Extra Exercise Problems for Exam I

1. For a give matrix A, MATLAB shows that

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   Let B be a 3×1 matrix. Determine if each of the following statements is true.

   (a) For every B, the system $AX = B$ is solvable.

   (b) For some B, the system $AX = B$ is not uniquely solvable.

   (c) The system $AX = 0$ has infinitely many solutions.

   (d) The system $AX = 0$ has a nontrivial solution.

   True or False $2 - 18$.

2. A homogeneous system of four equations in five unknowns has a non-trivial solution.

3. If A has more rows than columns then the system $Ax = b$ has a unique solution for all constant columns b.

4. If A is a nonsingular matrix then $A^{-1}x = b$ has a unique solution.

5. If A and B are of the same size then $A^2B^2 = ABAB$.

6. If A is nonsingular then $(A^T)^{-1} = (A^{-1})^T$.

7. If u_1 and u_2 are solutions to the homogeneous system $Ax = 0$, then $2u_1 - 5u_2$ is also a solution.

8. If A and B are nonsingular $n \times n$ matrices, then AB is also nonsingular.

9. If A is nonsingular then A^4 is also nonsingular.

10. If A, B are nonzero matrices, then $A \cdot B$ is also nonzero.

11. If A is a square matrix then $A + A^T$ is symmetric.

12. If W is a subspace of a vector space V then W contains the zero vector $\mathbb{O}$.

13. If A is an $n \times n$ matrix and the system $Ax = b$ has a unique solution, then the homogeneous system $Ax = 0$ has a nontrivial solution.

14. Every homogeneous system has a solution.

15. If the tail of the vector $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ is $(1, 2)$, then its head is $(3, 8)$.

16. If A is a 2×2 matrix such that $A^2 = I_2$, then $A = I_2$.

17. The sum of two symmetric matrices is symmetric.

18. If you input the following line in MATLAB, you will get an error message: [1 2 3;4 5 6]*[−1 −1;−1 −1].

19. (a) Write down a linear system of equations which must be solved to find all scalars c_1, c_2, c_3 satisfying

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.$$

(b) Solve the above linear system by row reducing the augmented matrix corresponding to the system.

20. For this problem it is given that MATLAB command

$$\text{rref}\left(\begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 7 \\ -1 & -2 & -4 \end{bmatrix}\right) \text{ gives } \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

If possible, find scalars c_1, c_2, and c_3, not all zero, so that

$$c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 7 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

21. Let V be the set of all ordered triples (x, y, z) and let the operation $\oplus$ and $\odot$ be defined by

$$(x, y, z) \oplus (x', y', z') = (x + x', y + y', z + z')$$

$$c \odot (x, y, z) = (cx, y, cz)$$

(a) Does the set V and the operation $\oplus$ have the property that for all u and v in V, $u \oplus v = v \oplus v$? Justify your answer.

(b) Does the set V and the given operations have the operations have the property that, for all u and v in V and for all real number c, $c \odot (u + v) = c \odot u \oplus c \odot v$? Justify your answer.

22. Find non-zero, non-identity, 2×2 matrices A and B such that $A^2 = I_2$ and $B^2 = O_{2\times 2}$.

23. The augmented matrix of a linear system is equivalent to the following matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & a^2 \\ 0 & 1 & 2a & 1-a & 2 \\ 0 & 0 & 0 & 0 & a^2 - a - 6 \end{array}\right]$$

where a is a parameter.

(a) Determine all choices for a for which the system has no solutions.

(b) Determine all choices for a for which the system has infinitely many solutions, and write down the solutions.

(c) Determine all choices for a for which the system has exactly one solution, and write down the solution.

24. For a given system

$$\begin{cases} x - y + z &= 2 \\ -x + 3y - 5z &= t \\ 3x + y - z &= 2 \end{cases}$$

(a) For which values of t does the system has a unique solution? Find the solution for these t.

(b) For which values of t is the y-component of the solution equal to -3?

25. If $A = \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}$, for which values of t is it possible to find a nonzero vector $x = \begin{pmatrix} x \\ y \end{pmatrix}$ such that $A \cdot x = t \cdot x$?
(Hint: solve the linear system that x is supposed to satisfy and find those t that give infinitely many solutions.)

26. For which values a and b does the system of linear equations (a) have infinitely many solutions; (b) have no solution; (c) have a single solution. (d) Find the solution for $a = b = 1$.

$$\begin{cases} x_1 + & a^2 x_2 + & ab x_3 &= a \\ & a x_2 + & b x_3 &= a \\ & & b x_3 &= a^2 \end{cases}$$

27. Determine whether the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -3 \\ 0 & -1 & 0 \end{pmatrix}$$

has an inverse. Solve $Ax = b$, where

$$b = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}.$$

Remark: You should also be aware of straightforward problems like those in the homework assignments. This collection does not cover Section 2.3 and 2.4.