Exact Solutions of Various Boussinesq Systems

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Abstract—It was shown in [1,2] that surface water waves in a water tunnel can be described by systems of the form

\[ \begin{align*}
\eta_t + u_x + (\eta u)_x + au_{xx} - b\eta_{xx} &= 0, \\
u_t + \eta_x + u u_x + c\eta_{xx} - du_{xx} &= 0,
\end{align*} \]

where \( a, b, c, \) and \( d \) are real constants. In this paper, we show that to find an exact traveling-wave solution of the system, it is suffice to find a solution of an ordinary differential equation, and the solution of the ordinary differential equation in a prescribed form can be found by solving a system of nonlinear algebraic equation. The exact solutions for some of the systems are presented at the end of the paper. © 1998 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

To describe small amplitude and long waves in a water channel, systems in the form of (1), which include the classical Boussinesq system (cf. [3]), were derived by Bona, Saut and Toland in [1], where \( a, b, c, \) and \( d \) are real constants and determined by three parameters \( \lambda, \mu, \) and \( 0 \leq \theta \leq 1 \) in the following way:

\[ \begin{align*}
a &= \frac{1}{2} \left( \theta^2 - \frac{1}{3} \right) \lambda, \\
b &= \frac{1}{2} \left( \theta^2 - \frac{1}{3} \right) (1 - \lambda), \\
c &= \frac{1}{2} \left( 1 - \theta^2 \right) \mu, \\
d &= \frac{1}{2} \left( 1 - \theta^2 \right) (1 - \mu).
\end{align*} \]

The dimensionless variables \( x \) and \( t \) are scaled, respectively, by \( h \) and \((h/g)^{1/2}\) where \( h \) denotes the undisturbed water depth and \( g \) denotes the acceleration of gravity. The variable \( \eta(x,t) \) is the nondimensional deviation of the water surface (scaled by \( h \)) from its undisturbed position and \( u(x,t) \) is the nondimensional horizontal velocity (scaled by \( \sqrt{gh} \)) at a height \( \theta h \) with \( 0 \leq \theta \leq 1 \) above the bottom of the channel. These three parameter family of systems are formally equivalent and correct through first order with regard to the small parameter \( \epsilon = \sup \{ \eta(x,t) \} \). In this paper, we concentrate on finding exact traveling-wave solutions of (1) which approach constants at infinities. The existence of these solutions is useful in the theoretical and numerical studies of

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the model systems. In fact, one of the exact solution we found here for the regularized Boussinesq system \((a = c = 0, b = d = 1/6)\) has been used in [2] to demonstrate the convergence rate of a numerical algorithm.

2. MAIN RESULTS

Denoting \(\xi = x + x_0 - C_s t\) with \(x_0\) and \(C_s\) being constants, we first present the result on the existence of traveling-wave solution

\[
\eta(x, t) = \eta(\xi), \quad u(x, t) = u(\xi),
\]  

(2)

that \(\eta(\xi)\) and \(u(\xi)\) are asymptotically small at large \(\xi\) and proportional to each other, so

\[
\lim_{\xi \to \pm \infty} (\eta(\xi), u(\xi)) = 0, \quad \eta(x, t) = Bu(x, t),
\]

(3)

with \(B\) being a constant. Substituting (2) and (3) into system (1) and using the fact that the resulting two equations are consistent, one can prove the following.

THEOREM. For a given system in the form of (1), if the constants \(a, b, c, d\) satisfy one of the following conditions:

(i) \(a - b + 2d \neq 0, p = (-b + c + 2d)/(a - b + 2d) > 0,\) and \((p - 1/2)((b - a)p - b) > 0;\)

(ii) \(a = b = c > 0, d = 0;\)

(iii) \(a = b = c < 0, d = 0;\)

(iv) \(a - b + 2d = 0, a = c, d > 0;\)

(v) \(a - b + 2d = 0, a = c, d < 0;\)

then the given system has solitary-wave solutions. Moreover, the exact solitary-wave solutions are of the form

\[
\eta(x, t) = \eta_0 \text{sech}^2(\lambda(x + x_0 - C_s t)),
\]

\[
u(x, t) = \pm \sqrt{\frac{3}{\eta_0 + 3 \eta_0 \text{sech}^2(\lambda(x + x_0 - C_s t))}},
\]

where

\[
C_s = \frac{3 + 2\eta_0}{\pm \sqrt{3 + 3 \eta_0}}, \quad \lambda = \frac{1}{2} \sqrt{\frac{2\eta_0}{3(a - b) + 2b(\eta_0 + 3)}},
\]

and \(\eta_0\) can be any constant satisfies

- in Case (i), \(\eta_0 = (3(1 - 2p))/2p;\)
- in Case (ii), \(0 < \eta_0 < +\infty;\)
- in Case (iii), \(-3 \leq \eta_0 < 0;\)
- in Case (iv), \(\eta_0 > -3\) and \(3/(\eta_0 + 3)\) is not in the closed interval between 1 and \(b/d;\)
- in Case (v), \(\eta_0 > -3\) and \(3/(\eta_0 + 3)\) is in the closed interval between 1 and \(b/d.\)

With a more general approach, one can find exact solutions where \(u(\xi)\) and \(\eta(\xi)\) are not proportional to each other and they do not approaches zero at infinity. Assuming that the traveling-wave solution \((u(\xi), \eta(\xi))\) tends to \((u_\infty, \eta_\infty)\) as \(\xi\) tends to \(\pm \infty.\) Substituting functions

\[
h(\xi) = \eta(\xi) - \eta_\infty, \quad \nu(\xi) = u(\xi) - u_\infty,
\]

(4)

into (1) and integrating the system once, one obtains

\[
-C_s h + v + \nu h + \eta_\infty \nu + u_\infty h + av'' + bC_s h'' = 0,
\]

\[
-C_s v + h + \frac{1}{2} v^2 + u_\infty v + ch'' + dC_s v'' = 0.
\]

(5)
Eliminating one of the dependent variables, one can find that \( v(\xi) \) (or \( h(\xi) \)) satisfies a fourth-order ordinary differential equations (cf. [4]). For instance, in the case that \( c \neq 0 \), one can eliminate \( h(\xi) \) and obtain an ordinary differential equation on \( v(\xi) \) as follows. Notice from (5) that \( h \) and \( h'' \) can be expressed as a function of \( v(\xi) \),

\[
\begin{align*}
h &= \frac{g_1(v)}{f(v)} \quad \text{and} \quad h'' &= \frac{g_2(v)}{f(v)},
\end{align*}
\]

where

\[
\begin{align*}
f(v) &= c(-C_s + v + u_\infty) - bC_s, \\
g_1(v) &= c(-v - av'' - \eta_\infty v) - bC_s \left(C_s v - \frac{1}{2}v^2 - dC_s v'' - u_\infty v\right), \\
g_2(v) &= v + av'' + \eta_\infty v + (-C_s + v + u_\infty) \left(C_s v - \frac{1}{2}v^2 - dC_s v'' - u_\infty v\right).
\end{align*}
\]

Differentiating the first equation in (6) twice with respect to \( \xi \) and using the second equation, one finds

\[
f^2 g_2 = g_1' f^2 - g_1 f f' - 2g_1 f f'' + 2g_1 (f')^2,
\]

which is an ordinary differential equation with dependent variable \( v(\xi) \). One can therefore established the fact that in order to find a traveling-wave solution of (1), it is suffice to find a solution \( v(\xi) \) satisfying the ordinary differential equation.

Notice again that the ordinary differential equation (7) involves only

\[
\begin{align*}
&v''', \quad v'v'', \quad (v'')^2, \quad v'', \quad (v')^2
\end{align*}
\]

terms, the Ansatz equation

\[
(v')^2 = \rho v^2 + \sigma v^3, \quad \rho \geq 0,
\]

(8) can be used to find solutions in the form of

\[
\begin{align*}
v(\xi) &= -\frac{\rho}{\sigma} \text{sech}^2 \left(\frac{1}{2} \sqrt{\rho} \xi\right)
\end{align*}
\]

(cf. [6]). Substituting (8) into (9) yields a polynomial equation on \( v(\xi) \) where the coefficients depend on \( \rho, \sigma, C_s, u_\infty, \) and \( \eta_\infty \). By requiring the coefficients to be zero, one obtains a system of algebraic equations and the solution \( \rho, \sigma, C_s, u_\infty, \) and \( \eta_\infty \) provides the solution of ordinary differential equation in the form of (9), which in turn yields the exact traveling-wave solution of the system with the help of (6) and (4).

The method described above is used on a large class of the systems in (1) which includes the system in [5] (formula (13.101)), the systems in [6], regularized Boussinesq system in [1], Boussinesq's original system (cf. [3]), and the integrable version of the Boussinesq system (cf. [7]). The exact traveling-wave solutions founded are listed in next section. The method presented in this paper is quite general and it recovered the solutions founded in [8,9], where a homogeneous balance method was used.

Other Ansatz equations can be used to find solutions in different forms [10].

### 3. EXACT TRAVELING-WAVE SOLUTIONS FOR SYSTEMS IN (1)

Denote \( \xi = x + x_0 - C_s t \), where \( x_0 \) and \( C_s \) are arbitrary constants, one can find the exact traveling-wave solutions for the following systems (\( \rho \geq 0 \) is an arbitrary constant).

- **a = 0:**

  \[
  \begin{align*}
  v(\xi) &= (1 - d\rho) C_s + 3 dC_s \rho \ \text{sech}^2 \left(\frac{1}{2} \sqrt{\rho} \xi\right), \\
  \eta(\xi) &= -1.
  \end{align*}
  \]
• $a = 0, \ c \neq 0$:

$$u(\xi) = \frac{C_s}{2c} (-b + 2c + 2d - b\rho) + \frac{3}{2} C_s b\rho \text{sech}^2 \left( \frac{1}{2} \sqrt{\rho} \xi \right),$$

$$\eta(\xi) = -1 + \frac{C_s^2}{4c^2} (b^2 - 4bd + 4d^2 - b^2\rho + 2b\xi\rho) + \frac{3C_s^2}{4c} b (b - 2d) \rho \text{sech}^2 \left( \frac{1}{2} \sqrt{\rho} \xi \right).$$

• $a = c = 0$:

$$u(\xi) = \frac{C_s}{3} (3 - 5b\rho) + 5C_s b\rho \text{sech}^2 \left( \frac{1}{2} \sqrt{\rho} \xi \right),$$

$$\eta(\xi) = -1 + \frac{C_s^2}{9} b (10b - 6d) \rho^2$$

$$+ \frac{5}{6} C_s^2 b (5b - 3d) \rho^2 \left( 2 \text{sech}^2 \left( \frac{1}{2} \sqrt{\rho} \xi \right) - 3 \text{sech}^4 \left( \frac{1}{2} \sqrt{\rho} \xi \right) \right).$$

• $b = c = 0, \ d \neq 0$:

$$u(\xi) = -\frac{a + 2dC_s^2 - 2d^2C_s^2 \rho}{2dC_s} + 3dC_s \rho \text{sech}^2 \left( \frac{1}{2} \sqrt{\rho} \xi \right),$$

$$\eta(\xi) = -1 + \frac{a}{4d^2C_s^2} \left( a - 2d^2C_s^2 \rho \right) + \frac{3}{2} a\rho \text{sech}^2 \left( \frac{1}{2} \sqrt{\rho} \xi \right).$$

• $c = d = 0, \ b \neq 0$:

$$u(\xi) = \frac{-a}{4bC_s} + C_s - \frac{5}{3} C_s b\rho + 5C_s b\rho \text{sech}^2 \left( \frac{1}{2} \sqrt{\rho} \xi \right),$$

$$\eta(\xi) = -1 + \frac{a}{16b^2C_s^2} - \frac{5}{12} a\rho + \frac{10}{9} C_s^2 b^2 \rho^2 + \left( \frac{5}{4} a\rho + \frac{25}{3} C_s^2 b^2 \rho^2 \right) \text{sech}^2 \left( \frac{1}{2} \sqrt{\rho} \xi \right)$$

$$- \frac{25}{2} b^2 k^2 \rho^2 \text{sech}^4 \left( \frac{1}{2} \sqrt{\rho} \xi \right).$$

• $b = c = d = 0, \ a > 0$:

$$u(\xi) = C_s \pm \sqrt{a\rho} \tanh \left( \frac{1}{2} \sqrt{\rho} \xi \right),$$

$$\eta(\xi) = -1 + \frac{1}{2} a\rho \text{sech}^2 \left( \frac{1}{2} \sqrt{\rho} \xi \right).$$

• $b = c = d = 0, \ a < 0$:

$$u(\xi) = C_s \pm \sqrt{-a\rho} \text{sech} \left( \frac{1}{2} \sqrt{\rho} \xi \right),$$

$$\eta(\xi) = -1 - \frac{1}{4} a\rho + \frac{1}{2} a\rho \text{sech}^2 \left( \frac{1}{2} \sqrt{\rho} \xi \right).$$

• $b = d = 0, \ a = c$:

$$u(\xi) = \mp \sqrt{\frac{1}{2} (1 + c\rho) + 2C_s} \pm \frac{3c}{\sqrt{2}} \text{sech}^2 \left( \frac{1}{2} \sqrt{\rho} \xi \right),$$

$$\eta(\xi) = -\frac{1 + c}{2} + \frac{3c}{2} \text{sech}^2 \left( \frac{1}{2} \sqrt{\rho} \xi \right).$$
REFERENCES