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Hardy–Sobolev Type Inequalities with Sharp Constants in Carnot–Carathéodory Spaces

Donatella Danielli · Nicola Garofalo · Nguyen Cong Phuc

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Abstract We prove a generalization with sharp constants of a classical inequality due to Hardy to Carnot groups of arbitrary step, or more general Carnot–Carathéodory spaces associated with a system of vector fields of Hörmander type. Under a suitable additional assumption (see Eq. 1.6 below) we are able to extend such result to the nonlinear case $p \neq 2$. We also obtain a sharp inequality of Hardy–Sobolev type.

Keywords Hardy type inequalities · Carnot groups · Carnot–Carathéodory spaces · Horizontal p-Laplacian

Mathematics Subject Classifications (2010) 35H20 · 35H30

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Nguyen Cong Phuc supported in part by NSF Grant DMS-0901083.

D. Danielli · N. Garofalo (✉)
Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA
e-mail: garofalo@math.purdue.edu

D. Danielli
e-mail: danielli@math.purdue.edu

N. C. Phuc
Department of Mathematics, Louisiana State University,
303 Lockett Hall, Baton Rouge, LA 70803, USA
e-mail: pcnguyen@math.lsu.edu

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1 Introduction

In [22] the following Hardy type inequality was proved for the Heisenberg group $\mathbb{H}^n$

$$\int_{\mathbb{H}^n} \frac{|\phi|^2}{N^2}\, |\nabla_H N|^2\, dg \;\le\; \left(\frac{2}{Q-2}\right)^2 \int_{\mathbb{H}^n} |\nabla_H \phi|^2\, dg\,, \qquad \phi \in C_o^\infty(\mathbb{H}^n \setminus \{e\})\,, \tag{1.1}$$

where we have indicated with $N = (|z|^4 + 16t^2)^{1/4}$ the Koranyi–Folland non-isotropic gauge and with $Q = 2n + 2$ the homogeneous dimension of $\mathbb{H}^n$. The constant in the right-hand side of Eq. 1.1 is optimal, see [29]. Following ideas introduced in [23] for uniformly elliptic operators, the inequality 1.1 was used in [22] to establish some strong unique continuation properties for singular perturbations of the Kohn–Spencer sub-Laplacian on $\mathbb{H}^n$. The inequality 1.1 was extended to the nonlinear case $p \neq 2$ in [45].

On the other hand, when $\mathbb{G}$ is a Carnot group with homogeneous dimension Q, then Folland and Stein [20] proved the following basic result which constitutes a subelliptic Sobolev embedding theorem: let $1 < p < Q$, and set $p^* = pQ/(Q - p)$. There exists $S_p(\mathbb{G}) > 0$ such that

$$\left(\int_{\mathbb{G}} |\phi|^{p^*}\, dg\right)^{\frac{1}{p^*}} \;\le\; S_p(\mathbb{G}) \left(\int_{\mathbb{G}} |\nabla_H \phi|^p\, dg\right)^{\frac{1}{p}}\,, \qquad \phi \in C_0^\infty(\mathbb{G})\,. \tag{1.2}$$

In connection with Eq. 1.2, we mention that Vassilev has shown in [49] that an extremal function always exists, and thereby the sharp constant in Eq. 1.2 is attained.

Recently, J. Goldstein asked the second named author the question of whether an inequality of Hardy type such as Eq. 1.1 be valid in the setting of Carnot groups. Understanding such problem was the original motivation of this paper. In fact, we will provide a general positive answer to Goldstein's question in the more general context of a system of smooth vector fields satisfying the finite rank condition rank Lie$[X_1, ..., X_m] \equiv n$, see the case $p = 2$ of Theorem 1.2 below. Furthermore, under the additional technical assumption Eq. 1.6, Theorem 1.2 states that the result for $p = 2$ can be extended when $p \neq 2$ to a nonlinear Hardy type inequality with sharp constants.

We emphasize that, in our approach, the role of the hypothesis 1.6 is purely instrumental. Its main function is that it allows us to apply the coarea formula in the version established in [43]. It is also worth stressing that Eq. 1.6 appears as a very natural and plausible assumption which, as we remark in Proposition 1.1 below, for general Carnot groups would follow from an equally plausible homogeneity property of the fundamental solution. Furthermore, Eq. 1.6 represents the differentiated version of the fundamental estimate obtained in [8], see Eq. 1.4 below. With all this being said, in the nonlinear case $p \neq 2$ the hypothesis 1.6 is, to the best of our knowledge, presently only known to be satisfied in groups of Heisenberg type. This (almost trivially) follows from the explicit fundamental solutions discovered by two of us and Capogna in [8], see Eq. 1.23 below (and also [31] for the case $p = Q$ of such result). Therefore, when $p \neq 2$, the value of our Theorem 1.2 is presently confined to providing an alternative proof of results already appeared in the papers [10, 45] and [28].

In fact, as far as we are aware of, that of groups of Heisenberg type is the most general setting in which when $p \neq 2$ the Hardy inequality with sharp constants Eq. 1.25 below is presently known. In this connection we should mention that in

[10] and [28] extensions of such nonlinear Hardy inequality have been stated for the class of the so-called *polarizable groups*. However, we note that there presently exist no known examples of polarizable groups, besides those of Heisenberg type. In our opinion our proof has the advantage of shifting the attention from special symmetry properties which can only be valid in groups of Heisenberg type (such as, for instance, the fact that the Folland–Kaplan gauge is a solution of the horizontal ∞-Laplacian), to working directly with the horizontal p-Laplacian. This is the spirit of the hypothesis 1.6.

As we have already mentioned, Theorem 1.2 is new already for the linear case $p = 2$. It is worth emphasizing here that, even in this linear setting, our approach allows to cover situations in which there is no underlying group or homogeneous structure, and which could not be dealt with by the existing techniques. This point will be better clarified in the sequel.

To introduce our main result consider a system $X = \{X_1, ..., X_m\}$ of C^∞ vector fields in $\mathbb{R}^n$, $n \geq 3$, satisfying the Chow–Hörmander condition on the Lie algebra rank Lie$[X_1, ..., X_m] \equiv n$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $K \subset \Omega$ be a fixed compact set. Denote by $\Lambda(x, r)$ the Nagel–Stein–Wainger polynomial relative to K, see Definition 2.1, with pointwise and local homogeneous dimensions $Q(x) \leq Q$ given by Eq. 2.1 below. For $1 < p < \infty$ we consider the horizontal p-Laplacian defined by Eq. 2.6, see [7, 8]. We denote by $\Gamma_p(x, \cdot)$ its fundamental solution with singularity at x and zero boundary values (a Green function). By this we mean that: (i) $\Gamma_p(x, \cdot) \in \mathcal{L}^{1,p}_{loc}(\Omega \setminus \{x\}$; (ii) $|\nabla_H \Gamma_p(x, \cdot)|^{p-1} \in L^1_{loc}(\Omega)$; (iii) $\phi \Gamma_p(x, \cdot) \in \mathcal{L}^{1,p}_0(\Omega)$, for any $\phi \in C^\infty_0(\overline{\Omega} \setminus \{x\})$; finally, $\Gamma_p(x, \cdot)$ satisfies the equation $\Delta_{H,p}\Gamma_p(x, \cdot) = -\delta_x$ in $\Omega \setminus \{x\}$ in the weak sense, i.e.,

$$\int_\Omega |\nabla_H \Gamma_p(x, \cdot)|^{p-2} < \nabla_H \Gamma_p(x, \cdot), \nabla_H \phi > \, dy \; = \; \phi(x) \, , \qquad \text{for every} \quad \phi \in C^\infty_o(\Omega). \tag{1.3}$$

Using the results in [7, 8], the existence of such a fundamental solution can be established following the ideas in [13, 24, 34, 35], but see also Danielli and Garofalo (personal communication), where a similar construction was carried in the case of Carnot groups. The following basic estimates of the fundamental solution were obtained in [8], Theorem 7.2: *Given $x \in K$ and $1 < p < Q(x)$, there exist constants $C, R_o > 0$, depending only on $K \subset \Omega \subset \mathbb{R}^n$, p, and $X_1, ..., X_m$, such that for every $0 < R < R_o$ and $y \in B(x, R) \setminus \{x\}$ one has*

$$C \left(\frac{d(x, y)^p}{\Lambda(x, d(x, y))} \right)^{\frac{1}{p-1}} \; \leq \; \Gamma_p(x, y) \; \leq \; C^{-1} \left(\frac{d(x, y)^p}{\Lambda(x, d(x, y))} \right)^{\frac{1}{p-1}} \, . \tag{1.4}$$

In Eq. 1.4 we have denoted by $d(x, y)$ the Carnot–Carathéodory distance associated with the vector fields $X_1, ..., X_m$, see [44]. Moreover, we have let $B(x, R) = \{y \in \mathbb{R}^n \mid d(x, y) < R\}$. The estimate 1.4 generalizes a fundamental result for the linear case $p = 2$, first obtained independently in [44] and in [48]. These authors also proved, with $\Gamma = \Gamma_2$,

$$|\nabla_H \Gamma(x, y)| \; \leq \; C^{-1} \, \frac{d(x, y)}{\Lambda(x, d(x, y))} \, . \tag{1.5}$$

Throughout this paper we will request that for any $p > 1$, with $p \neq 2$, the fundamental solution (or Green function) of the operator $\Delta_{H,p}$ on Ω satisfies the following generalization of Eq. 1.5.

Hypothesis There exist $C, R_0 > 0$, depending on K and $X_1, \ldots, X_m$, such that for every $x \in K$, $0 < R < R_o$ for which $B(x, 4R) \subset \Omega$, and a.e. $y \in B(x, R) \setminus \{x\}$ one has

$$|\nabla_H \Gamma_p(x, y)| \leq C^{-1} \left(\frac{d(x, y)}{\Lambda(x, d(x, y))} \right)^{\frac{1}{p-1}} . \tag{1.6}$$

Such hypothesis is fulfilled for every $1 < p < \infty$ in every Carnot group of Heisenberg type, see Eq. 1.23 below. From that identity, established in [8], one easily proves Eq. 1.6. We stress that the above assumption Eq. 1.6 is not the weakest one that could be made, and that to the expenses of additional technicalities, we could have chosen much weaker hypothesis. Since, however, we are primarily interested in illustrating the ideas, we have decided to stick with Eq. 1.6.

Furthermore, we emphasize that Eq. 1.6 is quite plausible, and we strongly feel that such property will eventually change its status from hypothesis to theorem. To corroborate this statement, let us observe that for a Carnot group the estimate 1.6 would follow from the following proposition for whose proof we refer to Danielli and Garofalo (personal communication).

Proposition 1.1 *Let $\mathbb{G}$ be a Carnot group of arbitrary step. Suppose that the horizontal p-Laplacian on $\mathbb{G}$ (defined by Eq. 2.6 below) has a fundamental solution which is homogeneous of degree $(p - Q)/(p - 1)$. Then, Eq. 1.6 is fulfilled.*

Here Q denotes the homogeneous dimension of $\mathbb{G}$ induced by the nonisotropic dilations associated with the grading of the Lie algebra. We mention that for the case $p = 2$ Folland proved in [20] that the fundamental solution vanishing at infinity of a sub-Laplacian is homogeneous of degree $2 - Q$, but such property is presently not known for $p \neq 2$.

We are thus ready to introduce the main result in this paper. Inspired by the estimate 1.4 for any fixed $x \in \Omega$, and for $1 < p < Q(x)$, we introduce the function

$$E_p(x, r) = \left(\frac{\Lambda(x, r)}{r^p} \right)^{\frac{1}{p-1}} , \tag{1.7}$$

which is strictly increasing in $r > 0$, and denote by $F_p(x, \cdot) = E_p(x, \cdot)^{-1}$ its inverse. We define the regularized pseudo-distance based at x,

$$\rho_x(y) = \rho_{p,x}(y) = F_p \left(x, \frac{1}{\Gamma_p(x, y)} \right), \qquad y \in \Omega \setminus \{x\}. \tag{1.8}$$

Henceforth, for the sake of simplifying the notation, we will routinely omit the subscript p, and simply write $E(x, r)$ and $F(x, s)$, instead of $E_p(x, r)$, $F_p(x, s)$.

We now recall the classical one-dimensional Hardy inequality [30]: let $1 < p < \infty$, $u(t) \geq 0$, and $\phi(t) = \int_0^t u(s) ds$, then

$$\int_0^\infty \left(\frac{\phi(t)}{t} \right)^p dt \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty \phi'(t)^p dt .$$

Here is our main result.

Theorem 1.2 *Given a compact set $K \subset \Omega$, let $x \in K$, and suppose that $1 < p < Q(x)$, where $Q(x)$ is the homogeneous dimension at x of the system $X = \{X_1, ..., X_m\}$, see Eq. 2.1 below. There exists $R_o > 0$ such that for every $0 < R < R_o$ for which $B_X(x, 4R) \subset \Omega$, one has for $\phi \in C_o^\infty(B_X(x, R) \setminus \{x\})$*

$$\int_{B_X(x,R)} |\phi|^p \left\{ \frac{E'(x, \rho_x)}{E(x, \rho_x)} \right\}^p |\nabla_H \rho_x|^p dy \leq \left(\frac{p}{p-1} \right)^p \int_{B_X(x,R)} |\nabla_H \phi|^p dy, \tag{1.9}$$

where we have let $B_X(x, R) = \{y \in \mathbb{R}^n \mid \rho_x(y) < R\}$. When $\Lambda(x, r)$ is a monomial (thus, e.g., in the case of a Carnot group) the constant in the right-hand side of Eq. 1.9 is best possible.

It is worth noting here that Theorem 1.2 cannot be deduced either from Theorem 1.6 in [12] or from Corollary 4.7 in the recent paper [16]. The reason is twofold. On one hand, these results do not provide a sharp constant. On the other hand, it is not known whether the complement of the set $B_X(x, R)$ is uniformly (X, p)-fat.

The main new ideas in the proof of Theorem 1.2 are: (i) the introduction of the function $E(x, r)$; and (ii) working with the regularized pseudo-distance ρ_x associated with the horizontal p-Laplacian $\Delta_{H,p}$. The introduction of the function $E(x, r)$ was inspired by the fundamental estimate 1.4. Notice that, as a consequence of Eq. 1.4, we obtain the following estimate: Given $x \in K$ and $1 < p < Q(x)$, there exist positive constants C, R_o, depending on $X_1, ..., X_m$, p, and K, such that for every $0 < R < R_o$ and $y \in B(x, R) \setminus \{x\}$ one has

$$C\, E(x, d(x, y)) \ \leq\ \Gamma_p(x, y)^{-1} \ \leq\ C^{-1}\, E(x, d(x, y))\,. \tag{1.10}$$

From Eqs. 1.10 and 1.8 we thus obtain

$$C\, d(x, y) \ \leq\ \rho_x(y) \ \leq\ C^{-1}\, d(x, y)\,. \tag{1.11}$$

We remark that, for any fixed $x \in \Omega$, thanks to the Hölder regularity result in [7], and to Eq. 1.11, if we let $\rho_x(x) = 0$, then $\rho_x \in C(\Omega)$. As we will see in Lemma 2.3, the assumption Eq. 1.6 guarantees that $|\nabla_H \rho_x|$ is locally bounded in Ω, hence thanks to the sub-Riemannian Rademacher–Stepanov theorem in [21, 26] we infer that ρ_x is locally Lipschitz continuous with respect to the Carnot–Carathéodory distance $d(x, y)$. Moreover, Eq. 1.11 implies that $\rho_x^{-1} \in L^s(K)$, provided that $1 < s < Q$. In particular, since in Theorem 1.2 we have chosen $1 < p < Q(x) \leq Q$, we conclude $\rho_x^{-1} \in L^p(B_X(x, R))$. Furthermore, Lemma 2.4 below implies that

$$\left(\frac{Q(x) - p}{p-1} \right)^p \frac{1}{\rho_x^p} \ \leq\ \left\{ \frac{E'(x, \rho_x)}{E(x, \rho_x)} \right\}^p \ \leq\ \left(\frac{Q - p}{p-1} \right)^p \frac{1}{\rho_x^p}\,. \tag{1.12}$$

As a consequence of Eq. 1.12, we thus obtain the following corollary of Theorem 1.2.

Corollary 1.3 *In the same hypothesis of Theorem 1.2 one has for $\phi \in C_o^\infty(B_X(x, R) \setminus \{x\})$*

$$\int_{B_X(x,R)} \frac{|\phi|^p}{\rho_x^p} |\nabla_H \rho_x|^p\, dy \ \leq\ \left(\frac{p}{Q(x) - p} \right)^p \int_{B_X(x,R)} |\nabla_H \phi|^p\, dy\,. \tag{1.13}$$

Interestingly, our approach allows to bypass some of the geometric obstructions which arise when one tries to extend the classical Hardy type inequalities to a sub-Riemannian setting. In this connection we mention that, after the completion of this paper, we became aware of the recent interesting work [4], in which the authors present a unified approach to improved L^p Hardy type inequalities (see their Section 4, in particular). For the case of a point singularity in $\mathbb{R}^m$, the point of view of our paper is somewhat close to that of [4]. It would be very interesting to investigate possible generalizations of the ideas in [4] to the setting of the present paper for singularities which are concentrated on a given compact set, for instance the case of a smooth surface when the weight is the Carnot–Carathéodory distance from the latter. Some recent results in this direction have been obtained in [16] in the case when the compact set is the smooth boundary of the ground domain Ω. However, this subject is at the present mostly *terra incognita*. For an interesting study of the properties of the distance function to a compact surface in the Heisenberg group, we refer the reader to the paper [2]. Another intriguing aspect of our approach is the fact that, in the end, the choice of the weight function in the left-hand side of Eq. 1.9 derives from solving a Bernoulli differential equation involving the Nagel–Stein–Wainger polynomial $\Lambda(x, r)$, see the proof of Theorem 1.2.

It is worth mentioning here that Theorem 1.2 contains as special cases the classical Hardy type inequality, with one point singularity, as well as Eq. 1.1 and those proved in [10, 28, 45] . For instance, in the (for us) trivial case when the group is Abelian, and therefore $\mathbb{G} \cong \mathbb{R}^m$, then formula 1.8 with $x = 0$ gives $\rho_0(y) = |y|$, whereas one has $E(x, r) \equiv c(m, p) r^{(m-p)/(p-1)}$. We thus find $E'(x, r)/E(x, r) \equiv (m - p)/(p - 1) r^{-1}$, and Eq. 1.9 gives for any $u \in C_o^\infty(\mathbb{R}^m \setminus \{0\})$

$$\int_{\mathbb{R}^m} \frac{|u(x)|^p}{|x|^p} dx \le \left(\frac{p}{m-p}\right)^p \int_{\mathbb{R}^m} |\nabla u(x)|^p dx ,$$

with the sharp constant, see [46].

When $\mathbb{G}$ is a (non-Abelian) Carnot group with homogeneous dimension Q, then a fundamental solution $\Gamma_p(g, g')$ for the horizontal p-Laplacian

$$\Delta_{H,p} u = div_H(|\nabla_H u|^{p-2} \nabla_H u) , \tag{1.14}$$

was constructed in [35] and Danielli and Garofalo (personal communication). Here, the symbols ∇_H and div_H denote the horizontal gradient and divergence with respect to a fixed orthonormal basis of the bracket generating layer of the Lie algebra, see Section 2. When $1 < p < Q$ we say that $\Gamma_p \in \mathcal{D}'(\mathbb{G})$ is a fundamental solution of $\Delta_{H,p}$ if: (i) $\Gamma_p \in \mathcal{L}^{1,p}_{loc}(\mathbb{G} \setminus \{e\})$; (ii) $\Delta_{H,p}\Gamma_p = -\delta$ in $\mathcal{D}'(\mathbb{G})$; (iii) $|\nabla_H \Gamma_p|^{p-1} \in L^1_{loc}(\mathbb{G})$; (iv) Γ_p satisfies the conditions

$$\lim_{g \to e} \Gamma_p(g) = +\infty , \qquad \lim_{d(g,e) \to \infty} \Gamma_p(g) = 0 . \tag{1.15}$$

The equation (ii) must be interpreted in the weak sense

$$\int_{\mathbb{G}} |\nabla_H \Gamma_p|^{p-2} < \nabla_H \Gamma_p, \nabla_H \phi > dg = \phi(e) , \qquad \text{for every} \quad \phi \in C_o^\infty(\mathbb{G}) . \tag{1.16}$$

We stress here that, unless we are in the linear case $p = 2$ (see [20]), or $\mathbb{G}$ is a group of Heisenberg type (see Eq. 1.23 below), it is *not* known in general if such fundamental solution possesses the appropriate homogeneity $(p - Q)/(p - 1)$ with respect to the non-isotropic dilations. Nonetheless, for a Carnot group the pointwise homogeneous dimension $Q(x)$ is constant, and equals Q. It follows that the Nagel–Stein–Wainger polynomial is constant with respect to the variable $x \in \mathbb{G}$ and it is given by the monomial $\Lambda(x,r) = C(\mathbb{G})r^Q$. As a consequence, in such situation we obtain

$$E(x,r) \;\equiv\; C(\mathbb{G},\,p)\, r^{\frac{Q-p}{p-1}}\;,$$

and therefore

$$\left\{\frac{E'(x,r)}{E(x,r)}\right\}^p \;\equiv\; \left(\frac{Q-p}{p-1}\right)^p\,\frac{1}{r^p}\,. \tag{1.17}$$

Also, formula 1.8 presently gives with $x = e \in \mathbb{G}$, the group identity,

$$\rho(g) \;\stackrel{def}{=}\; \Gamma_p(g)^{\frac{1-p}{Q-p}}\;, \qquad\qquad g \neq e\,. \tag{1.18}$$

We note further that, thanks to the results in [8, 17], there exist positive constants $\alpha = \alpha(\mathbb{G},\,p)$ and $\beta = \beta(\mathbb{G},\,p)$ such that for every $g \in \mathbb{G}$, with $g \neq e$, one has

$$\frac{\alpha}{d(g,e)^{\frac{Q-p}{p-1}}} \;\le\; \Gamma_p(g) \;\le\; \frac{\beta}{d(g,e)^{\frac{Q-p}{p-1}}}\;, \tag{1.19}$$

where we have denoted by $d(g,g')$ the Carnot–Carathéodory distance on $\mathbb{G}$ associated with a basis of the horizontal bundle. From the basic estimates 1.19 it is clear that for every $g \in \mathbb{G} \setminus \{e\}$ one has

$$\alpha^*\, d(g,e) \;\le\; \rho(g) \;\le\; \beta^*\, d(g,e)\,. \tag{1.20}$$

With ρ as in Eq. 1.18, and in view of Eq. 1.17, combining Theorem 1.2 with the Folland–Stein embedding Eq. 1.2, we now obtain the following sharp inequality of Hardy–Sobolev type.

Theorem 1.4 *Let $\mathbb{G}$ be a Carnot group with homogeneous dimension Q, and let $1 < p < Q$. For $0 \le s \le p$ we define the critical exponent relative to s as follows*

$$p^*(s) \;=\; p\,\frac{Q-s}{Q-p}\,.$$

For every $\phi \in C_o^\infty(\mathbb{G} \setminus \{e\})$, where e is the group identity, one has

$$\int_{\mathbb{G}} \frac{|\phi|^{p^*(s)}}{\rho^s}\,|\nabla_H \rho|^s\, dg \;\le\; \left(\frac{p}{Q-p}\right)^s\, S_p(\mathbb{G})^{\frac{p^*}{p}(p-s)}\,\left(\int_{\mathbb{G}} |\nabla_H \phi|^p\, dg\right)^{\frac{p^*(s)}{p}}\,. \tag{1.21}$$

In particular, when $s = 0$, then Eq. 1.21 is just the Folland–Stein embedding Eq. 1.2, whereas for $s = p$ we obtain the Hardy type inequality

$$\int_{\mathbb{G}} \frac{|\phi|^p}{\rho^p}\,|\nabla_H \rho|^p\, dg \;\le\; \left(\frac{p}{Q-p}\right)^p \int_{\mathbb{G}} |\nabla_H \phi|^p\, dg\,. \tag{1.22}$$

The constant $\left(\frac{p}{Q-p}\right)^p$ in the right-hand side of Eq. 1.22 is sharp.

In Theorem 1.5 we have denoted with dg the bi-invariant Haar measure on $\mathbb{G}$ obtained by pushing forward the standard Lebesgue measure on the Lie algebra $\mathfrak{g}$ via the exponential mapping.

When $\mathbb{G}$ is a Carnot group of Heisenberg type (see Section 2 for the relevant definitions), then the horizontal p-Laplacian $\Delta_{H,p}$ possesses a remarkable explicit fundamental solution. This was discovered in [8], where it was found that

$$\Gamma_p(g) \;=\; \begin{cases} \frac{p-1}{Q-p}\sigma_p^{-\frac{1}{p-1}} \; N(g)^{-\frac{Q-p}{p-1}} \,, & p \neq Q \,, \\ \\ -\sigma_Q^{-\frac{1}{Q-1}} \; \log N(g) \,, & p = Q \,, \end{cases} \tag{1.23}$$

is a fundamental solution of $\Delta_{H,p}$ (such function continues to be a fundamental solution also in the range $Q < p < \infty$, except that in such case it does not have a singularity at e). In Eq. 1.23 we have denoted by $N(g) = (|x(g)|^4 + 16|y(g)|^2)^{\frac{1}{4}}$ Kaplan's renormalized gauge on $\mathbb{G}$, see [38], whereas we have let $\sigma_p = Q\omega_p$, with

$$\omega_p \;=\; \int_{\{g\in\mathbb{G}\,|\,N(g)<1\}} |\nabla_H N(g)|^p \, dg \,.$$

We note that the case $p = 2$ of Eq. 1.23 was first discovered by Folland [19] for the Heisenberg group, and subsequently generalized by Kaplan [38] to groups of Heisenberg type. The conformal case $p = Q$ was also found in [31]. Returning to Theorem 1.4, from the latter, from Eq. 1.23 and from Eq. 1.18, we thus obtain the following interesting corollary.

Corollary 1.5 *Let $\mathbb{G}$ be a group of Heisenberg type with homogeneous dimension Q, and let $1 < p < Q$. For $0 \le s \le p$ and with $p^*(s) = p\frac{Q-s}{Q-p}$ one has for every $\phi \in C_o^\infty(\mathbb{G}\setminus\{e\})$,*

$$\int_{\mathbb{G}} \frac{|\phi|^{p^*(s)}}{N^s} \, |\nabla_H N|^s \, dg \;\le\; \left(\frac{p}{Q-p}\right)^s \, S_p(\mathbb{G})^{\frac{p^*}{p}(p-s)} \, \left(\int_{\mathbb{G}} |\nabla_H \phi|^p \, dg\right)^{\frac{p^*(s)}{p}} \,. \tag{1.24}$$

In particular, when $s = p$ we obtain the Hardy type inequality

$$\int_{\mathbb{G}} \frac{|\phi|^p}{N^p} \, |\nabla_H N|^p \, dg \;\le\; \left(\frac{p}{Q-p}\right)^p \int_{\mathbb{G}} |\nabla_H \phi|^p \, dg \,. \tag{1.25}$$

We mention that after this paper was completed and circulated we have received from L. D'Ambrosio his preprint [10] in which, with a different method (see also [9]), he establishes among other things the Hardy type inequality 1.25 in Corollary 1.5. An extensively revised version of [10] has subsequently appeared in print in the interesting paper [11]. Subsequently, we have also received from I. Kombe his interesting preprint (Kombe, submitted for publication), in which he establishes in the case $p = 2$ various weighted Hardy inequalities with optimal constants in groups of Heisenberg type. In this setting, he also obtains Eq. 1.24. We mention that Eq. 1.25 has also subsequently appeared in [28].

Finally, it is interesting to observe that the critical exponent $p^*(s)$ in the inequality 1.21 coincides with that in the trace embedding theorems proved in [14, 15]. As we next explain, this is not a mere coincidence. We recall the following special case of Theorem 1.9 in [14].

Theorem 1.6 *Let $\mathbb{G}$ be a Carnot group with homogeneous dimension Q, and let $1 < p < Q$. Let μ be a nonnegative Borel measure on $\mathbb{G}$ such that for some $M > 0$, and $0 \le s < p$ one has*

$$\mu(B(g,r)) \le M \frac{|B(g,r)|}{r^s}, \qquad g \in \mathbb{G}, \quad r > 0. \tag{1.26}$$

There exist positive constants, $C = C(\mathbb{G}, p, s)$ and $\sigma = \sigma(\mathbb{G}) \ge 1$, such that if $u \in C_0^\infty(\sigma B)$, with $\sigma B = B(g, \sigma r)$, then

$$\int_B |\phi|^{p^*(s)} \, d\mu \le C\, M \left(\int_{\sigma B} |\nabla_H \phi|^p \, dg \right)^{\frac{p^*(s)}{p}} .$$

Finally, Eq. 1.26 is also necessary for the latter inequality to hold.

It is clear that if we define a nonnegative Borel measure on $\mathbb{G}$ by letting

$$d\mu \overset{def}{=} \frac{|\nabla_H \rho|^s}{\rho^s} \, dg ,$$

then thanks to Lemma 2.3 below we have $|\nabla_H \rho| \le C(\mathbb{G})$. In view of Eq. 1.20, for $0 \le s < Q$ we would thus have that μ satisfies Eq. 1.26. As a consequence, the inequality 1.21 in Theorem 1.4 (although perhaps not with the sharp constant) could be obtained as a special case of the trace Theorem 1.6. However, this is not the case, since as we will see the proof of Theorem 1.4 relies crucially on first proving the end-point result 1.22. Such result, in turn, cannot be obtained from Theorem 1.6 since, as it is explained in [14], the latter fails at the end-point $s = p$, which is what is needed for Eq. 1.22.

The relevance of inequalities of Hardy–Sobolev type is well-known, and there is a large literature on this subject. Although this is only a very partial list, the reader can consult the papers [1, 3–6, 17, 18, 27, 33, 39–42, 46], and the references therein. In the classical setting, the L^2 Hardy inequality implies via Plancherel theorem Heisenberg's uncertainty principle from quantum mechanics. For the uncertainty inequality in groups of Heisenberg type the reader should see [28], or also [11]. The L^2 Hardy inequality is also crucial in the study of self-adjointness and the bottom of the spectrum of the time-independent Schrödinger operator with inverse square potential $V(x) = c(n)|x|^{-2}$, see [47]. We have already mentioned a basic application to questions of unique continuation for elliptic operators with singular lower order terms, see e.g. [23]. Another interest in Hardy type inequalities is to ascertain whether a given function in the Sobolev space $W^{k,p}(\Omega)$ actually belongs to $W_o^{k,p}(\Omega)$. In fact, Theorem V.3.4 in [18] states that for a given open subset $\Omega \subset \mathbb{R}^n$, with $\Omega \ne \mathbb{R}^n$, and $1 < p < \infty$,

$$u \in W^{k,p}(\Omega) , \ \frac{u}{\delta^k} \in L^p(\Omega) \implies u \in W_o^{k,p}(\Omega) . \tag{1.27}$$

Here, we have let $\delta(x) = dist(x, \partial\Omega)$. Thus, if $u \in W^{1,p}(\Omega)$ satisfies the Hardy type inequality

$$\int_\Omega \frac{|u|^p}{\delta^p} \, dx \le C \int_\Omega |\nabla u|^p \, dx , \tag{1.28}$$

then we conclude that $u \in W_o^{1,p}(\Omega)$. Vice-versa, it was proved in [36] that if Ω is a sufficiently regular bounded open set, for instance if Ω is a Lipschitz domain, then every function in $W_o^{1,p}(\Omega)$ satisfies Eq. 1.28. In fact, in Eq. 1.27 with $k = 1$ one can replace the condition $u/\delta \in L^p(\Omega)$ with $u/\delta \in L^{p,\infty}(\Omega)$, the weak L^p space, see [37, 50].

2 Proof of Theorem 1.2

In this section we prove Theorem 1.2. For $n \geq 3$ we consider a system of C^∞ vector fields $X_1, ..., X_m$ on $\mathbb{R}^n$ satisfying Hörmander finite rank condition: rank $\mathrm{Lie}[X_1, ..., X_m] \equiv n$. We denote by $d(x, y)$ the Carnot–Carathéodory distance generated by $X_1, ..., X_m$, and for every $x \in \mathbb{R}^n$ and $r > 0$ we let $B(x, r) = \{y \in \mathbb{R}^n \mid d(x, y) < r\}$. Let $Y_1, ..., Y_l$ denote the collection of the X_j's and of those commutators which are needed to generate $\mathbb{R}^n$. A "degree" is assigned to each Y_i, namely the corresponding order of the commutator. If $I = (i_1, ..., i_n)$, $1 \leq i_j \leq l$, is an n-tuple of integers, following [44] one defines $d(I) = \sum_{j=1}^n deg(Y_{i_j})$, and $a_I(x) = \det\,(Y_{i_1}, ..., Y_{i_n})$.

Definition 2.1 The *Nagel–Stein–Wainger polynomial* is defined by

$$\Lambda(x, r) \;=\; \sum_I \,|a_I(x)|\, r^{d(I)}, \qquad\qquad r > 0.$$

For a given compact set $K \subset \mathbb{R}^n$, we let

$$Q \;=\; \sup\,\{d(I) \mid \; |a_I(x)| \neq 0, x \in K\}, \qquad Q(x) \;=\; \inf\,\{d(I) \mid |a_I(x)| \neq 0\}, \tag{2.1}$$

and notice that from the work in [44] we know

$$3 \;\leq\; n \;\leq\; Q(x) \;\leq Q\,.$$

It is immediate that for every $x \in K$, and every $r > 0$, one has

$$t^Q\,\Lambda(x, r) \;\leq\; \Lambda(x, tr) \;\leq\; t^{Q(x)}\,\Lambda(x, r)\,, \qquad 0 \leq t \leq 1\,. \tag{2.2}$$

The numbers Q and $Q(x)$ are respectively called the *local homogeneous dimension* of K, and the *homogeneous dimension at* x, with respect to the system $X_1, ..., X_m$. One easily sees that for any $x \in K$ and $r > 0$

$$Q(x) \;\leq \frac{r\Lambda'(x, r)}{\Lambda(x, r)} \;\leq\; Q\,. \tag{2.3}$$

The following fundamental result is due to Nagel, Stein and Wainger [44]: *For every compact set* $K \subset \mathbb{R}^n$ *there exist constants* C, $R_o > 0$ *such that, for any* $x \in K$, *and* $0 < r \leq R_o$, *one has*

$$C\,\Lambda(x, r) \;\leq\; |B(x, r)| \;\leq\; C^{-1}\,\Lambda(x, r). \tag{2.4}$$

As a consequence, with $C_1 = 2^Q$, *one has for every* $x \in K$, *and any* $0 < r \leq R_o$

$$|B(x, 2r)| \;\leq\; C_1\,|B(x, r)|\,. \tag{2.5}$$

We now introduce the relevant nonlinear operator. For any $1 < p < \infty$ we consider the horizontal p-Laplacian

$$\Delta_{H,p}u \;=\; div_H^*(|\nabla_H u|^{p-2}\nabla_H u) \;\overset{def}{=}\; -\sum_{i=1}^{m} X_i^*(|\nabla_H u|^{p-2}X_i u) \;=\; 0 \qquad \text{in } \Omega\,, \tag{2.6}$$

where X_i^* denotes the formal adjoint of X_i, and we have let $|\nabla_H u|^2 = \sum_{i=1}^m (X_i u)^2$, see [7, 8]. Given an open set $\Omega \subset \mathbb{R}^n$, we denote by $\Gamma_p(x,\cdot)$ a fundamental solution of $\Delta_{H,p}$ with zero boundary values on $\partial\Omega$ and with singularity at $x \in \Omega$.

We recall the following fundamental estimate, which is Theorem 7.2 in [8]: Let $K \subset\subset \Omega$, then for every $x \in K$, and any $1 < p < Q(x)$, there exist positive constants C, R_o, depending on $X_1, ..., X_m$, p and K, such that for any $0 < r < R_o$, and $y \in B(x,r)$ one has

$$C\left(\frac{d(x,y)^p}{\Lambda(x,d(x,y))}\right)^{\frac{1}{p-1}} \;\le\; \Gamma_p(x,y) \;\le\; C^{-1}\left(\frac{d(x,y)^p}{\Lambda(x,d(x,y))}\right)^{\frac{1}{p-1}}\,. \tag{2.7}$$

When $X_1, ..., X_m$ constitutes an orthonormal basis of bracket generating vector fields in a Carnot group $\mathbb{G}$, then a global (in all of $\mathbb{G}$) fundamental solution Γ_p for $\Delta_{H,p}$ has been constructed in [17, 35]. Using similar ideas, for any bounded open set Ω one can construct a positive fundamental solution with generalized zero boundary values, i.e., a Green function, in the more general situation of a Carnot–Carathéodory space, see [13, 24]. Henceforth, for a fixed $x \in \Omega$ we will denote by $\Gamma_p(x,\cdot)$ such a fundamental solution with singularity at x. This means that $\Gamma_p(x,\cdot)$ satisfies the equation

$$\int_\Omega |\nabla_H\Gamma_p(x,\cdot)|^{p-2} < \nabla_H\Gamma_p(x,\cdot), \nabla_H\phi > dy \;=\; \phi(x)\,, \tag{2.8}$$

for every $\phi \in C_o^\infty(\Omega)$. The estimate 2.7 suggests that the following nonlinear function should play a key role in the study of Hardy type inequalities. For any given $x \in K \subset\subset \Omega$, we fix a number $p = p(x)$ such that $1 < p < Q(x)$, and introduce the function

$$E(x,r) \;\overset{def}{=}\; \left(\frac{\Lambda(x,r)}{r^p}\right)^{\frac{1}{p-1}}\,. \tag{2.9}$$

Because of the constraint imposed on $p = p(x)$, we see that, for every fixed $x \in K$, the function $r \to E(x,r)$ is strictly increasing, and thereby invertible. We denote by $F(x,\cdot) \;=\; E(x,\cdot)^{-1}$, the inverse function of $E(x,\cdot)$, so that

$$E(x, F(x,r)) \;=\; F(x, E(x,r)) \;=\; r.$$

We now define for every $x \in K$

$$\rho_x(y) \;=\; F\left(x, \frac{1}{\Gamma(x,y)}\right)\,. \tag{2.10}$$

Remark 2.2 We emphasize that in a Carnot group $\mathbb{G}$ one has, for every $x \in \mathbb{G}$, $Q(x) \equiv Q$, the homogeneous dimension of the group, and therefore the Nagel–Stein–Wainger polynomial is in fact just a monomial, i.e., $\Lambda(x,r) \equiv C(\mathbb{G})r^Q$. As a consequence, up to a universal factor, the function $E(x,r)$ is constant with respect to $x \in \mathbb{G}$ and equals $r^{(Q-p)/(p-1)}$. In this situation, we see that when $x = e$, then the function ρ_x in Eq. 2.10 coincides with the function in Eq. 1.18.

Using the function $E(x, r)$ in Eq. 2.9 it should be clear that we can recast the estimate 2.7 in the following more suggestive form

$$\frac{C}{E(x, d(x, y))} \leq \Gamma_p(x, y) \leq \frac{C^{-1}}{E(x, d(x, y))} . \tag{2.11}$$

As a consequence of Eq. 2.11 and of Eq. 2.10, we obtain the following estimate: *Given $x \in K$ and $1 < p < Q(x)$, there exist positive constants C, R_o, depending on $X_1, \ldots, X_m$, p, and K, such that every $0 < r < R_o$, one has for $y \in B(x, r)$*

$$C\, d(x, y) \leq \rho_x(y) \leq C^{-1}\, d(x, y) . \tag{2.12}$$

We can thus think of the function ρ_x as a *regularized pseudo-distance*. The reason for such name comes from the following lemma.

Lemma 2.3 *The function ρ_x is locally Lipschitz continuous in Ω with respect to the Carnot–Carathéodory metric $d(x, y)$. In the case $p = 2$, one also has $\rho_x \in C^\infty(\Omega \setminus \{x\})$.*

Proof First of all, $\rho_x \in C(\Omega)$. To see this, we recall that in [7] it was proved that weak solutions of $\Delta_{H,p}$ satisfy a uniform Harnack inequality, and are locally Hölder continuous. As a consequence, $\Gamma_p(x, \cdot)$ is strictly positive and locally Hölder continuous in $\Omega \setminus \{x\}$, and thus such is also ρ_x. Furthermore, from Eq. 2.7 we obtain

$$\lim_{y \to x} \Gamma_p(x, y) = +\infty ,$$

hence

$$\lim_{y \to x} \rho_x(y) = 0 .$$

As a consequence, if we define $\rho_x(x) = 0$, we obtain a continuous function in Ω. Next, from Eq. 2.10 we obtain

$$\Gamma_p(x, y) = E(x, \rho_x(y))^{-1} .$$

The chain rule for the spaces $\mathcal{L}^{1,p}$ now gives

$$\nabla_H \Gamma_p(x, y) = \frac{E'(x, \rho_x(y))}{E(x, \rho_x(y))^2} \nabla_H \rho_x(y) ,$$

which gives

$$|\nabla_H \rho_x(y)| = \left\{ \frac{E(x, \rho_x(y))}{\rho_x(y) E'(x, \rho_x(y))} \right\} \rho_x(y) E(x, \rho_x(y))\, |\nabla_H \Gamma_p(x, y)| .$$

We now notice that, thanks to Eq. 2.12, the assumption 1.6 can be reformulated as follows

$$|\nabla_H \Gamma_p(x, y)| \leq C \frac{1}{\rho_x(y) E(x, \rho_x(y))} .$$

Substituting the latter inequality in the previous one, we reach the conclusion

$$|\nabla_H \rho_x(y)| \;\leq\; C \left\{ \frac{E(x,\rho_x(y))}{\rho_x(y)\, E'(x,\rho_x(y))} \right\}, \tag{2.13}$$

for a.e. $y \in B(x, R)$ with $R > 0$ small enough. From Eq. 2.9 we now obtain

$$\begin{aligned} \frac{E'(x,r)}{E(x,r)} &= \frac{1}{p-1}\, \frac{r^p}{\Lambda(x,r)} \left\{ \frac{\Lambda'(x,r)}{r^p} - \frac{p}{r}\, \frac{\Lambda(x,r)}{r^p} \right\} \\ &= \frac{1}{p-1} \left\{ \frac{\Lambda'(x,r)}{\Lambda(x,r)} - \frac{p}{r} \right\}. \end{aligned} \tag{2.14}$$

Taking Eq. 2.3 into account, we obtain from Eq. 2.14

$$\frac{Q(x)-p}{p-1} \;\leq\; \frac{r\,E'(x,r)}{E(x,r)} \;\leq\; \frac{Q-p}{p-1}. \tag{2.15}$$

Using Eq. 2.15 in Eq. 2.13 we conclude

$$|\nabla_H \rho_x(y)| \;\leq\; C\, \frac{p-1}{Q(x)-p}, \qquad \text{for a.e.} \quad y \in B(x,R). \tag{2.16}$$

Thanks to the subelliptic Rademacher–Stepanov theorem in [21, 26], we reach the desired conclusion that

$$\rho_x \;\in\; \Gamma_d^{0,1}(B(x,R)), \tag{2.17}$$

where $\Gamma_d^{0,1}$ denotes the space of functions which are Lipschitz continuous with respect to the metric $d(x, y)$.

Finally, in the case $p = 2$ it follows from the hypoellipticity result in [32] that $\rho_x \in C^\infty(\Omega \setminus \{x\})$. □

The estimate 2.15 is interesting enough to deserve to be stated separately.

Lemma 2.4 *Let $K \subset \mathbb{R}^n$ be a compact set, with local parameters Q and R_0, for any $x \in K$ and any $0 < r < R_0$ one has*

$$\left(\frac{Q(x)-p}{p-1}\right)^p \frac{1}{r^p} \;\leq\; \left\{\frac{E'(x,r)}{E(x,r)}\right\}^p \;\leq\; \left(\frac{Q-p}{p-1}\right)^p \frac{1}{r^p}.$$

To prove Theorem 1.2 we will need the following lemmas.

Lemma 2.5 *Let $h \in C^2((0,\infty))$, with $h', h'' \in L^\infty_{loc}((0,\infty))$, and set $v = h \circ \Gamma_p$, where $\Gamma_p = \Gamma_p(x,\cdot)$. Then v is a weak solution in $\Omega \setminus \{x\}$ of the equation*

$$\Delta_{H,p} v \;=\; (p-1)\, |h'(\Gamma_p)|^{p-2}\, h''(\Gamma_p)\, |\nabla_H \Gamma_p|^p.$$

Proof First of all, we observe that $\Gamma_p = \Gamma_p(x,\cdot)$ has no singularity in $\Omega \setminus \{x\}$. In fact, by the results in [7] we know that $\Gamma_p \in \Gamma^{0,\delta}_{loc}(\Omega \setminus \{x\})$ for some $\delta > 0$ depending on Ω, $X_1, ..., X_m$ and p. By the chain rule for the spaces $\mathcal{L}^{1,p}$, we obtain $v \in \mathcal{L}^{1,p}_{loc}(\Omega \setminus \{x\})$ and the weak derivative of v is given by

$$\nabla_H v \;=\; h'(\Gamma_p)\, \nabla_H \Gamma_p.$$

We thus find for any $\phi \in C_o^\infty(\Omega \setminus \{x\})$

$$\int_\Omega |\nabla_H v|^{p-2} < \nabla_H v, \nabla_H \phi > dy = \int_\Omega |h'(\Gamma_p)|^{p-2}\, h'(\Gamma_p)\, |\nabla_H \Gamma_p|^{p-2} < \nabla_H \Gamma_p, \nabla_H \phi > dy\,.$$

We next define

$$\psi \;=\; |h'(\Gamma_p)|^{p-2}\, h'(\Gamma_p)\, \phi\,,$$

and observe that $\psi \in \mathcal{L}_o^{1,p}(\Omega)$, with

$$\nabla_H \psi \;=\; |h'(\Gamma_p)|^{p-2}\, h'(\Gamma_p)\, \nabla_H \phi \;+\; (p-1)\, |h'(\Gamma_p)|^{p-2}\, h''(\Gamma_p)\, \phi\, \nabla_H \Gamma_p\,,$$

and that *supp* $\psi \subset \Omega \setminus \{x\}$. We can thus use such ψ as a test function in Eq. 2.8, obtaining

$$\begin{aligned} 0 \;=\; & \int_\Omega |h'(\Gamma_p)|^{p-2}\, h'(\Gamma_p)\, |\nabla_H \Gamma_p|^{p-2} < \nabla_H \Gamma_p, \nabla_H \phi > dy \\ & + (p-1) \int_\Omega |h'(\Gamma_p)|^{p-2}\, h''(\Gamma_p)\, |\nabla_H \Gamma_p|^p\, \phi\, dy\,. \end{aligned}$$

We conclude

$$\int_\Omega |\nabla_H v|^{p-2} < \nabla_H v, \nabla_H \phi > dy \;=\; -\,(p-1) \int_\Omega |h'(\Gamma_p)|^{p-2}\, h''(\Gamma_p)\, |\nabla_H \Gamma_p|^p\, \phi\, dy\,,$$

which completes the proof. □

Lemma 2.6 *Let $x \in \Omega$ and let ρ_x be the function defined in Eq. 2.10, then one has in the weak sense in $\Omega \setminus \{x\}$*

$$\Delta_{H,p} \rho_x \;=\; (p-1)|\nabla_H \rho_x|^p \left\{ 2\, \frac{E'(x,\rho_x)}{E(x,\rho_x)} \;-\; \frac{E''(x,\rho_x)}{E'(x,\rho_x)} \right\}\,.$$

Proof In what follows we drop for simplicity the dependence on x and simply write ρ instead of ρ_x, and Γ_p instead of $\Gamma_p(x,\cdot)$. We are going to apply Lemma 2.5 with the choice $h(t) = F(x, t^{-1})$, so that $v = h(\Gamma_p)$. All differentiations are performed with respect to the variable y. Since

$$h'(t) = -t^{-2} F'(x, t^{-1})\,, \quad h''(t) \;=\; 2t^{-3} F'(x, t^{-1}) + t^{-4} F''(x, t^{-1})\,,$$

Lemma 2.5 gives

$$\Delta_{H,p} \rho = (p-1) \left(\Gamma_p^{-2} F'(x, \Gamma_p^{-1}) \right)^{p-2} |\nabla_H \Gamma_p|^p \left\{ 2\, \Gamma_p^{-3} F'(x, \Gamma_p^{-1}) + \Gamma_p^{-4} F''(x, \Gamma_p^{-1}) \right\}.$$

Using

$$|\nabla_H \Gamma_p|^p \;=\; |\nabla_H \rho|^p\, \Gamma_p^{2p}\, F'(x, \Gamma_p^{-1})^{-p}\,,$$

we conclude

$$\Delta_{H,p} \rho \;=\; (p-1)\, |\nabla_H \rho|^p \left\{ 2\, \frac{F'(x, \Gamma_p^{-1})^{-1}}{\Gamma_p^{-1}} \;+\; \frac{F''(x, \Gamma_p^{-1})}{F'(x, \Gamma_p^{-1})^2} \right\}$$

The inverse function theorem now gives

$$2\,\frac{F'(x,\Gamma_p^{-1})^{-1}}{\Gamma_p^{-1}} + \frac{F''(x,\Gamma_p^{-1})}{F'(x,\Gamma_p^{-1})^2} = 2\,\frac{E'(x,\rho)}{E(x,\rho)} - \frac{E''(x,\rho)}{E'(x,\rho)}\,.$$

Substituting this equation in the previous one, we obtain the desired conclusion.

□

A remarkable consequence of Lemma 2.6 is the following generalization of the classical formula

$$\Delta_p\, r = \frac{m-1}{r}\,,$$

valid in $\mathbb{R}^m$, for the standard p-Laplacian and with $r = r(x) = |x|$. The reader should notice the appearance of the pointwise homogeneous dimension $Q(x)$.

Corollary 2.7 *There exists $R_o > 0$, depending on the local parameters of $K \subset\subset \Omega$, such that for every $x \in K$ one has*

$$\Delta_{H,p}\rho_x = \frac{Q(x)-1}{\rho_x}\left(1+\omega_x\right)|\nabla_H \rho_x|^p \qquad in \quad B(x,R_o)\setminus\{x\}\,,$$

where $|\omega_x(y)| \le C\,\rho_x(y)$ as $\rho_x(y)\to 0$, uniformly in $x \in K$.

Proof Keeping in mind Eq. 2.9, a computation (whose details are left to the interested reader) gives

$$2\,\frac{E'(x,\rho)}{E(x,\rho)} - \frac{E''(x,\rho)}{E'(x,\rho)} = \frac{p}{p-1}\,\frac{d}{d\rho}\log\left(\frac{\Lambda(x,\rho)}{\rho^p}\right) - \frac{d}{d\rho}\log\left(\frac{d}{d\rho}\left(\frac{\Lambda(x,\rho)}{\rho^p}\right)\right) \tag{2.18}$$

Definition 2.1 gives

$$\frac{\Lambda(x,r)}{r^p} = |a_{Q(x)}(x)|r^{Q(x)-p} + \ldots . + |a_Q(x)|r^{Q-p}$$

from which one easily recognizes that the right-hand side in Eq. 2.18 equals $\frac{Q(x)-1}{p-1}\frac{(1+\omega(\rho))}{\rho}$. Substitution of this information in Lemma 2.6 completes the proof.

□

In the rest of this section we will use repeatedly tools from sub-Riemannian calculus such as a generalization of Federer's coarea formula. We will need to apply these results to product of functions for which one of the two factors is $|\nabla_H \rho_x|$ and the other one is a smooth function compactly supported in the punctured pseudo-ball $B_X(x,R)\setminus\{x\}$. This is where we use the fact that ρ_x is a Lipschitz continuous function with respect to the CC metric associated with the system $X = \{X_1, \ldots, X_m\}$. As we have seen, this property is a consequence of the estimates in [8], and of the hypothesis 1.6.

We recall the relevant result, which is Theorem 4.2 in [43]. In what follows $d(x, y)$ denotes the CC distance associated with a given system X of Hörmander type . We denote with $\Gamma_d^{0,1}$ the space of functions which are Lipschitz continuous with respect to $d(x, y)$.

Theorem 2.8 *Let $f \in \Gamma^{0,1}_d$, and $g \in L^1(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} g(x)|\nabla_H f(x)|dx = \int_{-\infty}^{\infty}\int_{\{f=t\}} g(x)d\sigma_H(x)dt\,,$$

where σ_H indicates the horizontal perimeter measure on the level set $\{f=t\}$.

We note that with $g \equiv 1$ and $f \in BV_X$ (functions of bounded variation with respect to the system X) Theorem 2.8 was first independently proved in [25] and [21].

Corollary 2.9 *Let $g \in C_0^\infty(B_X(x,R)\setminus\{x\})$, then one has*

$$\int_{B_X(x,R)} g(y)\,|\nabla_H\rho_x(y)|\,dy \;=\; \int_0^R \int_{\partial B_X(x,t)} g(y)\,d\sigma_H\,dt\,.$$

Lemma 2.10 *Let $\phi \in C_0^\infty(B_X(x,R)\setminus\{x\})$, then for a.e. $0 < s < R$ one has*

$$\begin{aligned}\frac{d}{ds}&\int_{\partial B_X(x,s)} |\phi|^p|\nabla_H\rho_x|^{p-1}d\sigma_H\\ &= (p-1)\int_{\partial B_X(x,s)} |\phi|^p\,|\nabla_H\rho_x|^{p-1}\left\{2\,\frac{E'(x,\rho_x)}{E(x,\rho_x)} - \frac{E''(x,\rho_x)}{E'(x,\rho_x)}\right\}d\sigma_H\\ &\quad + p\int_{\partial B_X(x,s)} |\phi|^{p-2}\phi|\nabla_H\rho_x|^{p-3} < \nabla_H\phi, \nabla_H\rho_x > d\sigma_H\,.\end{aligned}$$

Proof Fix a function $\phi \in C_o^\infty(B_X(x,R))\setminus\{x\})$. To simplify the notation we let $\Omega_s = B_X(x,s)$, $0 < s < R$, and set $\rho = \rho_x$. Noting that wherever it exists on $\partial\Omega_s$ the outer horizontal unit normal ν_H is given by $\nu_H = \nabla_H\rho/|\nabla_H\rho|$. The divergence theorem then gives

$$\begin{aligned}\int_{\partial\Omega_s} |\phi|^p\,|\nabla_H\rho|^{p-1}\,d\sigma_H \;&= \int_{\partial\Omega_s} |\phi|^p\,|\nabla_H\rho|^{p-2} < \nabla_H\rho, \nu_H > d\sigma_H\\ &= \int_{\Omega_s} div_H^*\left(|\phi|^p|\nabla_H\rho|^{p-2}\nabla_H\rho\right)\,dy\\ &= \int_{\Omega_s} |\phi|^p\,\Delta_{H,p}\rho\,dx\\ &\quad + \int_{\Omega_s} |\nabla_H\rho|^{p-2} < \nabla_H\rho, \nabla_H(|\phi|^p) > \,dy\,. \qquad (2.19)\end{aligned}$$

Next, substituting the conclusion of Lemma 2.6 in the first integral in the right-hand side of Eq. 2.19, and using again the divergence theorem we can write

$$\begin{aligned}\int_{\partial\Omega_s}|\phi|^p\,|\nabla_H\rho|^{p-1}\,d\sigma_H &= (p-1)\int_0^s\int_{\partial\Omega_\tau}|\phi|^p|\nabla_H\rho|^{p-1}\left\{2\,\frac{E'(x,\rho_x)}{E(x,\rho_x)} - \frac{E''(x,\rho_x)}{E'(x,\rho_x)}\right\}d\sigma_H d\tau\\ &\quad + p\int_0^s\int_{\partial\Omega_\tau} |\phi|^{p-2}\phi|\nabla_H\rho|^{p-3} < \nabla_H\phi, \nabla_H\rho > d\sigma_H d\tau\,.\end{aligned}$$

Differentiating the latter identity, we reach the desired conclusion. □

We are now ready to present the

Proof of Theorem 1.2 Fix a function $\phi \in C_0^\infty(B_X(x,R)) \setminus \{x\})$. From now on, to simplify the notation we let $\Omega_r = B_X(x,r)$, $\rho = \rho_x$, and $E(\rho) = E(x,\rho_x)$. Thanks to Eq. 2.12 there exists $0 < \epsilon < r < R$ such that

$$supp\,\phi \;\subset\; \Omega_r \;\setminus \Omega_\epsilon \;.$$

We consider a non-negative and non-increasing function $f \in C^1(0,\infty)$, to be determined appropriately. One has

$$\begin{aligned}\int_{B_X(x,R)} |\phi|^p(-f'(\rho))|\nabla_H\rho|^p\,dy &= \int_{\Omega_r\setminus\Omega_\epsilon} |\phi|^p(-f'(\rho))|\nabla_H\rho|^p\,dy\\ &= \int_\epsilon^r (-f'(s))\int_{\partial\Omega_s} |\phi|^p\;|\nabla_H\rho|^{p-1}d\sigma_H ds\\ &= \int_\epsilon^r f(s)\,\frac{d}{ds}\int_{\partial\Omega_s} |\phi|^p\;|\nabla_H\rho|^{p-1}d\sigma_H ds\,. \end{aligned} \tag{2.20}$$

We now use Lemma 2.10 to obtain from Eq. 2.20

$$\begin{aligned}&\int_{B_X(x,R)} |\phi|^p(-f'(\rho))|\nabla_H\rho|^p\,dy\\ &\quad= \int_\epsilon^R f(s)(p-1)\int_{\partial\Omega_s} |\phi|^p\;|\nabla_H\rho|^{p-1}\left\{2\,\frac{E'(\rho)}{E(\rho)} - \frac{E''(\rho)}{E'(\rho)}\right\} d\sigma_H ds\\ &\qquad+ p\int_\epsilon^R f(s)\int_{\partial\Omega_s} |\phi|^{p-2}\phi|\nabla_H\rho|^{p-3} < \nabla_H\phi, \nabla_H\rho > d\sigma_H ds\\ &\quad= \int_{B_X(x,R)} |\phi|^p|\nabla_H\rho|^p\;(p-1)\left\{2\,\frac{E'(\rho)}{E(\rho)} - \frac{E''(\rho)}{E'(\rho)}\right\} f(\rho)dy\\ &\qquad+ \int_{B_X(x,R)} p\;f(\rho)|\phi|^{p-2}\phi|\nabla_H\rho|^{p-2} < \nabla_H\phi, \nabla_H\rho > dy\;, \end{aligned} \tag{2.21}$$

where we have used the co-area formula one more time. Applying Hölder inequality we obtain from Eq. 2.21

$$\begin{aligned}&\int_{B_X(x,R)} |\phi|^p|\nabla_H\rho|^p\left\{(p-1)\left\{2\,\frac{E'(\rho)}{E(\rho)} - \frac{E''(\rho)}{E'(\rho)}\right\} f(\rho) + f'(\rho)\right\} dy\\ &\quad\le \int_{B_X(x,R)} p\;f(\rho)|\phi|^{p-1}|\nabla_H\rho|^{p-1}|\nabla_H\phi|dy\\ &\quad\le \left(\int_{B_X(x,R)} (p\;f(\rho))^{\frac{p}{p-1}}|\phi|^p|\nabla_H\rho|^p dy\right)^{\frac{p-1}{p}}\left(\int_{B_X(x,R)} |\nabla_H\phi|^p dy\right)^{\frac{1}{p}}\;. \end{aligned} \tag{2.22}$$

At this point we choose the function $f(\rho)$ in such a way that the integrand in the first factor of the right-hand side of Eq. 2.22 coincides with the integrand in the left-hand side. Precisely, we choose $f(\rho)$ so that it satisfies the differential equation

$$f'(\rho) + (p-1)\left\{2\,\frac{E'(\rho)}{E(\rho)} - \frac{E''(\rho)}{E'(\rho)}\right\} f(\rho) = (p\, f(\rho))^{\frac{p}{p-1}} . \tag{2.23}$$

Remarkably, this is a Bernoulli equation which, via the substitution $y(\rho) = f(\rho)^{-\frac{1}{p-1}}$, can be integrated in an elementary fashion. Keeping in mind that

$$2\,\frac{E'(\rho)}{E(\rho)} - \frac{E''(\rho)}{E'(\rho)} = \frac{d}{d\rho}\left[\log\,\frac{E(\rho)^2}{E'(\rho)}\right] ,$$

we obtain

$$y(\rho) = \frac{p^{\frac{p}{p-1}}}{p-1}\,\frac{E(\rho)}{E'(\rho)} ,$$

and therefore,

$$f(\rho) = \frac{(p-1)^{p-1}}{p^p}\left(\frac{E'(\rho)}{E(\rho)}\right)^{p-1} . \tag{2.24}$$

With this choice of the function $f(\rho)$ we thus obtain

$$f'(\rho) + (p-1)\left\{2\,\frac{E'(\rho)}{E(\rho)} - \frac{E''(\rho)}{E'(\rho)}\right\} f(\rho) = (p\, f(\rho))^{\frac{p}{p-1}} = \left(\frac{p-1}{p}\right)^p \left(\frac{E'(\rho)}{E(\rho)}\right)^p . \tag{2.25}$$

Substituting Eq. 2.25 in Eq. 2.22, we finally reach the desired conclusion. □

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