

1. It is given that $A = \begin{bmatrix} 1 & 0 & -2 & 1 & 3 \\ -1 & 1 & 5 & -1 & -3 \\ 0 & 2 & 6 & 0 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{bmatrix}$, $\mathbf{rref}(A) = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and

$$\mathbf{rref}(A^T) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

- Find the rank of A .
- Find the nullity of A .
- Find a basis for the column space of A . We require that you choose the vectors for the basis from the column vectors of A .
- Find another basis for the column space of A .
- Find a basis for the row space of A . We require that you choose the vectors for the basis from the row vectors of A .
- Find another basis for the row space of A .
- Find a basis for the null space of A .
- Find a basis for the orthogonal complement of the row space of A .
- Write the third column of A as a linear combination of the other columns.

2. It is given that $A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 1 & -4 \\ -1 & -2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 3 & 6 & 2 & -5 \end{bmatrix}$, $\mathbf{rref}(A^T) = \begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and

$$\mathbf{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

- Find the rank of A .
- Find the nullity of A .
- Find a basis for the column space of A . We require that you choose the vectors for the basis from the column vectors of A .
- Find another basis for the column space of A .
- Find a basis for the row space of A . We require that you choose the vectors for the basis from the row vectors of A .

- (f) Find another basis for the row space of A .
- (g) Find a basis for the null space of A .
- (h) Find a basis for the orthogonal complement of the row space of A .
- (i) Do the row vectors of A span \mathbb{R}^4 ?

3. Let a be a real number (parameter). Consider the linear system corresponding to the following augmented matrix for the system (variables are x, y , and z)

$$[A|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 2a & a & 1 \\ 1 & 2a & 1 & 0 \\ 0 & 1 & a & 1 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

- (a) Find all values of a for which the system has a unique solution.
- (b) For what value of a does the system have the solution $x = 3, y = -1, z = 1$?

4. Determine the values of a so that the linear system

$$\begin{cases} x + y + z = 2 \\ 2x + 3y + z = 5 \\ 2x + 3y + (a^2 - 1)z = a + 1 \end{cases}$$

has (a) no solution, (b) a unique solution, and (c) infinitely many solutions.

5. Let $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation such that

$$L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \quad L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \quad L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}.$$

Find the standard matrix for L .

6. Let $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation such that

$$L\left(\begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \quad L\left(\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

- (a) Write $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$.

(b) Find $L \left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right)$.

7. Find the standard matrix of the linear transformation L defined by

$$L \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 2x - z \\ x + 2y + z \\ 3x - y \end{bmatrix}.$$

8. If A is a 5×5 matrix and $\det A = 3$ find $A^2(\text{adj}A)^2$.

9. Compute the determinant.

(a) $\begin{bmatrix} 2 & -1 & 3 \\ -2 & 1 & 2 \\ 1 & 0 & 4 \end{bmatrix}$; (b) $\begin{bmatrix} 2 & -1 & 3 \\ -2 & 1 & 2 \\ 4 & -2 & 1 \end{bmatrix}$.

10. (a) Find a basis for $S = \{t^2 + 2t, 3t^2 + t - 1\}$. Does $6t^2 - 1$ belong to $\text{span}S$?

(b) Find a basis for $W = \left\{ \begin{bmatrix} a & b \\ 0 & 3b \end{bmatrix}; a, b \in \mathbb{R} \right\}$. What is the dimension of W ?

11. For what values of d are the vectors $\begin{bmatrix} 1 \\ 3 \\ d \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ linearly independent?

12. Given $A = \begin{bmatrix} 2 & 0 & 1 \\ -2 & 1 & 3 \\ 2 & -1 & 1 \end{bmatrix}$. Find $\text{adj} A$.

13. Find the inverse of the matrix $A = \begin{bmatrix} 2 & -2 & 1 \\ -3 & 2 & 0 \\ 4 & 1 & 1 \end{bmatrix}$.

14. Find all values of a for which the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 5 \\ 1 & a & 6 \end{bmatrix}$ is nonsingular.

15. Consider the set of all pairs (x, y) of real numbers, with the following operations:

a) $(x, y) \oplus (w, z) = (x + w, y + z)$, for all pairs (x, y) , (w, z) .

b) $c \odot (x, y) = (cx, y)$, for all pairs (x, y) and every number c .

Is this a vector space? If not, state which conditions fail and show why they fail.

16. Suppose A is a 3×3 matrix with eigenvalues 1, 3, 5. What are the eigenvalues of A^3 ?

Suppose $v = \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}$ is an eigenvector of A . Find an eigenvector of A^3 .

17. Use the Gram-Schmidt process to construct an orthonormal basis for the subspace W of \mathbb{R}^4 spanned by

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ -3 \end{bmatrix} \right\}.$$

18. Let W be the subspace of \mathbb{R}^3 spanned by the vector $w = [1 \ 1 \ -1]$.

(a) Find a basis for the orthogonal complement W^\perp of W .

(b) Find an orthonormal basis for W^\perp .

19. Let W be the subspace of \mathbb{R}^4 spanned by the set of vectors $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$,

and $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 4 \end{bmatrix}$. (a) Find the orthogonal projection of \mathbf{v} onto W . (b) Write \mathbf{v} as $\mathbf{w} + \mathbf{u}$,

with $\mathbf{w} \in W$ and $\mathbf{u} \in W^\perp$. (c) Find the distance from \mathbf{v} to W .

20. Let A be a 5×7 matrix with $\text{rank}(A) = 3$, and let \mathbf{b} be a 5×1 matrix. Determine if each of the following statements is true.

(a) For every \mathbf{b} , the system $A\mathbf{x} = \mathbf{b}$ is uniquely solvable. True False

(b) For some \mathbf{b} , the system $A\mathbf{x} = \mathbf{b}$ is uniquely solvable, and for some \mathbf{b} , it is not solvable. True False

(c) For every \mathbf{b} , the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions. True False

(d) For some \mathbf{b} , the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions, and for some \mathbf{b} , it is not solvable. True False

(e) For some \mathbf{b} , the system $A\mathbf{x} = \mathbf{b}$ is not solvable. True False

21. A , B and C are 3×3 matrices and k is a scalar. determine if each of the following statements is *always* true.

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|--|------|-------|
| (a) If $A^2 = I_3$, then $A = I_3$ or $-I_3$. | True | False |
| (b) $\det(AB) = \det(A)\det(B)$. | True | False |
| (c) $\det(kA) = k^3 \det(A)$. | True | False |
| (d) If C is invertible, the characteristic polynomial of A is the same as that of $C^{-1}AC$. | True | False |
| (e) $(A - A^T)^T = A - A^T$. | True | False |
| (f) $\det(\text{adj}(A)) = \det(A)$. | True | False |

22. For each of the following sets of vectors, determine if it is a vector (sub)space:

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|---|-----|----|
| (a) The set of all vectors in \mathbb{R}^4 with the property $2x_1 + x_2 - 3x_3 + x_4 = 0$; | Yes | No |
| (b) The set of all matrices which have the form $\begin{bmatrix} a & b \\ c & 2 \end{bmatrix}$; | Yes | No |
| (c) The set of all vectors in \mathbb{R}^3 , which have the form $(0, a - b + c, 3b + c)$ where a , b and c are arbitrary real numbers; | Yes | No |
| (d) The set of all polynomials P in the space of all polynomials of degree at most 7, with the property $P(2) = 0$; | Yes | No |
| (e) The set of all nonsingular matrices in the space of 3×3 matrices; | Yes | No |

23. Determine if each of the following sets of vectors is linearly independent or linearly dependent:

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|---|-------------|-----------|
| (a) $\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}$; | Independent | Dependent |
| (b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} \right\}$; | Independent | Dependent |
| (c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} \right\}$; | Independent | Dependent |

- (d) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix} \right\};$ Independent Dependent
- (e) $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$ Independent Dependent

24. We have a 3×3 matrix $A = \begin{bmatrix} a & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix}$ with $\det(A) = 3$. Compute the determinant of the following matrix

(a) $\begin{bmatrix} a-2 & 1 & 2 \\ b-4 & 3 & 4 \\ c-6 & 5 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} 7a & 7 & 14 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} a & 1 & 1 \\ b & 3 & 3 \\ c & 5 & 5 \end{bmatrix}$

(d) A^T

(e) $(5A)^{-1}$

25. Find the determinants of the following matrices:

(a) $\begin{bmatrix} 1 & 3 & 2 & -1 \\ 10 & 2 & -1 & 3 \\ 7 & 0 & 10 & 0 \\ 1 & 3 & 2 & -1 \end{bmatrix}.$

(b) $\begin{bmatrix} 12 & 3 & 2 & 3 \\ 5 & 2 & -1 & 2 \\ 5 & 4 & 10 & 4 \\ 0 & 1 & 2 & 1 \end{bmatrix}.$

(c) $\begin{bmatrix} 6 & 3 & -4 & 3 \\ 1 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 4 & 1 & 2 & 1 \end{bmatrix}.$

$$(d) \begin{bmatrix} 2 & 3 & 2 & -1 & 1 \\ 0 & 2 & -1 & 3 & 8 \\ 0 & 0 & 5 & 1 & -4 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

$$(e) \begin{bmatrix} 2 & 3 & 2 & -1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

26. Find all eigenvalues of $A = \begin{bmatrix} 2 & 1 & 2 \\ -3 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

27. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

- (a) Find all eigenvalues of A .
- (b) For each eigenvalue as above, find an eigenvector of A associated to it.
- (c) Find a diagonal matrix D and an orthogonal matrix P such that $P^T A P = D$.

28. Determine an invertible matrix \hat{A} and a vector \hat{b} such that the solution to $\hat{A}\hat{X} = \hat{b}$ is the least squares solution to $AX = b$, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

29. Find the least squares fit line for the points

$$(1, 2), (2, 1), (3, 3), (4, 3).$$

30. Consider the homogeneous linear system of differential equations $\mathbf{x}'(t) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 2 & -1 \\ 8 & 8 \end{bmatrix}.$$

- (a) Find all eigenvalues of A .
- (b) Find the general solution to the linear system of differential equations.

(c) Solve the initial value problem for the given conditions $x_1(0) = 3$ and $x_2(0) = -10$.

31. Consider the homogeneous linear system of differential equations $\mathbf{x}'(t) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}.$$

(a) Find all eigenvalues of A .

(b) Find the real-valued general solution to the linear system of differential equations.

(c) Solve the initial value problem for the given conditions $x_1(0) = 5$ and $x_2(0) = 1$.

32. Consider the homogeneous linear system of differential equations $\mathbf{x}'(t) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}.$$

(a) Find all eigenvalues of A .

(b) Find the general solution to the linear system of differential equations.

(c) Solve the initial value problem for the given conditions

$$x_1(0) = -2, \quad x_2(0) = 1, \quad x_3(0) = 2$$

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