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MA303

FINAL EXAMINATION

Practice

Name \_\_\_\_\_ ID # \_\_\_\_\_ Section # \_\_\_\_\_

There are 20 Problems on this booklet. For All problems, mark the answers clearly.

1. Consider the differential equation

$$y' = 2t - y^2, \quad y(0) = 0.$$

Compute an approximation to  $y(2)$  using the improved Euler scheme with  $h = 1$ . The improved Euler scheme is:

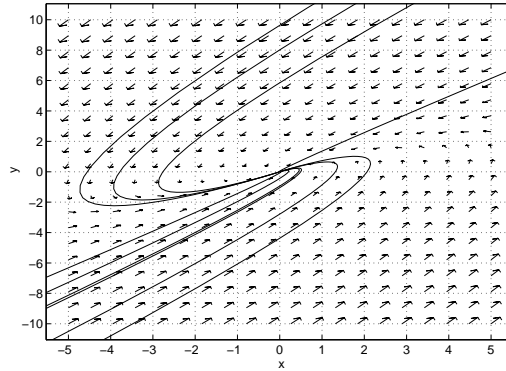
$$\begin{aligned} f_n &= f(t_n, y_n); & t_{n+1} &= t_n + h; \\ y_{n+1} &= y_n + \frac{h}{2}(f_n + f(t_n + h, y_n + hf_n)). \end{aligned}$$

The approximate value of  $y(2)$  is \_\_\_\_\_.

2. The phase portrait for a linear system of the form

$$\bar{x}' = A\bar{x},$$

where  $A$  is a  $2 \times 2$  matrix is as follows.



(i) Which of the following best describes the eigenvalues of  $A$ :

- (a) two real distinct positive eigenvalues
- (b) two real distinct negative eigenvalues
- (c) two real eigenvalues of opposite signs
- (d) two complex eigenvalues with positive real part
- (e) two complex eigenvalues with negative real part
- (f) one positive repeated eigenvalue
- (g) one negative repeated eigenvalue

(ii) Describe the type and the stability of the critical point  $(0, 0)$ :

- (a) a unstable spiral point
- (b) an asymptotically stable spiral point
- (c) a unstable saddle point
- (d) an asymptotically stable improper node
- (e) a unstable improper node
- (f) a stable center

3. Find all values of  $k$  for which the equilibrium point  $(0, 0)$  of the system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -2 & k \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

is a saddle point.

a)  $k > 0$

b)  $k > 1$

c)  $k < 0$

d)  $k < 1$

e)  $k \geq 1$

4. Find the solution of the linear system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

a)  $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{2}e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{2}e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

b)  $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{2}e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

c)  $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{2}e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

d)  $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

e)  $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{2}e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{2}e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

5. Find the solution of the linear system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

a)  $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{2}e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{2}e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

b)  $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{2}e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

c)  $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{2}e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

d)  $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

e)  $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-3t} \begin{pmatrix} 3 + 4t \\ 2 + 4t \end{pmatrix}$

6. The general solution of  $\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} + t \begin{pmatrix} -4 \\ -3 \end{pmatrix}$  is

a)  $\mathbf{x}(t) = C_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 4 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

b)  $\mathbf{x}(t) = C_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 4 \\ 3/2 \end{pmatrix} + \begin{pmatrix} 4 \\ 3/4 \end{pmatrix}$

c)  $\mathbf{x}(t) = C_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -4 \\ -3 \end{pmatrix}$

d)  $\mathbf{x}(t) = C_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -4 \\ 3/2 \end{pmatrix} + \begin{pmatrix} -4 \\ 3/4 \end{pmatrix}$

e)  $\mathbf{x}(t) = C_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 4/3 \\ 3/2 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

7. If  $y'' + 2y' + 5y = 0$ ,  $y(0) = 1$ ,  $y'(0) = 2$ , then  $y(\frac{\pi}{2}) =$

- a) 1.
- b) 0.
- c)  $e^\pi$ .
- d)  $e^{\frac{\pi}{2}}$ .
- e)  $-e^{-\frac{\pi}{2}}$ .

8. What is the solution of initial value problem

$$y'' = \delta(t - \pi) - \delta(t - 2\pi); \quad y(0) = y'(0) = 0.$$

- a)  $y(t) = u_\pi(t)t - u_{2\pi} + \pi$
- b)  $y(t) = \frac{1}{t^2}(e^{-\pi t} - e^{-2\pi t})$
- c)  $y(t) = c_1 + c_2 t$
- d)  $y(t) = u_\pi(t)(t - \pi) - u_{2\pi}(t)(t - 2\pi)$
- e)  $y(t) = u_\pi(t)t - u_{2\pi}t$

9. What is the solution of initial value problem

$$y'' + 2y' + 2y = \delta(t - \pi); \quad y(0) = 1, \quad y'(0) = 0.$$

- a)  $y(t) = e^{-t} \cos(t) + e^{-t} \sin(t) + u_{\pi}(t)e^{\pi-t} \sin(t - \pi)$
- b)  $y(t) = e^{-t} \cos(t) + 2e^{-t} \sin(t) + u_{\pi}(t)e^{\pi-t} \sin(t - \pi)$
- c)  $y(t) = u_{\pi}(t)e^{t-\pi} \sin(t - \pi)$
- d)  $y(t) = e^{-t} \cos(t) + 2e^{-t} \sin(t) + u_{\pi}(t)e^{\pi-t} \sin(t - \pi) + u_{\pi}(t)e^{\pi-t} \cos(t - \pi)$
- e)  $y(t) = e^{-t} \cos(t) + e^{-t} \sin(t) + u_{\pi}(t)e^{\pi-t} \cos(t - \pi)$

10. Let  $y(x) = \sum_{n=0}^{\infty} a_n x^n$  be a series solution for

$$(1 - x)y'' + y = 0$$

near  $x = 0$ . The recurrence relation for  $a_n$  is

- a)  $(n + 2)(n + 1)a_{n+2} - n(n + 1)a_{n+1} + a_n = 0, \quad n \geq 0$
- b)  $(n + 2)(n + 1)a_{n+2} + a_n = 0, \quad n \geq 0$
- c)  $(n + 2)(n + 1)a_n - n(n + 1)a_{n+1} + a_{n+2} = 0, \quad n \geq 0$
- d)  $(n + 2)(n + 1)a_{n+2} + n(n + 1)a_{n+1} + a_n = 0, \quad n \geq 0$
- e)  $(n + 2)(n + 1)a_{n+2} + n(n + 1)a_{n+1} - 4a_n = 0, \quad n \geq 0$

11. (continue from the previous problem) Find the first four non-zero terms of the series solution for

$$(1 - x)y'' + y = 0, \quad y(0) = 1, \quad y'(0) = -1.$$

$$y(x) = \underline{\hspace{15cm}}.$$

12. Find all  $T(t)$  such that  $u(x, t) = e^x T(t)$  is a solution of

$$4u_{xx} = u_{tt}.$$

$$T(t) = \underline{\hspace{15cm}}.$$

13. Consider the equation

$$xu_x + yu_y = 0.$$

- (a) Find a pair of ordinary differential equations which can be used to replace the partial differential equation.
- (b) Find a **non-constant solution**  $u(x, y) = \underline{\hspace{2cm}}$

14. A silver bar one meter long is initially heated to a temperature of  $40^{\circ}\text{C}$  on its left half and  $60^{\circ}\text{C}$  on its right half. It is perfectly insulated at its ends. For silver, the thermal diffusivity is known to be  $1.71 \text{ cm}^2/\text{sec}$ .

**Without solving**, write down the problem that must be solved to determine the temperature  $u(x, t)$  at a point  $x$  centimeters from the left end, at time  $t$  seconds. That is, write down the heat equation, boundary conditions and initial temperature distribution.

15. The eigenvalues and the associated eigenfunctions of the following initial value problem:

$$y''(x) + \lambda y(x) = 0, \quad y'(0) = 0, \quad y(1) = 0$$

are

a)  $\lambda_n = (n + \frac{1}{2})^2 \pi^2, \quad y_n(x) = \sin(n + \frac{1}{2})\pi x, \quad n = 0, 1, 2, \dots$

b)  $\lambda_n = (n + \frac{1}{2})^2, \quad y_n(x) = \cos(n + \frac{1}{2})x, \quad n = 0, 1, 2, \dots$

c)  $\lambda_n = (n + \frac{1}{2})^2, \quad y_n(x) = \sin(n + \frac{1}{2})x, \quad n = 0, 1, 2, \dots$

d)  $\lambda_n = (n + \frac{1}{2})^2 \pi^2, \quad y_n(x) = \cos(n + \frac{1}{2})\pi x, \quad n = 0, 1, 2, \dots$

e)  $\lambda_n = n^2 \pi^2, \quad y_n(x) = \cos n\pi x, \quad n = 0, 1, 2, \dots$

16. The Fourier series for the following periodic function:

$$f(x) = \begin{cases} 1, & \text{if } -1 \leq x < 0, \\ 2, & \text{if } 0 \leq x < 1, \end{cases}, \quad f(x+2) = f(x) \text{ for all } x$$

is

a)  $\frac{3}{2} + \sum_{n=1}^{\infty} (1 - (-1)^n) \frac{1}{n\pi} \sin n\pi x$

b)  $\frac{3}{2} + \sum_{n=1}^{\infty} (1 - (-1)^n) \frac{1}{n\pi} \cos n\pi x$

c)  $1 + \sum_{n=1}^{\infty} (1 - (-1)^n) \frac{1}{n\pi} (\cos n\pi x + \sin n\pi x)$

d)  $1 + \sum_{n=1}^{\infty} (1 - (-1)^n) \frac{1}{n\pi} (\cos nx + \sin nx)$

e)  $\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\pi x$

17. A string  $L = \pi$  m long and fixed at both ends is giving an initial velocity of  $g(x) = 1$  m/s from equilibrium. The string has no initial displacement. Find an expression for the displacement function  $u(x, t)$  valid for all  $x$  in  $0 \leq x \leq \pi$  and all  $t \geq 0$ . (Assume  $\alpha = 1$ , where  $\alpha$  is the constant in the wave equation.)

Recall: The solution of the wave equation

$$\alpha^2 u_{xx} = u_{tt}, \quad 0 < x < l, \quad t > 0,$$

satisfying the homogeneous boundary conditions  $u(0, t) = u(l, t) = 0$  for  $t > 0$  and initial conditions

$$u(x, 0) = f(x) \quad \text{and} \quad u_t(x, 0) = g(x) \quad \text{for} \quad 0 \leq x \leq l$$

has the general form

$$u(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \left( c_n \cos \frac{n\pi \alpha t}{l} + k_n \sin \frac{n\pi \alpha t}{l} \right)$$

where

$$c_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx \quad \text{and} \quad k_n = \frac{2}{n\pi\alpha} \int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) dx.$$

- a)  $\sum_{n=1}^{\infty} (1 - (-1)^n) \frac{2}{n^2\pi^2} \sin(n\pi x) \sin(n\pi t)$
- b)  $\sum_{n=1}^{\infty} (1 - (-1)^n) \frac{2}{n^2\pi} \sin(nx) \sin(nt)$
- c)  $\sum_{n=1}^{\infty} (1 - (-1)^n) \frac{2}{n^2\pi} \sin(nx) \cos(nt)$
- d)  $\sum_{n=1}^{\infty} (1 - (-1)^n) \frac{2}{n^2\pi^2} \sin(n\pi x) \cos(n\pi t)$
- e)  $\sum_{n=1}^{\infty} (1 - (-1)^n) \frac{2}{n^2\pi^2} \sin(nx) (\sin nt + \cos(n\pi t))$

18. Consider

$$9u_{xx} = u_t$$

- Find  $T(t)$  such that  $u(x, t) = e^x T(t)$  is a solution
- Find  $X(x)$  such that  $u(x, t) = X(x)e^t$  is a solution and  $u(0, t) = 0$
- Find  $X(x)$  and  $T(t)$  such that their product  $u(x, t)$  is a solution and  $u(0, t) = u(5, t) = 0$
- Find a solution such that  $u(0, t) = u(5, t) = 0, u(x, 0) = \sum_{k=1}^{\infty} k^{-2} \sin(k\pi x)$ . Compute  $u(3, 4)$
- Find solution with  $u(x, 0) = 2$

19. Consider the equation

$$u_{xx} + u_{yy} = 0, \quad 0 < x < 1, \quad 0 < y < 1$$

which satisfies the boundary conditions

$$\begin{aligned} u(0, y) &= 0, & u(1, y) &= 0, & 0 \leq x \leq 1, \\ u(x, 0) &= 0, & u(x, 1) &= 1, & 0 \leq y \leq 1. \end{aligned}$$

Find the solution by using the following steps:

- Let  $u(x, y) = X(x)Y(y)$ . Find the two ordinary differential equations satisfied by  $X(x)$  and  $Y(y)$ .
- solving  $X(x)$ , namely finding the eigenvalues and eigenfunctions;
- solving  $Y(y)$  and write out the general solution for the homogeneous problem;
- find the general solution for  $u(x, 1) = f(x)$ ;
- find the solution with  $f(x) = 1$ .

20.

**EXAMINATION (Answer)**

1. 1.5

2. G; D

3. C

4. D

5. E

6. B

7. E

8. D

9. A

10. A

11.  $1 - x - \frac{1}{2}x^2 + \frac{1}{24}x^4$

12.  $T(t) = c_1e^{2t} + c_2e^{-2t}$

13. (a)  $xX' + \lambda X = 0$ ,  $yY' - \lambda Y = 0$ , (b)  $x^{-1}y$

14.  $(1.71)u_{xx} = u_t; u_x(0, t) = u_x(100, t) = 0; u(x, 0) = \begin{cases} 40 & 0 < x < 50 \\ 60 & 50 < x < 100 \end{cases}$

15. D

16. A

17. B

18. (a)  $T(t) = c_1e^{9t}$ , (b)  $X(x) = c_2 \sinh(x/3)$ ,

(c)  $X(x) = \sin(n\pi x/5)$  and  $T(t) = e^{-\frac{9n^2\pi^2}{25}t}$ ,

(d)  $\sum_{k=1}^{\infty} k^{-2} \sin(k\pi x)e^{-9k^2\pi^2 t}$ ,  $u(3, 4) = 0$ ,

(e)  $\sum_{n=1}^{\infty} \frac{4}{n\pi} (1 - (-1)^n) \sin(n\pi x/5)e^{-\frac{9n^2\pi^2}{25}t}$

19. (a)  $X'' + \lambda X = 0; X(0) = X(1) = 0; Y'' - \lambda Y = 0; Y(0) = 0$

(b)  $\lambda_n = n^2\pi^2, X_n = \sin(n\pi x), n = 1, 2, \dots$

(c)  $Y_n = e^{n\pi y} - e^{-n\pi y}; u(x, y) = \sum_{n=1}^{\infty} C_n \sin(n\pi x)(e^{n\pi y} - e^{-n\pi y})$

(d)  $C_n = \frac{2}{e^{n\pi} - e^{-n\pi}} \int_0^1 f(x) \sin(n\pi x) dx$

(e)  $u(x, y) = \sum_{n=1}^{\infty} \frac{2(1 - (-1)^n)}{n\pi(e^{n\pi} - e^{-n\pi})} \sin(n\pi x)(e^{n\pi y} - e^{-n\pi y})$