MA 16200 Exam 2 Study Guide, Fall 2014

LESSON 10

Chapter 7.2 Trigonometric Integrals

1. Basic trig integrals you should know.

$$\int \sin x \; dx = -\cos x + C$$
$$\int \cos x \; dx = \sin x + C$$
$$\int \sec^2 x \; dx = \tan x + C$$
$$\int \sec x \;\tan x \; dx = \sec x + C$$
$$\int \csc^2 x \; dx = -\cot x + C$$
$$\int \csc x \;\cot x \; dx = -\csc x + C$$

$$\int \tan x \; dx = \ln|\sec x| + C = -\ln|\cos x| + C$$
$$\int \sec x \; dx = \ln|\sec x + \tan x| + C$$
$$\int \cot x \; dx = \ln|\sin x| + C$$
$$\int \csc x \; dx = \ln|\csc x - \cot x| + C$$

2. Products and powers of trig functions.

Usually the idea is to change the integral to the form $\displaystyle\int u^n \; du$ (or the integral of linear combinations of u^n).

Integrands containing powers of sine and cosine functions. $\int \sin^m x \cos^n x \; dx$.

- Odd powers of $\sin x$ – factor odd factor of $\sin x$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to convert remaining even powers of $\sin x$ to $\cos x$. Then let $u = \cos x$.
- Odd powers of $\cos x$ – factor odd factor of $\cos x$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to convert remaining even powers of $\cos x$ to $\sin x$. Then let $u = \sin x$.
- No odd powers of $\sin x$ or $\cos x$ – use $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$ or $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$.

  After integrating the trig identity $\sin 2x = \sin x \cos x$ may be useful.

Integrands containing tangent and secant functions. $\int \tan^m x \sec^n x \; dx$.

- factor $\sec^2 x$ and change remaining factors of $\sec^2 x$ to tangent using $\sec^2 x = \tan^2 x + 1$.
- factor $\tan x \;\sec x$ and change remaining factors of $\tan^2 x$ to secant using $\tan^2 x = \sec^2 x - 1$.

LESSONS 11 and 12

Chapter 7.3 Trig Subsitution

If integrand contains*	Use this substitution	And use this identity
$\sqrt{a^2-(u(x))^2}$	$u(x)=a\sin\theta$	$1-\sin^2\theta=\cos^2\theta$
$\sqrt{a^2+(u(x))^2}$	$u(x)=a\tan\theta$	$1+\sin^2\theta=\sec^2\theta$
$\sqrt{(u(x))^2-a^2}$	$u(x)=a\sec\theta$	$\sec^2\theta-1=\tan^2\theta$

* Sometimes the integrand will contain powers of these roots.

Completing the square is sometimes necessary to get the forms $a^2-(u(x))^2,\ a^2+(u(x))^2,$ or $(u(x))^2-a^2$.

LESSONS 13 and 14

Chapter 7.3 Partial Fractions

- Let $\dfrac{P(x)}{Q(x)}$ be a rational function ($P(x)$ and $Q(x)$ are both polynomials) where P and Q have no common factors and the degree of P is less than the degree of Q. If P and Q have common factors, cancel them. If degree of P not less than degree of Q, then use long division to divide P by Q.
- Factor the denominator $Q(x)$ into linear factors $(ax+b)$ and irreducible quadratic factors (ax^2+bx+c, where $b^2-4ac<0$).
- m linear factors $(ax+b)^m$ give rise to m partial fractions of the form

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\frac{A_3}{(ax+b)^3}+\cdots+\frac{A_{m-1}}{(ax+b)^{m-1}}+\frac{A_m}{(ax+b)^m}$$

$A_1, A_2, A_3, \ldots, A_n$ are constants. When you set up your partial fractions use $A, B, C, D, \ldots$ in the numerators instead. With the examples we use, you won't run out of letters!

- n irreducible quadratic factors $(ax^2+bx+c)^n$ give rise to n partial fractions of the form

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\frac{A_3x+B_3}{(ax^2+bx+c)^3}+\cdots+\frac{A_{n-1}x+B_{n-1}}{(ax^2+bx+c)^{n-1}}+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

$A_1, A_2, A_3, \ldots, A_n$ and $B_1, B_2, B_3, \ldots, B_n$ are constants. When you set up your partial fractions use $Ax+B, Cx+D, Ex+F, \ldots$ in the numerators instead. With the examples we use, you won't run out of letters!

Set the rational function $\dfrac{P(x)}{Q(x)}$ equal to its sum of partial fractions. Next multiply both sides of this equation by the common denominator $Q(x)$. Then solve for the constants by either choosing values of x that will eliminate all but one constant, and/or equating the coefficients of like powers of x to get a system of linear equations where the unknowns are the constants. Solve that system using your "favorite" method.

- When integrating the partial fractions, you will usually be dealing with the following kinds of integrals:

$$\int \frac{1}{x-a}\, dx = \ln|x-a| + C$$

$$\int \frac{1}{(x-a)^n}\, dx = \frac{(x-a)^{n+1}}{n+1} + C, \qquad n \neq -1$$

$$\int \frac{1}{x^2+a}\, dx = \frac{1}{\sqrt{a}} \tan^{-1} \frac{x}{\sqrt{a}} + C$$

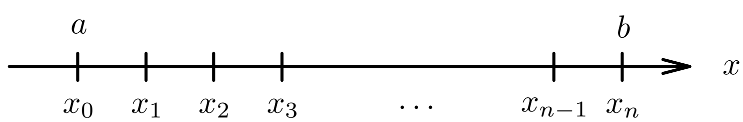
LESSON 15

Chapter 7.6 Integation using tables

When using a table of integrals to evaluate $\int f(x)\, dx$, remember that x is not always equal to u. Sometimes you first must use a substitution $u = u(x)$. If you don't, your answer will be incorrect by a constant factor.

Chapter 7.7 Approximate Integration

To approximate $\int f(x)\, dx$, subdivide the interval $[a, b]$ into n equal subintervals. Let $x_i = a + i\left(\frac{b-a}{n}\right)$. Note that $x_0 = a$ and $x_n = b$.



- Midpoint Rule:
$$\int_a^b f(x)dx \approx M_n = \left(\frac{b-a}{n}\right)\left(f\left(\frac{x_0+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right) + f\left(\frac{x_2+x_3}{2}\right) + \cdots + f\left(\frac{x_{n-1}+x_n}{2}\right)\right)$$
- Trapeziodal Rule:
$$\int_a^b f(x)dx \approx T_n = \left(\frac{b-a}{2n}\right)(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)).$$
- Simpson's Rule: **Note: n must be even**
$$\int_a^b f(x)dx \approx$$
$$S_n = \left(\frac{b-a}{3n}\right)(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)).$$

LESSON 16

Chapter 7.8 Improper Integrals

The integral $\int_a^b f(x)dx$ is "improper" if at least one of the following is true:

- $a = \infty$
- $b = -\infty$
- f has an infinite discontinuity at some point c in the interval $[a, b]$ (the graph of f has a vertical asymptote $x = c$).

We determine whether an "improper" integral has a value by evaluating the limit of "proper" integrals as follows: **(remember to evaluate the proper integral on the right–hand side <u>before</u> evaluating the limit)**

Improper integral of Type I

- $\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx.$
- $\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx.$

Improper integral of Type II

- If f has a discontinuity at a, then $\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx.$
- If f has a discontinuity at b, then $\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx.$

If the limit exists, then the limit value is the value of the improper integral. If the limit does not exist, then the improper integral is divergent.

If f has an infinite discontinuity at c where $a < c < b$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

provided BOTH improper integrals on the right hand side exist. If either diverges, then $\int_a^b f(x)dx$ diverges.

An important integral: $\int_1^\infty \frac{1}{x^p}dx$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Comparison Test for Improper Integrals When you can't determine an anti-derivative for an improper integral, a comparison with a known convergent or divergent integral is useful.

- Suppose that f and g are continuous functions with $0 \leq g(x) \leq f(x)$ for $x \geq a$.

If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.

If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

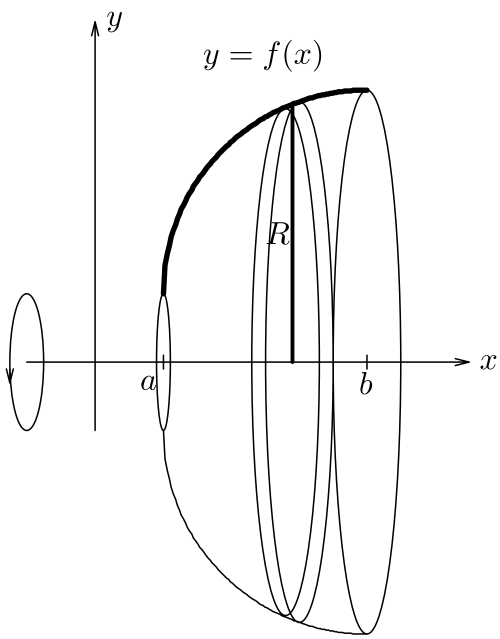
LESSON 17

Chapter 8.1 Arc length

- If $y = f(x)$ and $a \le x \le b$, then $L = \int_a^b \sqrt{1 + (f'(x))^2}\ dx$
- If $x = g(y)$ and $c \le y \le d$, then $L = \int_c^d \sqrt{1 + (g'(y))^2}\ dy$

Chapter 8.2 Area of a surface of revolution

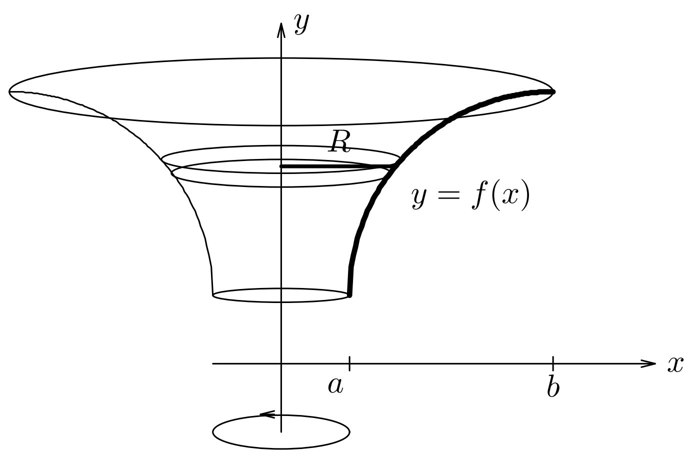
Rotate about the x–axis



$R = \text{radius} = f(x)$

$$\text{Surface Area} = \int_a^b 2\pi(f(x))\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

Rotate about the y–axis



$R = \text{radius} = x$

$$\text{Surface Area} = \int_a^b 2\pi(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$
