HOMEWORK #7 - MA 504

PAULINHO TCHATCHATCHA

Chapter 3, problem 8. If $\sum a_n$ converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converges.

Solution.
Since $\{b_n\}$ is monotonic and bounded, $\{b_n\}$ converges by Theorem 3.14, say $b_n \to b \in \mathbb{R}$. Since $\{b_n\}$ is monotonic, we either $b_{n+1} \geq b_n$ for all n (non-decreasing) or $b_{n+1} \leq b_n$ for all n (non-increasing). Assume, without loss of generality, that $\{b_n\}$ is monotonic non-decreasing, otherwise consider $-b_n$. If $\{b_n\}$ is non-decreasing, then $b_n \leq b$ for all n. Then consider $c_n = b - b_n$.
We have that since $\sum a_n$ converges, the partial sums A_n of $\sum a_n$ form a bounded sequence; $c_0 \geq c_1 \geq c_2 \geq \cdots$, and $c_n \to 0$.
Therefore by Theorem 3.42 $\sum a_n b_n$ converges.

Chapter 3, problem 9. Find the radius of convergence of each of the following power series:

(a) $\sum n^3 z^n$,
(b) $\sum \frac{2^n}{n!} z^n$,
(c) $\sum \frac{2^n}{n^2} z^n$,
(d) $\sum \frac{n^3}{3^n} z^n$.

Solution.

By Theorem 3.39, the radius of convergence of $\sum c_n z^n$ is $R = \dfrac{1}{\alpha}$, where $\alpha = \limsup\limits_{n\to\infty} \sqrt[n]{|c_n|}$.
Note that by Theorem 3.37, if $\lim\limits_{n\to\infty} \dfrac{c_{n+1}}{c_n}$ exists, then $\lim\limits_{n\to\infty} \dfrac{c_{n+1}}{c_n} = \alpha$.

(a) We have by theorems 3.3 and 3.20

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|n^3|} = \limsup_{n\to\infty} (\sqrt[n]{n})^3 = (\limsup_{n\to\infty} \sqrt[n]{n})^3 = 1.$$

Hence $R = 1$.

(b) We have

$$\lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \lim_{n\to\infty} \frac{2}{n} = 0.$$

Hence $R = \infty$.

(c) We have

$$\lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)^2}}{\frac{2^n}{n^2}} = \lim_{n\to\infty} \frac{2n^2}{(n+1)^2} = 2.$$

Hence $R = \frac{1}{2}$.

(d) We have

$$\lim_{n\to\infty} \frac{\frac{(n+1)^3}{3^{n+1}}}{\frac{n^3}{3^n}} = \lim_{n\to\infty} \frac{1}{3}\left(1 + \frac{1}{n}\right)^3 = \frac{1}{3}.$$

Hence $R = 3$.

Chapter 3, problem 11. Suppose $a_n > 0$, $s_n = a_1 + ... + a_n$, and $\sum a_n$ diverges.

(a) Prove that $\sum \frac{a_n}{1+a_n}$ diverges.
(b) Prove that

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$$

and deduce that $\sum \frac{a_n}{s_n}$ diverges.
(c) Prove that

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and deduce that $\sum \frac{a_n}{s_n^2}$ converges.
(d) What can be said about

$$\sum \frac{a_n}{1+na_n} \quad \text{and} \quad \sum \frac{a_n}{1+n^2a_n} \ ?$$

Solution.
(a) If $\{a_n\}$ is not bounded, then $a_n \to \infty$. Then

$$\lim_{n\to\infty}\left(1 - \frac{a_n}{1+a_n}\right) = \lim_{n\to\infty} \frac{1}{1+a_n} = 0.$$

Hence $\frac{a_n}{1+a_n} \to 1$ and $\sum \frac{a_n}{1+a_n}$ diverges by theorem 3.23.
So assume that $\{a_n\}$ is bounded. We have then that there exists $M > 0$ such that $a_n \leq M$.

Then $\dfrac{1}{1+a_n} \geq \dfrac{1}{1+M}$, and

$$\sum \frac{a_n}{1+a_n} \geq \sum \frac{a_n}{1+M} = \frac{1}{1+M} \sum a_n.$$

Since $\sum a_n$ diverges, by theorem 3.25 $\sum \dfrac{a_n}{1+a_n}$ diverges.

(b) Since $a_n > 0$, $s_{n+1} > s_n$ for all n. Then

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq \frac{a_{N+1}}{s_{N+k}} + \cdots + \frac{a_{N+k}}{s_{N+k}} = \frac{a_{N+1} + \cdots a_{N+k}}{s_{N+k}} = \frac{s_{N+k} - s_N}{s_{N+k}} = 1 - \frac{s_N}{s_{N+k}}.$$

Suppose that $\dfrac{a_n}{s_n}$ converges. Then given $0 < \epsilon < 1$, there exists $N \in \mathbb{N}$ such that

$$\epsilon > \frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}},$$

for all k.
Since $\sum a_n$ diverges, $\{s_n\}$ is not bounded, so $s_n \to \infty$. Hence since in the inequality above k is arbitrary, if we let $k \to \infty$, we get $\epsilon \geq 1$ a contradiction with $\epsilon < 1$. Therefore $\sum \frac{a_n}{s_n}$ diverges.

(c) Similarly as before, since $s_n > s_{n-1}$ for all $n \geq 1$, we have

$$\frac{a_n}{s_n^2} \leq \frac{a_n}{s_{n-1}s_n} = \frac{s_n - s_{n-1}}{s_{n-1}s_n} = \frac{1}{s_{n-1}} - \frac{1}{s_n}.$$

Hence

$$\sum_{n=0}^{N} \frac{a_n}{s_n^2} \leq 1 + \sum_{n=1}^{N} \left(\frac{1}{s_{n-1}} - \frac{1}{s_n} \right) = 1 + \frac{1}{a_0} - \frac{1}{s_N} \leq 1 + \frac{1}{a_0}.$$

Then we get that all the partial sums of $\sum_{n=0}^{N} \frac{a_n}{s_n^2}$ are bounde by $1+1/a_0$, so $\displaystyle\sum_{n=0}^{N} \frac{a_n}{s_n^2}$ converges.

(d) We have that

$$\frac{a_n}{1+n^2 a_n} \leq \frac{a_n}{n^2 a_n} = \frac{1}{n^2}.$$

Since $\sum \dfrac{1}{n^2}$ converges, $\sum \dfrac{a_n}{1+n^2 a_n}$ converges.
For $a_n = \dfrac{1}{n}$, we have that

$$\sum \frac{a_n}{1+n a_n} = \sum \frac{1}{2n} \text{ diverges.}$$

Now consider $a_n = 1$ when $n = 2^k$ for some $k \in \mathbb{N}$, and $a_n = 2^{-n}$ otherwise. Then

$$\sum a_n \geq \sum_{k=0}^{\infty} 1 \text{ diverges,}$$

but

$$\sum \frac{a_n}{1+na_n} \leq \sum_{k=0}^{\infty} \frac{1}{1+2^k} + \sum_{n=1}^{\infty} \frac{1}{2^n+n} \text{ coverges.}$$

Therefore $\dfrac{a_n}{1+na_n}$ might converge or diverge.

Chapter 3, problem 13. Prove that the Cauchy product of two absolutely convergent series converges absolutely.

Solution.
Let $\sum a_n$ and $\sum b_n$ be absolutely convergent series. Consider $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. We want to show that $\sum c_n$ converges absolutely.
Indeed we have

$$\sum_{n=0}^{N} |c_n| \leq \sum \sum_{k=0}^{n} |a_k||b_{n-k}| = |a_0|B_N + |a_1|B_{N-1} + \cdots + |a_N|B_0 \leq (|a_0| + \cdots |a_N|)B_N = A_N B_N,$$

where $A_n = \sum_{k=0}^{n} |a_k|$, $B_n = \sum_{k=0}^{n} |b_k|$.
Since $\sum a_n$ and $\sum b_n$ are absolutely convergent, $\{A_n\}$ and $\{B_n\}$ are bounded. Therefore by the inequality above $\sum |c_n|$ converges.

Problem A. Prove Theorem 4.4 using: 1) The definition of limit; 2) Theorems 4.2 and 3.3.

Solution.
Suppose $E \subset X$, a metric space, p is a limit point of E, f and g are complex functions on E, and

$$\lim_{x \to p} f(x) = A, \quad \lim_{x \to p} g(x) = B.$$

First we want to show

$$\lim_{x \to p} (f+g)(x) = A + B.$$

Indeed, given $\epsilon > 0$, there exists $\delta_1 > 0$ such that

$$|f(x) - A| < \epsilon, \quad \text{if} \quad |x - p| < \delta_1.$$

Similarly, there exists $\delta_2 > 0$ such that

$$|g(x) - B| < \epsilon, \quad \text{if} \quad |x - p| < \delta_2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then

$$|(f+g)(x) - (A+B)| \leq |f(x) - A| + |g(x) - B| < 2\epsilon \quad \text{if} \quad |x - p| < \delta.$$

Since $\epsilon > 0$ is arbitrary, we have that

$$\lim_{x \to p} (f+g)(x) = A + B.$$

Or take a sequence $\{p_n\}$ such that $p_n \to p$ with $p_n \neq p$.
Then by theorem 4.2

$$\lim_{n\to\infty} f(p_n) = A, \quad \lim_{n\to\infty} g(p_n) = B.$$

So by theorem 3.3

$$\lim_{n\to\infty}(f+g)(p_n) = \lim_{n\to\infty} f(p_n) + \lim_{n\to\infty} g(p_n) = A + B.$$

Since $\{p_n\}$ is an arbitrary sequence such that $p_n \to p$ with $p_n \neq p$, it follows from theorem 4.2 that

$$\lim_{x\to p}(f+g)(x) = A + B.$$

Now we want to show

$$\lim_{x\to p}(fg)(x) = AB.$$

Indeed, let $\epsilon > 0$ be given. Similarly as before we can take $\delta > 0$ such that

$$|f(x) - A| < \epsilon \text{ and } |g(x) - B| < \epsilon \quad \text{if } |x - p| < \delta.$$

We have then

$$|(fg)(x)-AB| = |(f(x)-A)(g(x)-B)+A(g(x)-B)+B(f(x)-A)| \leq |f(x)-A||g(x)-B|+|A||g(x)-B|+|B|$$

Since $\epsilon > 0$ is arbitrary, we can take $1 > \epsilon > 0$ as small as we want, so that $\epsilon(\epsilon+|A|+|B|) < \epsilon(1+|A|+|B|)$ can be as small as we want. So

$$\lim_{x\to p}(fg)(x) = AB.$$

Or take take a sequence $\{p_n\}$ such that $p_n \to p$ with $p_n \neq p$.
Then by theorem 4.2

$$\lim_{n\to\infty} f(p_n) = A, \quad \lim_{n\to\infty} g(p_n) = B.$$

So by theorem 3.3

$$\lim_{n\to\infty}(fg)(p_n) = (\lim_{n\to\infty} f(p_n))(\lim_{n\to\infty} g(p_n)) = AB.$$

As, before since $\{p_n\}$ is an arbitrary sequence such that $p_n \to p$ with $p_n \neq p$, it follows from theorem 4.2 that

$$\lim_{x\to p}(fg)(x) = AB.$$

Finally we want to show

$$\lim_{x\to p}\left(\frac{f}{g}\right)(x) = \frac{A}{B}, \text{ if } B \neq 0.$$

Indeed, let $\epsilon > 0$ be given. Similarly as before we can take $\delta > 0$ such that

$$|f(x) - A| < \epsilon \text{ and } |g(x) - B| < \epsilon \quad \text{if } |x - p| < \delta.$$

Since $B \neq 0$, we might assume $\epsilon < |B|/2$. Then

$$|g(x)| \geq |B| - |g(x) - B| > |B| - \epsilon > \frac{|B|}{2} \quad \text{if } |x - p| < \delta.$$

So
$$\left|\frac{1}{g(x)} - \frac{1}{B}\right| = \left|\frac{B - g(x)}{Bg(x)}\right| \leq \frac{2}{|B|^2}|B - g(x)| < \frac{2\epsilon}{|B|^2}, \quad |x - p| < \delta.$$
$$\left|\frac{f(x)}{g(x)} - \frac{A}{B}\right| = \left|\frac{1}{g(x)}(f(x) - A) + A\left(\frac{1}{g(x)} - \frac{1}{B}\right)\right| \leq \frac{|f(x) - A|}{|g(x)|} + |A|\left|\frac{1}{g(x)} - \frac{1}{B}\right|$$
$$\Rightarrow \left|\frac{f(x)}{g(x)} - \frac{A}{B}\right| \leq \frac{2\epsilon}{|B|} + \frac{2|A|\epsilon}{|B^2|}.$$
Since $\epsilon > 0$ is arbitrary,
$$\lim_{x\to p}\left(\frac{f}{g}\right)(x) = \frac{A}{B}.$$
Or take take a sequence $\{p_n\}$ such that $p_n \to p$ with $p_n \neq p$.
Then by theorem 4.2
$$\lim_{n\to\infty} f(p_n) = A, \quad \lim_{n\to\infty} g(p_n) = B.$$
So by theorem 3.3
$$\lim_{n\to\infty}\left(\frac{f}{g}\right)(p_n) = \frac{\lim_{n\to\infty} f(p_n)}{\lim_{n\to\infty} g(p_n)} = \frac{A}{B}.$$
As, before since $\{p_n\}$ is an arbitrary sequence such that $p_n \to p$ with $p_n \neq p$, it follows from theorem 4.2 that
$$\lim_{x\to p}\left(\frac{f}{g}\right)(x) = \frac{A}{B}.$$