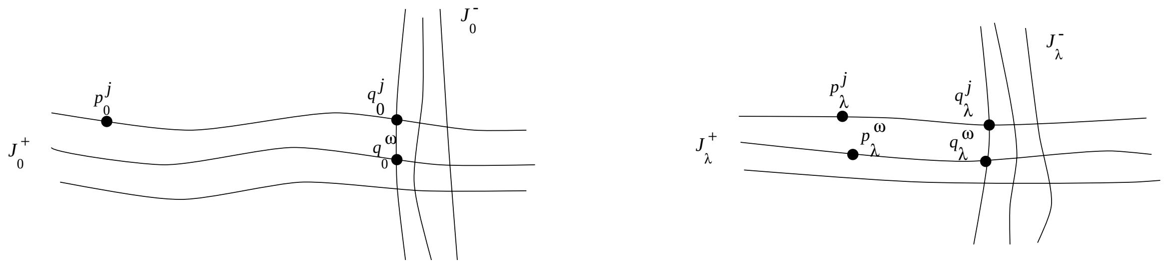

FIGURE 2. Corresponding points under the map Ψ_λ.

to $q_0^\infty \in J_0^-$. See figure 2. The continuity of Ψ_λ implies that $q_\lambda^j = \Psi_\lambda(q_0^j)$ converges to $q_\lambda^\infty = \Psi_\lambda(q_0^\infty)$. Moreover, $p_\lambda^j = \Psi_\lambda(p_0^j)$ is contained in the same leaf as q_λ^j, and dropping to a further subsequence, we may assume that p_λ^j converges to a point p_λ^∞ in $U_\lambda^+ \cup J_\lambda^+$ in the same leaf as q_λ^∞. Since Ψ_λ is a homeomorphism on J_0^-, a neighborhood, Y, of q_0^∞ in J_0^- maps onto a neighborhood of q_λ^∞ in J_λ^-. Since Ψ_λ maps each leaf of the lamination of $J_0^+ \cup U_0^+$ bijectively to a leaf of the lamination of $J_\lambda^+ \cup U_\lambda^+$, the image of the leaves through points in Y contains a neighborhood in $J_\lambda^+ \cup U_\lambda^+$ of the point p_λ^∞. Theorem 6.6 implies that Ψ_λ converges uniformly when approaching a limit leaf. Together, these facts imply that Ψ_λ maps a small neighborhood in $J_0^+ \cup U_0^+$ of p_0^∞ to a neighborhood of p_λ^∞ in $J_\lambda^+ \cup U_\lambda^+$. In particular, this image includes p_λ^j for all large j, so the preimage, the small neighborhood, includes p_0^j for all large j, which contradicts $\|p_0^j\| \to \infty$. ■

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