1. Define, for $x \in \mathbb{R}^n$,
\[
f(x) = \begin{cases} \frac{1}{|x|^{n+1}} & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}
\]
Prove that $f$ is integrable outside any ball $B(0, \varepsilon)$, and that there exists a constant $C > 0$ such that
\[
\int_{B(0,\varepsilon)^c} f(x) \, dx \leq \frac{C}{\varepsilon}.
\]

2. Let $\{f_k\}$ be a sequence of nonnegative measurable functions on $\mathbb{R}^n$, and assume that $f_k$ converges pointwise almost everywhere to a function $f$. If
\[
\int_{\mathbb{R}^n} f = \lim_{k \to \infty} \int_{\mathbb{R}^n} f_k < \infty,
\]
show that
\[
\int_E f = \lim_{k \to \infty} \int_E f_k
\]
for all measurable subsets $E$ of $\mathbb{R}^n$. Moreover, show that this is not necessarily true if
\[
\int_{\mathbb{R}^n} f = \lim_{k \to \infty} \int_{\mathbb{R}^n} f_k = \infty.
\]

3. Assume that $E$ is a measurable subset of $\mathbb{R}^n$, with $|E| < \infty$. Prove that a nonnegative function $f$ defined on $E$ is integrable if, and only if,
\[
\sum_{k=0}^{\infty} \left| \{x \in E \mid f(x) \geq k\} \right| < \infty.
\]

4. Suppose that $E$ is a measurable subset of $\mathbb{R}^n$, with $|E| < \infty$. If $f, g$ are measurable functions on $E$, define
\[
\rho(f, g) = \int_E \frac{|f - g|}{1 + |f - g|}.
\]
Prove that $\rho(f_k, f) \to 0$ as $k \to \infty$ if, and only if, $f_k$ converges in measure to $f$ as $k \to \infty$.

5. Define the gamma function $\Gamma : \mathbb{R}^+ \to \mathbb{R}$ by
\[
\Gamma(y) = \int_0^\infty e^{-u} u^{y-1} \, du,
\]
and the beta function $B : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$
\[
B(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1} \, dt.
\]
(a) Prove that the definition of gamma function is well posed, i.e. the function $u \mapsto e^{-u} u^{y-1}$ is in $L(\mathbb{R}^+)$ for all $y \in \mathbb{R}^+$.
(b) Show that
\[
B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}.
\]
6. Let \( f \in L(\mathbb{R}^n) \), and for \( h \in \mathbb{R}^n \) define \( f_h : \mathbb{R}^n \rightarrow \mathbb{R}, \ f_h(x) = f(x - h) \). Prove that
\[
\lim_{h \to 0} \int_{\mathbb{R}^n} |f_h - f| = 0.
\]

7. (a) If \( f, g, f, g \in L(\mathbb{R}^n) \), \( f_k \to f \) and \( g_k \to g \) a.e. in \( \mathbb{R}^n \), \( |f_k| \leq g_k \), and
\[
\int_{\mathbb{R}^n} g_k \to \int_{\mathbb{R}^n} g,
\]
prove that
\[
\int_{\mathbb{R}^n} f_k \to \int_{\mathbb{R}^n} f.
\]
(b) Using part (a), show that if \( f_k, f \in L(\mathbb{R}^n) \), \( f_k \to f \) a.e in \( \mathbb{R}^n \), then
\[
\int_{\mathbb{R}^n} |f_k - f| \to 0 \text{ as } k \to \infty
\]
if, and only if,
\[
\int_{\mathbb{R}^n} |f_k| \to \int_{\mathbb{R}^n} |f| \text{ as } k \to \infty.
\]