Chapter 2, problem 19.

(a) If $A$ and $B$ are disjoint closed sets in some metric space $X$, prove that they are separated.

(b) Prove the same for disjoint open set.

(c) Fix $p \in X$, $\delta > 0$, define $A$ to be the set of all $q \in X$ for which $d(p, q) < \delta$, define $B$ similarly, with $>$ in place of $<$. Prove that $A$ and $B$ are separated.

(d) Prove that every connected metric space with at least two points is uncountable. Hint: Use (c).

Solution.

(a) Let $A$ and $B$ are disjoint closed sets in some metric space $X$. We want to prove that they are separated. Since $A$ and $B$ are disjoint and closed, we have

$A \cap B = A \cap B = \emptyset, \quad A \cap \overline{B} = A \cap B = \emptyset.$

Hence by definition 2.45, $A$ and $B$ are separated.

(b) Let $A$ and $B$ are disjoint open sets in some metric space $X$. We want to prove that they are separated. Since $A$ and $B$ are open, the complements of $A$ and $B$, say $A^c$ and $B^c$ are closed. Suppose that $A$ and $B$ are not separated, ie, either $A \cap B$ or $A \cap B^c$ is nonempty. Assume without loss of generality that $A \cap B \neq \emptyset$. Let $x \in A \cap B$. Then, since $B$ is open, there exists a neighborhood $N$ of $x$ such that $N \subseteq B$. But since $x \in A$, $N \cap A \neq \emptyset$. Therefore, since $A \cap N \subseteq A \cap B$, $A \cap B \neq \emptyset$, a contradiction. Hence $A$ and $B$ are separated.

(c) Fix $p \in X$, $\delta > 0$, define

$A = \{q \in X : d(p, q) < \delta\}, \quad B = \{q \in X : d(p, q) > \delta\}.$

First it not hard to see that $A$ and $B$ are open. Indeed, if $q \in A$, then consider the neighborhood

$N_r(q) = \{z \in X : d(q, z) < r\}, \quad 0 < r < \delta - d(p, q).$

Then if $z \in N_r(q)$, we have

$d(p, z) \leq d(p, q) + d(z, q) < d(p, q) + \delta - d(p, q) = \delta.$

So $z \in A$, and since $z \in N_r(q)$ is arbitrary, $N_r(q) \subseteq A$ and $A$ is open.

Similarly one can show that for any $q \in B$, $N_r(q) \subseteq B$ for any $0 < r < d(p, q) - \delta$, using that for any $z \in N_r(q)$

$d(p, z) \geq d(p, q) - d(q, z) > d(p, q) - d(p, q) + \delta = \delta.$

Clearly $A$ and $B$ are disjoint open sets in $X$, so by (b) they are separated.

(d) Let $X$ be a connected metric space with at least two points. We want to show that $X$ can not be countable.
For each \( t \in (0, 1) \), let \( r_t = td(x_1, x_2) \). We have that since \( X \) has at least two points, say \( x_1, x_2 \in X \), then \( d(x_1, x_2) > 0 \), and \( r_t > 0 \). For each \( t \in (0, 1) \) consider
\[
A_t = \{ q \in X | d(x_1, q) < r_t \}, \quad B_t = \{ q \in X | d(x_2, q) > r_t \}.
\]

We have that \( x_1 \in A \) and \( x_2 \in B \), so \( A \) and \( B \) are nonempty open sets. So either \( X = A_t \cup B_t \) or there exists \( x_t \in X \) such that \( d(x_1, x_t) = r_t \). Since \( X \) is connected, \( X \neq A_t \cup B_t \), and for each \( t \in (0, 1) \), there exists \( x_t \in X \) s.t. \( d(x_1, x_t) = r_t \), but this gives an injective correspondence \( f : (0, 1) \to X, f(t) = x_t \). Since \( (0, 1) \) is uncountable, since we have this injective correspondence \( f \), \( X \) is also uncountable.

Q.E.D.

**Chapter 2, problem 22.** A metric space is called separable if it contains a countable dense subset. Show that \( \mathbb{R}^k \) is separable.

Solution.
Let \( Q^k \) be the set of points of \( R^k \) which have only rational coordinates, ie,
\[
Q^k = \mathbb{Q} \times \cdots \times \mathbb{Q} = \{ (q_1, \ldots, q_k) \in \mathbb{R}^k | q_j \in \mathbb{Q}, j = 1, \ldots, k \}.
\]
Clearly \( Q^k \) is countable since it is a finite product of countable sets.
We now want to show that \( Q^k \) is dense in \( \mathbb{R}^k \), ie, for any \( x = (x_1, \ldots, x_k) \in \mathbb{R}^k \) and any \( \epsilon > 0 \), there exists \( q \in Q^k \) such that
\[
|q - x| < \epsilon.
\]
Since \( \mathbb{Q} \) is dense in \( \mathbb{R} \), for each \( j = 1, \ldots, k \), there exists \( q_j \in \mathbb{Q} \) such that
\[
|q_j - x_j| < \frac{\epsilon}{\sqrt{k}}, \quad j = 1, \ldots, k.
\]
Let \( q = (q_1, q_2, \ldots, q_k) \). Then
\[
|q - x|^2 = (q_1 - x_1)^2 + \cdots + (q_k - x_k)^2 < \frac{\epsilon^2}{k} + \cdots + \frac{\epsilon^2}{k} = \epsilon^2.
\]
Hence \( Q^k \) is dense in \( \mathbb{R}^k \) and \( \mathbb{R}^k \) is separable.

**Chapter 2, problem 29.** Prove that every open set in \( \mathbb{R} \) is the union of an at most countable collection of disjoint segments.

Solution.
Let \( O \subset \mathbb{R} \) be open. Assume that \( O \) is nonempty.
For each \( q \in O \cap \mathbb{Q} \), let \( R_q = \{ r > 0 | (q - r, q + r) \subset O \} \). Since \( O \) is open, by what we showed above \( R_q \neq \emptyset \) and if \( r_0 \in R_q \), then \( r \in R_q \) for every \( 0 < r \leq r_0 \).
Note that if \( \cup_{r \in R_q} (q - r, q + r) = \mathbb{R} \), then \( O = \mathbb{R} \), otherwise
\[
r_q = \sup R_q < \infty.
\]
So assume that \( \sup R_q < \infty \), and consider \( r_q = \sup R_q \). We see that
\[
I_q = (q - r_q, q + r_q) = \bigcup_{r \in R_q} (q - r, q + r) \subset O.
\]
We claim that
\[
O = \bigcup_{q \in \mathbb{Q} \cap O} I_q.
\]
Since \( O \cap \mathbb{Q} \subset \mathbb{Q} \), then \( O \cap \mathbb{Q} \) is countable, so the union above is also countable.

Clearly, \( \bigcup_{q \in \mathbb{Q} \cap O} I_q \subset O \).

Now if \( x \in O \), there exists \( \epsilon > 0 \), such that \( (x - \epsilon, x + \epsilon) \subset O \). By the density of \( \mathbb{Q} \) in \( \mathbb{R} \), there exists \( q \in O \cap \mathbb{Q} \) such that \( 0 < q - x < \epsilon/2 \). We see that \( (q - \epsilon/2, q + \epsilon/2) \subset O \). Indeed, if \( z \in (q - \epsilon/2, q + \epsilon/2) \), then
\[
|z - x| \leq |z - q| + |q - x| < \epsilon/2 + \epsilon/2 = \epsilon,
\]
ie, \( z \in (x - \epsilon, x + \epsilon) \subset O \). So \( \epsilon/2 \leq r_q \) by the definition of \( r_q \), and we have that
\[
x \in (q - \epsilon/2, q + \epsilon/2) \subset (q - r_q, q + r_q) = I_q.
\]
Since \( x \in O \) is arbitrary, we have that
\[
O \subset \bigcup_{q \in \mathbb{Q} \cap O} I_q,
\]
and hence \( O = \bigcup_{q \in \mathbb{Q} \cap O} I_q \). Since \( O \cap \mathbb{Q} \) is countable, say \( O \cap \mathbb{Q} = \{ q_1, q_2, \ldots, q_n, \ldots \} \) and \( I_{q_j} = I_j \). Then
\[
O = \bigcup_{j=1}^{\infty} I_j.
\]
Let \( E_n = I_n \setminus \bigcup_{j=1}^{n-1} I_j \), \( n = 2, 3, \ldots, E_1 = I_1 \). We have that
\[
O = \bigcup_{j=1}^{\infty} I_j = \bigcup_{n=1}^{\infty} E_n.
\]
Note that by construction, each \( E_n \) is either a segment, a finite disjoint union of segments or empty, and \( E_n \cap E_m = \emptyset \) if \( n \neq m \). Therefore the equality above proves that \( O \) is the union of an at most countable collection of disjoint segments.

**Chapter 3, problem 1.** Prove that convergence of \( \{ n \} \) implies convergence of \( \{ |s_n| \} \). Is the converse true?

**Solution.**
Suppose that \( s_n \rightarrow s \). We have
\[
||s_n| - |s|| \leq |s_n - s|.
\]
Since \( s_n \rightarrow s \), given \( \epsilon > 0 \), there exists \( N \) such that \( |s_n - s| < \epsilon \) for all \( n \geq N \). By the inequality above we see that \( ||s_n| - |s|| < \epsilon \) for all \( n \geq N \). Since \( \epsilon > 0 \) is arbitrary, we see that \( |s_n| \rightarrow s \).

The converse is not true. Indeed, consider \( s_n = (-1)^n \). Then \( \{ s_n \} \) does NOT converge, but \( |s_n| = 1 \) converge to 1.

**Chapter 3, problem 2.** Calculate \( \lim_{n \to \infty} (\sqrt{n^2 + n} - n) \).
Solution.
We have
\[
\sqrt{n^2 + n} - n = \frac{\sqrt{n^2 + n} - n}{\sqrt{n^2 + n} + n} (\sqrt{n^2 + n} + n) = \frac{n^2 + n - n^2}{\sqrt{n^2 + n} + n} = \frac{n}{\sqrt{n^2 + n} + n}, \quad \forall n > 0.
\]
Now note that
\[
\frac{n}{\sqrt{n^2 + n} + n} = \frac{1}{\sqrt{1 + \frac{1}{n}} + 1} \to \frac{1}{2} \quad \text{as} \quad n \to \infty.
\]
Therefore
\[
\lim_{n \to \infty} (\sqrt{n^2 + n} - n) = \frac{1}{2}.
\]

Chapter 3, problem 3. If \( s_1 = \sqrt{2}, \) and
\[
s_{n+1} = \sqrt{2 + \sqrt{s_n}}, \quad (n = 2, 3, \ldots),
\]
prove that \( \{s_n\} \) converges, and that \( s_n < 2 \) for \( n = 1, 2, 3, \ldots. \)

Solution.
Let \( s \in \mathbb{R} \) be such that \( s = \sqrt{2 + \sqrt{s}}. \) We see that such \( s \) exists since the function
\[
f(s) = s - \sqrt{2 + \sqrt{s}}
\]
is continuous and \( f(4) > 0, \) \( f(\sqrt{2}) < 0, \) so there must be a \( s, \sqrt{2} < s < 4, \) such that \( f(s) = 0. \)
We have that
\[
|s_{n+1} - s| = |\sqrt{2 + \sqrt{s_n}} - \sqrt{2 + \sqrt{s}}| = \frac{|\sqrt{2 + \sqrt{s_n}} - \sqrt{2 + \sqrt{s}}|}{\sqrt{2 + \sqrt{s_n}} + \sqrt{2 + \sqrt{s}}} (\sqrt{2 + \sqrt{s_n}} + \sqrt{2 + \sqrt{s}}) =
\]
\[
\frac{|\sqrt{s_n} - \sqrt{s}|}{\sqrt{2 + \sqrt{s_n}} + \sqrt{2 + \sqrt{s}}} \leq \frac{|\sqrt{s_n} - \sqrt{s}|}{2\sqrt{2}},
\]
The inequality follows form the fact that \( s, s_n > 0, \) so \( \sqrt{2 + \sqrt{s_n}, \sqrt{2 + \sqrt{s}} > \sqrt{2}, \) so \( \sqrt{2 + \sqrt{s_n}} + \sqrt{2 + \sqrt{s}} > 2\sqrt{2} \) and \( \frac{1}{\sqrt{2 + \sqrt{s_n}} + \sqrt{2 + \sqrt{s}}} < \frac{1}{2\sqrt{2}}. \)
We have now that
\[
\frac{|\sqrt{s_n} - \sqrt{s}|}{2\sqrt{2}} = \frac{|\sqrt{s_n} - \sqrt{s}|}{2\sqrt{2}} \frac{\sqrt{s_n} + \sqrt{s}}{\sqrt{s_n} + \sqrt{s}} = \frac{|s_n - s|}{2\sqrt{s_n + \sqrt{s}}(\sqrt{s_n} + \sqrt{s})}.
\]
Now note that, by definition, \( s_n \geq \sqrt{2} \) for all \( n, \) and \( s > \sqrt{2}, \) so
\[
\frac{|s_n - s|}{2\sqrt{2}(\sqrt{s_n} + \sqrt{s})} \leq \frac{|s_n - s|}{2\sqrt{2}(2^{1/4} + 2^{1/4})} \leq \frac{|s_n - s|}{4}.
\]
Therefore we showed
\[
|s_{n+1} - s| \leq \frac{|s_n - s|}{4}.
\]
If we continue this process we have
\[|s_{n+1} - s| \leq \frac{|s_n - s|}{4} \leq \frac{|s_{n-1} - s|}{2^4} \leq \ldots \leq \frac{|s_{n+k} - s|}{2^{2k}}, \quad k \leq n.\]

So for \( k = n \), we have
\[|s_{n+1} - s| \leq \frac{|s_1 - s|}{2^{2n}} = |s_n - s| \leq \frac{\sqrt{2} - s}{2^{2n}} \leq \frac{4}{2^{2n}} = \frac{1}{2^{2n-2}}, \quad n \geq 2.\]

Since \( \frac{1}{2^{2n-2}} \to 0 \), as \( n \to \infty \), we have that \( s_n \to s \).

Since \( \sqrt{x} \leq x \) for any \( x \geq 1 \), and \( \sqrt{x^{1/2}} \leq \sqrt{x} \leq x, x \geq 1 \). Clearly one has that \( s_2 = \sqrt{2 + \sqrt{2}} < \sqrt{2 + 2} = 2 \). Now using induction, assuming \( s_n < 2 \), \( s_{n+1} = \sqrt{2 + \sqrt{s_n}} < \sqrt{2 + 2} = 2 \). Therefore \( s_n < 2 \) for \( n = 1, 2, 3, \ldots \).

**Problem A.** Show that a sequence \( \{p_n\} \) is converging to a point \( p \) if, and only if, every subsequence of \( \{pn\} \) converges to \( p \).

Solution.

Suppose that \( p_n \to p \). Let \( \{p_{n_j}\} \) be a subsequence of \( \{p_n\} \). Since \( p_n \to p \), for every \( \epsilon > 0 \), there exists \( N \) such that
\[|p_n - p| < \epsilon, \quad \forall n \geq N.\]

In particular
\[|p_{n_j} - p| < \epsilon, \quad \forall n_j \geq N.\]

Hence \( \{p_{n_j}\} \) converges to \( p \).

Now assume that \( \{p_n\} \) does not converges to \( p \). Then given \( \epsilon > 0 \), for every \( k \in \mathbb{N} \) there exists \( n_k \geq k \) such that \( |p_{n_k} - p| \geq \epsilon \). Then \( \{p_{n_k}\} \) does not converge to \( p \). Therefore if every subsequence of \( \{pn\} \) converges to \( p \), then \( \{p_n\} \) is converging to a point \( p \).

**Problem B.** Show that a sequence \( \{p_n\} \) is Cauchy if, and only if, \( \text{diam}(E_N) \to 0 \) as \( N \to \infty \) (here, \( E_N = \{p_N, p_{N+1}, \ldots\} \).

Solution.

Let \( E_N = \{p_N, p_{N+1}, \ldots\} \). We have
\[\text{diam}(E_N) = \sup\{|p_n - p_m| : n, m \geq N\}.\]

So if \( \{p_n\} \) is Cauchy, then for every \( \epsilon > 0 \) there exists \( M \) such that
\[|p_n - p_m| < \epsilon, \quad \forall n, m \geq M.\]

This implies that \( \text{diam}(E_N) \leq \epsilon \), for all \( N \geq M \). Since \( \epsilon > 0 \) is arbitrary, we have that \( \text{diam}(E_N) \to 0 \) as \( N \to \infty \).

Now if \( \text{diam}(E_N) \to 0 \) as \( N \to \infty \), then we have for every \( \epsilon > 0 \), there exists \( M \) such that
\[\text{diam}(E_N) = \sup\{|p_n - p_m| : n, m \geq N\} < \epsilon, \quad \forall N \geq M,\]

in particular
\[|p_n - p_m| < \epsilon, \quad \forall n, m \geq M,\]

ie, since \( \epsilon > 0 \) is arbitrary \( \{p_n\} \) is Cauchy.