Chapter 3, problem 8. If \( \sum a_n \) converges, and if \( \{b_n\} \) is monotonic and bounded, prove that \( \sum a_n b_n \) converges.

Solution.
Since \( \{b_n\} \) is monotonic and bounded, \( \{b_n\} \) converges by Theorem 3.14, say \( b_n \to b \in \mathbb{R} \). Since \( \{b_n\} \) is monotonic, we either \( b_{n+1} \geq b_n \) for all \( n \) (non-decreasing) or \( b_{n+1} \leq b_n \) for all \( n \) (non-increasing). Assume, without loss of generality, that \( \{b_n\} \) is monotonic non-decreasing, otherwise consider \(-b_n\). If \( \{b_n\} \) is non-decreasing, then \( b_n \leq b \) for all \( n \). Then consider \( c_n = b - b_n \).
We have that since \( \sum a_n \) converges, the partial sums \( A_n \) of \( \sum a_n \) form a bounded sequence; \( c_0 \geq c_1 \geq c_2 \geq \cdots \), and \( c_n \to 0 \).
Therefore by Theorem 3.42 \( \sum a_n b_n \) converges.

Chapter 3, problem 9. Find the radius of convergence of each of the following power series:

(a) \( \sum n^3 z^n \),
(b) \( \sum \frac{2^n}{n!} z^n \),
(c) \( \sum \frac{n^2}{2^n} z^n \),
(d) \( \sum \frac{n^3}{3^n} z^n \).

Solution.

By Theorem 3.39, the radius of convergence of \( \sum c_n z^n \) is \( R = \frac{1}{\alpha} \), where \( \alpha = \limsup_{n \to \infty} \sqrt[n]{|c_n|} \).
Note that by Theorem 3.37, if \( \lim_{n \to \infty} \frac{c_{n+1}}{c_n} \) exists, then \( \lim_{n \to \infty} \frac{c_{n+1}}{c_n} = \alpha \).

(a) We have by theorems 3.3 and 3.20

\[ \alpha = \limsup_{n \to \infty} \sqrt[n]{|n^3|} = \limsup_{n \to \infty} (\sqrt[n]{n})^3 = (\limsup_{n \to \infty} \sqrt[n]{n})^3 = 1. \]

Hence \( R = 1 \).
(b) We have
\[ \lim_{n \to \infty} \frac{2^{n+1}}{(n+1)!} = \lim_{n \to \infty} \frac{2}{n} = 0. \]
Hence \( R = \infty \).

(c) We have
\[ \lim_{n \to \infty} \frac{2^{n+1}}{(n+1)^2} = \lim_{n \to \infty} \frac{2n^2}{(n+1)^2} = 2. \]
Hence \( R = \frac{1}{2} \).

(d) We have
\[ \lim_{n \to \infty} \frac{(n+1)^3}{3^{n+1}} = \lim_{n \to \infty} \frac{1}{3} \left(1 + \frac{1}{n}\right)^3 = \frac{1}{3}. \]
Hence \( R = 3 \).

Chapter 3, problem 11. Suppose \( a_n > 0, s_n = a_1 + \ldots + a_n \), and \( \sum a_n \) diverges.

(a) Prove that \( \sum \frac{a_n}{1 + a_n} \) diverges.

(b) Prove that
\[ \frac{a_{N+1}}{s_{N+1}} + \ldots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}} \]
and deduce that \( \sum \frac{a_n}{s_n} \) diverges.

(c) Prove that
\[ \frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n} \]
and deduce that \( \sum \frac{a_n}{s_n^2} \) converges.

(d) What can be said about
\[ \sum \frac{a_n}{1 + na_n} \quad \text{and} \quad \sum \frac{a_n}{1 + n^2 a_n} ? \]

Solution.

(a) If \( \{a_n\} \) is not bounded, then \( a_n \to \infty \). Then
\[ \lim_{n \to \infty} \left(1 - \frac{a_n}{1 + a_n}\right) = \lim_{n \to \infty} \frac{1}{1 + a_n} = 0. \]
Hence \( \frac{a_n}{1 + a_n} \to 1 \) and \( \sum \frac{a_n}{1 + a_n} \) diverges by theorem 3.23.

So assume that \( \{a_n\} \) is bounded. We have then that there exists \( M > 0 \) such that \( a_n \leq M \).
Then \( \frac{1}{1 + a_n} \geq \frac{1}{1 + M} \), and
\[
\sum \frac{a_n}{1 + a_n} \geq \sum \frac{a_n}{1 + M} = \frac{1}{1 + M} \sum a_n.
\]
Since \( \sum a_n \) diverges, by theorem 3.25 \( \sum \frac{a_n}{1 + a_n} \) diverges.

(b) Since \( a_n > 0 \), \( s_{n+1} > s_n \) for all \( n \). Then
\[
\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq \frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} = \frac{a_{N+1} + \cdots + a_{N+k}}{s_{N+k}} = \frac{s_{N+k} - s_N}{s_{N+k}} = 1 - \frac{s_N}{s_{N+k}}.
\]
Suppose that \( \frac{a_n}{s_n} \) converges. Then given \( 0 < \epsilon < 1 \), there exists \( N \in \mathbb{N} \) such that
\[
\epsilon > \frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}},
\]
for all \( k \).
Since \( \sum a_n \) diverges, \( \{s_n\} \) is not bounded, so \( s_n \to \infty \). Hence since in the inequality above \( k \) is arbitrary, if we let \( k \to \infty \), we get \( \epsilon \geq 1 \) a contradiction with \( \epsilon < 1 \). Therefore \( \sum \frac{a_n}{s_n} \) diverges.

(c) Similarly as before, since \( s_n > s_{n-1} \) for all \( n \geq 1 \), we have
\[
\frac{a_n}{s_n^2} \leq \frac{a_n}{s_{n-1}s_n} = \frac{s_n - s_{n-1}}{s_{n-1}s_n} = \frac{1}{s_{n-1}} - \frac{1}{s_n}.
\]
Hence
\[
\sum_{n=0}^{N} \frac{a_n}{s_n^2} \leq 1 + \sum_{n=1}^{N} \left( \frac{1}{s_{n-1}} - \frac{1}{s_n} \right) = 1 + \frac{1}{a_0} - \frac{1}{s_N} \leq 1 + \frac{1}{a_0}.
\]
Then we get that all the partial sums of \( \sum_{n=0}^{N} \frac{a_n}{s_n^2} \) are bounded by \( 1 + 1/a_0 \), so \( \sum_{n=0}^{N} \frac{a_n}{s_n^2} \) converges.

(d) We have that
\[
\frac{a_n}{1 + n^2a_n} \leq \frac{a_n}{n^2a_n} = \frac{1}{n^2}.
\]
Since \( \sum \frac{1}{n^2} \) converges, \( \sum \frac{a_n}{1 + n^2a_n} \) converges.

For \( a_n = \frac{1}{n} \), we have that
\[
\sum \frac{a_n}{1 + na_n} = \sum \frac{1}{2n} \text{ diverges}.
\]
Now consider \( a_n = 1 \) when \( n = 2^k \) for some \( k \in \mathbb{N} \), and \( a_n = 2^{-n} \) otherwise. Then
\[
\sum a_n \geq \sum_{k=0}^{\infty} 1 \text{ diverges},
\]
but
\[ \sum \frac{a_n}{1 + na_n} \leq \sum_{k=0}^{\infty} \frac{1}{1 + 2^k} + \sum_{n=1}^{\infty} \frac{1}{2^n + n} \]
covers. Therefore \( \frac{a_n}{1 + na_n} \) might converge or diverge.

**Chapter 3, problem 13.** Prove that the Cauchy product of two absolutely convergent series converges absolutely.

Solution.
Let \( \sum a_n \) and \( \sum b_n \) be absolutely convergent series. Consider \( c_n = \sum_{k=0}^{n} a_k b_{n-k} \). We want to show that \( \sum c_n \) converges absolutely.
Indeed we have
\[ \sum_{n=0}^{N} |c_n| \leq \sum_{k=0}^{n} |a_k||b_{n-k}| = |a_0|B_N + |a_1|B_{N-1} + \cdots + |a_N|B_0 \leq (|a_0| + \cdots + |a_N|)B_N = A_NB_N, \]
where \( A_n = \sum_{k=0}^{n} |a_k|, B_n = \sum_{k=0}^{n} |b_k| \).
Since \( \sum a_n \) and \( \sum b_n \) are absolutely convergent, \( \{A_n\} \) and \( \{B_n\} \) are bounded. Therefore by the inequality above \( \sum |c_n| \) converges.

**Problem A.** Prove Theorem 4.4 using: 1) The definition of limit; 2) Theorems 4.2 and 3.3.

Solution.
Suppose \( E \subset X \), a metric space, \( p \) is a limit point of \( E \), \( f \) and \( g \) are complex functions on \( E \), and
\[ \lim_{x \to p} f(x) = A, \quad \lim_{x \to p} g(x) = B. \]
First we want to show
\[ \lim_{x \to p} (f + g)(x) = A + B. \]
Indeed, given \( \epsilon > 0 \), there exists \( \delta_1 > 0 \) such that
\[ |f(x) - A| < \epsilon, \quad \text{if} \quad |x - p| < \delta_1. \]
Similarly, there exists \( \delta_2 > 0 \) such that
\[ |g(x) - B| < \epsilon, \quad \text{if} \quad |x - p| < \delta_2. \]
Let \( \delta = \min\{\delta_1, \delta_2\} \). Then
\[ |(f + g)(x) - (A + B)| \leq |f(x) - A| + |g(x) - B| < 2\epsilon \quad \text{if} \quad |x - p| < \delta. \]
Since \( \epsilon > 0 \) is arbitrary, we have that
\[ \lim_{x \to p} (f + g)(x) = A + B. \]
Or take a sequence \( \{p_n\} \) such that \( p_n \to p \) with \( p_n \neq p \). Then by theorem 4.2
\[
\lim_{n \to \infty} f(p_n) = A, \quad \lim_{n \to \infty} g(p_n) = B.
\]

So by theorem 3.3
\[
\lim_{n \to \infty} (f + g)(p_n) = \lim_{n \to \infty} f(p_n) + \lim_{n \to \infty} g(p_n) = A + B.
\]
Since \( \{p_n\} \) is an arbitrary sequence such that \( p_n \to p \) with \( p_n \neq p \), it follows from theorem 4.2 that
\[
\lim_{x \to p} (f + g)(x) = A + B.
\]

Now we want to show
\[
\lim_{x \to p} (fg)(x) = AB.
\]
Indeed, let \( \epsilon > 0 \) be given. Similarly as before we can take \( \delta > 0 \) such that
\[
|f(x) - A| < \epsilon \quad \text{and} \quad |g(x) - B| < \epsilon \quad \text{if} \quad |x - p| < \delta.
\]
We have then
\[
|(fg)(x)| = |(f(x) - A)(g(x) - B) + A(g(x) - B) + B(f(x) - A)| 
\leq |f(x) - A||g(x) - B| + |A||g(x) - B| + |B|
\]
Since \( \epsilon > 0 \) is arbitrary, we can take \( 1 > \epsilon > 0 \) as small as we want, so that \( \epsilon(\epsilon + |A| + |B|) < \epsilon(1 + |A| + |B|) \) can be as small as we want. So
\[
\lim_{x \to p} (fg)(x) = AB.
\]

Or take a sequence \( \{p_n\} \) such that \( p_n \to p \) with \( p_n \neq p \). Then by theorem 4.2
\[
\lim_{n \to \infty} f(p_n) = A, \quad \lim_{n \to \infty} g(p_n) = B.
\]
So by theorem 3.3
\[
\lim_{n \to \infty} (fg)\{p_n\} = (\lim_{n \to \infty} f(p_n))(\lim_{n \to \infty} g(p_n)) = AB.
\]
As, before since \( \{p_n\} \) is an arbitrary sequence such that \( p_n \to p \) with \( p_n \neq p \), it follows from theorem 4.2 that
\[
\lim_{x \to p} (fg)(x) = AB.
\]
Finally we want to show
\[
\lim_{x \to p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}, \quad \text{if} \quad B \neq 0.
\]
Indeed, let \( \epsilon > 0 \) be given. Similarly as before we can take \( \delta > 0 \) such that
\[
|f(x) - A| < \epsilon \quad \text{and} \quad |g(x) - B| < \epsilon \quad \text{if} \quad |x - p| < \delta.
\]
Since \( B \neq 0 \), we might assume \( \epsilon < |B|/2 \). Then
\[
|g(x)| \geq |B| - |g(x) - B| > |B| - \epsilon > \frac{|B|}{2} \quad \text{if} \quad |x - p| < \delta.
\]
So
\[
\left| \frac{1}{g(x)} - \frac{1}{B} \right| = \left| \frac{B - g(x)}{Bg(x)} \right| \leq \frac{2}{|B|^2} |B - g(x)| < \frac{2\epsilon}{|B|^2}, \quad |x - p| < \delta.
\]
\[
f\left( \frac{f(x) - A}{g(x)} \right) = \left| \frac{1}{g(x)} \right| (f(x) - A) + A \left( \frac{1}{g(x)} - \frac{1}{B} \right) \leq \frac{|f(x) - A|}{|g(x)|} + |A| \left| \frac{1}{g(x)} - \frac{1}{B} \right|
\Rightarrow \left| \frac{f(x)}{g(x)} - \frac{A}{B} \right| \leq \frac{2\epsilon}{|B|} + \frac{2|A|\epsilon}{|B|^2}.
\]
Since \( \epsilon > 0 \) is arbitrary,
\[
\lim_{x \to p} \left( \frac{f}{g} \right)(x) = \frac{A}{B}.
\]
Or take a sequence \( \{p_n\} \) such that \( p_n \to p \) with \( p_n \neq p \).
Then by theorem 4.2
\[
\lim_{n \to \infty} f(p_n) = A, \quad \lim_{n \to \infty} g(p_n) = B.
\]
So by theorem 3.3
\[
\lim_{n \to \infty} \left( \frac{f}{g} \right)(p_n) = \frac{\lim_{n \to \infty} f(p_n)}{\lim_{n \to \infty} g(p_n)} = \frac{A}{B}.
\]
As, before since \( \{p_n\} \) is an arbitrary sequence such that \( p_n \to p \) with \( p_n \neq p \), it follows from theorem 4.2 that
\[
\lim_{x \to p} \left( \frac{f}{g} \right)(x) = \frac{A}{B}.
\]