Math 265, Spring 2002

Solution for Final Sample Problems

1.

(a) The rank of A is 3 since there are three leading ones in **rref**A.

(b) The nullity of A is 2. nullity + rank = # of columns $\Longrightarrow$ nullity + 3 = 5 $\Longrightarrow$ nullity = 2.
Or, nullity = # of columns without leading one = 2.

(c) The leading ones in **rref**A lie on the first, second, and fifth columns. Therefore, the corresponding columns from **A** (not **rref**A,) form a basis of the column space.

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 1 \\ 4 \end{bmatrix} \right\}.$$

(d) The other basis for the column space of A can be obtained by the survived rows in **rref**A^T. Of course, we need to take the transpose again to make them as column vectors.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

(e) The leading ones in **rref**A^T lie on the first, second, and third columns. Therefore, the corresponding rows from **A** (not **rref**A^T,) form a basis of the row space.

$$\left\{ \begin{bmatrix} 1 & 0 & -2 & 1 & 3 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 5 & -1 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 6 & 0 & 1 \end{bmatrix} \right\}.$$

(f) The other basis for the row space of A can be obtained by the survived columns in **rref**A. Of course, we need to take the transpose again to make them as row vectors.

$$\left\{ \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 3 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right\}.$$

(g) Look at **rref**A. The null space of A can be expressed as

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \left\{ \begin{aligned} x_1 &= 2x_3 - x_4 \\ x_2 &= -3x_3 \\ x_5 &= 0 \end{aligned} \right. \right\} \Longrightarrow \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_3 - x_4 \\ -3x_3 \\ x_3 \\ x_4 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} = \text{span} \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(h) The orthogonal complement of the row space of A is the same as the null space of A. Therefore,

$$\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(i) Set up the homogeneous system $A\mathbf{x} = \mathbf{0}$. If we would like to write the third column as a linear combination of the other columns, it is the same as we set $x_3 = 1$. Then

$$\begin{cases} x_1 = & 2x_3 - x_4 \\ x_2 = & -3x_3 \\ x_5 = & 0 \end{cases} \Longrightarrow \begin{cases} x_1 = & 2 - x_4 \\ x_2 = & -3 \\ x_5 = & 0 \end{cases}.$$

For instance, we can set $x_4 = 0$. Then $x_1 = 2, x_2 = -3$. It means that

$$2 \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} - 3 \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} -2 \\ 5 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} -2 \\ 5 \\ 6 \\ 1 \end{bmatrix} = -2 \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + 3 \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}.$$

2.

(a) The rank of A is 2 since there are two leading ones in **rref**A.

(b) The nullity of A is 2. nullity + rank = # of columns $\Longrightarrow$ nullity + 2 = 4 $\Longrightarrow$ nullity = 2.
Or, nullity = # of columns without leading one = 2.

(c) The leading ones in **rref**A lie on the first and third columns. Therefore, the corresponding columns from **A** (not **rref**A,) form a basis of the column space.

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

(d) The other basis for the column space of A can be obtained by the survived rows in **rref**A^T. Of course, we need to take the transpose again to make them as column vectors.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

(e) The leading ones in **rref**A^T lie on the first, second, and third columns. Therefore, the corresponding rows from **A** (not **rref**A^T,) form a basis of the row space.

$$\left\{ \begin{bmatrix} 1 & 2 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 4 & 1 & -4 \end{bmatrix} \right\}.$$

(f) The other basis for the row space of A can be obtained by the survived columns in **rref**A. Of course, we need to take the transpose again to make them as row vectors.

$$\left\{ \begin{bmatrix} 1 & 2 & 0 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 2 \end{bmatrix} \right\}.$$

(g) Look at **rref**A. The null space of A can be expressed as

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \begin{cases} x_1 = & 0 \\ x_2 = & x_4 \\ x_3 = & -x_4 \end{cases} \right\} \Longrightarrow \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ x_4 \\ -x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\} = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

(h) The orthogonal complement of the row space of A is the same as the null space of A. Therefore,

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

(i) Since the rank of A is 2, the dimension of the row space of A is 2. If row vectors span $\mathbb{R}^4$, then the dimension of the row space must be 4. This is not true.

3.

(a) The $\mathbf{ref}[A|\mathbf{b}]$ is

$$\left[\begin{array}{ccc|c} 1 & 2a & a & 1 \\ 1 & 2a & 1 & -1 \\ 0 & 1 & a & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 2a & a & 1 \\ 0 & 0 & 1-a & -1 \\ 0 & 1 & a & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 2a & a & 1 \\ 0 & 1 & a & 1 \\ 0 & 0 & 1-a & -1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

If $1-a=0$, then the third equation becomes $0=1$, which is inconsistent. This is not the case that we want. Assume $1-a \neq 0$. Then we can solve $z=-1/(1-a)$, and backward substitute to solve y and x; the system has a unique solution. Therefore, the condition on a such that the system has a unique solution is $a \neq 1$.

There is the other way to do it. A system has a unique solution if and only of the rank is equal to the number of variables; in this case, the rank is 3. This leads that the determinant of the coefficient matrix is non-zero.

$$\begin{vmatrix} 1 & 2a & a \\ 1 & 2a & 1 \\ 0 & 1 & a \end{vmatrix} \neq 0 \Longrightarrow -(a-1) \neq 0 \Longrightarrow a \neq 1.$$

(b) If $x=3, y=-1, z=1$ is a solution, then

$$\begin{cases} 1\cdot 3+(2a)(-1)+a\cdot 1 = & 1 \\ 1\cdot 3+(2a)(-1)+1\cdot 1 = & 0 \\ 0\cdot 3+1\cdot(-1)+a\cdot 1 = & 1 \\ 0\cdot 3+0\cdot(-1)+0\cdot 1 = & 0 \end{cases} \Longrightarrow \begin{cases} -a = & -2 \\ -2a = & -4 \\ a = & 2 \\ 0 = & 0 \end{cases} \Longrightarrow \begin{cases} a = & 2 \\ a = & 2 \\ a = & 2 \\ 0 = & 0 \end{cases} \Longrightarrow a=2.$$

4.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 5 \\ 2 & 3 & (a^2-3) & a+3 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & (a^2-5) & a-1 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & (a^2-4) & a-2 \end{array}\right].$$

Look at the last equation.

$$(a^2-4)z = a-2.$$

If $a^2-4=0, a=-2,2$, we have a consistent problem.

(a) If $a=-2$, then the equation becomes $0=-2-2=-4$; the system is not consistent. Hence, there is no solution.

(b) If $a \neq 2,-2$, we can solve $z=1/(a+2)$ and backward substitute to get y, x. Therefore, the system has a unique solution.

There is the other way to do it. A system has a unique solution if and only of the rank is equal to the number of variables; in this case, the rank is 3. This leads that the determinant of the coefficient matrix is non-zero.

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 2 & 3 & (a^2-3) \end{vmatrix} \neq 0 \Longrightarrow a^2-4 \neq 0 \Longrightarrow a \neq 2,-2.$$

(c) If $a = 2$, then the equation becomes $0 = 2 - 2 = 0$; the system is consistent and z is arbitrary. Hence, there are infinitely many solutions.

5. The standard matrix $A$for L is

$$A = \begin{bmatrix} L(\mathbf{e}_1) & L(\mathbf{e}_2) & L(\mathbf{e}_3) \end{bmatrix} = \begin{bmatrix} L\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) & L\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) & L\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) \end{bmatrix} = \begin{bmatrix} -3 & 2 & -1 \\ 1 & -2 & 2 \\ 2 & 1 & 4 \end{bmatrix}.$$

6.

(a) The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 3 & -1 \\ 1 & 2 & -3 \\ -2 & -1 & 3 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & 2 & -3 \\ -2 & 3 & -1 \\ -2 & -1 & 3 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & 3 & -3 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right] \Longrightarrow \begin{bmatrix}-1\\-3\\3\end{bmatrix} = -\begin{bmatrix}-2\\1\\2\end{bmatrix} - \begin{bmatrix}3\\2\\1\end{bmatrix}.$$

(b)

$$L\left(\begin{bmatrix}-1\\-3\\3\end{bmatrix}\right) = L\left(-\begin{bmatrix}-2\\1\\2\end{bmatrix} - \begin{bmatrix}3\\2\\1\end{bmatrix}\right) = -L\left(\begin{bmatrix}-2\\1\\2\end{bmatrix}\right) - L\left(\begin{bmatrix}3\\2\\1\end{bmatrix}\right) = -\begin{bmatrix}-3\\1\\2\end{bmatrix} - \begin{bmatrix}2\\-2\\1\end{bmatrix} = \begin{bmatrix}1\\1\\-3\end{bmatrix}.$$

7. The standard matrix $A$for L is

$$A = \begin{bmatrix} L(\mathbf{e}_1) & L(\mathbf{e}_2) & L(\mathbf{e}_3) \end{bmatrix} = \begin{bmatrix} L\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) & L\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) & L\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 2 & 1 \\ 3 & -1 & 0 \end{bmatrix}.$$

8.

$$A \cdot \operatorname{adj} A = (\det A) \cdot I_5 \Longrightarrow A^2 \cdot (\operatorname{adj} A)^2 = A \cdot A \cdot \operatorname{adj} A \cdot \operatorname{adj} A$$

$$= A \cdot \det AI_5 \cdot \operatorname{adj} A = (\det AI_5) \cdot A \cdot \operatorname{adj} A = (\det AI_5) \cdot (\det AI_5) = (\det A)^2 I_5.$$

$$\det\left(A^2 \cdot (\operatorname{adj} A)^2\right) = \det\left((\det A)^2 I_5\right) = \left((\det A)^2\right)^5 \cdot \det I_5 = \left(3^2\right)^5 \cdot 1 = 3^{10}.$$

9.

(a) $8 + 0 + (-2) - 3 - 8 - 0 = -5$.

(b) $2 + 12 + (-8) - 12 - (-8) - 2 = 0$.

10.

(a) The vectors $t^2 + 2t, 3t^2 + t - 1$ are linearly independent. Therefore, S is a basis. There is no solution for the system $a_1(t^2 + 2t) + a_2(3t^2 + t - 1) = 6t^2 - 1$. Hence, the vector $6t^2 - 1$ dose not belongs to span S.

(b) At first, we check W is a subspace. We only need to check two conditions; one is W closed under addition, and the other is W closed under scalar multiplication.

(1) Addition.

$$\mathbf{w_1} = \begin{bmatrix} a_1 & b_1 \\ 0 & 3b_1 \end{bmatrix}, \mathbf{w_2} = \begin{bmatrix} a_2 & b_2 \\ 0 & 3b_2 \end{bmatrix} \Longrightarrow \mathbf{w_1} + \mathbf{w_2} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ 0 & 3(b_1 + b_2) \end{bmatrix}.$$

Let $a = a_1 + a_2, b = b_1 + b_2$.

$$\mathbf{w_1} + \mathbf{w_2} = \begin{bmatrix} a & b \\ 0 & 3b \end{bmatrix} \in \mathbf{W}.$$

Therefore, W is closed under addition.

(2) Scalar multiplication.

$$c \in \mathbb{R}, \mathbf{w} = \begin{bmatrix} a' & b' \\ 0 & 3b' \end{bmatrix} \in W \Longrightarrow c\mathbf{w} = \begin{bmatrix} ca' & cb' \\ 0 & 3cb' \end{bmatrix}.$$

Choose $a = ca', b = cb'$. We have

$$c\mathbf{w} = \begin{bmatrix} ca' & cb' \\ 0 & 3cb' \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 3b \end{bmatrix} \in W.$$

Therefore, W is closed under scalar multiplication.

$$\begin{bmatrix} a & b \\ 0 & 3b \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} \Longrightarrow W = \text{span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}\right\}.$$

These two matrices are linearly independent. Therefore they form a basis for W and $\dim W = 2$.

11. Let A be the 3×3 matrix. The column vectors of A are linearly independent if and only if the determinant of A is non-zero.

$$\begin{vmatrix} 1 & 1 & 0 \\ 3 & 1 & 1 \\ d & 0 & 1 \end{vmatrix} \neq 0 \Longrightarrow 1 + 0 + d - 0 - 0 - 3 \neq 0 \Longrightarrow d \neq 2.$$

12.

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 4, \quad A_{12} = (-1)^{1+2}\begin{vmatrix} -2 & 3 \\ 2 & 1 \end{vmatrix} = 8, \quad A_{13} = (-1)^{1+3}\begin{vmatrix} -2 & 1 \\ 2 & -1 \end{vmatrix} = 0,$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} = -1, \quad A_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = 0, \quad A_{23} = (-1)^{2+3}\begin{vmatrix} 2 & 0 \\ 2 & -1 \end{vmatrix} = 2,$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix} = -1, \quad A_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 1 \\ -2 & 3 \end{vmatrix} = -8, \quad A_{33} = (-1)^{3+3}\begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix} = 2.$$

$$\text{adj}\, A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 4 & -1 & -1 \\ 8 & 0 & -8 \\ 0 & 2 & 2 \end{bmatrix}.$$

13.

$$A^{-1} = \frac{1}{\det A}\,\text{adj}\, A = -\frac{1}{13}\begin{bmatrix} 2 & 3 & -2 \\ 3 & -2 & -3 \\ -11 & -10 & -2 \end{bmatrix}.$$

14. A is non-singular if and only if $\det A \neq 0$.

$$\begin{vmatrix} 1 & 2 & 1 \\ 0 & -3 & 5 \\ 1 & a & 6 \end{vmatrix} \neq 0 \Longrightarrow -5a - 5 \neq 0 \Longrightarrow a \neq -1.$$

15. It is not a vector space. It fails the following condition:

$$(c_1 + c_2) \odot (x, y) = \big(c_1 \odot (x, y)\big) \oplus \big(c_2 \odot (x, y)\big).$$

Let $c_1 = 2, c_2 = 3, x = 1, y = 1$

$$(2+3) \odot (1,1) = 5 \odot (1,1) = (5 \cdot 1, 1) = (5,1)$$

On the other hand,

$$\big(2 \odot (1,1)\big) \oplus \big(3 \odot (1,1)\big) = (2,1) \oplus (3,1) = (5,2) \neq (5,1).$$

16. Let λ be an eigenvalue of A and $\mathbf{v}$ an eigenvector associated with λ. Then

$$A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) = \lambda(\lambda\mathbf{v}) = \lambda^2\mathbf{v}.$$

This means that λ^2 is an eigenvalue of A^2 and $\mathbf{v}$ an eigenvector associated with λ^2. The same argument can imply that λ^3 is an eigenvalue of A^3 and $\mathbf{v}$ an eigenvector associated with λ^3.

Hence, $1^3, 3^3, 5^3$ are eigenvalues for A^3 and $\mathbf{v}$ is an eigenvector for A^3.

17. Let

$$\mathbf{u}_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}2\\-1\\0\\1\end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix}3\\-3\\0\\-2\end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix}1\\-2\\0\\-3\end{bmatrix}.$$

$$\mathbf{v}_1 = \mathbf{u}_1 \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \mathbf{v}_2 = \mathbf{u}_2 - \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 = \begin{bmatrix}2\\-1\\0\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} = \begin{bmatrix}3/2\\-3/2\\0\\1\end{bmatrix}.$$

It is no hurt that we use $2\begin{bmatrix}3/2\\-3/2\\0\\1\end{bmatrix} = \begin{bmatrix}3\\-3\\0\\2\end{bmatrix}$ as $\mathbf{v}_2$.

$$\mathbf{v}_3 = \mathbf{u}_3 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 = \begin{bmatrix}3\\-3\\0\\-2\end{bmatrix} - \left(\frac{0}{2}\right)\begin{bmatrix}1\\1\\0\\0\end{bmatrix} - \left(\frac{14}{22}\right)\begin{bmatrix}3\\-3\\0\\2\end{bmatrix} = \frac{4}{11}\begin{bmatrix}1\\-1\\0\\-3\end{bmatrix}.$$

Again, it is no hurt that we use $\begin{bmatrix}1\\-1\\0\\-3\end{bmatrix}$ as $\mathbf{v}_3$.

$$\mathbf{v}_4 = \mathbf{u}_4 - \left(\frac{\mathbf{u}_4 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{u}_4 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_3 - \left(\frac{\mathbf{u}_4 \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3}\right)\mathbf{v}_3 = \begin{bmatrix}1\\-2\\0\\-3\end{bmatrix} - \left(\frac{-1}{2}\right)\begin{bmatrix}1\\1\\0\\0\end{bmatrix} - \left(\frac{3}{22}\right)\begin{bmatrix}3\\-3\\0\\2\end{bmatrix} - \left(\frac{12}{11}\right)\begin{bmatrix}1\\-1\\0\\-3\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}.$$

$$\left\{\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}3\\-3\\0\\2\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\-3\end{bmatrix}\right\}.$$

Note that the fourth vector is the zero vector; we can not put it into a basis.

18.

(a)

$$W^{\perp} = \left\{ \mathbf{w} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, x_1 + x_2 - x_3 = 0 \right\} \Longrightarrow x_1 = -x_2 + x_3$$

$$\Longrightarrow \mathbf{w} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Longrightarrow W^{\perp} = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(b) Use the Gram-Schmidt process to produce an orthogonal basis and then normalize them to be orthonormal.

$$\left\{ \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix} \right\}.$$

19.

(a) First, use the Gram-Schmidt to find an orthogonal basis for W first. An orthogonal basis for W is

$$\left\{ \mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

$$\text{Proj}_W \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}} \mathbf{v_1} + \frac{\mathbf{v} \cdot \mathbf{v_2}}{\mathbf{v_2} \cdot \mathbf{v_2}} \mathbf{v_2} + \frac{\mathbf{v} \cdot \mathbf{v_3}}{\mathbf{v_3} \cdot \mathbf{v_3}} \mathbf{v_3} = \frac{-9}{9} \begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \end{bmatrix} + \frac{3}{6} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} + \frac{9}{18} \begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 2 \end{bmatrix}.$$

(b)

$$\mathbf{w} = \text{Proj}_{\mathbf{W}} \mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 2 \end{bmatrix}. \quad \mathbf{u} = \text{Proj}_{\mathbf{W}^{\perp}} \mathbf{v} = \mathbf{v} - \mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix}.$$

(c) The distance d from $\mathbf{v}$ to W is the length of $\text{Proj}_{W^{\perp}} \mathbf{v} = \mathbf{u}$.

$$d = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}.$$

20.

(a) False. The uniquely solvable case only happens when the rank is the same as the number of variables (columns).
(b) False. The uniquely solvable case only happens when the rank is the same as the number of variables (columns).
(c) False. The system always has solutions if and only if the rank is equal to the number of equations (rows).
(d) True. Since the rank is less than the number of variables, we have extra freedom. Therefore, if there is a solution, we have infinitely many solutions.

(e) True. Since the rank is less than the number of equations (rows), there are some **b** such that the system is not solvable.

21.

(a) False. For example $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(b) True.

(c) True.

(d) True.

(e) False. $(A - A^T)^T = A^T - A \neq A - A^T$.

(f) False. $\det(\operatorname{adj} A) = (\det A)^{n-1} = (\det A)^2$.

22.

(a) Yes.

(b) No. No zero vector here.

(c) Yes.

(d) No. No zero vector here.

(e) No. No zero vector here.

23.

(a) Yes.

(b) Yes

(c) No. $\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 1 & 4 \end{vmatrix} = 0.$

(d) No. The number of vectors is greater than the dimension.

(e) Yes.

24.

(a) 3.

(b) $21 = 3 \cdot 7$.

(c) 0, since two columns are the same.

(d) 3.

(e) $\dfrac{1}{5^3 \cdot 3}$.

$$\det\left((5A)^{-1}\right) = \frac{1}{\det(5A)} = \frac{1}{5^3 \det(A)} = \frac{1}{5^3 \cdot 3}.$$

25.

(a) 0. Two rows are the same.

(b) 0. Two columns are the same.

(c) 0. There is a zero row.

(d) $-40 = 2 \cdot 2 \cdot 5 \cdot (-1) \cdot 2$.

(e) -24.

$$\begin{vmatrix} 2 & 3 & 2 & -1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} \cdot \begin{vmatrix} 5 & 1 \\ 1 & -1 \end{vmatrix} = 4 \cdot (-5-1) = -24.$$

26.

$$p(\lambda) = \det(\lambda I_3 - A) = \begin{vmatrix} \lambda-2 & -1 & -2 \\ 3 & \lambda-1 & 1 \\ 0 & 0 & \lambda-1 \end{vmatrix} = (\lambda-1)\begin{vmatrix} \lambda-2 & -1 \\ 3 & \lambda-1 \end{vmatrix}$$

$$= (\lambda-1)\left((\lambda-2)(\lambda-1)+3\right) = (\lambda-1)(\lambda^2-3\lambda+5).$$

The characteristic equation is $(\lambda-1)(\lambda^2-3\lambda+5)=0$. $\lambda = 1, \dfrac{3 \pm \sqrt{11}i}{2}$.

27.

(a) $\lambda = 1, 1, 4$.

(b) For $\lambda = 1$,

$$\lambda I_3 - A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{cases} x_1 = & -x_2 - x_3 \\ x_2, x_3 & \text{arb.} \end{cases} \Longrightarrow x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

For $\lambda = 4$,

$$\lambda I_3 - A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{cases} x_1 = & x_3 \\ x_2 = & x_3 \\ x_3 & \text{arb.} \end{cases} \Longrightarrow x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$1 \rightsquigarrow \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad 4 \rightsquigarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(c)

$$1 \rightsquigarrow \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, 4 \rightsquigarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Longrightarrow 1 \rightsquigarrow \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, 4 \rightsquigarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Longrightarrow 1 \rightsquigarrow \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}, 4 \rightsquigarrow \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}.$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix},$$

where the diagonal entries of D are from the eigenvalues of A, and the column vectors of P are from the associated orthonormal eigenvectors.

28.

$$\hat{A} = A^T \cdot A = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}, \quad \hat{\mathbf{b}} = A^T \cdot \mathbf{b} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

29. Consider a line defined by

$$y = a + bx.$$

We would like to determine the best a and b.

$$\begin{cases} a + b = & 2 \\ a + 2b = & 1 \\ a + 3b = & 3 \\ a + 4b = & 3 \end{cases} \Longrightarrow \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 3 \end{bmatrix}.$$

Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}., \quad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 3 \end{bmatrix}.$$

The new system is

$$(A^T \cdot A)\hat{\mathbf{x}} = A^T \cdot \mathbf{b} \Longrightarrow \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 9 \\ 25 \end{bmatrix} \Longrightarrow \hat{\mathbf{x}} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}.$$

Hence, the least square fit line is

$$y = 1 + \frac{1}{2}x.$$

30.

(a) $\lambda = 6, 4$

(b)

$$\mathbf{x}(t) = c_1 \begin{bmatrix} -1 \\ 4 \end{bmatrix} e^{6t} + c_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{4t},$$

where c_1, c_2 are arbitrary.

(c) Set

$$\begin{bmatrix} 3 \\ -10 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} \Longrightarrow \begin{cases} -c_1 - c_2 = & 3 \\ 4c_1 + 2c_2 = & -10 \end{cases} \Longrightarrow c_1 = -2, c_2 = -1.$$

Therefore,

$$\mathbf{x}(t) = -2 \begin{bmatrix} -1 \\ 4 \end{bmatrix} e^{6t} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{4t}.$$

31.

(a) The characteristic equation is $\lambda^2 + 3 = 0 \Longrightarrow \lambda = \pm\sqrt{3}i$.

(b) For $\lambda = i$, assume that $\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector associated to i. Then

$$(\lambda I_2 - A)\mathbf{v} = \mathbf{0} \Longrightarrow \begin{bmatrix} \sqrt{3}i - 1 & 2 \\ -2 & \sqrt{3}i + 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The augmented matrix is

$$\left[\begin{array}{cc|c} \sqrt{3}i - 1 & 2 & 0 \\ -2 & \sqrt{3}i + 1 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} -2 & \sqrt{3}i + 1 & 0 \\ \sqrt{3}i - 1 & 2 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & -(\sqrt{3}i + 1)/2 & 0 \\ 0 & 0 & 0 \end{array}\right] \Longrightarrow x_1 = \frac{\sqrt{3}i + 1}{2} x_2$$

$$\Longrightarrow \mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}i + 1}{2} x_2 \\ x_2 \end{bmatrix} = \frac{x_2}{2} \begin{bmatrix} \sqrt{3}i + 1 \\ 2 \end{bmatrix}.$$

Therefore, an eigenvector can be

$$\begin{bmatrix} \sqrt{3}i + 1 \\ 2 \end{bmatrix}.$$

For the complex root cases, we do not need to compute the other eigenvalue and eigenvector. The real part and the imaginary part of the solution $\mathbf{v}e^{\lambda t}$ will be two linearly independent solutions.

$$\mathbf{v}e^{\lambda t} = \begin{bmatrix} \sqrt{3}i + 1 \\ 2 \end{bmatrix} e^{\sqrt{3}it} = \begin{bmatrix} \sqrt{3}i + 1 \\ 2 \end{bmatrix} (\cos\sqrt{3}t + i\sin\sqrt{3}t) = \begin{bmatrix} \cos\sqrt{3}t - \sqrt{3}\sin\sqrt{3}t + i(\sqrt{3}\cos\sqrt{3}t + \sin\sqrt{3}t) \\ 2\cos\sqrt{3}t + i2\sin\sqrt{3}t \end{bmatrix}$$

$$= \begin{bmatrix} \cos\sqrt{3}t - \sqrt{3}\sin\sqrt{3}t \\ 2\cos\sqrt{3}t \end{bmatrix} + i \begin{bmatrix} \sqrt{3}\cos\sqrt{3}t + \sin\sqrt{3}t \\ 2\sin\sqrt{3}t \end{bmatrix}.$$

Two independent solutions are

$$\begin{bmatrix} \cos\sqrt{3}t - \sqrt{3}\sin\sqrt{3}t \\ 2\cos\sqrt{3}t \end{bmatrix}, \quad \begin{bmatrix} \sqrt{3}\cos\sqrt{3}t + \sin\sqrt{3}t \\ 2\sin\sqrt{3}t \end{bmatrix}.$$

The general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} \cos\sqrt{3}t - \sqrt{3}\sin\sqrt{3}t \\ 2\cos\sqrt{3}t \end{bmatrix} + c_2 \begin{bmatrix} \sqrt{3}\cos\sqrt{3}t + \sin\sqrt{3}t \\ 2\sin\sqrt{3}t \end{bmatrix},$$

where c_1, c_2 are arbitrary.

(c) The initial value is

$$\mathbf{x}(0) = \begin{bmatrix} 5 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} \cos\sqrt{3}\cdot 0 - \sqrt{3}\sin\sqrt{3}\cdot 0 \\ 2\cos\sqrt{3}\cdot 0 \end{bmatrix} + c_2 \begin{bmatrix} \sqrt{3}\cos\sqrt{3}\cdot 0 + \sin\sqrt{3}\cdot 0 \\ 2\sin\sqrt{3}\cdot 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} \sqrt{3} \\ 0 \end{bmatrix}$$

$$\Longrightarrow \begin{cases} c_1 + \sqrt{3}c_2 = & 5 \\ 2c_1 = & 1 \end{cases} \Longrightarrow c_1 = \frac{1}{2}, c_2 = \frac{3\sqrt{3}}{2} \Longrightarrow \mathbf{x}(t) = \frac{1}{2} \begin{bmatrix} \cos\sqrt{3}t - \sqrt{3}\sin\sqrt{3}t \\ 2\cos\sqrt{3}t \end{bmatrix} + \frac{3\sqrt{3}}{2} \begin{bmatrix} \sqrt{3}\cos\sqrt{3}t + \sin\sqrt{3}t \\ 2\sin\sqrt{3}t \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 10\cos\sqrt{3}t + 2\sqrt{3}\sin\sqrt{3}t \\ 2\cos\sqrt{3}t + 6\sqrt{3}\sin\sqrt{3}t \end{bmatrix} = \begin{bmatrix} 5\cos\sqrt{3}t + \sqrt{3}\sin\sqrt{3}t \\ \cos\sqrt{3}t + 3\sqrt{3}\sin\sqrt{3}t \end{bmatrix}.$$

$$\Longrightarrow x_1(t) = 5\cos\sqrt{3}t + \sqrt{3}\sin\sqrt{3}t,\ x_2(t) = \cos\sqrt{3}t + 3\sqrt{3}\sin\sqrt{3}t.$$

32.

(a) $\lambda = 1, 2, 3$

(b)

$$\mathbf{x(t)} = \mathbf{c_1} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \mathbf{e^t} + \mathbf{c_2} \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} \mathbf{e^{2t}} + \mathbf{c_3} \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} \mathbf{e^{3t}},$$

where c_1, c_2, c_3 are arbitrary.

(c) Set

$$\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} \Longrightarrow c_1 = 1, c_2 = 1, c_3 = -1$$

Therefore,

$$\mathbf{x(t)} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \mathbf{e^t} + \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} \mathbf{e^{2t}} - \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} \mathbf{e^{3t}}.$$