LAB #2

Escape Velocity

Goal: Determine the initial velocity an object is shot upward from the surface of the earth so as to never return; illustrate scaling variables to simplify differential equations.

Required tools: ***dfield*** and ***pplane*** ; second order differential equations; converting a second order differential equation with missing independent variable to a first order differential equation; converting a second order differential equation to a system of differential equations.

Discussion

Newton's Law of Gravitational Attraction tells us that in general an object of mass m which is x miles above the surface of the earth will feel a force of approximately

$$F = -\frac{0.0061m}{\left(1 + \frac{x}{4000}\right)^2}$$

pounds with the number "4000" an approximation to the radius of the earth in miles and "0.0061" the acceleration due to gravity measured in miles/sec^2.

On the other hand, according to Newton's 2nd Law

$$F = m\frac{d^2x}{dt^2}\,.$$

Thus equating these expressions for F leads to the differential equation

$$\frac{d^2x}{dt^2} = -\frac{0.0061}{\left(1 + \frac{x}{4000}\right)^2}\,.$$

To study this equation using the Student Edition of Matlab, we need to change the units to avoid overly large intervals. Rather than measuring distance in miles, we will use multiples of the radius of the earth. This amounts to making the substitution

$$y = \frac{x}{4000}$$

(scaling the dependent variable) resulting in the equation

$$\frac{d^2y}{dt^2} = -\frac{0.0061}{4000}\,\frac{1}{(1+y)^2}\,. \qquad (\mathbf{A})$$

We can further simplify this equation by changing the units of time (scaling the independent variable). If we let

$$s = \sqrt{\frac{0.0061}{4000}}\;t$$

then by the Chain Rule

$$\frac{dy}{dt} = \frac{dy}{ds}\frac{ds}{dt} = \sqrt{\frac{0.0061}{4000}}\,\frac{dy}{ds}$$

and hence

$$\begin{aligned}
\frac{d^2y}{dt^2} &= \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\sqrt{\frac{0.0061}{4000}}\,\frac{dy}{ds}\right) \\
\frac{d^2y}{dt^2} &= \sqrt{\frac{0.0061}{4000}}\,\frac{d}{dt}\left(\frac{dy}{ds}\right) \\
\frac{d^2y}{dt^2} &= \sqrt{\frac{0.0061}{4000}}\,\left[\frac{d}{ds}\left(\frac{dy}{ds}\right)\,\frac{ds}{dt}\right] \\
\frac{d^2y}{dt^2} &= \frac{0.0061}{4000}\,\frac{d^2y}{ds^2}
\end{aligned}$$

Using the last expression, the differential equation **(A)** becomes

$$\frac{d^2y}{ds^2} = -\frac{1}{(1+y)^2}\,. \qquad \textbf{(B)}$$

Hence y is now viewed as a function of s, instead of t.

Equation **(B)** is a second order differential equation whose independent variable s is missing. We can convert this special type of 2nd order differential equation to a first order equation using a simple technique. Let

$$v = \frac{dy}{ds}\,.$$

This is simply the velocity of the projectile using s as the unit of time instead of t and y as the distance instead of x.

Since $\dfrac{dy}{ds} = v$, from the Chain Rule we obtain

$$\frac{d^2y}{ds^2} = \frac{dv}{ds} = \frac{dv}{dy}\,\frac{dy}{ds} = v\,\frac{dv}{dy}\,.$$

Thus, equation **(B)** can be written as

$$v\,\frac{dv}{dy} = -\frac{1}{(1+y)^2}\,. \qquad \textbf{(C)}$$

Equation **(C)** is a 1st order differential equation which may be solved to express v as a function of y.

Assignment

(1) Enter equation **(C)** into ***dfield*** and plot a few solutions starting at $y = 0$ and various initial values for the velocity v using the ***dfield*** default ranges for v and y. You will need to enter it in the form

$$\frac{dv}{dy} = -\frac{1}{v\,(1+y)^2}\,. \qquad \textbf{(D)}$$

(If you have trouble with ***dfield*** crashing, see the next exercise.)

(2) When the author did **(C)** using ***dfield5*** with the Student Edition of MATLAB, ***dfield*** crashed almost every time. The reason is that for this particular equation, the solution algorithm on which ***dfield*** is based requires many data points to produce an accurate picture. In fact, it can require so many as to exceed the maximum capacity of the Student Edition of MATLAB. There is another program, ***pplane*** , which works much better for this equation. To use ***pplane*** , note that equation **(B)** is equivalent to the system

$$\begin{cases} y' = v \\ v' = -\dfrac{1}{(1+y)^2} \end{cases}$$

This is a system of differential equations in 2 unknowns, y and v. You will study systems in detail later. For the moment, it is sufficient to know that ***pplane*** will plot one of the unknowns in the system as a function of the other. Specifically, start ***pplane*** (the command is "***pplane*** " or possibly "***pplane***N" where N is the version of MATLAB you are using) and then enter the above equations into the equation boxes of ***pplane*** . Use the ranges $-1 \le y \le 15$ and $-2 \le v \le 3$.

(3) Plot the solutions with initial values $y = 0$ and $v = b$ where b is $0.5, 1.0, 1.5$, and 2. You can either left click onto the appropriate points in the ***pplane*** "Display" window or use the "keyboard input" option in the "Solutions" pull down menu of the "Display" window. What is the maximum height of the object (in miles) in each case where there is a finite maximum ? (Recall that $4000y = x$.) In the cases where there is no maximum, approximate the limiting value of v as $t \longrightarrow \infty$. (This is the *terminal velocity*). What is the terminal velocity in *miles/sec* ?

<u>*Hint*</u> : Since $4000y = x$, we get $\dfrac{dx}{dt} = 4000\,\dfrac{dy}{dt} = 4000\,\dfrac{dy}{ds}\,\dfrac{ds}{dt} = 4000\,v\,\dfrac{ds}{dt}$.

(4) In Part (3), you should discover that the escape velocity lies between $v = 1$ and $v = 1.5$ (since object never reaches a finite maximum height, it never returns to earth). To estimate the escape velocity, plot the orbit that goes through the point $(y, v) = (10, 0)$ and note the velocity v_0 of this orbit corresponding to $x = 0$. This velocity will be somewhat below the escape velocity. Use the zoom feature to estimate v_0 to within 2 places after the decimal. What is the value of v_0 in *miles/sec* ?

(5) Solve the separable differential equation **(D)** with the initial condition $v(0) = v_0$ to obtain the following expression

$$y = \frac{2}{v^2 + (2 - v_0^2)} - 1\,.$$

(a) According to this formula, for $v_0 = 1$, what is the maximum value of y ? Explain your answer mathematically and demonstrate using a ***pplane*** plot.

(b) Use the above formula to prove that if $v_0 = \sqrt{2}$, then y tends to 1 as v approaches 1. The value $v = 1$ is then the terminal velocity. Check this using an appropriate ***pplane*** plot.

(c) What, according to the above formula, is the escape velocity ?
Hint : The escape velocity will be the smallest value of v_0 for which $y \longrightarrow \infty$ as v tends to some number.

(6) The above calculations ignore the effect of air resistance on the object. We assume that resistance is proportional to velocity and decreases with increasing height. Explicitly, we assume the frictional force to be

$$F_f = -\frac{kv}{(1+y)^2}$$

where k is a physical constant (the coefficient of friction) and we will initially take $k = 0.01$. We will also assume that our object has mass $m = 1$ in which case our differential equation becomes

$$\frac{d^2y}{ds^2} = -\frac{1}{(1+y)^2} - \frac{0.01v}{(1+y)^2} = -\frac{(1+0.01v)}{(1+y)^2}\,. \qquad (\mathbf{E})$$

Rewrite (**E**) as a system of equations as in Part (2). Use ***pplane*** to estimate the escape velocity as in Part (2) to find how much air resistance changes the calculations. Next, solve the equation analytically. You will not be able to explicitly solve for v. Instead, you should get

$$100v - 10^4 \ln(1+0.01v) - \frac{1}{(1+y)} = C$$

Show that $C = 100v_0 - 10^4 \ln(1+0.01v_0) - 1$ and then explain why the object escapes if and only if $100v_0 - 10^4 \ln(1+0.01v_0) > 1$. Exit ***pplane*** and then use MATLAB to plot the function $100v - 10^4 \ln(1+0.01v) - 1$ to find the escape velocity. Make sure that your answer is accurate to within two digits after the decimal. This may require doing several plots over increasingly smaller intervals. The appropriate sequence of commands is:

```
>> grid on
>> hold on
>> fplot('100*x-10^4*log(1+0.01*x)-1',[xmin,xmax,ymin,ymax])
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where *xmin, xmax* and *ymin, ymax* are the ranges on the horizontal and vertical axes, respectively. Note that the function to be plotted must be enclosed in single quotes. The command "grid on" causes the axes to be visible and "hold on" prevents them from being erased as subsequent graphs are drawn.

(7) Estimate the escape velocity as in Part (6) if $k = \dfrac{0.1}{\mathbf{seed}}$.