# HOMEWORK #4 - MA 504

PAULINHO TCHATCHATCHA

**Chapter 2, problem 19.**
(a) If $A$ and $B$ are disjoint closed sets in some metric space $X$, prove that they are separated.
(b) Prove the same for disjoint open set.
(c) Fix $p \in X$, $\delta > 0$, define $A$ to be the set of all $q \in X$ for which $d(p, q) < \delta$, define $B$ similarly, with $>$ in place of $<$. Prove that $A$ and $B$ are separated.
(d) Prove that every connected metric space with at least two points is uncountable. Hint: Use (c).

Solution.
(a) Let $A$ and $B$ are disjoint closed sets in some metric space $X$. We want to prove that they are separated. Since $A$ and $B$ are disjoint and closed, we have

$$\overline{A} \cap B = A \cap B = \emptyset, \quad A \cap \overline{B} = A \cap B = \emptyset.$$

Hence by definition 2.45, $A$ and $B$ are separated.
(b) Let $A$ and $B$ are disjoint open sets in some metric space $X$. We want to prove that they are separated. Since $A$ and $B$ are open, the complements of $A$ and $B$, say $A^c$ and $B^c$ are closed. Suppose that $A$ and $B$ are not separated, ie, either $\overline{A} \cap B$ or $A \cap \overline{B}$ is nonempty. Assume without loss of generality that $\overline{A} \cap B \neq \emptyset$. Let $x \in \overline{A} \cap B$. Then, since $B$ is open, there exists a neighborhood $N$ of $x$ such that $N \subset B$. But since $x \in \overline{A}$, $N \cap A \neq \emptyset$. Therefore, since $A \cap N \subset A \cap B$, $A \cap B \neq \emptyset$, a contradiction. Hence $A$ and $B$ are separated.
(c) Fix $p \in X$, $\delta > 0$, define

$$A = \{q \in X | d(p, q) < \delta\}, \quad B = \{q \in X | d(p, q) > \delta\}.$$

First it not hard to see that $A$ and $B$ are open. Indeed, if $q \in A$, then consider the neighborhood

$$N_r(q) = \{z \in X | d(q, z) < r\}, \quad 0 < r < \delta - d(p, q).$$

Then if $z \in N_r(q)$, we have

$$d(p, z) \leq d(p, q) + d(z, q) < d(p, q) + \delta - d(p, q) = \delta.$$

So $z \in A$, and since $z \in N_r(q)$ is arbitrary, $N_r(q) \subset A$ and $A$ is open.
Similarly one can show that for any $q \in B$, $N_r(q) \subset B$ for any $0 < r < d(p, q) - \delta$, using that for any $z \in N_r(q)$

$$d(p, z) \geq d(p, q) - d(q, z) > d(p, q) - d(p, q) + \delta = \delta.$$

Clearly $A$ and $B$ are disjoint open sets in $X$, so by (b) they are separated.
(d) Let $X$ be a connected metric space with at least two points. We want to show that $X$ can not be countable.

For each $t \in (0,1)$, let $r_t = td(x_1, x_2)$. We have that since $X$ has at least two points, say $x_1, x_2 \in X$, then $d(x_1, x_2) > 0$, and $r_t > 0$. For each $t \in (0,1)$ consider

$$A_t = \{q \in X | d(x_1, q) < r_t\}, \quad B_t = \{q \in X | d(x_2, q) > r_t\}.$$

We have that $x_1 \in A$ and $x_2 \in B$, so $A$ and $B$ are nonempty open sets. So either $X = A_t \cup B_t$ or there exists $x_t \in X$ such that $d(x_1, x_t) = r_t$. Since $X$ is connected, $X \neq A_t \cup B_t$, and for each $t \in (0,1)$, there exists $x_t \in X$ s.t. $d(x_1, x_t) = r_t$, but this gives an injective correspondence $f : (0,1) \to X$, $f(t) = x_t$. Since $(0,1)$ is uncountable, since we have this injective correspondence $f$, $X$ is also uncountable.
Q.E.D.

**Chapter 2, problem 22.** A metric space is called separable if it contains a countable dense subset. Show that $\mathbb{R}^k$ is separable.

Solution.
Let $\mathbb{Q}^k$ be the set of points of $R^k$ which have only rational coordinates, ie,

$$\mathbb{Q}^k = \underbrace{\mathbb{Q} \times \cdots \times \mathbb{Q}}_{k \text{ times}} = \{(q_1, ..., q_k) \in \mathbb{R}^k | q_j \in \mathbb{Q}, j = 1, ..., k\}.$$

Clearly $\mathbb{Q}^k$ is countable since it is a finite product of countable sets.
We now want to show that $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$, ie, for any $x = (x_1, ..., x_k) \in \mathbb{R}^k$ and any $\epsilon > 0$, there exists $q \in \mathbb{Q}^k$ such that

$$|q - x| < \epsilon.$$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, for each $j = 1, ..., k$, there exists $q_j \in \mathbb{Q}$ such that

$$|q_j - x_j| < \frac{\epsilon}{\sqrt{k}}, \quad j = 1, ..., k.$$

Let $q = (q_1, q_2, ..., q_k)$. Then

$$|q - x|^2 = (q_1 - x_1)^2 + \cdots + (q_k - x_k)^2 < \frac{\epsilon^2}{k} + ... + \frac{\epsilon^2}{k} = \epsilon^2.$$

Hence $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$ and $\mathbb{R}^k$ is separable.

**Chapter 2, problem 29.** Prove that every open set in $\mathbb{R}$ is the union of an at most countable collection of disjoint segments.

Solution.
Let $O \subset \mathbb{R}$ be open. Assume that $O$ is nonempty.
For each $q \in O \cap \mathbb{Q}$, let $R_q = \{r > 0 | (q - r, q + r) \subset O\}$. Since $O$ is open, by what we showed above $R_q \neq \emptyset$ and if $r_0 \in R_q$, then $r \in R_q$ for every $0 < r \leq r_0$.
Note that if $\cup_{r \in R_q}(q - r, q + r) = \mathbb{R}$, then $O = \mathbb{R}$, otherwise

$$r_q = \sup R_q < \infty.$$

So assume that $\sup R_q < \infty$, and consider $r_q = \sup R_q$. We see that

$$I_q = (q - r_q, q + r_q) = \cup_{r \in R_q}(q - r, q + r) \subset O.$$

We claim that

$$O = \cup_{q \in O \cap \mathbb{Q}} I_q.$$

Since $O \cap \mathbb{Q} \subset \mathbb{Q}$, then $O \cap \mathbb{Q}$ is countable, so the union above is also countable.
Clearly,

$$\cup_{q \in O \cap \mathbb{Q}} I_q \subset O.$$

Now if $x \in O$, there exists $\epsilon > 0$, such that $(x - \epsilon, x + \epsilon) \subset O$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $q \in O\mathbb{Q}$ such that $0 < x - q < \epsilon/2$. We see that $(q - \epsilon/2, q + \epsilon/2) \subset O$. Indeed, if $z \in (q - \epsilon/2, q + \epsilon/2)$, then

$$|z - x| \leq |z - q| + |q - x| < \epsilon/2 + \epsilon/2 = \epsilon,$$

ie, $z \in (x - \epsilon, x + \epsilon) \subset O$. So $\epsilon/2 \leq r_q$ by the definition of $r_q$, and we have that

$$x \in (q - \epsilon/2, q + \epsilon/2) \subset (q - r_q, q + r_q) = I_q.$$

Since $x \in O$ is arbitrary, we have that

$$O \subset \cup_{q \in O \cap \mathbb{Q}} I_q,$$

and hence $O = \cup_{q \in O \cap \mathbb{Q}} I_q$. Since $O \cap \mathbb{Q}$ is countable, say $O \cap \mathbb{Q} = \{q_1, q_2, ..., q_n, ...\}$ and $I_{q_j} = I_j$. Then

$$O = \cup_{j=1}^{\infty} I_j.$$

Let $E_n = I_n \setminus \cup_{j=1}^{n-1} I_j$, $n = 2, 3, ...$, $E_1 = I_1$. We have that

$$O = \cup_{j=1}^{\infty} I_j = \cup_{n=1}^{\infty} E_n.$$

Note that by construction, each $E_n$ is either a segment, a finite disjoint union of segments or empty, and $E_n \cap E_m =$ if $n \neq m$. Therefore the equility above proves that $O$ is the union of an at most countable collection of disjoint segments.

**Chapter 3, problem 1.** Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true?

Solution.
Suppose that $s_n \to s$. We have

$$||s_n| - |s|| \leq |s_n - s|.$$

Since $s_n \to s$, given $\epsilon > 0$, there exists $N$ such that $|s_n - s| < \epsilon$ for all $n \geq N$. By the inequality above we see that $||s_n| - |s|| < \epsilon$ for all $n \geq N$. Since $\epsilon > 0$ is arbitrary, we see that $|s_n| \to s$.
The converse is not true. Indeed, consider $s_n = (-1)^n$. Then $\{s_n\}$ does NOT converge, but $|s_n| = 1$ converge to 1.

**Chapter 3, problem 2.** Calculate $\lim\limits_{n \to \infty} (\sqrt{n^2 + n} - n)$.

Solution.
We have

$$\sqrt{n^2+n}-n=\frac{\sqrt{n^2+n}-n}{\sqrt{n^2+n}+n}(\sqrt{n^2+n}+n)=\frac{n^2+n-n^2}{\sqrt{n^2+n}+n}=\frac{n}{\sqrt{n^2+n}+n},\quad \forall n>0.$$

Now note that

$$\frac{n}{\sqrt{n^2+n}+n}=\frac{1}{\sqrt{1+\frac{1}{n}}+1}\to\frac{1}{2}\quad\text{as}\quad n\to\infty.$$

Therefore

$$\lim_{n\to\infty}(\sqrt{n^2+n}-n)=\frac{1}{2}.$$

**Chapter 3, problem 3.** If $s_1=\sqrt{2}$, and

$$s_{n+1}=\sqrt{2+\sqrt{s_n}},\quad (n=2,3,...),$$

prove that $\{s_n\}$ coverges, and that $s_n<2$ for $n=1,2,3,....$

Solution.
Let $s\in\mathbb{R}$ be such that $s=\sqrt{2+\sqrt{s}}$. We see that such $s$ exists since the function $f(s)=s-\sqrt{2+\sqrt{s}}$ is continuous and $f(4)>0$, $f(\sqrt{2})<0$, so there must be a $s$, $\sqrt{2}<s<4$, such that $f(s)=0$.
We have that

$$|s_{n+1}-s|=|\sqrt{2+\sqrt{s_n}}-\sqrt{2+\sqrt{s}}|=\frac{|\sqrt{2+\sqrt{s_n}}-\sqrt{2+\sqrt{s}}|}{\sqrt{2+\sqrt{s_n}}+\sqrt{2+\sqrt{s}}}(\sqrt{2+\sqrt{s_n}}+\sqrt{2+\sqrt{s}})=$$

$$=\frac{|\sqrt{s_n}-\sqrt{s}|}{\sqrt{2+\sqrt{s_n}}+\sqrt{2+\sqrt{s}}}\leq\frac{|\sqrt{s_n}-\sqrt{s}|}{2\sqrt{2}},$$

The inequality follows form the fact that $s,s_n>0$, so $\sqrt{2+\sqrt{s_n}},\sqrt{2+\sqrt{s}}>\sqrt{2}$, so $\sqrt{2+\sqrt{s_n}}+\sqrt{2+\sqrt{s}}>2\sqrt{2}$ and $\dfrac{1}{\sqrt{2+\sqrt{s_n}}+\sqrt{2+\sqrt{s}}}<\dfrac{1}{2\sqrt{2}}$.

We have now that

$$\frac{|\sqrt{s_n}-\sqrt{s}|}{2\sqrt{2}}=\frac{|\sqrt{s_n}-\sqrt{s}|}{2\sqrt{2}}\frac{\sqrt{s_n}+\sqrt{s}}{\sqrt{s_n}+\sqrt{s}}=\frac{|s_n-s|}{2\sqrt{2}(\sqrt{s_n}+\sqrt{s})}.$$

Now note that, by definition, $s_n\geq\sqrt{2}$ for all $n$, and $s>\sqrt{2}$, so

$$\frac{|s_n-s|}{2\sqrt{2}(\sqrt{s_n}+\sqrt{s})}\leq\frac{|s_n-s|}{2\sqrt{2}(2^{1/4}+2^{1/4})}\leq\frac{|s_n-s|}{4}.$$

Therefore we showed

$$|s_{n+1}-s|\leq\frac{|s_n-s|}{4}.$$

If we continue this process we have

$$|s_{n+1}-s| \leq \frac{|s_n-s|}{4} \leq \frac{|s_{n-1}-s|}{2^4} \leq \cdots \leq \frac{|s_{n+1-k}-s|}{2^{2k}}, \quad k \leq n.$$

So for $k=n$, we have

$$|s_{n+1}-s| \leq \frac{|s_1-s|}{2^{2n}} = |s_{n+1}-s| \leq \frac{|\sqrt{2}-s|}{2^{2n}} \leq \frac{4}{2^{2n}} = \frac{1}{2^{2n-2}}, \quad n \geq 2.$$

Since $\frac{1}{2^{2n-2}} \to 0$, as $n \to \infty$, we have that $s_n \to s$.

Since $\sqrt{x} \leq x$ for any $x \geq 1$, and $\sqrt{x^{1/2}} \leq \sqrt{x} \leq x, x \geq 1$. Clearly one has that $s_2 = \sqrt{2+\sqrt{2^{1/2}}} < \sqrt{2+2} = 2$. Now using induction, assuming $s_n < 2$, $s_{n+1} = \sqrt{2+\sqrt{s_n}} < \sqrt{2+2} = 2$. Therefore $s_n < 2$ for $n = 1, 2, 3, ...$ .

**Problem A.** Show that a sequence $\{p_n\}$ is converging to a point $p$ if, and only if, every subsequence of $\{pn\}$ converges to $p$.
Solution.
Suppose that $p_n \to p$. Let $\{p_{n_j}\}$ be a subsequence of $\{p_n\}$. Since $p_n \to p$, for every $\epsilon > 0$, there exists $N$ such that

$$|p_n - p| < \epsilon, \quad \forall n \geq N.$$

In particular

$$|p_{n_j} - p| < \epsilon, \quad \forall n_j \geq N.$$

Hence $\{p_{n_j}\}$ coverges to $p$.
Now assume that $\{p_n\}$ does not converges to $p$. Then given $\epsilon > 0$, for every $k \in \mathbb{N}$ there exists $n_k \geq k$ such that $|p_{n_k} - p| \geq \epsilon$. Then $\{p_{n_k}\}$ does not converge to $p$. Therefore if every subsequence of $\{pn\}$ converges to $p$, then $\{p_n\}$ is converging to a point $p$.

**Problem B.** Show that a sequence $\{p_n\}$ is Cauchy if, and only if, $\text{diam}(E_N) \to 0$ as $N \to \infty$ (here, $E_N = \{p_N, p_{N+1}, ...\}$).
Solution.
Let $E_N = \{p_N, p_{N+1}, ...\}$. We have

$$\text{diam}(E_N) = \sup\{|p_n - p_m| : n, m \geq N\}.$$

So if $\{p_n\}$ is Cauchy, then for every $\epsilon > 0$ there exists $M$ such that

$$|p_n - p_m| < \epsilon, \quad \forall n, m \geq M.$$

This implies that $\text{diam}(E_N) \leq \epsilon$, for all $N \geq M$. Since $\epsilon > 0$ is arbitrary, we have that $\text{diam}(E_N) \to 0$ as $N \to \infty$.
Now if $\text{diam}(E_N) \to 0$ as $N \to \infty$, then we have for every $\epsilon > 0$, there exists $M$ such that

$$\text{diam}(E_N) = \sup\{|p_n - p_m| : n, m \geq N\} < \epsilon, \quad \forall N \geq M,$$

in particular

$$|p_n - p_m| < \epsilon, \quad \forall n, m \geq M,$$

ie, since $\epsilon > 0$ is arbitrary $\{p_n\}$ is Cauchy.