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The sub-elliptic obstacle problem: $C^{1,\alpha}$ regularity of the free boundary in Carnot groups of step two

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Abstract

The sub-elliptic obstacle problem arises in various branches of the applied sciences, e.g., in mechanical engineering and robotics, mathematical finance, image reconstruction and neurophysiology. In the recent paper [Donatella Danielli, Nicola Garofalo, Sandro Salsa, Variational inequalities with lack of ellipticity. I. Optimal interior regularity and non-degeneracy of the free boundary, Indiana Univ. Math. J. 52 (2) (2003) 361–398; MR1976081 (2004c:35424)] it was proved that weak solutions to the sub-elliptic obstacle problem in a Carnot group belong to the Folland–Stein (optimal) Lipschitz class $\Gamma^{1,1}_{\mathrm{loc}}$ (the analogue of the well-known $C^{1,1}_{\mathrm{loc}}$ interior local regularity for the classical obstacle problem). However, the regularity of the free boundary remained a challenging open problem. In this paper we prove that, in Carnot groups of step $r = 2$, the free boundary is (Euclidean) $C^{1,\alpha}$ near points satisfying a certain thickness condition. This constitutes the sub-elliptic counterpart of a celebrated result due to Caffarelli [Luis A. Caffarelli, The regularity of free boundaries in higher dimensions, Acta Math. 139 (3–4) (1977) 155–184; MR0454350 (56 #12601)].

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1. Introduction and main results

Introduction

The study of variational inequalities occupies a central position in calculus of variations and in the applied sciences. The basic prototype of such inequalities is represented by the so-called *obstacle problem*, in which, given $f \in H^{-1}(\Omega)$, one seeks a solution u to the minimization problem

$$a(v, v-u) \geqslant \langle f, v-u \rangle \quad \text{for all } v \in \mathbf{K}_\psi, \tag{1.1}$$

where for a given open set $\Omega \subset \mathbf{R}^n$

$$a(u, v) := \int_\Omega \langle A(x)\nabla u, \nabla v \rangle \, dx \tag{1.2}$$

denotes the quadratic form on $H^1(\Omega)$ associated with an assigned uniformly elliptic matrix $A(x) = (a_{ij}(x))$ with entries in $L^\infty(\Omega)$. Given an *obstacle* $\psi \in H^1(\Omega)$, satisfying $\psi \leqslant 0$ on $\partial\Omega$ (in the sense of $H^1(\Omega)$), one defines the convex set $\mathbf{K}_\psi := \{v \in H_0^1(\Omega)\colon\ v \geqslant \psi \text{ on } \Omega \text{ in } H^1(\Omega)\}$.

The problem (1.1) has a unique variational solution u, see [21]. It is well known that under additional regularity assumptions on $A(x)$ such solution possesses optimal interior smoothness properties. For instance, in the isotropic case when $a_{ij}(x) \equiv \delta_{ij}$, then the optimal interior regularity of u is expressed in terms of its membership to $H^{2,\infty}_{\mathrm{loc}}(\Omega)$, or, equivalently, $u \in C^{1,1}_{\mathrm{loc}}(\Omega)$. Once such interior smoothness is obtained, one can study the regularity of the so-called free boundary, i.e., the boundary of the set where u touches the obstacle ψ. In 1977 Kinderlehrer and Nirenberg [20] proved that if one knows a priori that the free boundary is a C^1 manifold, then in fact it is real analytic. In the same year, in his ground-breaking paper [5], Caffarelli proved that the free boundary is locally $C^{1,\alpha}$ in the neighborhood of any point of positive density of the same, thus bridging the gap with [20] and completing the study of the regularity of the free boundary for the classical obstacle problem.

The central objective of this paper is to establish a result similar to Caffarelli's for a class of variational inequalities whose distinctive new feature is that only appropriate families of directions are permissible, resulting in lack of ellipticity for the problem at hand at every point of the ambient space (i.e., lack of the ellipticity of the matrix A in (1.2)). The study of the relevant obstacle problem in such setting leads to challenging new directions and is motivated by problems in mechanical engineering and robotics, mathematical finance, image reconstruction and neurophysiology. For these aspects we refer the reader to the papers [4,11–13,22–28]. Of special interest for us are the cited recent works of Citti and Sarti who have formulated a remarkable model, which the authors call roto-translation space and which is based on the three-dimensional Heisenberg group $\mathbf{H}^1$, of perceptual completion and formation of subjective surfaces inspired by the architecture of the visual cortex of the brain.

Statement of main results

The geometric framework of our work is that of graded, nilpotent Lie groups, also known as *Carnot groups*, for whose definition and main properties we refer the reader to Section 2. In mathematics and in the applied sciences such Lie groups have a long history, which started with the foundational paper by Carathéodory [10] on Carnot thermodynamics. For instance, they play an important role in control theory, in particular robotics and mechanical engineering [11]. The systematic investigation of the analytic and geometric properties of these Lie groups, however, approximately begun only thirty years ago and continues at a sustained pace presently, especially thanks to exciting recent developments in PDE's and geometric measure theory.

Given a Carnot group $\mathbf{G}$, we let $X_1, \ldots, X_m$ denote the left-invariant smooth vector fields on $\mathbf{G}$ associated with a fixed orthonormal basis of the bracket-generating layer of its Lie algebra $\mathfrak{g}$. We denote with ∇_H and Δ_H respectively the left-invariant differential operators defined as in (2.1). We say that a nonnegative function u in an open set $\Omega \subset \mathbf{G}$ is a solution of the *sub-elliptic obstacle problem* if $u \in \mathcal{L}^{1,2}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ and

$$\Delta_H u = \chi_{\{u>0\}} \quad \text{in } \mathcal{D}'(\Omega). \tag{1.3}$$

Such u arises when one minimizes the horizontal energy functional

$$\mathcal{E}(u; \Omega) := \int_\Omega \left\{ |\nabla_H u|^2 + 2u \right\} dg,$$

subject to the constraints

$$u \geqslant 0, \qquad u - \varphi \in \mathcal{L}^{1,2}_0(\Omega),$$

where $\varphi \in \mathcal{L}^{1,2}(\Omega)$ is a nonnegative boundary datum.

To put our results in the proper perspective we recall that in the recent paper [14], the first two named authors and S. Salsa have investigated the optimal interior regularity of the weak solution u of (1.3) in an arbitrary Carnot group. One of the main results there states that such u belongs to the Folland–Stein class $\Gamma^{1,1}_{\mathrm{loc}}$, i.e., for any $\Omega' \Subset \Omega$, and dg-almost everywhere in Ω', one has

$$|X_i X_j u| \leqslant C, \quad i, j = 1, \ldots, m, \tag{1.4}$$

where C depends only on $\|u\|_{L^\infty(\Omega)}$, and on the (CC)-distance $\operatorname{dist}(\Omega', \partial\Omega)$. This regularity is best possible since, as proved by an example in [14], the second horizontal derivatives of u cannot in general be continuous across the free boundary $\partial\{u > 0\} \cap \Omega$. From (1.4) one obtains in particular, thanks to the sub-elliptic Rademacher–Stepanov theorem, that the horizontal derivatives $X_j u$ are Lipschitz continuous with respect to the (CC)-distance, and therefore they vanish continuously on the free boundary. Since the grading assumption on the Lie algebra gives $[V_1, V_1] = V_2$, then (1.4) implies that also the derivatives in the second layer (vertical derivatives) $Y_l u$, $l = 1, \ldots, k$, are locally bounded. However, this basic information is insufficient for the analysis of the regularity properties of the *free boundary* $\partial\{u > 0\} \cap \Omega$. To make further progress into the problem one needs to know that also the vertical derivatives (and therefore, all commutators of horizontal derivatives) are continuous across the free boundary. This is the subject of our first result.

Theorem I *(Continuity of vertical derivatives). Let u be a solution of the obstacle problem* (1.3) *in a Carnot group* $\mathbf{G}$ *of step $r = 2$, then every vertical derivative $Y_l u$, $l = 1, \dots, k$, vanishes continuously on the free boundary $\partial\{u > 0\} \cap \Omega$; i.e., one has for any $g_0 \in \partial\{u > 0\} \cap \Omega$*

$$\lim_{\substack{g \to g_0 \\ g \in \{u > 0\}}} Y_l u(g) = 0.$$

Once the key issue of the continuity of the second layer derivatives has been settled, we turn to the regularity properties of the free boundary. Our main result in this paper states the free boundary $\partial\{u > 0\} \cap \Omega$ is $C^{1,\alpha}$ regular (with respect to the Euclidean metric) near points where the coincidence set $\{u = 0\}$ is sufficiently thick. We measure the thickness in the terms of the quantity introduced in the following definition.

Definition 1.1 *(Thickness function).* Given a set $E \subset \mathbf{G}$, let

$$\delta_r(g, E) := \frac{1}{r} \min \operatorname{diam}_g \bigl(E \cap H_g \cap B_r(g)\bigr),$$

where, we recall, $H_g = L_g(\exp(V_1))$ is the horizontal plane through the point g, spanned by $X_j(g)$, $j = 1, \dots, m$. For a set $S \subset H_g$, the *minimal diameter* $\min \operatorname{diam}_g(S)$ is defined as the Euclidean minimal diameter in the Lie algebra $\mathfrak{g}$ of the set $\Sigma = \exp^{-1}(g^{-1} S) \subset V_1 \cong \mathbf{R}^m$, which is the minimum distance between a pair of parallel hyperplanes in V_1 that contain the set Σ in the strip between them.

Theorem II *($C^{1,\alpha}$ regularity of the free boundary). Let u be a solution of the obstacle problem* (1.3) *in a Carnot group* $\mathbf{G}$ *of step $r = 2$. For every $\Omega' \Subset \Omega$ there exists a modulus of continuity $\sigma(r)$ depending only on $M = \|u\|_{L^\infty(\Omega)}$, $d = \operatorname{dist}(\Omega', \partial\Omega)$ and* $\mathbf{G}$*, such that if for some $g_0 \in \partial\{u > 0\} \cap \Omega'$, and $r > 0$,*

$$\delta_r\bigl(g_0, \{u = 0\}\bigr) > \sigma(r),$$

then

$$\partial\{u > 0\} \cap B_{r/2}(g_0)$$

is a Euclidean $C^{1,\alpha}$ hypersurface. More specifically, there exists $\kappa = \kappa(r, M, d, \mathbf{G}) > 0$ such that for every $g \in \partial\{u > 0\} \cap B_{r/2}(g_0)$, after a possible rotation of horizontal coordinate axes, we have the representation

$$\bigl(g^{-1} \cdot \partial\{u > 0\}\bigr) \cap B_{\kappa r} = \bigl\{(x, y)\colon x_m = f(x', y)\bigr\} \cap B_{\kappa r},$$

for a $C^{1,\alpha}$ function $f(x', y)$, with $\|\nabla_{x',y} f\|_{C^\alpha} \leqslant C(r, M, d, \mathbf{G})$.

By a *modulus of continuity* in the theorem above we understand a nondecreasing function $\sigma : (0, \infty) \to [0, \infty]$, such that $\sigma(0+) = 0$.

Remark 1.2. In fact, one can relax the thickness condition in Theorem II by replacing the function $\delta_r(g_0, \{u=0\})$ with

$$\delta_r^*\bigl(g_0, \{u=0\}\bigr) := \sup_{g \in B_{r/2}(g_0)} \delta_r\bigl(g, \{u=0\}\bigr).$$

The conclusion will remain the same, perhaps with different constants and modulus of continuity.

Theorem II generalizes a similar result in the classical obstacle problem, see e.g. [6]. It is also similar in the spirit to a free boundary regularity result in the Stefan problem [5]. In particular, the thickness of the free boundary in the Stefan problem is measured in terms of the density of the spatial sections of the zero set $\{u=0\}$, while in our case it is measured in terms of "horizontal sections."

Outline of the paper

As we mentioned earlier, Section 2 contains some preliminary material on Carnot groups, with special emphasis on groups of step $r=2$, which constitute the relevant geometric framework.

The core of the paper really starts with Section 3, where we prove Theorem I, which states that in a Carnot group of step $r=2$, similarly to the first-layer derivatives, in fact, also the second-layer derivatives vanish continuously on the free boundary $\partial\{u>0\} \cap \Omega$. This crucial information opens the way to the study of the regularity of the free boundary.

In Section 4 we analyze the so-called global solutions to the obstacle problem. It is important to observe here that such solutions appear naturally in the blow-up of local solutions in the proof of Theorem I. In Theorem 4.2 we prove that, remarkably, the global solutions to the sub-elliptic obstacle problem, having quadratic growth at infinity (with respect to the (CC)-distance), do not depend on the variables in the second layer. Thanks to the Baker–Campbell–Hausdorff formula (2.4), this information implies that they must be global solutions to the classical obstacle problem in the horizontal variables. This crucial fact allows us to implement the whole arsenal developed by Caffarelli on global solutions for the classical obstacle problem.

In Section 5 we collect some remarks on differentiation along right-invariant vector fields. The key observation, here, is that any right-invariant derivative commutes with any left-invariant derivative. Thus, in particular, unlike what happens with left-invariant differentiation, any right-invariant first-layer derivative of a solution of the sub-Laplacian Δ_H defined in (2.1), is again a solution. Furthermore, since right-derivatives differ from left-derivatives by combinations of derivatives along the second layer directions, thanks to Theorem I also the right-invariant derivatives of a local solution to the obstacle problem are continuous across the free boundary, and thereby vanish on it. This fact has far reaching consequences in the remaining part of the paper. To the best of our knowledge this is the first instance in which right-invariant derivatives are used in a systematic way in the study of a sub-elliptic problem, and we feel that the ideas developed here will have applications in several other directions as well.

Using these properties of the right-invariant derivatives, in combination with the results from the previous sections, in Section 6 we prove that, under the density assumption, stated in Theorem II, the free boundary is a Lipschitz continuous graph (in the classical sense) with respect to the (non-characteristic) horizontal directions, see Theorem 6.3.

In Section 7 we collect some results about the so-called NTA (non-tangentially accessible) domains which play an important role in the proof of our main result. Namely, we use that the ratio of two nonnegative Δ_H-harmonic functions in a domain in $\Omega \subset \mathbf{G}$, vanishing continuously on a relatively open portion of $\partial\Omega$, is Hölder continuous up to that portion of the boundary, see Theorem 7.4. In the Euclidean setting, this property is very well known for the so-called NTA domains since the work of Jerison and Kenig [18] and uses Jones's localization theorem [19]. In the case of Carnot groups, no such localization theorem is known, and we need to compensate by imposing a local NTA condition, see Definition 7.1. This turns out to be sufficient for our purposes, since the epigraphs of Lipschitz functions in horizontal directions are locally NTA, see Theorem 7.6.

Finally, the proof of the $C^{1,\alpha}$-regularity of the free boundary is given in Section 8, by adapting the method originally due to Athanasopoulos and Caffarelli [1].

The essential novelty of our work consists in the successful treatment of the geometrically significant case of Carnot groups of step $r = 2$. It is important to clarify here that, although our results constitute a fundamental step in the study of the regularity of the free boundary for the sub-elliptic obstacle problem, they by no means exhaust it since we do not investigate here the counterpart of the cited $C^{1,\alpha} \Rightarrow C^{\omega}$ result of Kinderlehrer and Nirenberg [20]. The latter at the moment remains a challenging open question, to which we hope to come back in a future study. We believe that the ideas and methods developed here can be applied to more general variational inequalities and free boundary problems governed by sub-elliptic operators; one should thereby think of (1.3) as a basic model case.

2. Notation and preliminaries

In this section we collect some definitions and facts concerning Carnot groups, mainly with the purpose of fixing the notation.

To introduce the relevant geometric set-up we recall that a simply-connected (real) Lie group $\mathbf{G}$ is called a *Carnot group of step* $r \geqslant 1$ if its Lie algebra $\mathfrak{g}$ is stratified and r-nilpotent, i.e., $\mathfrak{g} = V_1 \oplus \cdots \oplus V_r$, $[V_1, V_j] = V_{j+1}$ for $r = 1, \ldots, r-1$, and $[V_1, V_r] = \{0\}$. A trivial (Abelian) example is when $r = 1$, in which case we can identify $\mathbf{G} \cong \mathfrak{g} \cong \mathbb{R}^m$. We will not, however, consider such case since in this setting the cited works [20] and [5] provide a complete answer. We will thus focus on non-Abelian groups. Given a Carnot group of step $r \geqslant 2$ we assume that $\mathfrak{g}$ is endowed with an inner product with respect to which the linear spaces $V_1, \ldots, V_r$ are mutually orthogonal. We fix an orthonormal basis $\{e_1, \ldots, e_m\}$ of the bracket generating layer V_1, an orthonormal basis $\{\varepsilon_1, \ldots, \varepsilon_k\}$ of the second layer V_2, etc. We denote by $\{X_1, \ldots, X_m\}$, $\{Y_1, \ldots, Y_k\}$, $\ldots$, the corresponding left-invariant vector fields in $\mathbf{G}$ defined by $X_j(g) := (L_g)_*(e_j)$, $j = 1, \ldots, m$, $Y_l(g) := (L_g)_*(\varepsilon_l)$, $l = 1, \ldots, k$, etc. Here, $L_g : \mathbf{G} \to \mathbf{G}$ denotes the operator of left-translation $L_g(g') = gg'$, and $(L_g)_*$ indicates its differential. The vector fields $X_1, \ldots, X_m$ form a basis of a subbundle (the *horizontal bundle*) of the tangent bundle $H\mathbf{G} \subset T\mathbf{G}$. We note explicitly that the fibers of $H\mathbf{G}$ are given by $H_g = L_g(\exp V_1)$ and that, given a function u on $\mathbf{G}$, one has

$$X_j u(g) = \lim_{t \to 0} \frac{u(g \exp(t e_j)) - u(g)}{t}, \quad j = 1, \ldots, m.$$

The left-invariant partial differential operators whose action on a function u is given by

$$\nabla_H u := \sum_{j=1}^{m} X_j u X_j, \qquad \Delta_H u := \sum_{j=1}^{m} X_j^2 u, \tag{2.1}$$

are respectively called the *sub-gradient* and the *sub-Laplacian* on $\mathbf{G}$ associated with the basis $\{e_1, \dots, e_m\}$. When the step of $\mathbf{G}$ is $r = 1$, then, as observed above, we can identify $\mathbf{G}$ with $\mathbf{R}^m$, and if e_j are the elements of the standard basis of $\mathbf{R}^m$, then $X_j = \partial_{x_j}$ and thereby ∇_H and Δ_H are just the ordinary gradient and Laplacian. However, as we have said we are only interested in the situation when $\mathbf{G}$ is non-Abelian, i.e., when $r > 1$. In such case the operator Δ_H fails to be elliptic at every point, yet it is hypoelliptic thanks to the grading assumption on $\mathfrak{g}$ and to Hörmander's well-known hypoellipticity theorem [17]. This fact will play a pervasive role in the present work.

Since our results are confined to groups of step $r = 2$, hereafter, we will focus the attention to this setting. For a more detailed discussion of those properties of general Carnot groups which are more closely connected to our work, we refer the reader to [14]. A Carnot group of step $r = 2$ is a simply connected Lie group $\mathbf{G}$ whose Lie algebra $\mathfrak{g}$ admits a nilpotent stratification of step 2, i.e., there exist linear subspaces V_1, V_2 such that

$$\mathfrak{g} = V_1 \oplus V_2, \qquad [V_1, V_1] = V_2, \qquad [V_1, V_2] = \{0\}. \tag{2.2}$$

We often call V_1 the *horizontal layer* and V_2 the *vertical layer* of $\mathfrak{g}$.

Perhaps the most important prototype of Carnot group (of step $r = 2$) is the *Heisenberg group* $\mathbf{H}^n$, with underlying manifold $\mathbf{R}^{2n+1}$, and non-Abelian group law given by

$$(x, y, t) \circ (x', y', t') := \left(x + x', y + y', t + t' + \frac{1}{2}\big(\langle x, y'\rangle - \langle x', y\rangle\big)\right).$$

With $g = (x, y, t)$, a basis for the Heisenberg algebra of left-invariant vector fields on $\mathbf{H}^n$ is given by

$$\begin{aligned} X_i(g) &:= (L_g)_*(e_i) = \partial_{x_i} - \frac{y_i}{2}\,\partial_t, \\ X_{n+i}(g) &:= (L_g)_*(e_{n+i}) = \partial_{y_i} + \frac{x_i}{2}\partial_t, \\ T(g) &:= (L_g)_*(e_{2n+1}) = \partial_t, \end{aligned} \tag{2.3}$$

where $i = 1, \dots, n$. We note that the grading assumption for the Lie algebra is fulfilled with $r = 2$. In fact, if we denote with $\mathfrak{h}^n$ the Heisenberg algebra one has $\mathfrak{h}^n = \mathbb{R}^{2n+1} = V_1 \oplus V_2$, with $V_1 = \mathbb{R}^{2n} \times \{0\}$, $V_2 = \{0\}_{\mathbb{R}^{2n}} \times \mathbb{R}$, and since $[X_i, X_{n+j}] = T\delta_{ij}$, one has $[V_1, V_1] = V_2$.

Returning to the setting of a Carnot group of step $r = 2$, we will assume throughout that its Lie algebra $\mathfrak{g}$ is endowed with a scalar product $\langle\cdot,\cdot\rangle$ with respect to which V_1 and V_2 are mutually orthogonal. With $m = \dim V_1$ and $k = \dim V_2$, we fix orthonormal bases $\{e_1, \dots, e_m\}$ of V_1 and $\{\varepsilon_1, \dots, \varepsilon_k\}$ of V_2. The exponential map $\exp : \mathfrak{g} \to \mathbf{G}$ provides a global analytic diffeomorphism [29]. Using such global coordinate chart we will routinely identify a point $g \in \mathbf{G}$ with its expression in the exponential coordinates

$$\big(x(g), y(g)\big) = \big(x_1(g), \dots, x_m(g), y_1(g), \dots, y_k(g)\big),$$

where we have defined $x_j : \mathbf{G} \to \mathbf{R}$, $y_l : \mathbf{G} \to \mathbf{R}$ by

$$x_j(g) := \big\langle \exp^{-1}(g), e_j\big\rangle, \qquad y_l(g) := \big\langle \exp^{-1}(g), \varepsilon_l\big\rangle.$$

We recall the important *Baker–Campbell–Hausdorff formula* [29], which for groups of step $r=2$ reads

$$\exp\xi\cdot\exp\eta=\exp\left(\xi+\eta+\frac{1}{2}[\xi,\eta]\right),\quad \xi,\eta\in\mathfrak{g}. \tag{2.4}$$

Using (2.4) one can express the group multiplication law in exponential coordinates

$$(x,y)\cdot(\tilde{x},\tilde{y})=\left(x+\tilde{x},y+\tilde{y}+\frac{1}{2}[x,\tilde{x}]\right), \tag{2.5}$$

where

$$[x,\tilde{x}]:=\sum_{l=1}^{k}\left(\sum_{i,j}b_{ij}^{l}x_i\tilde{x}_j\right)\varepsilon_l.$$

Here, we have indicated by b_{ij}^l the so-called *group constants*, defined by

$$b_{ij}^{l}:=\langle[e_i,e_j],\varepsilon_l\rangle,$$

so that $[e_i,e_j]=\sum_{l=1}^{k}b_{ij}^{l}\varepsilon_l$. Each element $\zeta\in\mathfrak{g}$ can be identified with the left-invariant vector field Z on $\mathbf{G}$ whose action on a function u is specified by

$$Zu(g):=\lim_{t\to 0}\frac{u(g\exp(t\zeta))-u(g)}{t}=\frac{d}{dt}u\big(g\exp(t\zeta)\big)\Big|_{t=0}. \tag{2.6}$$

We will always indicate with $\{X_1,\dots,X_m\}$ and $\{Y_1,\dots,Y_k\}$ the left-invariant vector fields on $\mathbf{G}$ defined by (2.6) corresponding to the orthonormal basis $\{e_1,\dots,e_m\}$, $\{\varepsilon_1,\dots,\varepsilon_k\}$ of V_1 and V_2, respectively. Using (2.4) one readily verifies that in the exponential coordinates

$$X_ju=\partial_{x_j}u+\frac{1}{2}\sum_{l=1}^{k}\left(\sum_{i=1}^{m}b_{ij}^{l}x_i\right)\partial_{y_l}u, \tag{2.7}$$

and

$$Y_lu=\partial_{y_l}u. \tag{2.8}$$

Every Carnot group $\mathbf{G}$ of step 2 is naturally equipped with a family of *non-isotropic dilations* $\delta_\lambda:\mathbf{G}\to\mathbf{G}$ given in the exponential coordinates by

$$\delta_\lambda g=\delta_\lambda(x,y):=\big(\lambda x,\lambda^2 y\big). \tag{2.9}$$

The (CC)-*distance* (Carnot–Carathéodory or control distance) $d(g,g')$ associated with the system $\{X_1,\dots,X_m\}$ is defined by minimizing over the time that it takes to move from one point to another along curves whose tangent vector is forced to lie in the span of $\{X_1,\dots,X_m\}$. For the general properties of such distance we refer the reader to [2]. We will denote by

$$B_R(g):=\big\{g'\in\mathbf{G}\colon\ d(g,g')<R\big\} \tag{2.10}$$

the (CC)-*ball* of radius R centered at g. We simply write $d(g) = d(g, e)$, and B_R for $B_R(e)$, where $e = (0, 0)$ is the group identity in $\mathbf{G}$. One can easily prove that for every $g, g', g'' \in \mathbf{G}$, and every $\lambda > 0$,

$$d(gg', gg'') = d(g', g''), \qquad d(\delta_\lambda g, \delta_\lambda g') = \lambda d(g, g'). \tag{2.11}$$

Observe that (2.11) immediately implies that $B_R(g) = g\delta_R(B_1)$, i.e., all (CC)-balls are obtained from B_1 by dilation and left-translation. We also use the notation

$$\operatorname{dist}(g, E) := \inf_{g' \in E} d(g, g')$$

for the (CC)-distance from a point g to a set E in $\mathbf{G}$.

It is also helpful for the reader to keep in mind that the (CC)-distance is equivalent to the so-called (more easily computable) *gauge pseudo-distance* defined by

$$\rho(g, g') := \left|g^{-1}g'\right|_{\mathbf{G}}, \tag{2.12}$$

where the gauge (or pseudo-norm), is defined by

$$|g|_{\mathbf{G}} := \left(|x|^4 + |y|^2\right)^{1/4}. \tag{2.13}$$

In order to distinguish the Euclidean balls in the exponential coordinate spaces from the (CC)-balls, we use superscript labels x and y:

$$B_r^x(x_0) := \left\{x \in \mathbf{R}^m \colon |x - x_0| < r\right\},$$
$$B_r^y(y_0) := \left\{y \in \mathbf{R}^k \colon |y - y_0| < r\right\}.$$

As usual, we omit x_0 and y_0 when these points coincide with the origin in their respective spaces. We will adopt the same superscript convention for the following cylindrical domains which will be used in Section 4 and thereafter: given $0 \leqslant s \leqslant 1$ and $h > 0$, we define

$$\mathcal{K}^x(r, s, h) := \left\{x = (x', x_m) \in \mathbf{R}^m \colon |x'| < r, -s < x_m < h\right\}. \tag{2.14}$$

We denote by dg the bi-invariant Haar measure on $\mathbf{G}$ obtained by pushing forward the standard Lebesgue measure on $\mathfrak{g}$ via the exponential map. Notice that such measure scales according to the formula $d(\delta_\lambda g) = \lambda^Q dg$, where $Q = m + 2k$ is the homogeneous dimension of the group $\mathbf{G}$.

The following functional spaces appear in this paper. Given an open set $\Omega \subset \mathbf{G}$, the horizontal Sobolev space $\mathcal{L}^{1,p}(\Omega)$, $1 \leqslant p \leqslant \infty$ is the Banach space of all functions $u \in L^p(\Omega)$ for which $X_j u \in L^p(\Omega)$, $j = 1, \ldots, m$, where the derivatives $X_j u$ are understood in the distributional sense. The norm in this space is given by

$$\|u\|_{\mathcal{L}^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \sum_{j=1}^{m} \|X_j u\|_{L^p(\Omega)}.$$

The anisotropic Hölder space $\Gamma^{0,\alpha}(\Omega)$, $0 < \alpha \leqslant 1$, is the Banach space of all functions $u \in L^\infty(\Omega)$ such that

$$\left|u(g) - u(g')\right| \leqslant L_\alpha d(g, g')^\alpha, \quad g, g' \in \Omega,$$

for some constant $L_\alpha > 0$. The norm is given by

$$\|u\|_{\Gamma^{0,\alpha}(\Omega)} := \|u\|_{L^\infty(\Omega)} + \sup_{g,g' \in \Omega} \frac{|u(g) - u(g')|}{d(g, g')^\alpha}.$$

The space $\Gamma^{1,\alpha}(\Omega)$ consists of $u \in \Gamma^{0,1}(\Omega)$, for which $X_j u$ exists for every $j = 1, \ldots, m$ and $X_j u \in \Gamma^{0,\alpha}$, and a similar definition is given for the space $\Gamma^{k,\alpha}(\Omega)$, when $k \geqslant 2$.

3. Continuity of vertical derivatives

The objective of this section is to establish Theorem I. The proof of such result will be a direct consequence of Lemma 3.4, which is in fact the main result of the section. We start with a definition of an appropriate class of local solutions of the obstacle problem.

Definition 3.1 *(Local solutions).* Let $\mathbf{G}$ be a Carnot group. Given $g_0 \in \mathbf{G}$, $M, r > 0$, we say that $u \in P_r(g_0, M)$ if $u \geqslant 0$, $u \in \mathcal{L}^{1,2}(B_r(g_0))$ and

(i) $\Delta_H u = \chi_{\{u>0\}}$ in $B_r(g_0)$;
(ii) $g_0 \in \partial\{u > 0\}$;
(iii) $|u| \leqslant M$ in $B_r(g_0)$.

Observe the following elementary translation and rescaling properties of the classes P_r: if $u \in P_r(g_0, M)$ then

$$u \circ L_{g_0} \in P_r(e, M), \qquad \frac{u \circ L_{g_0} \circ \delta_r}{r^2} \in P_1\big(e, M/r^2\big),$$

where $L_{g_0} : g \mapsto g_0 g$ is the left translation operation. In the sequel we will denote the class $P_r(e, M)$ also by $P_r(M)$.

Two basic properties of local solutions are the $\Gamma^{1,1}$-continuity and the non-degeneracy property proved in [14]. Here are the relevant statements.

Lemma 3.2. *Let $\mathbf{G}$ be a Carnot group, then there exists $C = C(\mathbf{G}) > 0$ such that if $u \in P_1(M)$ one has $u \in \Gamma^{1,1}(B_{1/2})$. As a consequence one has dg-almost everywhere in $B_{1/2}$,*

$$|X_i X_j u| \leqslant CM, \quad i, j = 1, \ldots, m.$$

In particular,

$$\left|X_i u(g)\right| \leqslant CMd(g), \quad 0 \leqslant u(g) \leqslant CMd(g)^2 \quad \textit{in } B_{1/2}.$$

Lemma 3.3. *Let* $\mathbf{G}$ *be a Carnot group, then there exists* $C = C(\mathbf{G}) > 0$ *such that if* $u \in P_1(M)$ *one has for every* $g_0 \in B_{1/2} \cap \overline{\{u > 0\}}$

$$\max_{B_r(g_0)} u \geqslant Cr^2, \quad 0 \leqslant r \leqslant 1/2.$$

The previous two results are surprisingly analogous, at least formally, to their classical counterparts (see e.g. [6]), except that Lemma 3.2 involves only control of the horizontal derivatives, and in Lemma 3.3 the non-degeneracy is measured with respect to the twisted geometry of the (CC)-balls. However, we immediately encounter the following difficulty in our case. Because of the grading assumption (2.2) on the Lie algebra, Lemma 3.2 implies that

$$|Y_l u| \leqslant CM \quad \text{in } B_{1/2},\ l = 1, \ldots, k, \tag{3.1}$$

for the derivatives along the vertical layer V_2 of $\mathfrak{g}$. Yet, we do not know whether $Y_l u$ is continuous, and therefore, in particular, if it vanishes on the free boundary. The positive answer to this crucial question is provided by our Theorem I (see Section 1) which we intend to prove in this section.

The proof of Theorem I will be an easy consequence of the following Lemma 3.4, which is really the key result of this section.

Lemma 3.4. *Let* $\mathbf{G}$ *be a Carnot group of step* $r = 2$. *Given* $M > 0$ *and* $\varepsilon > 0$ *there exists* $\rho = \rho(\varepsilon, M, \mathbf{G}) > 0$ *sufficiently small, such that for every* $u \in P_1(M)$ *one has for every* $l = 1, \ldots, k$

$$|Y_l u| < \varepsilon \quad \text{in } B_\rho.$$

Proof. Without loss of generality we prove the lemma only for $Y_k u$. Let

$$M_k := \lim_{\rho \to 0} \sup\bigl\{Y_k u(g)\colon\ u \in P_1(M),\ g \in B_\rho \cap \{u > 0\}\bigr\},$$

$$m_k := \lim_{\rho \to 0} \inf\bigl\{Y_k u(g)\colon\ u \in P_1(M),\ g \in B_\rho \cap \{u > 0\}\bigr\}.$$

Observe that, thanks to the basic information (3.1), we know that

$$-\infty < m_k \leqslant M_k < \infty.$$

Our goal is to establish the following

Claim. $m_k = M_k = 0$.

To prove the claim it will suffice to show that $M_k \leqslant 0$ and that $m_k \geqslant 0$. We prove the former statement in two steps.

Step 1: Blow-up. Assume by the contrary that $M_k > 0$, and let u_n and $g_n \in B_{1/2n} \cap \{u_n > 0\}$, $n = 1, 2, \ldots$, be maximizing sequences such that

$$\lim_{n \to \infty} Y_k u_n(g_n) = M_k. \tag{3.2}$$

We consider the functions

$$v_n(g) := \frac{u_n(g_n\delta_{d_n}(g))}{d_n^2},$$

where

$$d_n := \operatorname{dist}\big(g_n, \partial\{u_n > 0\} \cap B_1\big) = \inf_{g\in\partial\{u_n>0\}\cap B_1} d(g_n, g).$$

Since $g_n \in B_{1/2}$, we have that $B_{1/2}(g_n) \subset B_1$, and therefore v_n are defined in $B_{1/2d_n}$ and satisfy

$$\Delta_H v_n = \chi_{\{v_n>0\}} \quad \text{in } B_{1/2d_n}. \tag{3.3}$$

Moreover, since when $g \in B_1$ one has $g_n\delta_{d_n}(g) \in B_{d_n}(g_n)$, and since by the choice of d_n this latter set is contained in $\partial\{u_n > 0\} \cap B_1$, we have

$$\Delta_H v_n = 1 \quad \text{in } B_1, \tag{3.4}$$

and there exists at least one point $h_n \in \partial B_1 \cap \partial\{v_n > 0\}$. Then, from Lemma 3.2, we have the uniform estimates

$$\begin{gathered}
0 \leqslant v_n(g) \leqslant CM\big(1 + d(g)^2\big),\\
\big|X_j v_n(g)\big| \leqslant CM\big(1 + d(g)\big), \quad j = 1, \ldots, m,\\
\big|Y_l v_n(g)\big| \leqslant CM, \quad l = 1, \ldots, k,
\end{gathered}$$

for any g such that $d(g) \leqslant 1/4d_n$, with n sufficiently large. Hence, we can extract a subsequence, still denoted by v_n, converging uniformly to a globally defined function $v \in \Gamma^{1,1}(\mathbf{G})$ which satisfies

$$\Delta_H v = \chi_{\{v>0\}} \quad \text{in } \mathbf{G}, \tag{3.5}$$

and which has at most quadratic growth at infinity

$$0 \leqslant v(g) \leqslant CM\big(1 + d(g)^2\big). \tag{3.6}$$

Next, we claim that

$$Y_k v \leqslant M_k \quad \text{in } \{v > 0\}, \tag{3.7}$$

and that

$$Y_k v(e) = M_k. \tag{3.8}$$

Indeed, for any $g \in \{v > 0\}$, there exists a ball $B_\rho(g)$ such that $v \geqslant \delta > 0$ in $\overline{B_\rho(g)}$, so for large n we can assume $v_n > 0$ in $B_\rho(g)$. In particular, $\Delta_H v_n = 1$ in $B_\rho(g)$ and, by the sub-elliptic estimates in [17], we can assume that the convergence $v_n \to v$ in $B_\rho(g)$ is not only uniform, but also that all derivatives of v_n converge uniformly to the corresponding derivatives of v. Hence

$$Y_k v(g) = \lim_{n\to\infty} Y_k v_n(g) = \lim_{n\to\infty} Y_k u_n\big(g_n \delta_{d_n}(g)\big) \leqslant M_k.$$

This proves (3.7). Now, noticing that by the construction $\Delta_H v_n = 1$ in B_1, we also have

$$\Delta_H v = 1 \quad \text{in } B_1, \tag{3.9}$$

so repeating the argument above, we can assume that $Y_k v_n$ converges locally uniformly to $Y_k v$ in B_1 and hence

$$Y_k v(e) = \lim_{n\to\infty} Y_k v_n(e) = \lim_{n\to\infty} Y_k u_n(g_n) = M_k,$$

by (3.2). This proves (3.8).

Step 2: Contradiction. We notice next that in a Carnot group of step $r = 2$ the operators Δ_H and Y_k commute, we thus obtain from (3.9) that

$$\Delta_H(Y_k v) = 0 \quad \text{in } B_1.$$

In view of (3.7)–(3.8) we thus infer that $Y_k v$ is a Δ_H-harmonic function in B_1 having a local maximum at e. (Note: even though we do not know that $v > 0$ in B_1, we do know that $v(e) > 0$, otherwise we would have $M_k = Y_k v(e) = 0$.) But then

$$Y_k v = M_k \quad \text{in } B_1, \tag{3.10}$$

by Bony's strong maximum principle [3]. Moreover, observe that by the same argument $Y_k v = M_k$ not only in B_1, but also in the whole connected component Ω_0 of the set $\{v > 0\}$ containing the group identity e.

At this point, we claim that (3.10) implies that

$$X_j v = 0 \quad \text{in } B_1. \tag{3.11}$$

Assuming for a moment that (3.11) is valid, we would obtain that

$$\Delta_H v = \sum_{j=1}^m X_j(X_j v) = 0 \quad \text{in } B_1,$$

a contradiction with (3.9). Therefore to complete the proof of the fact that $M_k \leqslant 0$, we need to establish (3.11).

To this end, using the exponential coordinates, we fix $\bar g = (\bar x, \bar y) \in B_1$ and define

$$\gamma(t) := (\bar x, \bar y', \bar y_k - t).$$

Let also

$$t^* = t^*(\bar x, \bar y) := \sup\big\{t > 0\colon\ \gamma\big([0,t]\big) \subset \Omega_0\big\}.$$

We claim that t^* is finite. Indeed, since $Y_k v = M_k$ along the segment $\gamma([0,t])$, for any $t < t^*$ we have that

$$0 \leqslant v\big(\gamma(t)\big) = v(\bar{g}) - M_k t.$$

Hence, recalling that we are assuming $M_k > 0$, we infer

$$t^* \leqslant \frac{v(\bar{g})}{M_k}.$$

Consider now the point

$$g^* = \gamma(t^*) \in \partial\{v > 0\}.$$

Since g^* is on the free boundary, by the continuity of the derivatives $X_j v$ (see Lemma 3.2), we have

$$X_j v(g^*) = 0. \tag{3.12}$$

On the other hand

$$X_j v(g^*) = \lim_{t \to t^*-} X_j v\big(\gamma(t)\big) = X_j v(\bar{g}). \tag{3.13}$$

The latter equality follows from the fact that

$$X_j v\big(\gamma(t)\big) = const \quad \text{for } 0 \leqslant t < t^*. \tag{3.14}$$

This is a direct consequence of the invariance of the horizontal vector fields X_j under the vertical translations, see (2.7), and the identity (3.10). Indeed, the segment $\gamma([0,t])$ is contained in a cylindrical set

$$U = U' \times [\bar{y}_k - t, \bar{y}_k] \subset \Omega_0,$$

where U' is a neighborhood of the point $(\bar{x}, \bar{y}')$ in $\mathbf{R}^m \times \mathbf{R}^{k-1}$. But then $Y_k v = M_k$ in U implies the representation

$$v(x, y', y_k) = \varphi(x, y') + M_k y_k \quad \text{in } U.$$

Consequently,

$$X_j v = X_j \varphi(x, y') + M_k X_j y_k \quad \text{at } (x, y) = \gamma(t).$$

However, from the differentiation formula (2.7), it is clear that the derivatives $X_j \varphi(x, y')$ and $X_j y_k$ are the same at the points $\gamma(t)$ and $\bar{g}$, since they differ only in y_k-coordinate. This implies (3.14) and consequently (3.11).

Summing up, we have established that

$$M_k = \lim_{\rho \to 0} \sup\big\{Y_k u(g)\colon\ u \in P_1(M),\ g \in B_\rho \cap \{u > 0\}\big\} \leqslant 0.$$

In a completely analogous fashion one can establish

$$m_k = \lim_{\rho\to 0} \inf\bigl\{Y_k u(g)\colon\ u \in P_1(M),\ g \in B_\rho \cap \{u > 0\}\bigr\} \geqslant 0.$$

Combining the latter two inequalities we obtain the claim, thus reaching the desired conclusion. □

Proof of Theorem I. By performing a left-translation and a rescaling we may assume that $g_0 = e$, $B_1 \Subset \Omega$ and $u \in P_1(M)$. The statement will follow now from Lemma 3.5. The proof is complete. □

In Section 4 we will need the following reformulation of Lemma 3.4.

Lemma 3.5. *For any $M > 0$ and $\varepsilon > 0$ there exists a large $R_0 = R_0(\varepsilon, M, \mathbf{G}) > 0$ such that for every $R \geqslant R_0$, and any $u \in P_R(MR^2)$, one has*

$$|Y_l u| < \varepsilon \quad \text{in } B_1,\ l = 1, \dots, k.$$

To close this section we would like to draw a parallel between Theorem I and the continuity of the temperature in the *Stefan problem*, which is a time-dependent version of the obstacle problem, and can be formulated locally as

$$\Delta u - u_t = \chi_{\{u>0\}} \quad \text{in } \Omega \subset \mathbf{R}^n \times \mathbf{R}, \tag{3.15}$$

see Duvaut [15]. In (3.15) $u_t \geqslant 0$ has the meaning of the temperature of the melting ice. One can prove that the weak solutions u of (3.15) are $C^{1,1}$ regular in the spatial variables, which implies only that u_t is bounded. However, to prove the regularity of the free boundary $\partial\{u > 0\}$, one needs to know that the temperature u_t continuously vanishes on $\partial\{u > 0\}$. This was established by Caffarelli and Friedman [7], who showed that u_t has a logarithmic modulus of continuity. Their method uses in an essential way that $u_t \geqslant 0$ and is not applicable to our problem, since the corresponding assumptions $Y_l u \geqslant 0$ would be very unnatural in (1.3). Perhaps, one will get a better analogy by comparing (1.3) with the Stefan problem with no sign restriction on u_t. Such a problem was recently studied by Caffarelli, Shahgholian and the third named author [8].

4. Global solutions

In this section we establish some basic information about the so-called *global solutions* of the obstacle problem, i.e., solutions of (1.3) in $\Omega = \mathbf{G}$. More exactly, we study the global solutions with at most quadratic growth at infinity, since they are the ones that appear after the blow-up in the proof of Lemma 3.4.

Definition 4.1 *(Global solutions).* We say that $u \in P_\infty(M)$ if $u \in \mathcal{L}^{1,2}_{\mathrm{loc}}(\mathbf{G})$, and

(i) $\Delta_H u = \chi_{\{u>0\}}$ in $\mathbf{G}$;
(ii) $e \in \partial\{u > 0\}$;
(iii) $|u(g)| \leqslant M(1 + d(g)^2)$ for every $g \in \mathbf{G}$.

This kind of global solutions have already appeared in our proof of Lemma 3.2, see Eqs. (3.5), (3.6). More generally, if $u_n \in P_{R_n}(MR_n^2)$ with $R_n \to \infty$, then by Lemma 3.2 we will have that

$$\begin{aligned} |X_i X_j u_n(g)| &\leqslant CM, \\ |X_i u_n(g)| &\leqslant CMd(g), \\ |u_n(g)| &\leqslant CMd(g)^2 \end{aligned}$$

in $B_{R_n/2}$, for some constant $C = C(\mathbf{G}) > 0$. Thus, we can extract a subsequence, still denoted by u_n, such that

$$u_n \to u_0 \quad \text{in } \Gamma^{1,\alpha}_{\mathrm{loc}}(\mathbf{G}),$$

for some $0 < \alpha < 1$, where $u_0 \in \Gamma^{1,1}_{\mathrm{loc}}(\mathbf{G})$. We claim that, in fact,

$$u_0 \in P_\infty(CM).$$

Indeed, the conditions (i) and (iii) are straightforward, and (ii) follows from the non-degeneracy, see Lemma 3.3.

In order to motivate our next result we recall the following striking property of Δ_H-harmonic functions in Carnot groups:

Let u be a bounded Δ_H-harmonic function in a Carnot group $\mathbf{G}$ *of arbitrary step, then u depends only on the variables in the horizontal layer V_1 of its Lie algebra.*

As a consequence, using the Baker–Campbell–Hausdorff formula one recognizes that u is a standard harmonic function with respect to such variables, hence thanks to the classical theorem of Liouville–Cauchy for harmonic functions u is constant. The next theorem is a result in the same spirit, but for global solutions of the obstacle problem.

Theorem 4.2. *Let* $\mathbf{G}$ *be a Carnot group of step $r = 2$, and let $u \in P_\infty(M)$, then $Y_l u = 0$ for any $l = 1, \dots, k$.*

Proof. If $u \in P_\infty(M)$, then $u \in P_R(M(1 + R^2))$ for any $R > 0$. If we thus apply Lemma 3.5, by letting $R \to \infty$, we conclude that $Y_l u = 0$ in B_1. Since the same argument can be applied to the rescalings $u_r = (1/r^2)u \circ \delta_r$ we obtain that $Y_l u_r = 0$ in B_1, or equivalently, $Y_l u = 0$ in B_r for any $r > 0$. This concludes the proof of the theorem. □

Theorem 4.2 says that in the exponential coordinates (x, y) we have the representation

$$u(x, y) = \varphi(x), \tag{4.1}$$

where φ is a solution of the obstacle problem for the ordinary Laplacian in the Euclidean space $\mathbf{R}^m$:

$$\Delta\varphi = \chi_{\{\varphi > 0\}} \quad \text{in } \mathbf{R}^m. \tag{4.2}$$

This opens a possibility for applying the known results on global solutions for the classical obstacle problem. In particular, our further analysis will be based on the following three lemmas. For proofs, see [6].

Lemma 4.3. *Let* $\mathbf{G}$ *be a Carnot group of step* $r = 2$*, and suppose that* $u \in P_\infty(M)$*, and that* φ *be as in* (4.1)*. Then* $\varphi(x)$ *is a convex function in* $\mathbf{R}^m$*; i.e., for any unit vector* $e \in \mathbf{R}^m$ *one has* $\partial_{ee}\varphi \geqslant 0$ *a.e. in* $\mathbf{R}^m$*. In particular,* $\{\varphi = 0\}$ *is a convex set in* $\mathbf{R}^m$*.*

Lemma 4.4. *Let* φ *be as in Lemma* 4.3*. There exists* $c = c(m) > 0$ *such that if* $\min \operatorname{diam}(\{\varphi = 0\} \cap B_1^x) \geqslant \sigma > 0$*, then the set* $\{\varphi = 0\}$ *contains a ball* $B_\rho^x(x^*)$ *with* $|x^*| \leqslant 1$ *and* $\rho = c\sigma$*.*

Lemma 4.5. *Let* φ*,* x^* *and* ρ *be as in Lemma* 4.4*. Assume that* $x^* = -se_m$ *for some* $0 \leqslant s \leqslant 1$*, and for* $h > 0$ *let* $\mathcal{K}^x(r, s, h)$ *be defined as in* (2.14)*. One has*:

(i) *For any unit vector* e *with* $|e - e_m| < \rho/8$

$$\partial_e\varphi \geqslant 0 \quad \text{in } \mathcal{K}^x(\rho/8, s, 1).$$

(ii) *Moreover, there exists* $C_0 = C_0(m, M, \rho) > 0$ *such that*

$$C_0\partial_e\varphi - \varphi \geqslant 0 \quad \text{in } \mathcal{K}^x(\rho/16, s, 1/2).$$

(iii) *The free boundary* $\partial\{\varphi > 0\} \cap \mathcal{K}^x(\rho/32, s, 1/4)$ *is a Lipschitz graph*

$$x_m = f(x'),$$

where f *is convex in* x'*, and there exists* $C = C(m, M) > 0$ *such that*

$$|\nabla_{x'} f| \leqslant \frac{C}{\rho}.$$

We emphasize at this point that our proof of Theorem II will consist in generalizing Lemma 4.5 to the case of local solutions. This, however, requires a substantial amount of additional work. As a first step in that direction, we establish a lemma on approximation of local solutions by global ones.

Lemma 4.6. *For given* $M > 0$*,* $\sigma > 0$ *and* $\varepsilon > 0$*, there exist* $C = C(\mathbf{G}) > 0$*, and* $R_0 = R_0(\mathbf{G}, M, \varepsilon, \sigma)$ *such that, if* $R \geqslant R_0$*,* $u \in P_R(MR^2)$*, and* $\delta_1(e, \{u = 0\}) \geqslant \sigma$*, then there exists a global solution* $u_0 \in P_\infty(CM)$*, for which*:

(i) $\|u - u_0\|_{\Gamma^1(B_1^x \times B_1^y)} \leqslant \varepsilon$;
(ii) u_0 *vanishes on a set* $B_\rho^x(x^*) \times \mathbf{R}^k$*, with* $|x^*| \leqslant 1$ *and* $\rho = c\sigma$ *as in Lemma* 4.4;
(iii) u *vanishes on* $B_{\rho/2}^x(x^*) \times B_1^y$*.*

Proof. To prove the lemma we use a contradiction argument based on compactness. Assuming the contrary, we can find a sequence $R_n \to \infty$ and $u_n \in P_{R_n}(MR_n^2)$, such that there exist no global solution with the indicated properties. Arguing as after Definition 4.1, we can extract a subsequence, still denoted by u_n, such that $u_n \to u_0$ in $\Gamma^{1,\alpha}_{\mathrm{loc}}(\mathbf{G})$, where $u_0 \in P_\infty(CM)$. Thus (i) will be satisfied for large n.

Next, observe that u_0 will satisfy $\delta_1(e, \{u_0 = 0\}) \geqslant \sigma$. By Theorem 4.2 we have $u_0(x, y) = \varphi_0(x)$ and $\{u_0 = 0\} = \{\varphi_0 = 0\} \times \mathbf{R}^k$. Now, applying Lemma 4.4, we can find a ball $B^x_\rho(x^*) \subset \{\varphi_0 = 0\}$ with $|x^*| \leqslant 1$. In other words,

$$B^x_\rho(x^*) \times \mathbf{R}^k \subset \{u_0 = 0\}. \tag{4.3}$$

We now claim that for sufficiently large n

$$B^x_{\rho/2}(x^*) \times B^y_1 \subset \{u_n = 0\}.$$

Indeed, if the set $U = B^x_{\rho/2}(x^*) \times B^y_1$ contains a point from $\partial\{u_n > 0\}$ for arbitrary large n, then by the non-degeneracy Lemma 3.3, it will necessarily contain a point from $\partial\{u_0 > 0\}$. On the other hand, the set U cannot be completely contained in $\{u_n > 0\}$ for large n, since this would imply that $\Delta_H u_0 = 1$ in U, which is in contrast with (4.3). Thus, (ii) and (iii) are also satisfied, which contradicts our assumption. This completes the proof of the lemma. □

5. Remarks on right-invariant derivatives

One difficulty that arises in the sub-elliptic obstacle problem is that if u is a solution of (1.3), the derivatives $X_j u$ are not generally Δ_H-harmonic in $\{u > 0\}$, since the operators X_j and Δ_H do not commute. However, to prove the regularity of the free boundary we need a sufficiently large number of Δ_H-harmonic functions in $\{u > 0\}$, continuously vanishing on the free boundary $\partial\{u > 0\}$. Interestingly, this problem is resolved by considering the derivatives along right-invariant vector fields.

For $\zeta \in \mathfrak{g}$, we define the corresponding right-invariant derivative by

$$\widetilde{Z}u(g) := \frac{d}{dt} u\big(\exp(t\zeta)g\big)\big|_{t=0}, \tag{5.1}$$

similarly to (2.6).

Lemma 5.1. *For any two $\zeta_1, \zeta_2 \in \mathfrak{g}$ one has*

$$[Z_1, \widetilde{Z}_2] = 0. \tag{5.2}$$

In other words, every left-invariant vector field commutes with any right-invariant one.

Proof. It is easily verified by the following argument. Let

$$\varphi(s, t) = u\big(\exp(t\zeta_2) g \exp(s\zeta_1)\big)$$

for small $s, t \in \mathbf{R}$. One has

$$Z_1 \widetilde{Z}_2 u(g) = \partial_s \partial_t \varphi(0, 0) = \partial_t \partial_s \varphi(0, 0) = \widetilde{Z}_2 Z_1 u(g),$$

which proves (5.2). □

Note that the above argument is valid in every Lie group. Using the fact that we work in a Carnot group of step $r = 2$, we can apply the Baker–Campbell–Hausdorff formula (2.4) to derive the following analogues of Eqs. (2.7)–(2.8):

$$\widetilde{X}_j u = \partial_{x_j} u - \frac{1}{2} \sum_{l=1}^{k} \left(\sum_{i=1}^{m} b_{ij}^l x_i \right) \partial_{y_l} u, \tag{5.3}$$

$$\widetilde{Y}_l u = Y_l u = \partial_{y_l} u. \tag{5.4}$$

We sum up this section with the following lemma, which will play a crucial role in the sequel.

Lemma 5.2. *Let u be a solution of the obstacle problem* (1.3)*, then for any $\zeta \in \mathfrak{g}$ the function $\widetilde{Z}u$ is Δ_H-harmonic in $\{u > 0\}$ and continuously vanishes on $\partial\{u > 0\} \cap \Omega$.*

Proof. Since $\widetilde{Z}$ commutes with every X_j, $j = i, \dots, m$, $\widetilde{Z}$ also commutes with $\Delta_H = \sum_{j=1}^m X_j^2$. We thus have,

$$\Delta_H(\widetilde{Z}u) = \widetilde{Z}(\Delta_H u) = \widetilde{Z}1 = 0 \quad \text{in } \{u > 0\}.$$

To show that $\widetilde{Z}u$ continuously vanishes on $\partial\{u > 0\} \cap \Omega$, we first observe that in view of Theorem I and (2.8), for any $l = 1, \dots, k$, the derivative $\partial_{y_l} u$ continuously vanishes on $\partial\{u > 0\} \cap \Omega$. As a consequence of Lemma 3.2 and (2.7), so does also $\partial_{x_j} u$, for $j = 1, \dots, m$. At this point, we apply (5.3)–(5.4) to reach the desired conclusion. □

6. Lipschitz regularity of free boundary

In this section we generalize Lemma 4.5 to "almost global solutions," i.e., solutions $u \in P_R(MR^2)$ with large R. Our main result is Theorem 6.3, for whose proof we use in an essential way two facts: (i) that the second layer derivatives continuously vanish on the free boundary; (ii) that the right-invariant derivatives of u are Δ_H-harmonic in the positivity set of u itself.

The bridge that allows to carry over the Lipschitz regularity result from global to local solutions is given in the following lemma, which generalizes an argument, originally due to Caffarelli, for the classical obstacle problem.

Lemma 6.1. *Let $\mathbf{G}$ be a Carnot group of arbitrary step. There exists a* (*sufficiently small*) *number $\varepsilon_0 = \varepsilon_0(\mathbf{G}) > 0$ such that if w be a bounded nonnegative solution of the obstacle problem $\Delta_H w = \chi_{\{w>0\}}$ in B_1, and h a Δ_H-harmonic function in $\{w > 0\} \cap B_1$ for which*:

(i) $h \geqslant 0$ *on* $\partial\{w > 0\} \cap B_1$;
(ii) $h - w \geqslant -\varepsilon_0$ *in* $\{w > 0\} \cap B_1$,

then

$$h - w \geqslant 0 \quad \textit{in } \{w > 0\} \cap B_{1/2}.$$

Proof. To establish the lemma we use the following auxiliary function $\psi \in \Gamma^{1,1}_{\mathrm{loc}}(\mathbf{G})$ with the following properties:

$$\psi(e) = 0, \qquad \psi(g) \geqslant C d(g)^2, \qquad |\Delta_H \psi| \leqslant 1 \quad \text{in } \mathbf{G}.$$

For the construction of such a function we refer the reader to [14, Theorem 5.5]. Now, assume that the statement of the lemma is false and let $g_0 \in \{w > 0\} \cap B_{1/2}$ be such that $h(g_0) - w(g_0) < 0$. Consider then the function

$$\varphi(g) := h(g) - w(g) + \psi\big(g_0^{-1} \cdot g\big).$$

One readily verifies that

$$\varphi(g_0) < 0, \qquad \Delta_H \varphi \leqslant 0 \quad \text{in } \{w > 0\} \cap B_1. \tag{6.1}$$

Besides, we have that

$$\varphi \geqslant 0 \quad \text{on } \partial\{w > 0\} \cap B_1,$$

and

$$\varphi(g) \geqslant -\varepsilon_0 + C d\big(g_0^{-1} g\big)^2 \geqslant -\varepsilon_0 + C\delta^2 \quad \text{for any } g \in \partial B_1 \cap \{w > 0\},$$

where $\delta = \operatorname{dist}(B_{1/2}, \partial B_1) > 0$. Therefore, if $\varepsilon_0 < C\delta^2$, we obtain that

$$\varphi \geqslant 0 \quad \text{on } \partial\big(\{w > 0\} \cap B_1\big). \tag{6.2}$$

Since φ is Δ_H-superharmonic, applying Bony's maximum principle [3] we conclude that $\varphi \geqslant 0$ in $\{w > 0\} \cap B_1$, which is a contradiction with (6.1). □

We will actually need the following slightly more general version of the previous lemma.

Lemma 6.2. *Let $\mathbf{G}$ be a Carnot group and $E \subset \mathbf{G}$. Consider a bounded nonnegative solution w of the obstacle problem $\Delta_H w = \chi_{\{w>0\}}$ in the open set*

$$\mathcal{N}_\delta(E) := \big\{g\colon \operatorname{dist}(g, E) < \delta\big\}.$$

There exists a sufficiently small number $\varepsilon_0 = \varepsilon_0(\mathbf{G})$ such that if h is a Δ_H-harmonic function in $\mathcal{N}_\delta(E) \cap \{w > 0\}$, satisfying:

(i) *$h \geqslant 0$ on $\partial\{w > 0\} \cap \mathcal{N}_\delta(E)$; and*
(ii) *$h - w \geqslant -\varepsilon_0 \delta^2$ in $\{w > 0\} \cap \mathcal{N}_\delta(E)$,*

then

$$h - w \geqslant 0 \quad \textit{in } \{w > 0\} \cap \mathcal{N}_{\delta/2}(E).$$

Proof. Consider h and w in every ball $B_\delta(g)$ with $g \in E$, translate and rescale them to functions in B_1, then apply Lemma 6.1 to such functions. □

In the following theorem, given a Carnot group $\mathbf{G}$ of step $r = 2$, with Lie algebra $\mathfrak{g} = V_1 \oplus V_2$, and given elements $\xi \in V_1$ and $\eta \in V_2$, we will indicate with X and Y the left-invariant vector fields on $\mathbf{G}$ associated respectively with ξ and η and defined by Eq. (2.6). The symbols $\tilde{X}$, $\tilde{Y}$ will indicate the corresponding right-invariant vector fields defined by (5.1).

Theorem 6.3. *Let $u \in P_R(MR^2)$ with $\delta_1(e, \{u = 0\}) \geqslant \sigma$ and $R \geqslant R_0$ be such that the conclusions in Lemma* 4.6 *are satisfied. Assume $x^* = -se_m$, $0 \leqslant s \leqslant 1$, and let $\mathcal{K}^x(r, s, h)$ be as in* (2.14). *One has*:

(i) *For every $A > 0$ there exists $R_1 = R_1(\mathbf{G}, M, \rho, A) > 0$ such that if $R \geqslant R_1$, $\xi \in V_1$ with $|\xi - e_m| < \rho/8$ and $\eta \in V_2$ with $|\eta| \leqslant 1$, we have for some $C_0 = C_0(\mathbf{G}, M, \rho) > 0$*

$$C_0(\widetilde{X}u + AYu) - u \geqslant 0 \quad \text{in } \mathcal{K}^x(\rho/32, s, 1/4) \times B^y_{1/2}.$$

(ii) *There exists $R_2 = R_2(\mathbf{G}, M, \rho) > 0$ such that if $R \geqslant R_2$, then the free boundary $\partial\{u > 0\} \cap \mathcal{K}^x(\rho/32, s, 1/4) \times B^y_{1/2}$ is a graph*

$$x_m = f(x', y),$$

where f is Lipschitz continuous in x' and y, and there exists $C = C(M, \mathbf{G}) > 0$ such that

$$|\nabla_{x'} f| \leqslant \frac{C}{\rho}, \qquad |\nabla_y f| \leqslant C.$$

Proof. *Part* (i). Let $u_0(x, y) = \varphi_0(x)$ be a global solution as in Lemma 4.6, and $\xi \in V_1$ be such that $|\xi - e_m| < \rho/8$. We note that, in view of Theorem 4.2, we know that $u_0(x, y) = \varphi_0(x)$, see (4.1). Lemma 4.5 implies that

$$C_0 \partial_\xi \varphi_0 - \varphi_0 \geqslant 0 \quad \text{in } \mathcal{K}^x(\rho/16, s, 1/2),$$

where $\xi = \sum_{j=1}^m \langle \xi, e_j \rangle e_j$. (Note that since ξ is not necessarily a unit vector in $\mathbf{R}^m$, the constant C_0 here should be taken slightly larger than that in Lemma 4.5.) Since u_0 is y-independent, the latter inequality can be re-written as

$$C_0 \widetilde{X} u_0 - u_0 \geqslant 0 \quad \text{in } \mathcal{K}^x(\rho/16, s, 1/2) \times \mathbf{R}^k. \tag{6.3}$$

From (6.3) and from $\|u - u_0\|_{\Gamma^1(B^x_1 \times B^y_1)} \leqslant \varepsilon$ (see (i) of Lemma 4.6), we obtain that

$$C_0 \widetilde{X} u - u \geqslant -C_1 \varepsilon \quad \text{in } \mathcal{K}^x(\rho/16, s, 1/2) \times B^y_1.$$

Moreover, using Lemmas 3.2 and 3.5, we find for R sufficiently large

$$C_0(\widetilde{X}u + AYu) - u \geqslant -C_2 \varepsilon \quad \text{in } \mathcal{K}^x(\rho/16, s, 1/2) \times B^y_1.$$

Thus, taking ε small enough, we can apply Lemma 6.2 with $h = C_0(\widetilde{X}u + AYu)$, and $w = u$, and conclude that

$$C_0(\widetilde{X}u + AYu) - u \geqslant 0 \quad \text{in } \mathcal{K}^x(\rho/32, s, 1/4) \times B^y_{1/2}.$$

It is important to observe that the lemma is indeed applicable: $\widetilde{X}u$ and Yu are Δ_H-harmonic in $\{u > 0\}$ and, in view of Theorem I and Lemma 5.2, they continuously vanish on $\partial\{u > 0\}$. This proves the part (i) of the lemma.

Part (ii). Observe that for any $\xi \in V_1$ with $|\xi - e_m| \leqslant \rho/8$, and $\eta \in V_2$ with $|\eta| \leqslant 1$, we have

$$\widetilde{X}u + AYu \geqslant 0 \quad \text{in } \mathcal{K}^x(\rho/32, s, 1/4) \times B^y_{1/2}.$$

As one can see from (5.3), which gives

$$\widetilde{X}_j u = \partial_{x_j} u - \frac{1}{2}\sum_{i,l} b^l_{i,j} x_i \partial_{y_l} u,$$

if we choose $A = A(\mathbf{G}) > 0$ sufficiently large, then varying the vector $|\eta| \leqslant 1$ at every point, for any vectors $e \in \mathbf{R}^m$, $\varepsilon \in \mathbf{R}^k$, such that

$$|e - e_m| \leqslant \frac{\rho}{8}, \quad |\varepsilon| \leqslant 4, \tag{6.4}$$

we will obtain

$$\langle e, \nabla_x u\rangle + \langle \varepsilon, \nabla_y u\rangle \geqslant 0 \quad \text{in } \mathcal{K}^x(\rho/32, s, 1/4) \times B^y_{1/2}. \tag{6.5}$$

Consider now the cone $\mathcal{C}$ in $\mathbf{R}^m \times \mathbf{R}^k$ generated by all vectors $\zeta = (e, \varepsilon)$ with e and ε as in (6.4). Define

$$U := \mathcal{K}^x(\rho/32, s, 1/4) \times B^y_{1/2}.$$

We can re-write (6.5) as

$$\partial_\zeta u \geqslant 0 \quad \text{in } U,$$

for every $\zeta \in \mathcal{C}$. In particular, we obtain that

$$(x_0, y_0) \in \{u > 0\} \cap U \quad \Rightarrow \quad \big((x_0, y_0) + \mathcal{C}\big) \cap U \subset \{u > 0\} \cap U \tag{6.6}$$

and

$$(x_0, y_0) \in \{u = 0\} \cap U \quad \Rightarrow \quad \big((x_0, y_0) - \mathcal{C}\big) \cap U \subset \{u = 0\} \cap U. \tag{6.7}$$

Besides, (6.6)–(6.7) imply that

$$(x_0, y_0) \in \partial\{u > 0\} \cap U \quad \Rightarrow \quad \big((x_0, y_0) + \mathcal{C}^\circ\big) \cap U \subset \{u > 0\} \cap U, \tag{6.8}$$

where $\mathcal{C}^\circ$ is the interior of the cone $\mathcal{C}$. Since we assume $(0,0) \in \partial\{u>0\}$, we have in particular that

$$-\mathcal{C} \cap U \subset \{u=0\} \cap U, \qquad \mathcal{C}^\circ \cap U \subset \{u>0\} \cap U.$$

We next notice that for any choice of $\bar{x}' \in \mathbf{R}^{m-1}$ and $\bar{y} \in \mathbf{R}^k$ with

$$|\bar{x}'| < \frac{\rho}{32}, \qquad |\bar{y}| < \frac{1}{2},$$

the line interval

$$J_{\bar{x}',\bar{y}} := \{\bar{x}'\} \times (-s, 1/4) \times \{\bar{y}\} = \big(\{\bar{x}'\} \times \mathbf{R} \times \{\bar{y}\}\big) \cap U$$

has a nonempty intersection with both $\mathcal{C}^\circ$ and $-\mathcal{C}^\circ$. This is verified directly by checking the sizes of U and the opening of $\mathcal{C}$. Thus, $J_{\bar{x}',\bar{y}}$ has a nonempty intersection with both $\{u=0\}$ and $\{u>0\}$. Since also $\partial_{x_m} u \geqslant 0$ in U, we obtain that there exists a unique

$$\bar{x}_m = f(\bar{x}', \bar{y}), \qquad -s \leqslant \bar{x}_m < 1/4,$$

such that

$$(\bar{x}', \bar{x}_m, \bar{y}) \in \partial\{u>0\} \cap U.$$

This means precisely that $\partial\{u>0\} \cap U$ is given by the graph

$$x_m = f(x', y), \quad |x'| < \rho/32,\ |y| \leqslant 1/2.$$

The Lipschitz continuity of f and the estimates for the partial derivatives follow immediately from (6.6)–(6.8). This completes the proof of the lemma. □

7. Local NTA property

In this section we prepare the ground for the proof of the $C^{1,\alpha}$ regularity of the free boundary. The approach that we are going to use is based on the boundary Harnack principle (also known as the local comparison theorem) for the class of NTA (nontangentially accessible) domains with respect to the (CC) distance. The relevant Fatou theory for such domains was developed in [9], and we will use or adapt to our situation several basic results form that paper.

Given a domain Ω in a Carnot group $\mathbf{G}$ and a constant $M>0$, we say that a ball $B_r(g) \subset \Omega$ is *M-nontangential*, if

$$M^{-1} r \leqslant \operatorname{dist}\big(B_r(g), \partial\Omega\big) \leqslant Mr.$$

An *M-Harnack chain* joining two given points $g_1, g_2 \in \Omega$, is a finite sequence of M-nontangential balls such that the first one contains g_1, the last one contains g_2, and such that consecutive balls have nonempty intersection. The *length* of the chain is the number of balls in the chain.

Definition 7.1 *(Corkscrew condition).* We say that the domain $\Omega \subset \mathbf{G}$ satisfies the (M, r_0)-*corkscrew condition* at $g_0 \in \partial\Omega$ for constants $M, r_0 > 0$ if:

(1) (Interior corkscrew) For any $0 < r \leqslant r_0$ there exists $A_r(g_0) \in \Omega$ such that

$$B_{r/M}\big(A_r(g_0)\big) \subset \Omega, \quad \text{and}$$
$$M^{-1}r \leqslant d\big(A_r(g_0), g_0\big) \leqslant Mr;$$

(2) (Exterior corkscrew) $\Omega^c = \mathbf{G} \setminus \overline{\Omega}$ satisfies the previous condition.

Definition 7.2 *(NTA property).* We say that a domain $\Omega \subset \mathbf{G}$ is *nontangentially accessible* (NTA) if there exist $M, r_0 > 0$ such that

(1) Ω satisfies the (M, r_0)-corkscrew condition at every point $g_0 \in \partial\Omega$.
(2) (Harnack chain condition) For every $0 < r \leqslant r_0$ and $g_1, g_2 \in \Omega$ with $\operatorname{dist}(g_i, \partial\Omega) \geqslant \varepsilon$, $i = 1, 2$, and $d(g_1, g_2) \leqslant C\varepsilon$, there exists an M-Harnack chain joining g_1 and g_2 with a length depending only on C, M and r_0 (but independent ε).

To give a motivation for our next definition, we observe that we will need to deal with domains that have the NTA property only near a certain part of their boundary. In the Euclidean space, a deep geometric localization theorem by Jones [19] (see also [18]) shows that being NTA is essentially a local property. This theorem says that for an NTA domain Ω in $\mathbf{R}^n$ and $x_0 \in \partial\Omega$, for any $0 < r \leqslant r_0$ one can find an NTA domain $\Omega_r(x_0)$ (with constants independent of r) such that

$$B_r(x_0) \cap \Omega \subset \Omega_r(x_0) \subset B_{Mr}(x_0) \cap \Omega.$$

The analogue of this theorem for Carnot groups is unknown to the authors, however the following simple observation can be easily shown by using equivalent definitions of the NTA property, see for instance [16]: if $g_1, g_2 \in \Omega$ are as in the condition (ii) of Definition 7.2, and additionally $g_1, g_2 \in B_r(g_0)$ for some $g_0 \in \partial\Omega$ and $0 < r \leqslant r_0$, then one can construct a Harnack chain as in (ii) which lies completely within the ball $B_{Mr}(g_0)$. (Here, the constants M and r_0 can be different than before.) We take this fact as the basis for our next definition.

Definition 7.3 *(Local NTA property).* We say that a domain $\Omega \subset \mathbf{G}$ is *locally nontangentially accessible* (locally NTA) at a point $g_0 \in \partial\Omega$ if there exist $M, r_0 > 0$ such that:

(1) Ω satisfies the (M, r_0)-corkscrew condition at every point of $B_{r_0}(g_0) \cap \partial\Omega$.
(2) (Localized Harnack chain condition) For every $0 < r \leqslant r_0$ and $g_1, g_2 \in B_r(g_0) \cap \Omega$ with $\operatorname{dist}(g_i, \partial\Omega) \geqslant \varepsilon$, $i = 1, 2$, and $d(g_1, g_2) \leqslant C\varepsilon$, there exists an M-Harnack chain joining g_1 and g_2, fully contained in $B_{Mr}(g_0) \cap \Omega$, and with a length depending only on C, M and r_0 (in particular, independent of r and ε).

Theorem 7.4 *(Boundary Harnack Principle). Let Ω be locally NTA at $g_0 \in \partial\Omega$ with constants M and r_0. Let $u \geqslant 0$ and $v > 0$ be two Δ_H-harmonic functions in $B_r(g_0) \cap \Omega$, $r < r_0$, continuously vanishing on $B_r(g_0) \cap \partial\Omega$. Then*

$$\sup_{B_{r/K}(g_0)\cap\Omega} \frac{u}{v} \leqslant C \inf_{B_{r/K}(g_0)\cap\Omega} \frac{u}{v},$$

where $C, K > 1$ depend only on M and $\mathbf{G}$.

Proof. The proof is a *verbatim* repetition of the one for Theorem 3 in [9]. □

Corollary 7.5. *Let Ω and u, v be as in Theorem* 7.4. *There exists constants $C > 0$ and $0 < \alpha < 1$, depending only on M and $\mathbf{G}$, such that*

$$\operatorname*{osc}_{B_\rho(g_0)\cap\Omega} \frac{u}{v} \leqslant C\left(\frac{\rho}{r}\right)^\alpha \sup_{B_r(g_0)\cap\Omega} \frac{u}{v}. \tag{7.1}$$

Proof. The proof follows the standard "Harnack inequality implies Hölder continuity" scheme. Carefully observe that no localization theorem of Jones' type is needed. Let

$$m_\rho := \inf_{B_\rho(g_0)\cap\Omega} \frac{u}{v}, \qquad M_\rho := \sup_{B_\rho(g_0)\cap\Omega} \frac{u}{v}.$$

We claim that

$$M_{\rho/K} - m_{\rho/K} \leqslant \theta(M_\rho - m_\rho), \tag{7.2}$$

for a certain $0 < \theta < 1$, which by iteration implies (7.1). To prove this inequality, consider

$$\frac{u}{v} - m_\rho = \frac{u - m_\rho v}{v} \geqslant 0 \quad \text{in } B_\rho(g_0) \cap \Omega.$$

We can thus apply Theorem 7.4 to the pair of functions $u - m_\rho v$ and v in $B_\rho(g_0) \cap \Omega$, obtaining

$$M_{\rho/K} - m_\rho \leqslant C(m_{\rho/K} - m_\rho). \tag{7.3}$$

On the other hand, consider

$$M_\rho - \frac{u}{v} = \frac{M_\rho v - u}{v} \geqslant 0 \quad \text{in } B_\rho(g_0) \cap \Omega,$$

and apply Theorem 7.4 to the pair of functions $M_\rho v - u$ and v in $B_\rho(g_0) \cap \Omega$. We will have

$$M_\rho - m_{\rho/K} \leqslant C(M_\rho - M_{\rho/K}). \tag{7.4}$$

Combining (7.3)–(7.4), we obtain

$$(C+1)(M_{\rho/K} - m_{\rho/K}) \leqslant (C-1)(M_\rho - m_\rho),$$

which is equivalent to (7.2). $\square$

We will need to apply the Boundary Harnack Principle in a special case of epigraphs of Euclidean Lipschitz functions in horizontal directions. Such domains are indeed locally NTA.

Theorem 7.6. *Let $\mathbf{G}$ be a Carnot group of step 2. Given a function $f:\mathbf{R}^{m-1}\times\mathbf{R}^k\to\mathbf{R}$ which is (Euclidean) Lipschitz continuous in x' and y, and which satisfies $f(0,0)=0$, consider the open set $\Omega\subset\mathbf{G}$ which in the exponential coordinates (x,y) is described by*

$$\Omega=\{x_m>f(x',y)\}.$$

The domain Ω is locally NTA at the group identity with constants depending only on $\mathbf{G}$ and on the Lipschitz norm of f.

Proof. This is essentially contained in [9, Theorem 14]. This theorem establishes that $C^{1,1}$ domains with a certain cylindrical symmetry property near the characteristic points are NTA. However, the $C^{1,1}$ regularity is not necessary away from the characteristic points. Considering the epigraphs in horizontal directions, we effectively avoid characteristic points.

For any $g\in\partial\Omega$ define

$$\gamma_g(s):=g\cdot\delta_s(e_m,0)=g\cdot(se_m,0).$$

Then, from the Lipschitz continuity of f, it follows that $\gamma_g(s)\in\Omega$ and $\gamma_g(-s)\in\mathbf{G}\setminus\overline{\Omega}$ for any $s>0$ and $g\in\partial\Omega$ sufficiently close to the origin. Moreover, if

$$\Gamma_\kappa(g):=g\cdot\bigcup_{s>0}B_{\kappa s}(se_m,0),\qquad \Gamma_\kappa^-(g):=g\cdot\bigcup_{s>0}B_{\kappa s}(-se_m,0)$$

then

$$\Gamma_\kappa(g)\subset\Omega,\qquad \Gamma_\kappa^-(g)\subset\mathbf{G}\setminus\overline{\Omega}\quad\text{if } g\in B_{\rho_0}\cap\partial\Omega$$

for $\rho_0,\kappa>0$ sufficiently small. Then for $g\in B_{\rho_0}\cap\partial\Omega$ we can take $\gamma_g(r)$ and $\gamma_g(-r)$ as the interior and exterior corkscrew points, respectively.

To proceed, we will need the following two lemmas.

Lemma 7.7. *The points $(\varepsilon e_m,0)$ and $(C\varepsilon e_m,0)$ can be joined with a Harnack chain in $\Gamma_\kappa(e)$ of length $C(\kappa,\mathbf{G})C$.*

Proof. Consider the sequence of balls $B_{\kappa s_j/2}(s_je_m,0)$ for $s_j=\varepsilon(1+j\kappa h)$, $j=0,\dots,\lfloor(C-1)/(\kappa h)\rfloor$. Then for $h=h(\mathbf{G})>0$ small enough, any two consecutive balls intersect and we obtain the required Harnack chain. $\square$

Lemma 7.8. *For any $\alpha>0$, we have*

$$d\big(\gamma_g(s),\gamma_e(s)\big)\leqslant\alpha s\quad\text{for } s>C(\alpha,\mathbf{G})|g|_{\mathbf{G}}.$$

Proof. We will prove the estimate for the equivalent gauge pseudo-distance (2.12) instead of the (CC)-distance, since the former is more suitable for explicit computations.

Let $g = (\xi, \eta)$. Then

$$\gamma_e(s) = (se_m, 0), \qquad \gamma_g(s) = (\xi, \eta) \cdot (se_m, 0) = \left(\xi + se_m, \eta + \frac{s}{2}[\xi, e_m]\right)$$

and

$$\gamma_e(s)^{-1}\gamma_g(s) = \left(\xi, \eta + \frac{s}{2}[\xi, e_m] - \frac{s}{2}[e_m, \xi + se_m]\right) = \big(\xi, \eta + s[\xi, e_m]\big).$$

Thus,

$$\begin{aligned}\rho\big(\gamma_g(s), \gamma_e(s)\big) = \left|\gamma_e(s)^{-1}\gamma_g(s)\right|_{\mathbf{G}} &= \big(|\xi|^4 + \left|\eta + s[\xi, e_m]\right|^2\big)^{1/4} \\ &\leqslant K\big(|g|_{\mathbf{G}} + \sqrt{s|g|_{\mathbf{G}}}\big),\end{aligned}$$

where $K = K(\mathbf{G})$. Then taking, for instance,

$$s > 2K\left(\frac{1}{\alpha} + \frac{1}{\alpha^2}\right)|g|_{\mathbf{G}},$$

we will have

$$K\big(|g|_{\mathbf{G}} + \sqrt{s|g|_{\mathbf{G}}}\big) < \alpha s,$$

and this will complete the proof of the lemma. □

Now we resume the proof of Theorem 7.6. We verify the localized Harnack chain condition. First observe that the union of the cones $\Gamma_{\kappa/2}(g)$ with $|g|_{\mathbf{G}} < \rho_0$ covers a region $B_{r_0} \cap \Omega$ for some small $r_0 = r_0(\rho_0, \kappa, \mathbf{G}) > 0$. Next take two points $g_1, g_2 \in B_r \cap \Omega$ for $r < r_0$ and such that

$$\operatorname{dist}(g_i, \partial\Omega) \geqslant \varepsilon, \qquad d(g_1, g_2) < C\varepsilon.$$

Let $\tilde{g}_i \in \partial\Omega$ be such that

$$g_i \in \Gamma_{\kappa/2}(\tilde{g}_i), \quad i = 1, 2.$$

Without loss of generality, we can assume that $\operatorname{dist}(g_i, \partial\Omega) < 2C\varepsilon$, otherwise we could take as a Harnack chain a single ball of radius $C\varepsilon$ centered at one of g_i. (This ball will also be contained in B_{Mr} with $M = M(\mathbf{G})$.) Then we will have

$$d(\tilde{g}_1, \tilde{g}_2) \leqslant C(\kappa, \mathbf{G})C\varepsilon.$$

Consider now the curves $\gamma_i = \gamma_{\tilde{g}_i}$. Then from Lemma 7.8 we have

$$d\big(\gamma_1(s), \gamma_2(s)\big) \leqslant (\kappa/4)s \quad \text{for } s \geqslant s_0 = C(\kappa, \mathbf{G})C\varepsilon.$$

To construct a Harnack chain from g_1 to g_2 we use Lemma 7.7 to construct Harnack chains from g_i to $\gamma_i(s_0)$, $i = 1, 2$, and then join the points $\gamma_i(s_0)$ with a single ball of radius $(\kappa/2)s$.

This construction of the Harnack chain involves at most $C(\kappa, \mathbf{G})C$ balls. Besides all balls are contained in $B_{C(\kappa,\mathbf{G})\varepsilon} \subset B_{M(\kappa,\mathbf{G})r}$, since we have that $r \geqslant c(\mathbf{G})\varepsilon$. Thus, the localized Harnack chain condition is satisfied. $\square$

Corollary 7.9. *Let $\Omega = \{x_m > f(x', y)\}$ be as in Theorem* 7.6 *and $u \geqslant 0$ and $v > 0$ be two Δ_H-harmonic functions in $B_{\rho_0} \cap \Omega$, $0 < \rho_0 < 1$ continuously vanishing on $B_{\rho_0} \cap \partial\Omega$. Then there exist $K > 1$, $0 < \alpha < 1$, and $C > 0$ depending only on Lipschitz norm of f, and $\mathbf{G}$ such that*

$$\left\| \frac{u}{v} \right\|_{C^\alpha(B_{\rho_0/K} \cap \Omega)} \leqslant C \frac{u((r_0/2)e_m, 0)}{v((r_0/2)e_m, 0)}.$$

Proof. One just needs to observe that the set $g^{-1} \cdot \Omega$ admits a representation $x_m > f_g(x', y)$ in $B_{\rho_0/2}$ for a Lipschitz function f_g with a norm close to that of f, if g is sufficiently close to e. Then the statement of the corollary follows from Theorem 7.6, Corollary 7.5, the interior Harnack inequality for Δ_H-harmonic functions (to control the oscillation of u/v away from the boundary), as well as the estimate

$$\sup_{B_{\kappa\rho_0} \cap \Omega} \frac{u}{v} \leqslant C \frac{u((r_0/2)e_m, 0)}{v((r_0/2)e_m, 0)},$$

which follows from Theorem 7.4. $\square$

8. $C^{1,\alpha}$ regularity of free boundary

We use the results of the previous section to improve on the Lipschitz regularity of the free boundary of "almost global" solutions. The following lemma can be regarded as a continuation of Lemma 6.3.

Lemma 8.1. *Let $u \in P_R(MR^2)$ with $\delta_1(e, \{u = 0\}) \geqslant \sigma$ and $R \geqslant R_2$ such that the conclusion of Lemma* 6.3(ii) *is satisfied. Then there exists $R_3 = R_3(\rho, M, \mathbf{G})$ such that if $R \geqslant R_3$ then the free boundary $\partial\{u > 0\} \cap \mathcal{K}^x(\rho/K, s, 1/K) \times B^y_{1/K}$ is a graph*

$$x_m = f(x', y),$$

where f is a $C^{1,\alpha}$ function with

$$\|\nabla_{x',y} f\|_{C^\alpha} \leqslant C,$$

where $K > 1$, $0 < \alpha < 1$, and C depend only on ρ, M, and $\mathbf{G}$.

Proof. The idea, that goes back to Athanasopoulos and Caffarelli [1], is to show that the ratios

$$\frac{\partial_{x_j} u}{\partial_{x_m} u}, \quad j = 1, \ldots, m-1,$$

and

$$\frac{\partial_{y_l} u}{\partial_{x_m} u}, \quad l = 1, \dots, k,$$

are Hölder continuous in $\{u > 0\} \cap \mathcal{K}^x(\rho/K, s, 1/K) \times B^y_{1/K}$. This will imply that the level sets $\{u = \lambda\}$ are graphs $x_m = f_\lambda(x', y)$ of uniformly $C^{1,\alpha}$ continuous functions, which will imply the claim of this lemma.

First observe that there exists a constant $\mu = \mu(\rho, M, \mathbf{G}) > 0$ such that for large R we have

$$\widetilde{X}_m u\big((1/8)e_m, 0\big) \geqslant \mu.$$

Indeed, arguing by contradiction and using the compactness, we would obtain otherwise a global solution $u_0(x, y) = \varphi_0(x) \in P_\infty(CM)$ such that

$$\partial_{x_m}\varphi_0 \geqslant 0 \quad \text{in } \mathcal{K}^x(\rho/32, s, 1/2), \qquad \partial_{x_m}\varphi_0\big((1/8)e_m\big) = 0.$$

Then by the minimum principle $\partial_{x_m}\varphi_0 = 0$ and consequently $\varphi_0 = 0$ in $\mathcal{K}^x(\rho/32, s, 1/4)$, which is a contradiction with the fact that $0 \in \partial\{\varphi_0 > 0\}$.

Next consider the pairs of functions $\widetilde{Z}u$ and $\widetilde{X}_m u$, where

$$Z = X_m + \alpha_j X_j + \beta_l Y_l, \quad |\alpha_j| \leqslant \rho/16,\ |\beta_l| \leqslant 2,$$

$j = 1, \dots, m-1$, $l = 1, \dots, k$. Then for sufficiently large R, both are nonnegative Δ_H-harmonic functions in $\{u > 0\} \cap U$, where $U = \mathcal{K}^x(\rho/32, s, 1/4) \times B^y_{1/2}$ and both vanish continuously on $\partial\{u > 0\} \cap U$. Applying Corollary 7.9, we obtain that the ratios

$$\frac{\widetilde{Z}u}{\widetilde{X}_m u} \in C^\alpha\big(\{u > 0\} \cap V\big), \quad V = \mathcal{K}^x(\rho/K, s, 1/K) \times B^y_{1/K}$$

with the C^α norm, depending only on ρ, M, and $\mathbf{G}$. Varying parameters $|\alpha_j| \leqslant \rho/16$ and $|\beta_l| \leqslant 2$, we obtain, that

$$\xi_j := \frac{\widetilde{X}_j u}{\widetilde{X}_m u}, \qquad \eta_l := \frac{\widetilde{Y}_l u}{\widetilde{X}_m u} \in C^\alpha, \quad j = 1, \dots, m-1,\ l = 1, \dots, k,$$

again with C^α norms depending only on ρ, M, and $\mathbf{G}$. Then using the representation

$$\partial_{x_j} u = \widetilde{X}_j u + \frac{1}{2}\sum_{i,l} b^l_{ij} x_i Y_l u, \quad j = 1, \dots, m,$$

we obtain

$$\begin{aligned} \frac{\partial_{x_j} u}{\partial_{x_m} u} &= \frac{\widetilde{X}_j u - \frac{1}{2}\sum_{i,l} b^l_{ij} x_i Y_l u}{\widetilde{X}_m u - \frac{1}{2}\sum_{i,l} b^l_{im} x_i Y_l u} \\ &= \frac{\xi_j - \frac{1}{2}\sum_{i,l} b^l_{ij} x_i \eta_l}{1 - \frac{1}{2}\sum_{i,l} b^l_{im} x_i \eta_l} \in C^\alpha\big(\{u > 0\} \cap V\big), \end{aligned}$$

provided that the constant $K = K(\rho, M, \mathbf{G})$ is so large that the inequalities $|x_i| \leqslant \rho/K$, $i = 1, \dots, m$, imply that

$$\left|\sum_{i,l} b_{im}^l x_i \eta_l\right| < 1.$$

Similarly, we obtain that

$$\frac{\partial_{y_l} u}{\partial_{x_m} u} \in C^\alpha\big(\{u > 0\} \cap V\big)$$

which completes the proof of the lemma. □

Proof of Theorem II. By performing a left translation and rescaling if necessary, we may assume that $\Omega' = B_1$, $g_0 = e$ and $u \in P_1(M)$. For a given $\sigma > 0$, let $\rho = c\sigma$ and $R_3(\rho, M, \mathbf{G})$ be as in Lemma 8.1. Let also the mapping

$$r \mapsto \sigma(r)$$

be the inverse of

$$\sigma \mapsto r(\sigma) := \frac{1}{R_3(c\sigma, M, \mathbf{G})}.$$

We claim that this is the modulus of continuity that satisfies that conditions of the theorem. Indeed, consider the rescaled function

$$u_r(x, y) = \frac{u(rx, r^2 y)}{r^2},$$

which belongs to $P_{R_3}(M R_3^2)$, apply Lemma 8.1 and rescale back to the function u. We obtain that, after a rotation in x-coordinates, the free boundary $\partial\{u > 0\}$ is given in

$$U = B^{x'}_{c\sigma r/K} \times (-sr, r/K) \times B^{y}_{r^2/K},$$

for some $0 < s \leqslant 1$, as a graph

$$x_m = f(x', y), \quad |x'| \leqslant c\sigma r/K,\ |y| \leqslant r^2/K,$$

with

$$\|\nabla_{x',y} f\|_{C^\alpha} \leqslant C(r, \sigma, M, \mathbf{G}).$$

To complete the proof of the theorem, observe that

$$\delta_r\big(e, \{u > 0\}\big) > \sigma \quad \Rightarrow \quad \delta_{2r}\big(g, \{u > 0\}\big) > c\sigma, \quad g \in B_{r/2},$$

for some $c = c(G) > 0$, provided $0 < r < r_0(\mathbf{G})$. This can be easily seen from Lemma 4.6. Thus, the representation as above is also valid for the set $g^{-1} \cdot \partial\{u > 0\}$, for any $g \in B_{r/2}$, perhaps with different constants in the estimates. This completes the proof of the theorem. $\square$

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