Problem: Solve the intial value problem

$$\begin{cases} x' &= -2x + y + \cos t \\ y' &= x - 2y \end{cases} \qquad \begin{cases} x(0) &= 2 \\ y(0) &= -1 \end{cases}$$

Solution: The problem is $X' = AX + g(t)$ and $X(0) = X_0$ where

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \qquad A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \qquad X_0 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad \text{and} \quad g(t) = \begin{pmatrix} \cos t \\ 0 \end{pmatrix}$$

We begin by finding the eigenvalues and eigenvectors of A:

$$\det(A - \lambda I) = \det \begin{pmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{pmatrix} = 4 + 4\lambda + \lambda^2 - 1 = \lambda^2 + 4\lambda + 3$$

Solving $\lambda^2 + 4\lambda + 3 = 0$ gives $\lambda = -1$ and $\lambda = -3$. An eigenvector for $\lambda = -1$ is $(1, 1)$ and an eigenvector for $\lambda = -3$ is $(1, -1)$. Using these vectors as columns, the matrix

$$U = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

diagonalizes A, that is $A = UDU^{-1}$ where U is as above and

$$D = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix} \quad \text{and} \quad U^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Now, it follows that $e^{tA} = Ue^{tD}U^{-1}$ and since the exponential of a diagonal matrix is the diagonal matrix whose diagonal entries are the exponentials of the original diagonal matrix, we get

$$e^{tA} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{-t} + e^{-3t} & e^{-t} - e^{-3t} \\ e^{-t} - e^{-3t} & e^{-t} + e^{-3t} \end{pmatrix}$$

Thus, the solution of the homogeneous system $X' = AX$ with $X(0) = X_0$ is

$$X_h(t) = e^{tA}X_0 = \frac{1}{2} \begin{pmatrix} e^{-t} + e^{-3t} & e^{-t} - e^{-3t} \\ e^{-t} - e^{-3t} & e^{-t} + e^{-3t} \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{-t} + 3e^{-3t} \\ e^{-t} - 3e^{-3t} \end{pmatrix} \tag{1}$$

Now, in general, the solution of the non-homogeneous system $X' = AX + g$ that satisfies $X(0) = 0$ is

$$X(t) = e^{tA} \int_0^t e^{-sA} g(s)\, ds$$

so in our case, we have

$$X_p(t) = \frac{1}{2} \begin{pmatrix} e^{-t} + e^{-3t} & e^{-t} - e^{-3t} \\ e^{-t} - e^{-3t} & e^{-t} + e^{-3t} \end{pmatrix} \int_0^t \frac{1}{2} \begin{pmatrix} e^{s} + e^{3s} & e^{s} - e^{3s} \\ e^{s} - e^{3s} & e^{s} + e^{3s} \end{pmatrix} \begin{pmatrix} \cos s \\ 0 \end{pmatrix} ds$$

or

$$X_p(t) = \frac{1}{4} \begin{pmatrix} e^{-t} + e^{-3t} & e^{-t} - e^{-3t} \\ e^{-t} - e^{-3t} & e^{-t} + e^{-3t} \end{pmatrix} \begin{pmatrix} \int_0^t (e^{s} + e^{3s}) \cos s\, ds \\ \int_0^t (e^{s} - e^{3s}) \cos s\, ds \end{pmatrix}$$

This is

$$X_p(t) = \frac{1}{40}\begin{pmatrix} e^{-t}+e^{-3t} & e^{-t}-e^{-3t} \\ e^{-t}-e^{-3t} & e^{-t}+e^{-3t} \end{pmatrix}\begin{pmatrix} 5e^t\cos t+5e^t\sin t+3e^{3t}\cos t+e^{3t}\sin t-8 \\ 5e^t\cos t+5e^t\sin t-3e^{3t}\cos t-e^{3t}\sin t-2 \end{pmatrix}$$

or

$$X_p(t) = \frac{1}{40}\begin{pmatrix} 16\cos t+12\sin t-10e^{-t}-6e^{-3t} \\ 4\cos t+8\sin t-10e^{-t}+6e^{-3t} \end{pmatrix} \tag{2}$$

Putting the solution of the homogeneous equations with the given initial values, Equation (1), and the solution of the non-homogeneous equations with the zero intial values, Equation (2), gives the solution of the given system with the given intial values:

$$X(t) = X_h(t)+X_p(t) = \frac{1}{2}\begin{pmatrix} e^{-t}+3e^{-3t} \\ e^{-t}-3e^{-3t} \end{pmatrix} + \frac{1}{40}\begin{pmatrix} 16\cos t+12\sin t-10e^{-t}-6e^{-3t} \\ 4\cos t+8\sin t-10e^{-t}+6e^{-3t} \end{pmatrix}$$

which is

$$X(t) = \begin{pmatrix} .4\cos t+.3\sin t+.25e^{-t}+1.35e^{-3t} \\ .1\cos t+.2\sin t+.25e^{-t}-1.35e^{-3t} \end{pmatrix}$$

Checking this, we see that

$$x'(t) = -.4\sin t+.3\cos t-.25e^{-t}-4.05e^{-3t}$$

and

$$\begin{aligned} -2x+y+\cos t &= -.8\cos t-.6\sin t-.5e^{-t}-2.7e^{-3t}+.1\cos t+.2\sin t+.25e^{-t}-1.35e^{-3t}+\cos t \\ &= -.4\sin t+.3\cos t-.25e^{-t}-4.05e^{-3t} \end{aligned}$$

and $x(0) = .4+.25+1.35 = 2$. Also, we see that

$$y'(t) = -.1\sin t+.2\cos t-.25e^{-t}+4.05e^{-3t}$$

and

$$\begin{aligned} x-2y &= .4\cos t+.3\sin t+.25e^{-t}+1.35e^{-3t}-.2\cos t-.4\sin t-.5e^{-t}+2.7e^{-3t} \\ &= -.1\sin t+.2\cos t-.25e^{-t}+4.05e^{-3t} \end{aligned}$$

and $y(0) = .1+.25-1.35 = -1$. Thus, the computed solution actually does solve the given initial value problem.