MA 16200 Exam 3 Study Guide, Fall 2014

LESSON 18

Chapter 8.3 Moments and Center of Mass

The center of mass of a discrete system of masses $m_1, m_2, m_3, \ldots, m_n$ located at points $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)$, respectively, is the point $(\overline{x}, \overline{y})$ where

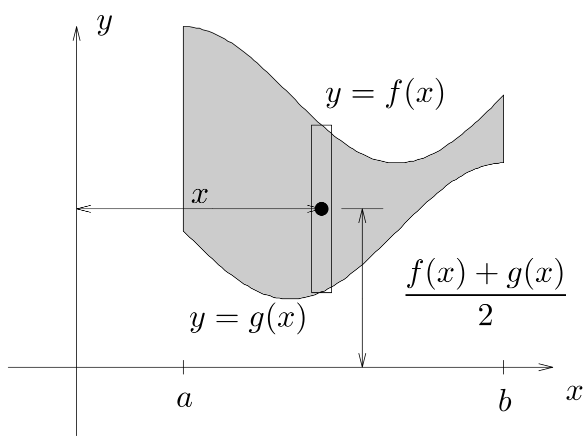
$$\overline{x} = \frac{M_y}{m} = \frac{\displaystyle\sum_{i=1}^{n} m_i x_i}{\displaystyle\sum_{i=1}^{n} m_i}, \qquad \overline{y} = \frac{M_x}{m} = \frac{\displaystyle\sum_{i=1}^{n} m_i y_i}{\displaystyle\sum_{i=1}^{n} m_i}$$

M_x is the moment about the y–axis, M_y is the moment about the x–axis, and m is the mass of the system.

The center of mass of a plate (lamina) bounded above by $y = f(x)$, below by $y = g(x)$, and by $x = a$ and $x = b$, with constant density ρ is the point $(\overline{x}, \overline{y})$, where

$$\overline{x} = \frac{M_y}{\text{mass}} = \frac{\displaystyle\int_a^b (\rho)(x)(f(x) - g(x))dx}{\displaystyle\int_a^b (\rho)(f(x) - g(x))dx} = \frac{\displaystyle\int_a^b (x)(f(x) - g(x))dx}{\displaystyle\int_a^b (f(x) - g(x))dx}$$

$$\overline{y} = \frac{M_x}{\text{mass}} = \frac{\displaystyle\int_a^b (\rho)\left(\frac{f(x) + g(x)}{2}\right)(f(x) - g(x))dx}{\displaystyle\int_a^b (\rho)(f(x) - g(x))dx} = \frac{\displaystyle\int_a^b \left(\frac{1}{2}\right)(f(x)^2 - g(x)^2)dx}{\displaystyle\int_a^b (f(x) - g(x))dx}$$



- **Theorem of Pappus**: Let R be a plane region that lies entirely on one side of a line l in a plane. If R is rotated about l, then the volume of the resulting solid is the product of the area A of R and the distance d traveled by the centroid of R.

LESSON 19

Chapter 11.1 Sequences

- Limit Laws for Sequences (page 693-695)
- Squeeze Theorem for Sequences
- **Theorem** if $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$.
- **Theorem** if $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$.
- Monotonic sequences (increasing or decreasing sequences). Bounded sequences.
- **Monotonic Sequence Theorem** Every bounded, monotonic sequence is convergent.

LESSON 20

Chapter 11.2 Series

1. Let $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$ be a series.
   - Let $s_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + a_3 + \cdots + a_N$ be the N–th partial sum.
   - If the sequence s_N converges so that $\lim_{N\to\infty} s_N = s$ is a real number, then the series $\sum_{n=1}^{\infty} a_n$ is **convergent** and $\sum_{n=1}^{\infty} a_n = s$.
   - If $\lim_{N\to\infty} s_N$ does not exist, then the series is **divergent**.
2. The **geometric series** $\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \dfrac{a}{1-r}, & \text{if } -1 < r < 1 \\ \text{divergent}, & \text{if } r \geq 1 \text{ or } r \leq -1 \end{cases}$
3. The **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is divergent.
4. **The Divergence Test**: If $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
5. The **algebra of convergent series**. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series and c is a constant, then so are $\sum_{n=1}^{\infty} ca_n$, $\sum_{n=1}^{\infty}(a_n + b_n)$ and $\sum_{n=1}^{\infty}(a_n - b_n)$, and
   - $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

- $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$
- $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

LESSON 21 Chapter 11.3 The Integral Test and p–series

1. **The Integral Test**: Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and $f(n) = a_n$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\, dx$ is convergent. In other words:
   - if $\int_1^{\infty} f(x)\, dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
   - if $\int_1^{\infty} f(x)\, dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.
2. The p–**series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and is divergent if $p \leq 1$.

LESSON 22 Chapter 11.4 The Comparison Tests

1. **The Comparison Test**: Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.
   - If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ is convergent.
   - If $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ is divergent.
2. **The Limit Comparison Test**: Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where c is a finite number and $c > 0$, then either both series converge or both diverge

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LESSON 23 Chapter 11.5 Alternating Series

- An **alternating series** is a series whose terms are alternately positive and negative.
- **The Alternating Series Test**: If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad b_n > 0$$

satisfies

$$\begin{array}{ll} \text{(i) } b_{n+1} \leq b_n & \text{for all } n \\ \text{(ii) } \lim_{n\to\infty} b_n = 0 & \end{array}$$

then the series is convergent.

- **Alternating Series Estimation Theorem**: If $s = \sum_{n=1}^{\infty}(-1)^{n-1}b_n$ is the sum of an alternating series that satisfies

$$\text{(i) } 0 \leq b_{n+1} \leq b_n, \qquad \text{and} \qquad \text{(ii) } \lim_{n\to\infty} b_n = 0$$

then

$$|s - s_n| \leq b_{n+1}$$

LESSONS 24 and 25 Chapter 11.6 Absolute Convergence and the Ratio and Root Tests

- A series $\sum^{\infty} a_n$ is called **absolutely convergent** if the series of absolute valued terms $\sum^{\infty} |a_n|$ is convergent.
- A series $\sum^{\infty} a_n$ is called **conditionally convergent** if it is convergent but the series $\sum^{\infty} |a_n|$ is divergent.
- If a series $\sum^{\infty} a_n$ is absolutely convergent, then it is convergent.
- **The Ratio Test**

  (i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

  (ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

  (iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then the Ratio Test is inconclusive, that is, no conclusion can be drawn about the convergence or divergence of the series $\sum_{n=1}^{\infty} a_n$.

- **The Root Test**

  (i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

  (ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

  (iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then the Root Test is inconclusive, that is, no conclusion can be drawn about the convergence or divergence of the series $\sum_{n=1}^{\infty} a_n$.

  - A useful limit to know for the Root Test is $\lim_{n\to\infty} (n)^{\frac{1}{n}} = 1$.

- Strategy for Testing Series. See page 739 in the text.

LESSONS 26 Chapter 11.8 Power Series

- A **power series** has the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

  where x is a variable and the c_n's are constants called the **coefficients** of the series.

- The sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots$$

  whose domain is the set of all numbers x for which the series converges.

- A **power series in** $(x-a)$ or a **power series centered at** a or a **power series about** a has the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3)x-a)^3 + \cdots$$

  .

- You usually determine the interval of convergence by using either the **Ratio** or **Root** tests. Convergence at the endpoints of the interval is determined separately by substituting each endpoint value of x into the power series.

- for a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, there are only three possibilities:

  (i) The series converges only when $x = a$.

  (ii) The series converges for all x.

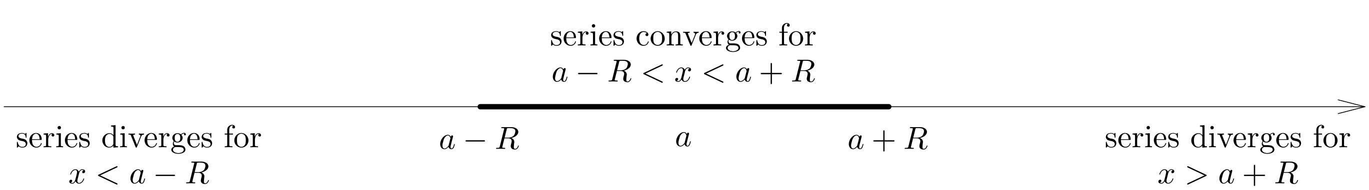
  (iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

- The number R in case (iii) is called the **radius of convergence**.

- The **interval of convergence** is the interval consisting of all values of x for which the series converges.

- In case (iii) there are 4 possible kinds of intervals:

$$(a-R, a+R) \qquad (a-R, a+R] \qquad [a-R, a+R) \qquad [a-R, a+R]$$



LESSONS 27 Chapter 11.9 Representation of Functions as Power Series

- You can substitute other powers of x for x in a power series. Of course you can also multiply a power series by a constant (or by powers of x) and you can add different convergent power series to create power series for different functions. Power Series can be integrated and differentiated (term by term). In this section, we exploit the geomtric series $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x| < 1$ by substitution, multiplication by constants (and by powers of x), differentiation and integration.

LESSONS 28 and 29 Chapter 11.10 Taylor and Maclaurin Series

- If f has a power series representation at $x = a$, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

that is, if f as a power series representation at $x = a$, then that power series must be
$f(x) = \sum_{n=0}^{\infty} \left(\dfrac{f^{(n)}(a)}{n!} \right) (x-a)^n = f(a) + \dfrac{f'(a)}{1!}(x-a) + \dfrac{f''(a)}{2!}(x-a)^2 + \dfrac{f'''(a)}{3!}(x-a)^3 + \cdots$
This power series is called the **Taylor Series** of f at $x = a$.

- The **Maclaurin Series** for f is the Taylor series at $x = 0$.
If f has a Maclaurin series, it is the series of the form
$f(x) = \sum_{n=0}^{\infty} \left(\dfrac{f^{(n)}(0)}{n!} \right) x^n = f(0) + \dfrac{f'(0)}{1!}x + \dfrac{f''(0)}{2!}x^2 + \dfrac{f'''(0)}{3!}x^3 + \cdots$

- Useful Maclaurin series and their radii of convergence:
$\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$
$e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots \qquad R = \infty$
$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots \qquad R = \infty$
$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots \qquad R = \infty$
