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The Mathematical Education of Prospective Teachers of Secondary School Mathematics: Old Assumptions, New Challenges

Joan Ferrini-Mundy, Michigan State University
Bradford Findell, National Research Council
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We would like to address two questions: What mathematics do prospective secondary school mathematics teachers need to know? In what context should they come to know it? Consideration of both matters has implications for the revision of the undergraduate program in mathematics.

What mathematics do prospective secondary school mathematics teachers need to know?

Clearly teachers must know the mathematics they teach. Deciding exactly what this means, and then determining what more mathematics do they need, are not simple matters. Typically, two perspectives have influenced the design of programs for the preparation of secondary teachers, and both are relevant to mathematics departments:

1. Prospective high school teachers should study essentially whatever *mathematics majors* study—because this will best equip them with a coherent picture of the discipline of mathematics and the directions in which it is heading, which should influence the school curriculum.
2. Prospective high school teachers' should study *mathematics education*—methods of teaching mathematics, pedagogical knowledge in mathematics, the 9-12 mathematics curriculum, etc.

Because this paper is intended to make recommendations on the undergraduate mathematics program, we will concentrate on the first of these components, with connections to the second where possible. Keep in mind, however, that there is much, in addition to mathematics and mathematics education, that secondary school teachers need to know, about students, about learning, about teaching, about curriculum, and about the contexts of schooling.

History of recommendations

The dominant approach to the mathematical preparation of secondary school teachers in the United States in recent years is to require that they complete an undergraduate major (or a near-equivalent) in mathematics. Interestingly enough, a quick review of the recommendations in this century about the mathematical preparation of teachers reveals that this trend toward a near-major has generally grown stronger with each set of recommendations. For instance, the 1911 report of the American subcommittee of the International Commission on the Teaching of Mathematics recommended preparation in several areas of pure mathematics, applied mathematics (e.g., mechanics, astronomy, physics), surveying, a "strong course on the teaching of secondary mathematics," other education, and "a course of an encyclopedic nature dealing critically with the field of elementary mathematics from the higher standpoint" (International Commission on the Teaching of Mathematics, 1911, pp. 13-14). There is no explicit call for a major in mathematics. Likewise, the 1935 recommendations of the Mathematical Association of

America's Commission on the Training and Utilization of Advanced Students of Mathematics calls for "minimum training in mathematics that goes as far as 6 hours of calculus, Euclidean geometry, theory of equations, and a history of mathematics courses." The courses that might have been more typical of a major at that time (advanced calculus, mechanics, projective geometry, additional algebra) are described as "desirable additional training." In reports from various groups in the late 50s and early 60s, the expectations for secondary teachers began to sound like a major, with calls for 24 semester-hours of mathematics courses (National Council of Teachers of Mathematics [NCTM], 1959), and 30 semester hours, including abstract algebra (AAAS, 1959). It was the Committee on the Undergraduate Program in Mathematics (CUPM), in 1961, that first recommended that "Prospective teachers of high school mathematics beyond the elements of algebra and geometry should complete a major in mathematics" (CUPM, 1961). Ten years later, this sentiment was still strongly held: "We regard it as a matter of great importance that a program for teachers should be identical to the one offered to other mathematics majors, except for a few courses peculiarly appropriate to prospective high school teachers" (CUPM, 1971, p. 170). The 1983 CUPM recommendations do not explicitly call for a mathematics major, but instead list 13 courses, including a 3-course calculus sequence, as the minimal preparation, with a call for additional work for teachers of calculus (CUPM, 1983). It is worth noting that 13 courses is *more* than a major in some institutions. In 1991, the MAA's Committee on the Mathematical Education of Teachers (COMET) assumed responsibility for the preparation of teachers: "These recommendations assume that those preparing to teach mathematics at the 9-12 level will complete the equivalent of a major in mathematics, but one quite different from that currently in place at most institutions" (Leitzel, 1991, p. 27). The recommendations list standards in seven content areas (e.g., geometry, continuous change, and mathematical structures) rather than specific courses.

Since the first CUPM recommendations, most major sets of national committee recommendations offered by the mathematics community, and most recommendations from the education community have recommended the equivalent of a major in mathematics as the fundamental preparation for the secondary teacher. Sometimes the recommendation is general and takes as a given that whatever is considered appropriate as a major is what should be expected. For instance, the new recommendations of the National Council of Accreditation of Teacher Education (NCATE), to go into effect next year, expect that candidates for teaching should "know the content of their field (a major or the substantial equivalent of a major)." The most current recommendations being developed for the mathematical education of teachers (the Conference Board of the Mathematical Sciences [CBMS] Mathematical Education of Teachers project), though reflecting some more current general issues about the undergraduate major, still make the same basic argument, as is evident in the following excerpt:

> The following outline of mathematics and supporting courses is one way to provide core knowledge for future high school teachers while satisfying many requirements in a standard mathematics major. (CBMS, in preparation)

Year One:	Calculus, Introduction to Statistics, Supporting Science
Year Two:	Calculus, Linear Algebra, and Introduction to Computer Science
Year Three:	Abstract Algebra, Geometry, Discrete Mathematics, and Statistics
Year Four:	Introduction to Real Analysis, Capstone, and Mathematics Education Courses

There is no question that teachers need to know mathematics in order to teach well in secondary schools—the logic in this seems unassailable. Yet at the same time, research studies have not been able to demonstrate a convincing relationship between teachers' knowledge of mathematics (often measured by the number of college mathematics courses taken) and their students' mathematical performance(see Begle, 1979; Monk, 1994). Perhaps teachers fail to learn the content of these courses, or they do learn it but find that it doesn't connect in any recognizable way with their classroom practice.

There are at least two problems with requiring the same preparation for mathematics teaching as for graduate school in mathematics. First, high school teachers are preparing for a professional practice that is completely different from that of conducting mathematical research. The mathematical demands they will face are different. But we are not arguing for less mathematical preparation for teachers. In fact, we would argue that with a typical major in mathematics, teachers may have *too little mathematical preparation of the kind they will need.* Second, by keeping content separate from pedagogy, prospective teachers may fail to acquire what Shulman (1987) called *pedagogical content knowledge*—"an understanding of how particular topics, problems, or issues are organized, presented, and adapted to the diverse interests and abilities of learners, and presented for instruction" (1987, p. 8).

The mathematics of the secondary school curriculum

Today's secondary school context is radically different from that of 20 years ago. First of all, secondary schools today take most seriously the commitment to educate all students to be prepared for a rapidly changing world—and thus *all* students need to be prepared for the possibility of higher education or a highly technical workplace. This has meant increasing trends away from vocational or general tracks, and toward a foundation of significant mathematics for all students (see, for instance, the recommendations of the NCTM 2000 *Principles and Standards for School Mathematics*). The range of options today in high school curricular materials reflects this shift. While the algebra 1- geometry-algebra 2-precalculus-calculus sequence that many of us experienced is still alive and well, today's materials also include substantial emphasis on data and statistics, on discrete mathematics, on dynamic geometry, and early treatment of functions and modeling. Some series are fully integrated, with titles like "math 1, 2, 3,4", or are completely organized around applications of mathematics and so-called contextual problems. The once unchallenged high school end-goal of advanced placement calculus has given way to other equally strong possibilities, such as advanced placement statistics, or sophisticated technical courses focusing on CAD-CAM technologies, finance, and applications of mathematics to the world of work. These trends are then at odds with what has been the traditional mathematics major, with its historic emphasis on abstract algebra and analysis as end-goals. And so, if one takes seriously the notion that being prepared to handle the mathematics of the secondary school is something crucial for teachers, then it seems that these shifts in the nature of secondary school mathematics education need to be taken quite seriously.

For secondary mathematics teachers, it is ironic that, except for occasional concepts that might be called upon in calculus, the entire four years of an undergraduate mathematics major address content that is, on the surface, unrelated to the topics of the high school curriculum. The only place where prospective secondary teachers are very likely to learn about such secondary school curriculum topics as the Law of Cosines, the Rational Root Theorem; the proof of Side-Angle-Side congruence, the Zero Product Principle or tests of divisibility is in the secondary school, as students themselves. More substantially, the kinds of integration of mathematical ideas, and connections that are necessary in teaching a coherent secondary program, are unlikely to be obvious to students on the basis of their undergraduate program. Consider the following example of a student teacher episode. This student had been a strong undergraduate mathematics major and a small state university; she had taken courses in abstract algebra, geometry at a junior level, and advanced calculus. She was conducting a lesson in algebra 2 class and was presenting the absolute value function. She showed the students the notation, $f(x) = |x|$, and drew the graph. A student said something like the following: "That graph reminds me of angles in geometry. Can we use the absolute value function as a way to write a formula for any angle?" The teacher was completely taken aback by the question. The question would have required the teacher to make a number of mathematical judgments on the spot, and also to connect ideas across content areas in unexpected ways.

As professors working with prospective high school teachers, are we confident that our students will be able to answer the following typical high school students' questions, in ways that are both mathematically sound and also accessible and compelling to a 15-year old?

- Why does a negative times a negative equal a positive?
- Why do I switch the direction of the less than symbol when I multiply both sides by a negative number?
- In every triangle that I tried in Sketchpad, the angles add up to 180. I don't need to do a proof, do I?
- I am not convinced that .99999… = 1.
- How do I know parallel lines never intersect?
- How do I know that the asymptote never hits the line? I mean, it crosses the line near 0.
- I think that 100 IS divisible by 3—the answer is 33 and one third.
- Why is it OK to use 22/7 for the value of pi, sometimes?
- I think that 1 has three different square roots: 1, -1, and .99999999. I am sure that .99999999 is a square root of 1 because when I multiply it by iteself on my calculator I get 1.00000000.

Mathematics for Teaching

So suppose we could construct a curriculum for secondary school teachers that, speaking strictly in terms of mathematical content, was in tune with the current secondary curriculum as headed, and genuinely would offer students a chance to see both where the concepts of the high school curriculum are embedded in a larger picture of mathematics and also to see elementary mathematics from an advanced standpoint or to develop "profound understanding of fundamental mathematics" (Ma, 1999). This would probably mean designing some courses specially for teachers, breaking tradition more explicitly with the sense that what's good for the math major is good for the prospective teacher. Both majors require substantial study of serious mathematics—but there may be reasons why that body of mathematical content is different in some ways.

But we suspect there is another body of knowledge that is mostly mathematical in character that high school teachers also need, and that is probably more within the purview of the mathematics department than it is the school of education. Researchers and mathematics educators are struggling with how to describe and talk about this knowledge. Usiskin calls it "applied mathematics for teaching." Ball and Bass have studied this notion in the elementary grades and call it "pedagogically useful mathematical understanding" (Ball & Bass, 2000). All of these ideas build on Shulman's notion of pedagogical content knowledge. In addition to having profound understanding of the content that is taught in the secondary grades, mathematics teachers at this level need to be able to draw on and use other knowledge that is mathematical in character, such as:

- finding the logic in someone else's argument or the meaning in someone else's representation;
- deciding which of several mathematical ideas has the most promise, and what to emphasize;
- making and explaining connections among mathematical ideas;
- situating a mathematical idea in a broader mathematical context;
- choosing representations that are mathematically profitable; and
- maintaining essential features of a mathematical idea while simplifying other aspects to help students understand the idea.

These kinds of mathematical activities, we would argue, are essential in teaching—certainly in the classroom while interacting directly with students around content, while answering their questions, while planning lessons, while designing tasks for class and for assessments, and while correcting student work. If this is the case, then this means that somewhere prospective teachers need to learn these mathematical skills—ideally, we contend, in mathematical material that is close to the material of high school classrooms, under the guidance of mathematicians in university mathematics departments.

There are at least three ways to create possibilities for connections between undergraduate mathematics and high school mathematics:

> **A mathematical approach.** Prospective teachers should study high school mathematics from an advanced standpoint (Usiskin, Peressini, Marchisotto, Stanley) so that they might have a "profound understanding of fundamental mathematics" (in the spirit of Ma, 1999). The approach is to find and exploit topics in the high school curriculum that can be extended and elaborated in ways that are sophisticated mathematically. An alternative is to find topics in the typical undergraduate curriculum and look for ways to connect them with key areas of the high school curriculum—this is the idea behind capstone courses or "shadow courses" that prospective teachers take alongside such courses as abstract algebra. Although these approaches may represent an improvement over some mathematics courses that have no connection to high school mathematics, the approach runs the risk of leaving prospective teachers without sufficient pedagogical content knowledge, which lies at the intersection of content and pedagogy.
>
> **An integrative approach.** Integrate the goals of the mathematics content and pedagogy courses so that teachers might be better able to see connections and later use them (see, e.g., Cooney et al., 1996).
>
> **An emergent approach.** Analyze the practice of teaching and determine what mathematical knowledge teachers draw upon in their practice. Then use real mathematical "problems" of mathematics teaching practice as sites for learning mathematics, taking advantage of the mathematical opportunities that emerge while working on the problems.

We believe that all of these approaches are worth pursuing. This emergent approach, however, is the most unusual and requires the most explanation. To understand this approach, it is useful to characterize the failure of the traditional approach as a failure at helping teachers transfer their mathematical knowledge into practice. Rather than constructing a solution apart from teaching, we start in the context of teaching practice and try to identify the interpreting, problem solving, and decision making activities in which a teacher actually engages, so that we may infer what mathematics is actually used. The next step would be to design a curriculum around such problems, in much the same way one might create a mathematics curriculum for engineers or social scientists by looking at the mathematical problems they have to solve.

A number of mathematics education researchers and professional developers (see, e.g., Ball & Cohen, 1999; Schifter, Bastable, & Russell, 1999; Shulman, 1992; Stein, Smith, Henningson, & Silver, 2000; Barnett, Goldenstein, & Jackson, 1994) have begun to explore this possibility through the use of videos of classrooms, student work, written cases, and student curriculum materials. The idea of using the actual work of teaching as a starting point for thinking about the mathematical preparation of teachers was explored further at the Teacher Preparation Mathematics Content Workshop hosted in 1999 by the Mathematical Sciences Education Board (see National Research Council, 2000). The ideas are just beginning to take shape in the mathematics education community, and so some concerted work at conceptualizing the notion more fully and at designing experiences in which prospective teachers might profitably acquire this mathematical knowledge is needed.

It is worth pointing out that this approach, although it affords some new opportunities, is not without potential pitfalls. One potential pitfall is that discussion might remain mired in school mathematics, never able to move toward higher mathematics. Another is that most of the discussion might be spent on more general teaching and learning issues. Our experience, however, is that with a well-chosen problem, the work here can be deep and substantive, often leading to mathematical territory that is unexplored or implicit in the traditional curriculum.

<u>Conclusions</u>

We wish to argue that some of the current challenges in secondary school mathematics education today, coupled with new insights from research in mathematics education, suggest that it may be time to move away from the relatively unquestioned assumption that has historically guided mathematics education in the past several decades—that a major in mathematics, or something that deviates from it only marginally—is the best mathematical preparation for prospective teachers of secondary school. Teachers need to understand mathematics deeply, and also to understand its applications, how ideas are integrated across subject matters, and be able to see mathematical possibilities in students' statements or written work. Could new majors be designed, specifically for prospective secondary school mathematics teachers, that bring together the three kinds of mathematical knowledge described here, in ways that would serve our secondary teachers well? Given that as many as half of the mathematics majors, at least in some research universities, are intending to become high school teachers, it seems that such development is warranted. The recommendations of the MET paper are timely in their recognition of the need to diversify the offerings in the undergraduate curriculum for the prospective secondary teacher—and increasingly, the set of recommended offerings diverges from what at least has been the mathematics major. But we need to prepare teachers to solve the kinds of mathematical problems that actually arise in teaching. This kind of thinking has not prevailed for secondary school teachers in mathematics.

In What Context Should Prospective Teachers Come to Know Mathematics?

The responsibility for the mathematical content preparation of secondary school mathematics teachers has been, historically, the responsibility of departments of mathematics—and that should continue to be the case, in our view. But, some rather serious issues need to be confronted in order for this to happen effectively. Within the faculty, who is responsible for the mathematical content knowledge of secondary teachers? What expertise do they need, and how do they acquire it? Do they need to be working in schools, conferring with secondary teachers, and staying current in their knowledge of mathematics education? And, if their primary background is in mathematics education, how do they remain current in mathematics?

Departmental environment

Little research has been conducted about the student learning environment for mathematics majors intending to teach secondary school mathematics, though there is much anecdotal information available. A frequent complaint among these students is that their experience in mathematics courses, in particular the nature of the mathematics instruction, is inconsistent with what they are learning in their education courses about the best ways to help students learn. So, in education courses about pedagogy, where they may be learning about the importance of actively engaging their future students, finding ways to make the subject matter meaningful and to connect it to other concepts, building on what students know, and using embedded assessments that call for explanation, they are developing certain knowledge and images about what effective mathematics teaching is like. In many upper level mathematics courses the instruction that is modeled does not include these elements. So, despite the fact that these upper level majors often are good mathematics students and have succeeded within the system, they sometimes are very conflicted about what should happen in their own mathematics teaching, because of the variation and dissonance they experience.

The use of technology in the undergraduate experience for prospective teachers is also an issue. NCTM and state standards tend to recommend the use of technology in secondary schools to support mathematics learning, and to implement this well in ways that advance their students' mathematical knowledge. Thus, prospective teachers need experience using technology as learners, in advanced mathematical settings. There are institutions, for instance, where "reform" calculus, or technology-rich calculus courses, are offered to students in the life sciences, or in some engineering and science tracks, but are not available to students majoring in mathematics, and therefore not available to prospective secondary school teachers. In this case the teachers, as mathematics majors, take the more theoretical, less applied calculus option and don't see the applications, connections, or experience the role of technology.

There are also aspects about the student learning environment (outside the classroom) that are problematic for strong students who have expressed interests in secondary school mathematics teaching. In conversations with their faculty advisors and mentors, such students report that sometimes mathematician advisors discourage them from teaching, with arguments about seeking more lucrative and prestigious options. Yet across the country, and especially in some urban areas, substantial numbers of middle school and secondary school children are being taught their mathematics by teachers without even the equivalent of a minor in the field. And, without good preparation of students in mathematics in high school, the pressure to offer remedial courses at the undergraduate level, and the lack of supply of strong students into mathematics will continue to be problems facing higher education. So it would seem logical that undergraduate mathematics faculty would be eager to encourage good and interested students into mathematics teaching.

A second difficult area of the student environment is related to advising. Students intending to be teachers need to meet the requirements of their major, as well as a set of course requirements and clinical experience requirements in professional education. Teacher education students have very full and demanding programs, typically, and so taking courses in the correct sequence, at the correct stage of their undergraduate career, is crucial to staying on track. And, with increasing numbers of teacher education programs being fifth year programs, or five year programs, or combined bachelors and masters degree programs, issues about when to apply to the program, deadlines for registering for student teaching, for internships, for various state and national exams, etc. become crucial to students' ability to complete their programs. Mathematics department advisors need to be aware that advising mathematics education students is complicated, and to work closely worth college of education advisors to be sure they have the most up-to-date guidance about what the overall program expectations are.

Accreditation

The preparation of teachers is professional preparation, and as such brings with it some features that are often unfamiliar to mathematics departments. Teachers in the nation's public schools must hold licenses; therefore, the programs that prepare them generally must be accredited, either through the state or through national organizations or both. Generally colleges of education bear the responsibility of maintaining accreditation, which involves periodic self-study reports, addressing the standards and expectations of the accrediting agencies, organizing site visits for outside reviewers, and responding to reviewer concerns. The activities of subject matter departments fall within the purview of these accreditation agencies and so faculty in mathematics departments are called upon to help prepare self-study reports and to meet with accreditation teams. This means that someone in the department needs to be aware on a continuing basis of the accreditation issues, and of changes and new trends that emerge..

The Need for Collaboration

The lack of mutual respect and cooperation between faculty in colleges of arts and sciences and faculty in education is a long-standing obstacle to the effective education of teachers.[1] It is unfortunately quite common for undergraduate students to hear of faculty in mathematics criticizing faculty in education for such things as lack of high standards, lack of understanding of mathematics, or teaching material that has no substance. And, conversely, students hear their education professors commiserate about poor teaching in the mathematics department, or lack of attention by mathematics faculty to current issues such as the role of technology. A variety of programs, conferences, and initiatives that are intended to bring together administrators and faculty in colleges of arts and science with those in education have been initiated over the years, although there is little evidence that such programs have effect. At the level of specific mathematics departments, some things can help: hiring faculty members whose professional scholarship is in mathematics education; arranging joint or adjunct appointments for mathematics faculty in education; including faculty from education in programmatic review or development efforts; holding regular meetings of those who advise prospective secondary school teachers in mathematics and those who do so in education; arranging to host visiting teachers-in-residence from local high schools; and facilitating joint efforts on specialized projects in research, curriculum, or teacher education. Deans and chairs

[1] Historically, it was in the university-based arenas that the liberal arts vs. professional e most strongly: “The collegiate [or university level] institutions entertained grave fea normal schools to properly equip teachers for the high schools, while the normal schoo to provide practice teaching and the need to adapting subject matter to the mind of the collegiate education inadequate,” (Pangburn, 1932, p. 52).

can enable such things to happen, but will need to make special efforts to monitor and discourage the very negative conversations that sometimes happen along these lines.

Mathematics Education

The body of research about mathematics teaching and learning is substantial and growing. The most recent major synthesis (Grouws, 1992) includes a chapter on advanced mathematical thinking, and a chapter on teacher education—both areas that bear upon issues in the aspects of the undergraduate major that contribute to teacher education.

The research on undergraduate mathematics education is rich in its documentation of student difficulties, in evidence about interventions that can support deep student understanding, and in portraying how technology can be used in the undergraduate arena to help support student learning (Dubinsky, Schoenfeld, & Kaput, 1994; Kaput, Schoenfeld, & Dubinsky, 1996; Schoenfeld, Kaput, & Dubinsky, 1998). Those who are concerned with the improvement of undergraduate education in general, and teacher education in particular, might find this literature useful. In fact, in most programs, prospective teachers read research about mathematics teaching and learning in their teacher education programs; perhaps such reading lists would be good background material for the mathematics faculty who are providing their content background.

Generally education research has little direct impact on practice, either at K-12 or in higher education. The impact of research tends to be more indirect; the theoretical perspectives and methodologies of research actually, in addition to et findings, can be useful. For instance, a standard research methodology used in gaining insights about student understandings of particular concepts is the clinical interview[2]. We have adapted this methodology and used it in mathematics courses for teachers. Asking prospective teachers to interview a high school student on some difficult topic.

Relationship with the Major

An important consideration for mathematics departments is the reality that prospective secondary school teachers comprise a growing and significant fraction of the set of math majors in many departments. Although data are not readily available, by many reports it seems plausible to guess that more than half of the mathematics majors nationally may be intending to teach secondary school. If this is the case, then the responsibility for mathematics departments to incorporate into their core conversation the serious consideration of what it takes to prepare a teacher mathematically. Will university departments be able to recognize that "teachers' mathematics" exists, is conceptually difficult, and should be offered through departments of mathematics? This is a non-trivial problem that deserves substantial intellectual and institutional resources.

Conclusion

At the core of departmental work relative to teacher education should be the very nature of the course and experiences for learning content that are provided to students. National

[2] A technique where the subject is presented with a mathematical problem and is asked to solve it and tell the interview what she/he is thinking along the way. The interviewer does some probing and prompting, but does not tutor the student or move the student toward a solution. The sole purpose is for the interviewer to elicit the students thinking, so that a "theory" can be built about the student's understanding of the concept at hand. Usually clinical interviews are tape record, transcribed, coded nd become part of a larger data set that can yield information about student understanding across a range of students, for a particular concept.

conversations along these lines are moving very quickly to develop a concept of "mathematics for teaching," as described in previous sections. Exploration of what this would mean at the secondary school level are less developed that at the elementary level, but if researchers begin to take up this line of work, there will ultimately be implications for, and challenges to, the traditional practice of having prospective teachers take only the mathematics courses taken by prospective graduate students. This is a big challenge, because the ideas are still nascent and the research is just taking shape; little is known about what this involves in practice, let alone in teaching students how to do it; the issues may vary considerably by content area; and methods for helping prospective teachers learn about this are highly underdetermined. We hope that this paper will serve to broaden the community of mathematicians, mathematics educators, and mathematics education researchers who are willing to contribute to this important area of work.

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