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Advances in Mathematics 215 (2007) 292–378
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Mathematics
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Sub-Riemannian calculus on hypersurfaces in Carnot groups

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Received 24 August 2006; accepted 3 April 2007

Available online 21 April 2007

Communicated by Luis Caffarelli

Abstract

We develop a sub-Riemannian calculus for hypersurfaces in graded nilpotent Lie groups. We introduce an appropriate geometric framework, such as horizontal Levi-Civita connection, second fundamental form, and horizontal Laplace–Beltrami operator. We analyze the relevant minimal surfaces and prove some basic integration by parts formulas. Using the latter we establish general first and second variation formulas for the horizontal perimeter in the Heisenberg group. Such formulas play a fundamental role in the sub-Riemannian Bernstein problem.

Keywords: Horizontal Levi-Civita connection; Horizontal second fundamental form; H-mean curvature; Intrinsic integration by parts; First and second variation of the horizontal perimeter

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[1] Supported in part by NSF grants DMS-0002801 and CAREER DMS-0239771.
[2] Supported in part by NSF Grant DMS-0300477.

0001-8708/$ – see front matter
doi:10.1016/j.aim.2007.04.004

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1. Introduction

The purpose of the present paper is to develop a sub-Riemannian calculus on smooth hypersurfaces in a class of nilpotent Lie groups which possess a rich geometry. Such groups arise as tangent spaces of Gromov–Hausdorff limits of Riemannian manifolds [4,13,75], and since they can be traced back to the foundational paper of Carathéodory [11] on Carnot thermodynamics, they have been christened Carnot groups by Gromov, see [53,54]. Our main motivation is, in a broad sense, the applications of the relevant calculus to the study of the non-characteristic Bernstein problem. We believe that, in perspective, our results will also prove useful to the study of the regularity theory of hypersurfaces of constant mean curvature in such settings. Both these problems have recently received increasing attention from several groups of mathematicians and there exists nowadays a wide literature. The following is only a partial list of references [1–3,6, 14–16,26,28,43–45,48,49,56,74,84,85,90]. For an extensive bibliography we refer the reader to the recent monographs [10,27] (see also the forthcoming book [47]), and to the papers [26,28]. Carnot groups play a pervasive role in analysis, geometry, and in various branches of the applied sciences, ranging from problems in optimal control and robotics, crystallography, mathematical finance, and neurophysiology of the brain. This latter aspect, in particular, has been recently brought to light in some very interesting works of Petitot and Tondut [80–82], and of Citti and Sarti [18,19], see also [20,21]. These latter works have shown that there exists a close link between the way in which the brain chooses to complete the missing visual data in the first layer of the cerebral cortex, V_1, and the minimal surfaces in a specific sub-Riemannian space, the so-called roto-translation group, arising in the mathematical modeling of the visual cortex V_1, see also [57].

To describe the content of this paper we recall that during the past century the study of minimal surfaces has been one of the main driving forces in mathematics. Such development was prompted by the study of the problems of Plateau and Bernstein which has led, as a by-product, to the development of the Geometric Measure Theory, see [39,72,73]. Minimal surfaces also play a central role in the positive mass theorem from relativity due to Schoen and Yau [89], see also the lecture notes [88]. Given the substantial progress which has occurred during the past decade in the theory of subelliptic equations, and in those closely connected aspects of geometric measure theory in sub-Riemannian spaces, it seems natural at this point to direct the attention to the understanding of those tools which are necessary for the development of a rich theory of minimal surfaces. As we mentioned above, in this paper we solely discuss hypersurfaces. Minimal manifolds of higher codimension are also of interest and we hope to investigate them in future studies.

In classical geometry a central notion is that of area of a (smooth) hypersurface. Such notion was extended by De Giorgi [29,30], with the introduction of his variational theory of perimeters which allowed to assign an "area" also to sets which are not a priori smooth. In a Carnot group $\boldsymbol{G}$ there exists a corresponding variational notion of perimeter adapted to the horizontal bundle $H\boldsymbol{G}$ (for a brief introduction to Carnot groups we refer the reader to Section 2). Given a distribution of smooth left-invariant vector fields $X = \{X_1, \dots, X_m\}$ which is an orthonormal basis of the horizontal bundle (and therefore it is bracket-generating for $T\boldsymbol{G}$), and an open set $\Omega \subset \boldsymbol{G}$, we let

$$\mathcal{F}(\Omega) = \left\{ \zeta = \sum_{i=1}^{m} \zeta_i X_i \in C_0^1(\Omega, H\boldsymbol{G}) \;\middle|\; |\zeta|_\infty = \sup_{\Omega} |\zeta| \leqslant 1 \right\}.$$

For a function $u \in L^1_{\mathrm{loc}}(\Omega)$, the H-variation of u with respect to Ω is defined by

$$\mathrm{Var}_H(u; \Omega) = \sup_{\zeta \in \mathcal{F}(\Omega)} \int_{\boldsymbol{G}} u \sum_{i=1}^{m} X_i \zeta_i \, dg.$$

A function $u \in L^1(\Omega)$ is called of bounded H-variation in Ω if $\mathrm{Var}_H(u; \Omega) < \infty$. The space $BV_H(\Omega)$ of functions with bounded H-variation in Ω, endowed with the norm

$$\|u\|_{BV_H(\Omega)} = \|u\|_{L^1(\Omega)} + \mathrm{Var}_H(u; \Omega),$$

is a Banach space. Similarly to the classical theory (for the latter, see for instance [52,97]), such space constitutes the appropriate replacement of the horizontal Sobolev $W^{1,1}_H(\Omega)$ space in the study of the relevant minimal surfaces, see [48]. Let now $E \subset \boldsymbol{G}$ be a measurable set, $\Omega \subset \boldsymbol{G}$ be an open set. The H-perimeter of E with respect to Ω is defined by the equation

$$P_H(E; \Omega) = \mathrm{Var}_H(\chi_E; \Omega), \tag{1.1}$$

where χ_E denotes the indicator function of E, see [9]. When E possesses sufficient regularity, e.g. when $\mathcal{S} = \partial E$ is a hypersurface of class C^2, then one finds that

$$P_H(E; \Omega) = \int_{\Omega \cap \partial E} d\sigma_H = \int_{\Omega \cap \partial E} \frac{|\boldsymbol{N}^H|}{|\boldsymbol{N}|} dH_{N-1}, \tag{1.2}$$

where we have denoted with $\boldsymbol{N}^H$ the projection of the (non-unit) Riemannian normal to ∂E onto the subbundle $H\boldsymbol{G}$. It is interesting to note that, in this situation, a useful alternative understanding of the H-perimeter (1.1) can be obtained by blowing-up the (suitably normalized) standard surface measure associated with the Riemannian regularization of the sub-Riemannian metric, see Theorem 8.5 below.

A "minimal surface" in Ω was defined in [48] as the boundary of a set of least H-perimeter, among all those with the same boundaries outside Ω. The existence of such "surfaces" (a priori, these are just sets of locally finite H-perimeter), and a measure theoretic solution of the Plateau problem, were also established in [48] following the classical approach of De Giorgi [29–31].

The natural question arises of whether such measure theoretic minimal surfaces have, at least when they are sufficiently smooth, vanishing "mean curvature." This prompts to investigate an appropriate notion of mean curvature adapted to the horizontal bundle HG. For level sets such a notion was proposed by one of us back in 1997, see [46]. For the Heisenberg group $\mathbb{H}^1$, another notion of mean curvature was introduced by Pauls in [79], who studied the solvability of the Plateau problem by means of the Riemannian regularization of the sub-Riemannian metric. For a surface in a three-dimensional CR manifold, yet another notion of mean curvature has been recently proposed in [15]. For instance, if the ambient manifold is the Heisenberg group $\mathbb{H}^1$, then the mean curvature of a surface $\mathcal{S} \subset \mathbb{H}^1$ is defined as the standard curvature of the curve of intersection of $\mathcal{S}$ with the horizontal plane passing through the base point. We note that, for surfaces in a Carnot group, this same notion of curvature was also already explicitly introduced in [24]. In this paper, given a C^2 hypersurface $\mathcal{S}$ in a Carnot group $\boldsymbol{G}$, we introduce a second fundamental form on $\mathcal{S}$ adapted to the horizontal subbundle $H\boldsymbol{G}$, and a geometric notion of mean curvature of $\mathcal{S}$, and we show that the latter coincides with either one of those proposed in [15,46,79], see Propositions 9.9, 9.13 and 9.14.

In a Carnot group $\boldsymbol{G}$, with grading of the Lie algebra $\mathfrak{g} = V_1 \oplus \cdots \oplus V_r$, we define a smooth left-invariant Riemannian metric $\langle\cdot,\cdot\rangle$ by imposing that the vector fields $X_1, \ldots, X_m, \ldots, X_{r,m_r}$, defined in (2.15), be orthonormal, see Section 5. We can thus consider the Riemannian connection ∇ on $\boldsymbol{G}$ induced by $\langle\cdot,\cdot\rangle$. We define the horizontal Levi-Civita connection ∇^H on $\boldsymbol{G}$ by projecting ∇ onto the horizontal bundle $H\boldsymbol{G}$, see Section 5. We note explicitly that ∇^H is, in essence, Cartan's non-holonomic connection introduced in his address at the Bologna International Congress of Mathematicians in 1928, see [12].

In Section 6, given an oriented C^2 hypersurface $\mathcal{S} \subset \boldsymbol{G}$, with Riemannian normal $\boldsymbol{N}$, we define the horizontal normal $\boldsymbol{N}^H$ to $\mathcal{S}$ as the projection of $\boldsymbol{N}$ onto the horizontal bundle, and the horizontal Gauss map as $\boldsymbol{\nu}^H = \boldsymbol{N}^H/|\boldsymbol{N}^H|$. Note that $|\boldsymbol{N}^H| \neq 0$ at every point which does not belong to the characteristic set $\Sigma_{\mathcal{S}}$ of $\mathcal{S}$. We recall that the latter is the collection of all points $g \in \mathcal{S}$ at which $H_g\boldsymbol{G} \subset T_g\mathcal{S}$. An important notion is that of horizontal tangent bundle $HT\mathcal{S}$ to $\mathcal{S}$, whose fiber $HT_g\mathcal{S}$ at each point $g \in \mathcal{S} \setminus \Sigma_{\mathcal{S}}$ is defined as the collection of all horizontal vectors which are orthogonal to $\boldsymbol{N}^H$. It can be easily recognized that $HT_g\mathcal{S} = T_g\mathcal{S} \cap H_g\boldsymbol{G}$. To obtain a connection on $HT\mathcal{S}$ we then project the horizontal Levi-Civita connection ∇^H on the horizontal tangent bundle $HT\mathcal{S}$. More explicitly, for every $X, Y \in C^1(\mathcal{S}; HT\mathcal{S})$ we define

$$\nabla_X^{H,\mathcal{S}} Y = \nabla_{\overline{X}}^H \overline{Y} - \langle \nabla_{\overline{X}}^H \overline{Y}, \boldsymbol{\nu}^H \rangle \boldsymbol{\nu}^H,$$

where $\overline{X}$, $\overline{Y}$ are any two horizontal vector fields on $\boldsymbol{G}$ such that $\overline{X} = X$, $\overline{Y} = Y$ on $\mathcal{S}$ (note that the above definition does not depend on the choice of the extensions). Unlike its Riemannian counterpart, the connection $\nabla_X^{H,\mathcal{S}} Y$ is not torsion free in general, and therefore it is not Levi-Civita in general. This is due to the fact that, given $X, Y \in C^1(\mathcal{S}; HT\mathcal{S})$, the projection $[X, Y]^H$ of $[X, Y]$ onto the horizontal bundle of $H\boldsymbol{G}$ does not in general belong to the horizontal tangent space to $\mathcal{S}$, $HT\mathcal{S}$. We note in passing that an interesting situation in which $\nabla_X^{H,\mathcal{S}} Y$ is Levi-Civita is that when $\boldsymbol{G} = \mathbb{H}^1$, the first Heisenberg group, or when $\boldsymbol{G} = \mathfrak{E}$, the four-dimensional Engel group, see Section 3.

Inspired by the Riemannian situation we next project ∇^H along the horizontal Gauss map $\boldsymbol{\nu}^H$. In this way we are able to introduce the following notion of horizontal second fundamental form on $\mathcal{S}$

$$II^{H,\mathcal{S}}(X, Y) = \langle \nabla_X^H Y, \boldsymbol{\nu}^H \rangle \boldsymbol{\nu}^H, \tag{1.3}$$

where $X, Y \in C^1(\mathcal{S}; HT\mathcal{S})$. Since $[X, Y]^H$ is not in general in $HT\mathcal{S}$, unlike its Riemannian predecessor (1.3) is not symmetric. One has in fact,

$$II^{H,\mathcal{S}}(X, Y) - II^{H,\mathcal{S}}(Y, X) = \langle [X, Y]^H, \boldsymbol{\nu}^H \rangle \boldsymbol{\nu}^H \neq 0.$$

At every point $g_0 \notin \Sigma_{\mathcal{S}}$, we define the horizontal mean curvature $\mathcal{H}$ (or H-mean curvature) of $\mathcal{S}$ as the negative of the trace of the (symmetrized) second fundamental form. If $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{m-1}\}$ is an orthonormal basis of $HT\mathcal{S}$, we thus have

$$\mathcal{H} = -\sum_{i=1}^{m-1} \langle \nabla^H_{\boldsymbol{e}_i} \boldsymbol{e}_i, \boldsymbol{\nu}^H \rangle. \tag{1.4}$$

If instead $g_0 \in \Sigma_{\mathcal{S}}$, then we define $\mathcal{H}(g_0)$ as the $\lim_{g \to g_0,\, g \notin \Sigma_{\mathcal{S}}} \mathcal{H}(g)$, whenever such limit exists. A C^2 hypersurface $\mathcal{S} \subset \boldsymbol{G}$ is said to have constant mean-curvature $c \in \mathbb{R}$ if $\mathcal{H} \equiv c$ as a continuous function on $\mathcal{S}$. We call $\mathcal{S}$ H-minimal if $\mathcal{H} \equiv 0$ on $\mathcal{S}$.

Having introduced the notion of H-mean curvature, and H-minimal surface, following the steps of the classical developments on the Bernstein problem, it is natural to study questions of stability, regularity, etc. It is well known that in the classical setting when $\mathcal{S} \subset \mathbb{R}^n$, with the standard surface measure $d\sigma$, an essential role in this program is played by the following integration by parts formula, see e.g. [52],

$$\int_{\mathcal{S}} \nabla f \, d\sigma = (n-1) \int_{\mathcal{S}} f H \boldsymbol{\nu} \, d\sigma, \tag{1.5}$$

where ∇ denotes the Levi-Civita connection on $\mathcal{S}$, $f \in C_0^2(\mathcal{S})$, and H is the mean curvature of $\mathcal{S}$. For instance, the fundamental a priori gradient estimates for minimal surfaces are derived from (1.5), see [5]. In Section 10 we establish an appropriate generalization of (1.5) to the case of a hypersurface in a Carnot group. The interesting feature of such intrinsic integration by parts formula is that the role of the surface measure is played by the H-perimeter. Furthermore, it links the horizontal connection $\nabla^{H,\mathcal{S}}$ on $\mathcal{S}$ to the H-mean curvature of $\mathcal{S}$. The relevant results states that for every $f \in C_0^1(\mathcal{S} \setminus \Sigma_{\mathcal{S}})$,

$$\int_{\mathcal{S}} \nabla^{H,\mathcal{S}} f \, d\sigma_H = \int_{\mathcal{S}} f \{ \mathcal{H} \boldsymbol{\nu}^H - \boldsymbol{c}^{H,\mathcal{S}} \} \, d\sigma_H, \tag{1.6}$$

where $\boldsymbol{c}^{H,\mathcal{S}}$ is a vector field on $\mathcal{S} \setminus \Sigma_{\mathcal{S}}$ with values in the horizontal tangent space $HT\mathcal{S}$, see Theorem 10.1. This result plays a central role in the establishment of the fundamental first and second variation formulas for the H-perimeter in Sections 14 and 15. Although (1.6) formally resembles (1.5), and in fact it encompasses its Riemannian predecessor, the presence of the vector field $\boldsymbol{c}^{H,\mathcal{S}}$ represents a new aspect which reflects the lack of torsion freeness of the connection $\nabla^{H,\mathcal{S}}$, see also Proposition 5.1 below. In the Abelian case when $\boldsymbol{G} \cong \mathbb{R}^n$, then $\boldsymbol{c}^{H,\mathcal{S}} \equiv 0$ and we recover (1.5). Another interesting situation in which $\boldsymbol{c}^{H,\mathcal{S}} \equiv 0$ is when $\mathcal{S}$ is a vertical cylinder on the horizontal layer, i.e., when $\mathcal{S}$ is locally described by a defining function which depends only on the horizontal variables.

Using the connection $\nabla^{H,\mathcal{S}}$ we define two differential operators on $\mathcal{S}$, see Definition 11.1. The former, denoted by $\Delta_{H,\mathcal{S}}$, is a sub-Riemannian version of the classical Laplace–Beltrami operator on a manifold. The latter, indicated by $\hat{\Delta}_{H,\mathcal{S}}$, contains an additional drift term, and is motivated by the intrinsic integration by parts formula (1.6). Its main raison d'être, in fact, is that a Stokes' type theorem holds for it, see Corollary 11.3. Formula (1.6) implies the following identity

$$\int_{\mathcal{S}} \langle \nabla^{H,\mathcal{S}} u, \nabla^{H,\mathcal{S}} \zeta \rangle \, d\sigma_H = -\int_{\mathcal{S}} u \hat{\Delta}_{H,\mathcal{S}} \zeta \, d\sigma_H$$

for every $u \in C^1(\mathcal{S})$, and every $\zeta \in C_0^2(\mathcal{S} \setminus \Sigma_{\mathcal{S}})$. Using this identity, we introduce a notion of sub-harmonicity on $\mathcal{S}$, see Definitions 11.13, 11.14. It is an interesting open question to study the properties of non-negative sub-harmonic functions on $\mathcal{S}$. For instance, when $\mathcal{S}$ is H-minimal, do such functions satisfy some kind of sub-mean value formula?

In Theorem 12.1 we connect the operator $\Delta_{H,\mathcal{S}}$ to the flow by horizontal mean curvature recently introduced by Bonk and Capogna [6]. We show that, similarly to its Riemannian counterpart, such flow satisfies the following interesting partial differential equation involving the horizontal tangential Laplacian $\Delta_{H,\mathcal{S}^t}$ on the hypersurfaces $\mathcal{S}^t = F(\mathcal{S}, t)$, images of $\mathcal{S}$ through the flow $F(\cdot, t)$, see Theorem 12.1,

$$\left\langle \frac{\partial F}{\partial t}, \boldsymbol{N} \right\rangle = \langle \Delta_{H,\mathcal{S}^t} F, \boldsymbol{N} \rangle.$$

Sections 13–15 are entirely devoted to a geometric study of C^2 surfaces in the Heisenberg group $\mathbb{H}^1$. In this setting, given a C^2 surface $\mathcal{S}$ with horizontal Gauss map $\boldsymbol{\nu}^H$, one easily recognizes that $HT\mathcal{S}$ is spanned by the single vector field $(\boldsymbol{\nu}^H)^\perp$. The triple $\{(\boldsymbol{\nu}^H)^\perp, \boldsymbol{\nu}^H, T\}$ forms an orthonormal moving frame on $\mathcal{S}$. In Section 13 we establish various geometric identities which connect horizontal covariant differentiation along such frame to geometric quantities such as the H-mean curvature and its derivatives.

In Section 14 we use such identities, in combination with some notable integration by parts formulas which follow from Theorem 10.1, see Lemma 14.8. This lemma plays a crucial role in establishing the first and second variation formulas for the H-perimeter measure which constitute the main results of the section, see Theorems 14.3 and 14.5. The former allows to give a positive answer to the question raised above: is a C^2 H-minimal surface a stationary point of the H-perimeter? In Theorem 14.3 we show that for $\mathcal{S} \subset \mathbb{H}^1$, the first-variation of the H-perimeter for a deformation of $\mathcal{S}$ along a vector field $\mathcal{X} \in C_0^2(\mathcal{S} \setminus \Sigma_{\mathcal{S}}, \mathbb{H}^1)$ is given by

$$\mathcal{V}_I^H(\mathcal{S}; \mathcal{X}) = \int_{\mathcal{S}} \mathcal{H} \frac{\langle \mathcal{X}, \boldsymbol{\nu} \rangle}{\langle \boldsymbol{\nu}^H, \boldsymbol{\nu} \rangle} \, d\sigma_H, \tag{1.7}$$

where $\boldsymbol{\nu} = \boldsymbol{N}/|\boldsymbol{N}|$ represents the Riemannian Gauss map on $\mathcal{S}$. In particular, $\mathcal{S}$ is stationary if and only if it is H-minimal (see also the less intrinsic first variation formula in Theorem 9.1 for deformations along the normal $\boldsymbol{N}$ and valid for hypersurfaces in an arbitrary Carnot group).

The central result of Section 14 is Theorem 14.5, which provides a second variation formula for the H-perimeter of $\mathcal{S}$. The proof of such formula is considerably more complex than that of (1.7), and obtaining it has required a substantial effort. Despite such effort we notice, however, that Theorem 14.5 is in practice not as useful as one would hope since it contains several terms whose geometric content is not transparent, and which are very difficult to handle. For the applications of the second variation formula to the fundamental question of stability it is crucial to be able to extract the geometry from Theorem 14.5. In order to do so one needs to eliminate in the integrals involved the various products of covariant derivatives of the projections of the testing vector field $\mathcal{X}$ along the moving frame $\{(\boldsymbol{\nu}^H)^\perp, \boldsymbol{\nu}^H, T\}$. In this endeavor one has to choose with extreme care the terms to play one against the other, so to be able to exploit the delicate cancelations deriving from the various Lagrangian quantities involved. Section 15 is devoted to this goal. In Theorem 15.2 we have succeeded in deriving the following geometric second variation formula:

$$\mathcal{V}_{II}^H(\mathcal{S};\mathcal{X}) = \int_{\mathcal{S}} \left\{ \left|\nabla^{H,\mathcal{S}} F\right|^2 + \left(2\mathcal{A} - \overline{\omega}^2\right) F^2 \right\} d\sigma_H, \tag{1.8}$$

where $\mathcal{S} \subset \mathbb{H}^1$ is an H-minimal surface, $\mathcal{X}$ is as in (1.7), and we have set

$$F = \frac{\langle \mathcal{X}, \boldsymbol{\nu}\rangle}{\langle \boldsymbol{\nu}^H, \boldsymbol{\nu}\rangle}.$$

The reader should compare (1.8) with the second variation formula in [8, p. 153]. The coefficient $2\mathcal{A} - \overline{\omega}^2$ of F^2 in (1.8) is a geometric quantity which involves the projection of $\boldsymbol{N}$ along T, and its horizontal covariant derivative along the vector field $(\boldsymbol{\nu}^H)^\perp$. With (1.8) in hands, one can attack the fundamental question of the stability. A non-characteristic H-minimal surface $\mathcal{S}$ is called *stable* if $\mathcal{V}_{II}^H(\mathcal{S};\mathcal{X}) \geqslant 0$ for any $\mathcal{X} \in C_0^2(\mathcal{S}, \mathbb{H}^1)$. In view of (1.8) we see that a surface $\mathcal{S}$ is stable if and only if the following stability inequality holds on $\mathcal{S}$

$$\int_{\mathcal{S}} \left(\overline{\omega}^2 - 2\mathcal{A}\right) F^2 \, d\sigma_H \leqslant \int_{\mathcal{S}} \left|\nabla^{H,\mathcal{S}} F\right|^2 d\sigma_H. \tag{1.9}$$

We emphasize that one can think of (1.9) as a Hardy type inequality on $\mathcal{S}$. One should compare (1.9) with its Riemannian counterpart, see e.g. inequality (1.105) in [22] for normal deformations.

We emphasize that the study of the stability is an important new aspect in the sub-Riemannian Bernstein problem. To clarify this point we recall the well-known fact that in the classical Bernstein problem, stability does not apparently play any role. This is due to the fact that the area functional for a graph $x_{n+1} = u(x)$, $x \in \Omega \subset \mathbb{R}^n$,

$$A(u) = \int_{\Omega} \sqrt{1 + |\nabla u|^2} \, dx,$$

is convex. As a consequence, a critical point of $A(u)$ is also a local minimizer, and therefore stable. By contrast, the sub-Riemannian area functional, the H-perimeter (1.1), is not convex, see [28], and the resulting Euler–Lagrange equation is not elliptic, but degenerate hyperbolic (-elliptic). Using the stability inequality (1.9), it has been recently shown in [26] that, contrarily to what was believed by several experts, the entire H-minimal graph $x = yt$ in $\mathbb{H}^1$, which has empty characteristic locus, is in fact unstable. This discovery has underscored the role of the stability in the sub-Riemannian Bernstein problem and opened the way to the solution of the latter. Subsequently, in fact, this result has been generalized in [28], where it has been proved the instability of every graph in $\mathbb{H}^1$ of the type $x = yG(t)$, with $y \in \mathbb{R}$, $t \in I \subset \mathbb{R}$, with $G \in C^2(I)$, and such that $G' > 0$ on some subinterval $J \subset I$. On the other hand, it has also been shown in [28] that every entire H-minimal graph in $\mathbb{H}^1$, with empty characteristic locus, and which is not itself a vertical plane $ax + by = \gamma$, after possibly a left-translation and a rotation about the t-axis, contains a graphical strip of the type $x = yG(t)$, with $G' > 0$ on some subinterval $J \subset \mathbb{R}$. Combining these two results, the authors have obtained a solution of the following sub-Riemannian Bernstein problem: *The only stable H-minimal entire graphs in $\mathbb{H}^1$, with empty characteristic locus, are the vertical planes*. Some of the ideas in [26] have also been used in the recent paper [3] to prove a similar Bernstein type theorem for the entire intrinsic graphs introduced in [45].

In closing we mention some recent papers that are connected to the present one. In [56] Hladky and Pauls have introduced a notion of mean curvature and derived the relevant minimal surface equation, but not the first and second variation formulas, for hypersurfaces in a class of sub-Riemannian spaces which encompasses that of Carnot groups. Similarly to ours, their results hold for C^2 hypersurfaces and away from the characteristic set. Their interesting approach can be seen as a generalization of the Webster–Tanaka geometric framework for CR manifolds, and systematically exploits the Lagrangian framework of Bryant, Griffiths and Grossmann [8]. Although the second fundamental form proposed in [56] is different from the one introduced in this paper, we notice that for Carnot groups their notion of mean curvature coincides with (1.4). An approach similar to that in [56] has been independently taken in the interesting paper by Montefalcone [74], who has also obtained some general first and second variation formulas similar to ours. Simultaneously, in her PhD dissertation C. Selby [90] has obtained first and second variation formulas with a completely different approach. Her study is based on a deep analysis of the asymptotic behavior of the left-invariant Riemannian metric obtained by blowing-up the non-horizontal directions.

2. Carnot groups

In this section we collect some of the basic geometric facts about Carnot groups. We particularly emphasize those properties which are useful in this paper. For more extensive sources we refer the reader to [4,34–36,41,47,53,54,58,75,77,78,86,92,94,96]. A *sub-Riemannian space* is a triple (M, HM, d) constituted by a connected Riemannian manifold M, with Riemannian distance $d_{\mathcal{R}}$, a subbundle of the tangent bundle $HM \subset TM$, and the Carnot–Carathéodory (CC) distance d generated by HM. Such distance is defined by minimizing only on those absolutely continuous paths γ whose tangent vector $\gamma'(t)$ belongs to $H_{\gamma(t)}M$, see [4,76]. Riemannian manifolds are a special example of sub-Riemannian spaces. They correspond to the case $HM = TM$. The tangent space of a sub-Riemannian space is itself a sub-Riemannian space (or a quotient of such spaces), but of a special type. It is a graded Lie group whose Lie algebra is nilpotent. These groups, which owe their name to the foundational paper of Carathéodory [11] on Carnot thermodynamics, occupy a central position in the study of hypoelliptic partial differential equations, harmonic analysis, sub-Riemannian geometry, CR geometric function theory, but also in the applied sciences such as mathematical finance, neurophysiology of the brain, mechanical engineering. They are called Carnot groups.

A Carnot group of step r is a connected, simply connected Lie group $\boldsymbol{G}$ whose Lie algebra $\mathfrak{g}$ admits a stratification $\mathfrak{g} = V_1 \oplus \cdots \oplus V_r$ which is r-nilpotent, i.e., $[V_1, V_j] = V_{j+1}$, $j = 1, \ldots, r-1$, $[V_j, V_r] = \{0\}$, $j = 1, \ldots, r$. We assume henceforth that $\mathfrak{g}$ is endowed with a scalar product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ with respect to which the V_j's are mutually orthogonal. A trivial example of (an Abelian) Carnot group is $\boldsymbol{G} = \mathbb{R}^n$, whose Lie algebra admits the trivial stratification

$\mathfrak{g} = V_1 = \mathbb{R}^n$. The simplest non-Abelian example of a Carnot group of step $r = 2$ is the $(2n+1)$-dimensional Heisenberg group $\mathbb{H}^n$, which is described in Section 3. Given a Carnot group $\boldsymbol{G}$, by the above assumptions on the Lie algebra one immediately sees that any basis of the *horizontal layer* V_1 generates the whole $\mathfrak{g}$. We will respectively denote by

$$L_g(g') = gg', \qquad R_g(g') = g'g, \tag{2.1}$$

the operators of left- and right-translation by an element $g \in \boldsymbol{G}$.

The exponential mapping $\exp : \mathfrak{g} \to \boldsymbol{G}$ defines an analytic diffeomorphism onto $\boldsymbol{G}$. We recall the important Baker–Campbell–Hausdorff formula, see, e.g., [95, Section 2.15],

$$\exp(\xi)\exp(\eta) = \exp\left(\xi + \eta + \frac{1}{2}[\xi, \eta] + \frac{1}{12}\big\{\big[\xi, [\xi, \eta]\big] - \big[\eta, [\xi, \eta]\big]\big\} + \cdots\right), \tag{2.2}$$

where the dots indicate commutators of order four and higher. Each element of the layer V_j is assigned the formal degree j. Accordingly, one defines dilations on $\mathfrak{g}$ by the rule

$$\Delta_\lambda \xi = \lambda \xi_1 + \cdots + \lambda^r \xi_r,$$

provided that $\xi = \xi_1 + \cdots + \xi_r \in \mathfrak{g}$, with $\xi_j \in V_j$. Using the exponential mapping $\exp : \mathfrak{g} \to \boldsymbol{G}$, these anisotropic dilations are then transferred to the group $\boldsymbol{G}$ as follows

$$\delta_\lambda(g) = \exp \circ \Delta_\lambda \circ \exp^{-1} g.$$

Throughout the paper we will indicate by dg the bi-invariant Haar measure on $\boldsymbol{G}$ obtained by lifting via the exponential map exp the Lebesgue measure on $\mathfrak{g}$. We let $m_j = \dim V_j$, $j = 1, \ldots, r$, and denote by $N = m_1 + \cdots + m_r$ the topological dimension of $\boldsymbol{G}$. One easily checks that

$$(d \circ \delta_\lambda)(g) = \lambda^Q \, dg, \quad \text{where } Q = \sum_{j=1}^{r} j m_j.$$

The number Q, called the *homogeneous dimension* of $\boldsymbol{G}$, plays an important role in the analysis of Carnot groups. In the non-Abelian case $r > 1$, one clearly has $Q > N$.

We denote by $d(g, g')$ the *CC distance* on $\boldsymbol{G}$ associated with the system X. It is well known that $d(g, g')$ is equivalent to the *gauge pseudo-metric* $\rho(g, g')$ on $\boldsymbol{G}$, i.e., there exists a constant $C = C(\boldsymbol{G}) > 0$ such that

$$C\rho(g, g') \leqslant d(g, g') \leqslant C^{-1}\rho(g, g'), \quad g, g' \in \boldsymbol{G}, \tag{2.3}$$

see [11,17,76,83,96]. The pseudo-distance $\rho(g, g')$ is defined as follows, see [41]. Let $|\cdot|$ denote the Euclidean distance to the origin on $\mathfrak{g}$. For $\xi = \xi_1 + \cdots + \xi_r \in \mathfrak{g}$, $\xi_j \in V_j$, one lets

$$|\xi|_{\mathfrak{g}} = \left(\sum_{j=1}^{r} |\xi_j|^{2r!/j}\right)^{1/2r!}, \qquad |g|_{\boldsymbol{G}} = \left|\exp^{-1} g\right|_{\mathfrak{g}}, \quad g \in \boldsymbol{G}, \tag{2.4}$$

and defines

$$\rho(g, g') = \left|g^{-1}g'\right|_{\boldsymbol{G}}. \tag{2.5}$$

Both d and ρ are invariant under left-translations

$$d\big(L_g(g'), L_g(g'')\big) = d(g', g''), \qquad \rho\big(L_g(g'), L_g(g'')\big) = \rho(g', g'') \tag{2.6}$$

and homogeneous of degree one

$$d\big(\delta_\lambda(g'), \delta_\lambda(g'')\big) = \lambda d(g', g''), \qquad \rho\big(\delta_\lambda(g'), \delta_\lambda(g'')\big) = \lambda\rho(g', g''). \tag{2.7}$$

Denoting respectively with

$$B(g, R) = \big\{g' \in \boldsymbol{G} \bigm| d(g', g) < R\big\}, \qquad B_\rho(g, R) = \big\{g' \in \boldsymbol{G} \bigm| \rho(g', g) < R\big\}, \tag{2.8}$$

the *CC* ball and the gauge pseudo-ball centered at g with radius R, one easily recognizes that there exist $\omega = \omega(\boldsymbol{G}) > 0$, and $\alpha = \alpha(\boldsymbol{G}) > 0$ such that

$$\big|B(g, R)\big| = \omega R^Q, \qquad \big|B_\rho(g, R)\big| = \alpha R^Q, \quad g \in \boldsymbol{G},\ R > 0. \tag{2.9}$$

Let $\pi_j : \mathfrak{g} \to V_j$ denote the projection onto the jth layer of $\mathfrak{g}$. Since the exponential map $\exp : \mathfrak{g} \to \boldsymbol{G}$ is a global analytic diffeomorphism, we can define analytic maps $\xi_j : \boldsymbol{G} \to V_j$, $j = 1, \ldots, r$, by letting $\xi_j = \pi_j \circ \exp^{-1}$. As a rule, we will use letters g, g', g'', g_0 for points in $\boldsymbol{G}$, whereas we will reserve the letters $\xi, \xi', \xi'', \xi_0, \eta$, for elements of the Lie algebra $\mathfrak{g}$. The notation $\{e_{j,1}, \ldots, e_{j,m_j}\}$, $j = 1, \ldots, r$, will indicate a fixed orthonormal basis of the jth layer V_j. For $g \in \boldsymbol{G}$, the projection of the *exponential coordinates* of g onto the layer V_j, $j = 1, \ldots, r$, are defined as follows

$$x_{j,s}(g) = \big\langle \xi_j(g), e_{j,s}\big\rangle_{\mathfrak{g}}, \quad s = 1, \ldots, m_j. \tag{2.10}$$

The vector $\xi_j(g) \in V_j$, $j = 1, \ldots, r$, will be routinely identified with the point

$$\big(x_{j,1}(g), \ldots, x_{j,m_j}(g)\big) \in \mathbb{R}^{m_j}.$$

Since Carnot groups of step $r = 2$ often play a special role in analysis and geometry, it will be convenient to have a simplified notation for objects in the horizontal layer V_1, and in the first vertical layer V_2. For simplicity, we set $m = m_1$, $k = m_2$, and let

$$\{e_1, \ldots, e_m\} = \{e_{1,1}, \ldots, e_{1,m_1}\}, \qquad \{\epsilon_1, \ldots, \epsilon_k\} = \{e_{2,1}, \ldots, e_{2,m_1}\}. \tag{2.11}$$

We indicate with

$$x_i(g) = \big\langle \xi_1(g), e_i\big\rangle_{\mathfrak{g}}, \quad i = 1, \ldots, m, \qquad t_s(g) = \big\langle \xi_2(g), \epsilon_s\big\rangle_{\mathfrak{g}}, \quad s = 1, \ldots, k, \tag{2.12}$$

the projections of the exponential coordinates of g onto V_1 and V_2, respectively. Whenever convenient, we will identify $g \in \boldsymbol{G}$ with its exponential coordinates

$$x(g) \overset{\text{def}}{=} \big(x_1(g), \ldots, x_m(g), t_1(g), \ldots, t_k(g), \ldots, x_{r,1}(g), \ldots, x_{r,m_r}(g)\big) \in \mathbb{R}^N, \tag{2.13}$$

and we will ordinarily drop in the latter the dependence on g, i.e., we will write $g = (x_1, \dots, x_{r,m_r})$.

For later purposes it will be useful to introduce the horizontal group constants of $\boldsymbol{G}$. By the grading assumption on the Lie algebra, we have $[V_1, V_1] = V_2$. Therefore, if $e_i, e_j \in \{e_1, \dots, e_m\}$, we let

$$b_{ij}^s \stackrel{\text{def}}{=} \big\langle [e_i, e_j], \epsilon_s \big\rangle_{\mathfrak{g}}, \quad \text{so that} \quad [e_i, e_j] = \sum_{s=1}^{k} b_{ij}^s \epsilon_s, \quad i, j = 1, \dots, m. \tag{2.14}$$

Consider the orthonormal basis $\{e_1, \dots, e_m, \epsilon_1, \dots, \epsilon_k, \dots, e_{r,1}, \dots, e_{r,m_r}\}$ of $\mathfrak{g}$. Using (2.1) we define left-invariant vector fields on $\boldsymbol{G}$ by letting

$$X_{j,s}(g) = (L_g)_*(e_{j,s}), \quad j = 1, \dots, r, \ s = 1, \dots, m_j, \tag{2.15}$$

where $(L_g)_*$ indicates the differential of L_g. As in (2.11) we use a special notation for the first two layers, and let

$$X_i(g) = (L_g)_*(e_i), \quad i = 1, \dots, m, \qquad T_s(g) = (L_g)_*(\epsilon_s) \quad s = 1, \dots, k, \ g \in \boldsymbol{G}. \tag{2.16}$$

Using the Baker–Campbell–Hausdorff formula (2.2) we can express (2.16) using the exponential coordinates (2.13), obtaining the following lemma.

Lemma 2.1. *For each $i = 1, \dots, m$, and $g = (x_1, \dots, x_{r,m_r})$, we have*

$$\begin{aligned} X_i &= \frac{\partial}{\partial x_i} + \sum_{j=2}^{r} \sum_{s=1}^{m_j} b_{j,i}^s (x_1, \dots, x_{j-1, m_{(j-1)}}) \frac{\partial}{\partial x_{j,s}} \\ &= \frac{\partial}{\partial x_i} + \sum_{j=2}^{r} \sum_{s=1}^{m_j} b_{j,i}^s (\xi_1, \dots, \xi_{j-1}) \frac{\partial}{\partial x_{j,s}}, \end{aligned} \tag{2.17}$$

where each $b_{j,i}^s$ is a homogeneous polynomial of weighted degree $j-1$. *In particular, if $\boldsymbol{G}$ has step $r = 2$, then for every $i = 1, \dots, m$, one has*

$$X_i = \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{s=1}^{k} \big\langle [\xi_1, e_i], \epsilon_s \big\rangle_{\mathfrak{g}} \frac{\partial}{\partial t_s} = \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{s=1}^{k} \sum_{\ell=1}^{m} b_{\ell i}^s x_\ell \frac{\partial}{\partial t_s}, \tag{2.18}$$

where $b_{\ell i}^s$ are the group constants defined by (2.14). *We notice that an immediate consequence of* (2.17) *is that*

$$\operatorname{div}_E X_i = 0, \quad i = 1, \dots, m, \tag{2.19}$$

where $\operatorname{div}_E X_i$ indicates the Euclidean divergence of X_i with respect to the exponential coordinates.

By weighted degree in the statement of Lemma 2.1 we mean that, as previously mentioned, the layer V_j, $j = 1, \dots, r$, in the stratification of $\mathfrak{g}$ is assigned the formal degree j. Correspondingly, each homogeneous monomial $\xi_1^{\alpha_1}\xi_2^{\alpha_2}\cdots\xi_r^{\alpha_r}$, with multi-indices $\alpha_j = (\alpha_{j,1}, \dots, \alpha_{j,m_j})$, $j = 1, \dots, r$, is said to have weighted degree k if

$$\sum_{j=1}^{r} j\left(\sum_{s=1}^{m_j}\alpha_{j,s}\right) = k.$$

3. Two basic models

In this section we describe two basic models of Carnot groups. The first example is the Heisenberg group $\mathbb{H}^n$ with step $r = 2$. Such group plays an ubiquitous role in analysis and geometry, see e.g. [4,10,42,51,59–62,65–68,75,92]. From the standpoint of geometry $\mathbb{H}^n$ constitutes the central prototype of a pseudoconvex CR manifold, with vanishing Webster–Tanaka curvature. In fact, via the Caley transform it can be identified with the boundary of the Siegel upper half-space

$$\mathcal{D}^+ = \left\{(z, z_{n+1}) \in \mathbb{C}^{n+1} \;\middle|\; \operatorname{Im} z_{n+1} > 2\sum_{j=1}^{n}|z_j|^2\right\},$$

see [93, Chapter 12]. The second example is the cyclic, or Engel group $\mathfrak{E}$, of step $r = 3$, see [23,75]. This is an interesting example to keep in mind since it represents the basic prototype of a group of step $r = 3$, and thereby constitutes the next level of difficulty with respect to the Heisenberg group. Some fundamental analytical and geometric properties are true for Carnot groups of step $r = 2$, but fail for groups of step $r \geqslant 3$. In this respect, $\mathfrak{E}$ is the simplest sub-Riemannian model in which to test whether conjectures which are true in step two continue to be valid in step three or higher.

3.1. The Heisenberg group $\mathbb{H}^n$

The underlying manifold of this Lie group is simply $\mathbb{R}^{2n+1}$, with the non-commutative group law

$$gg' = (x, y, t)(x', y', t') = \left(x + x', y + y', t + t' + \frac{1}{2}\big(\langle x, y'\rangle - \langle x', y\rangle\big)\right), \tag{3.1}$$

where we have let $x, x', y, y' \in \mathbb{R}^n$, $t, t' \in \mathbb{R}$. Let $(L_g)_*$ be the differential of the left-translation (3.1). A simple computation shows that

$$\begin{aligned}
(L_g)_*\left(\frac{\partial}{\partial x_i}\right) &\overset{\text{def}}{=} X_i = \frac{\partial}{\partial x_i} - \frac{y_i}{2}\frac{\partial}{\partial t}, \quad i = 1, \dots, n,\\
(L_g)_*\left(\frac{\partial}{\partial y_i}\right) &\overset{\text{def}}{=} X_{n+i} = \frac{\partial}{\partial y_i} + \frac{x_i}{2}\frac{\partial}{\partial t}, \quad i = 1, \dots, n,\\
(L_g)_*\left(\frac{\partial}{\partial t}\right) &\overset{\text{def}}{=} T = \frac{\partial}{\partial t}.
\end{aligned} \tag{3.2}$$

We note that the only non-trivial commutator is

$$[X_i, X_{n+j}] = \delta_{ij} T, \quad i, j = 1, \ldots, n,$$

therefore the vector fields $\{X_1, \ldots, X_{2n}\}$ generate the Lie algebra $\mathfrak{h}_n = \mathbb{R}^{2n+1} = V_1 \oplus V_2$, where $V_1 = \mathbb{R}^{2n} \times \{0\}_t$, $V_2 = \{0\}_{(x,y)} \times \mathbb{R}$. We notice that the sub-Laplacian (see (5.19)) associated with the orthonormal basis $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\}$ of V_1 is

$$\Delta_H = \sum_{j=1}^{2n} X_j^2 = \Delta_{x,y} + \frac{1}{4}\big(|x|^2 + |y|^2\big)\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t}\sum_{j=1}^{n}\left\{y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j}\right\}, \tag{3.3}$$

which coincides with the real part of the complex Kohn–Spencer Laplacian, see [93]. The non-isotropic group dilations are

$$\delta_\lambda(g) = \big(\lambda x, \lambda y, \lambda^2 t\big), \tag{3.4}$$

with homogeneous dimension $Q = 2n + 2$. A convenient renormalization of the gauge (2.4) is given by

$$N(g) = \Big(\big(|x|^2 + |y|^2\big)^2 + 16t^2\Big)^{1/4}. \tag{3.5}$$

The importance of such function is connected with the discovery due to Folland [40] that the fundamental solution of (3.3) is given by

$$\Gamma(g) = \Gamma(g, e) = \frac{C_Q}{N(g)^{Q-2}}, \tag{3.6}$$

where $C_Q < 0$ is an explicit constant.

As a useful illustration, we compute the metric tensor $g_{ij}\, d\xi_i \otimes d\xi_j$ associated with the smooth Riemannian product on $\mathbb{H}^1$ with respect to which $\{X_1, X_2, T\}$ is an orthonormal basis. From (3.2) we obtain

$$\frac{\partial}{\partial x} = X_1 + \frac{y}{2}T, \qquad \frac{\partial}{\partial y} = X_2 - \frac{x}{2}T, \qquad \frac{\partial}{\partial t} = T, \tag{3.7}$$

and therefore the metric coefficients are given by $g_{11} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x}\rangle = 1 + \frac{y^2}{4}$, $g_{12} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\rangle = -\frac{xy}{4}$, etc. One easily finds

$$(g_{ij}) = \begin{pmatrix} 1 + \frac{y^2}{4} & -\frac{xy}{4} & \frac{y}{2} \\ -\frac{xy}{4} & 1 + \frac{x^2}{4} & -\frac{x}{2} \\ \frac{y}{2} & -\frac{x}{2} & 1 \end{pmatrix}. \tag{3.8}$$

Notice that, since $\det(g_{ij}) = 1$, the volume form is given by the standard (Lebesgue) volume form $dx \wedge dy \wedge dt$ in $\mathbb{R}^3$. The inverse (g^{ij}) of the matrix (3.8) is given by

$$(g^{ij}) = \begin{pmatrix} 1 & 0 & -\frac{y}{2} \\ 0 & 1 & \frac{x}{2} \\ -\frac{y}{2} & \frac{x}{2} & 1 + \frac{x^2+y^2}{4} \end{pmatrix}. \tag{3.9}$$

Recall now the expression of the Riemannian gradient in local coordinates, see for instance [55, p. 387],

$$\nabla u = \sum_{i,j=1}^{N} g^{ij} \frac{\partial u}{\partial \xi_i} \frac{\partial}{\partial \xi_j}, \tag{3.10}$$

where we have denoted by $N = \dim(\boldsymbol{G})$. Keeping in mind (3.2), a simple calculation gives

$$(g^{ij}) \begin{pmatrix} u_x \\ u_y \\ u_t \end{pmatrix} = \begin{pmatrix} X_1 u \\ X_2 u \\ \frac{x u_y - y u_x}{2} + \left(1 + \frac{x^2+y^2}{4}\right) u_t \end{pmatrix} = \begin{pmatrix} X_1 u \\ X_2 u \\ \frac{x}{2} X_2 u - \frac{y}{2} X_1 u + T u \end{pmatrix}.$$

From this formula, and from (3.7), (3.9), we finally obtain

$$\nabla u = \left\langle (g^{ij}) \begin{pmatrix} u_x \\ u_y \\ u_t \end{pmatrix}, \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial t} \end{pmatrix} \right\rangle_{\mathbb{R}^3} = X_1 u X_1 + X_2 u X_2 + T u T, \tag{3.11}$$

which verifies (5.15). It is worth observing that the Laplace–Beltrami operator is given by

$$\Delta u = X_1 X_1 u + X_2 X_2 u + T T u.$$

3.2. The four-dimensional Engel group

We next describe the four-dimensional cyclic or Engel group. This group is important in many respects since it represents the next level of difficulty with respect to the Heisenberg group and provides an ideal framework for testing whether results which are true in step 2 generalize to step 3 or higher. The reader unfamiliar with the cyclic group can consult [23], or also [75]. The Engel group $\mathfrak{E} = K_3$, see [23, Example 1.1.3], is the Lie group whose underlying manifold can be identified with $\mathbb{R}^4$, and whose Lie algebra is given by the grading,

$$\mathfrak{e} = V_1 \oplus V_2 \oplus V_3,$$

where $V_1 = \operatorname{span}\{e_1, e_2\}$, $V_2 = \operatorname{span}\{e_3\}$, and $V_3 = \operatorname{span}\{e_4\}$, so that $m_1 = 2$ and $m_2 = m_3 = 1$. We assign the bracket relations

$$[e_1, e_2] = e_3, \qquad [e_1, e_3] = e_4, \tag{3.12}$$

all other brackets being assumed trivial. For the corresponding left-invariant vector fields on $\mathfrak{E}$ given by $X_i(g) = (L_g)_*(e_i)$, $i = 1, 2$, $T(g) = (L_g)_*(e_3)$, $S(g) = (L_g)_*(e_4)$, we obtain the corresponding commutator relations

$$[X_1, X_2] = T, \qquad [X_1, T] = \big[X_1, [X_1, X_2]\big] = S, \tag{3.13}$$

all other commutators being trivial. We observe that the homogeneous dimension of $\mathfrak{E}$ is

$$Q = m_1 + 2m_2 + 3m_3 = 7.$$

We will denote with (x, y), t and s respectively the variables in V_1, V_2 and V_3, so that any $\xi \in \mathfrak{e}$ can be written as $\xi = xe_1 + ye_2 + te_3 + se_4$. If $g = \exp(\xi)$, we will identify $g = (x, y, t, s)$. The group law in $\mathfrak{E}$ is given by the Baker–Campbell–Hausdorff formula (2.2). In exponential coordinates, if $g = \exp(\xi)$, $g' = \exp(\xi')$, we have

$$g \circ g' = \xi + \xi' + \frac{1}{2}[\xi, \xi'] + \frac{1}{12}\big\{\big[\xi, [\xi, \xi']\big] - \big[\xi', [\xi, \xi']\big]\big\}.$$

A computation based on (3.12) gives (see also [23, Example 1.2.5])

$$g \circ g' = (x + x', y + y', t + t' + P_3, s + s' + P_4),$$

where

$$P_3 = \frac{1}{2}(xy' - yx'),$$
$$P_4 = \frac{1}{2}(xt' - tx') + \frac{1}{12}\big(x^2y' - xx'(y + y') + yx'^2\big).$$

Using the Baker–Campbell–Hausdorff formula we find the following expressions for the vector fields $X_1, \ldots, X_4$:

$$\begin{cases} X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial t} - \left(\frac{t}{2} + \frac{xy}{12}\right)\frac{\partial}{\partial s}, \\ X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial t} + \frac{x^2}{12}\frac{\partial}{\partial s}, \\ T = \frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial s}, \\ S = \frac{\partial}{\partial s}. \end{cases} \tag{3.14}$$

We note that the action of X_1, X_2, T on a function on $\mathfrak{E}$ which is independent of the variable s reduces to the action of the corresponding vector fields in $\mathbb{H}^1$.

4. The subbundle of horizontal planes

Consider a Carnot group $\boldsymbol{G}$, with Lie algebra $\mathfrak{g} = V_1 \oplus \cdots \oplus V_r$, with an orthonormal basis $\{e_1, \ldots, e_m\}$ of the horizontal layer V_1, and corresponding system $X = \{X_1, \ldots, X_m\}$ of generators, where $X_i(g) = (L_g)_*(e_i)$, $g \in \boldsymbol{G}$. Henceforth, the fiber $H_g\boldsymbol{G}$ of the horizontal bundle at a point $g \in \boldsymbol{G}$ will be denoted by H_g, so that $H\boldsymbol{G} = \bigcup_{g \in \boldsymbol{G}} H_g$. We note explicitly that $H_g = g\exp(V_1)$, where $\exp : \mathfrak{g} \to \boldsymbol{G}$ denotes the exponential mapping. We will call H_g the *horizontal plane* through g. For example, when $\boldsymbol{G}$ is the Heisenberg group $\mathbb{H}^n$, then a simple computation shows that the horizontal plane through a point $g_0 = (x_0, y_0, t_0)$ is given by the hyperplane

$$H_g = \left\{(x, y, t) \in \mathbb{H}^n \;\middle|\; t = t_0 + \frac{1}{2}\big(\langle x_0, y\rangle - \langle y_0, x\rangle\big)\right\}. \tag{4.1}$$

More in general, we have the following result which is [24, Proposition 4.3].

Proposition 4.1. *Let $\boldsymbol{G}$ be a Carnot group of step* 2, *then for any given $g_0 \in \boldsymbol{G}$ the horizontal plane passing through g_0 is the collection of all points $g \in \boldsymbol{G}$ whose exponential coordinates $(x, t) = (x(g), t(g))$ verify the k linear equations*

$$\Psi_s(g) = t_s(g) - t_s(g_0) - \frac{1}{2}\sum_{i,j=1}^{m} b_{ij}^s x_i(g_0) x_j(g) = 0, \quad s = 1, \dots, k,$$

where b_{ij}^s represent the horizontal group constants defined by (2.14).

Another interesting example is provided by the four-dimensional Engel group $\mathfrak{E}$ described in the previous section. Identifying $\mathfrak{E}$ with $\mathbb{R}^4$, with coordinates $g = (x, y, t, s)$, given a point $g_0 = (x_0, y_0, t_0, s_0)$ we have that $H_{g_0} = \operatorname{span}\{X_1(g_0), X_2(g_0)\}$. A simple computation based on (3.14) shows that H_{g_0} is described by the two equations

$$\begin{cases} \Psi_1(x, y, t, s) = t - t_0 + \frac{xy_0 - x_0 y}{2} = 0, \\ \Psi_2(x, y, t, s) = s - s_0 + \frac{x(6t_0 + x_0 y_0) - x_0^2 y - 6x_0 t_0}{12} = 0. \end{cases} \tag{4.2}$$

From (2.17) in Lemma 2.1 we see that for a Carnot group $\boldsymbol{G}$ of step r, with $N = \dim(\boldsymbol{G})$, the horizontal plane H_{g_0} is described by a system of $N - m$ linear equations for the exponential variables, see (2.13),

$$\begin{cases} \Psi_1(g) = t_1(g) - t_1(g_0) - B_1(g) = 0, \\ \vdots \\ \Psi_k(g) = t_k(g) - t_k(g_0) - B_k(g) = 0, \\ \vdots \\ \Psi_{r,1}(g) = x_{r,1}(g) - x_{r,1}(g_0) - B_{r,1}(g) = 0, \\ \vdots \\ \Psi_{r,m_r}(g) = x_{r,m_r}(g) - x_{r,m_r}(g_0) - B_{r,m_r}(g) = 0, \end{cases} \tag{4.3}$$

with $B_j(g_0) = 0$ for $j = 1, \dots, k, \dots, B_{r,j}(g_0) = 0$, $j = 1, \dots, m_r$.

Definition 4.2. We say that $\mathcal{S} \subset \boldsymbol{G}$ is a C^k hypersurface if $\mathcal{S}$ is a co-dimension one immersed manifold of class C^k. If, in addition, $\mathcal{S}$ is embedded, then we say that it is an embedded hypersurface.

We note explicitly that, by the implicit function theorem, for every $g_0 \in \mathcal{S}$ there exist an open set $\mathcal{O} \subset \boldsymbol{G}$ and a function $\phi \in C^k(\mathcal{O})$ such that: (i) $|\nabla\phi(g)| \neq 0$ for every $g \in \mathcal{O}$; (ii) $\mathcal{S} \cap \mathcal{O} = \{g \in \mathcal{O} \mid \phi(g) = 0\}$. When we will need to use this local representation, we will always assume that $\mathcal{S}$ is oriented in such a way that for every $g_0 \in \mathcal{S}$ and ϕ as in (ii), one has $\boldsymbol{N}(g_0) = \nabla\phi(g_0)$, where $\boldsymbol{N}$ denotes the non-unit Riemannian normal to $\mathcal{S}$. The following notion plays a pervasive role in sub-Riemannian geometry, as well as in the study of subelliptic equations.

Definition 4.3. Given a C^1 hypersurface $\mathcal{S} \subset \boldsymbol{G}$, a point $g_0 \in \mathcal{S}$ is called *characteristic* if one has $H_{g_0} \subset T_{g_0}\mathcal{S}$. Notice that this is equivalent to saying that

$$X_j(g_0) \in T_{g_0}\mathcal{S}, \quad j = 1, \ldots, m. \tag{4.4}$$

The *characteristic locus of* $\mathcal{S}$, $\Sigma_{\mathcal{S}}$, is the collection of all characteristic points of $\mathcal{S}$.

Although we will not use in this paper the following two results, we recall them because of their interest. The first theorem is a special case of a result due to Derridj [32,33].

Theorem 4.4. *Let $\mathcal{S}$ be a C^∞ hypersurface in a sub-Riemannian space M of dimension N, then denoting with H^s the s-dimensional Hausdorff measure constructed with the Riemannian distance one has*

$$H^{N-1}(\Sigma_{\mathcal{S}}) = 0.$$

For Carnot groups one has the following sharper result first proved in codimension one by Balogh for the Heisenberg group [2], and subsequently extended to arbitrary Carnot groups and codimension by Magnani [69,70].

Theorem 4.5. *Let $\boldsymbol{G}$ be a Carnot group and denote by $\mathcal{H}^s$ the s-dimensional Hausdorff measure constructed with the Carnot–Carathéodory distance. For any C^1 manifold of codimension k one has*

$$\mathcal{H}^{Q-k}(\Sigma_{\mathcal{S}}) = 0.$$

In particular, the characteristic set of a C^1 hypersurface has zero $\mathcal{H}^{Q-1}$-measure.

Since for a C^2 hypersurface in a Carnot group $\boldsymbol{G}$ it was proved in [27] that the H-perimeter measure $P_H(\Omega; \cdot)$, introduced in Section 8 below, is mutually absolutely continuous with respect to the Hausdorff measure $\mathcal{H}^{Q-1}$, we conclude from Theorem 4.5 that for such domains the H-perimeter measure of the characteristic set is zero, i.e.,

$$\sigma_H(\Sigma_{\mathcal{S}}) = 0. \tag{4.5}$$

Proposition 4.6. *Let $\boldsymbol{G}$ be a Carnot group, and $\mathcal{S} \subset \boldsymbol{G}$ be a C^k hypersurface. If $g_0 \in \mathcal{S} \setminus \Sigma_{\mathcal{S}}$, denote by $\mathcal{S}_0 = \mathcal{S} \cap H_{g_0}$. There exists a sufficiently small open neighborhood $\mathcal{O}$ of g_0 such that $\mathcal{S}_0 \cap \mathcal{O}$ is a C^k submersed manifold of $\boldsymbol{G}$ of dimension $m - 1$.*

Proof. According to Definition 4.2, there exists a neighborhood $\mathcal{O}$ of g_0 such that $\mathcal{S} \cap \mathcal{O} = \{g \in \mathcal{O} \mid \phi(g) = 0\}$. Using the exponential coordinates (2.13), we now introduce the C^k function $F : \exp^{-1}(\mathcal{O}) \subset \mathbb{R}^N \to \mathbb{R}^{N-m+1}$ defined by

$$F \stackrel{\text{def}}{=} (\phi, \Psi_1, \ldots, \Psi_k, \ldots, \Psi_{r,1}, \ldots, \Psi_{r,m_r}),$$

where the $N - m$ functions Ψ_j are as in (4.3). Clearly, we have $F(g_0) = 0 \in \mathbb{R}^{N-m+1}$. Denoting by J_F the Jacobian matrix of F, we now claim that the hypothesis

$$g_0 \in \mathcal{S} \setminus \Sigma_{\mathcal{S}} \quad \Longrightarrow \quad \operatorname{rank} J_F(g_0) = N - m + 1.$$

Taking the claim for granted, we see that the conclusion of Proposition 4.6 immediately follows from the implicit function theorem (of course, by possibly restricting the neighborhood $\mathcal{O}$), since the latter guarantees that, locally around g_0, the set $\mathcal{S}_0$ is a submersed manifold of class C^k of dimension $N - (N - m + 1) = m - 1$.

We now prove the claim in two special situations, namely that of a Carnot group of step $r = 2$, and that of the Engel group $\mathfrak{E}$, leaving it to the interested reader to provide the (lengthy) details for a general Carnot group. Suppose then that $\boldsymbol{G}$ has step $r = 2$. Since $g_0 \notin \Sigma$, we know that $\nabla_H \phi(g_0) \neq 0$. Therefore, there exists $i \in \{1, \ldots, m\}$ such that $X_i \phi(g_0) \neq 0$. Without loss of generality, let us assume that $X_m \phi(g_0) \neq 0$. According to (2.18) we thus have

$$\phi_{x_m}(g_0) + \frac{1}{2} \sum_{s=1}^{k} \sum_{j=1}^{m} b_{jm}^s x_j(g_0) \phi_{t_s}(g_0) \neq 0. \tag{4.6}$$

The Jacobian matrix of $F = (\phi, \Psi_1, \ldots, \Psi_k)$ at g_0 is now given by

$$J_F(g_0) = \begin{pmatrix} \phi_{x_1} & \ldots & \phi_{x_m} & \phi_{t_1} & \phi_{t_2} & \ldots & \phi_{t_k} \\ -\frac{1}{2}\sum_{i=1}^m b_{i1}^1 x_i(g_0) & \ldots & -\frac{1}{2}\sum_{i=1}^m b_{im}^1 x_i(g_0) & 1 & 0 & \ldots & 0 \\ -\frac{1}{2}\sum_{i=1}^m b_{i1}^2 x_i(g_0) & \ldots & -\frac{1}{2}\sum_{i=1}^m b_{im}^2 x_i(g_0) & 0 & 1 & \ldots & 0 \\ . & \ldots & . & . & . & \ldots & . \\ . & \ldots & . & . & . & \ldots & . \\ -\frac{1}{2}\sum_{i=1}^m b_{i1}^k x_i(g_0) & \ldots & -\frac{1}{2}\sum_{i=1}^m b_{im}^k x_i(g_0) & 0 & 0 & \ldots & 1 \end{pmatrix},$$

where all derivatives of ϕ are evaluated at g_0. We consider the $(k+1) \times (k+1)$ minor

$$\tilde{J}_F(g_0) = \begin{pmatrix} \phi_{x_m} & \phi_{t_1} & \phi_{t_2} & \ldots & \phi_{t_k} \\ -\frac{1}{2}\sum_{i=1}^m b_{im}^1 x_i(g_0) & 1 & 0 & \ldots & 0 \\ -\frac{1}{2}\sum_{i=1}^m b_{im}^2 x_i(g_0) & 0 & 1 & \ldots & 0 \\ . & . & . & \ldots & . \\ . & . & . & \ldots & . \\ -\frac{1}{2}\sum_{i=1}^m b_{im}^k x_i(g_0) & 0 & 0 & \ldots & 1 \end{pmatrix},$$

of the matrix $J_F(g_0)$. A careful examination of the special structure of the matrix $\tilde{J}_F(g_0)$, and the cofactor expansion of its determinant, allow to conclude that

$$\det \tilde{J}_F(g_0) = \phi_{x_m} + \frac{1}{2}\sum_{i=1}^{m} b_{im}^1 x_i(g_0)\phi_{t_1} + \cdots + \frac{1}{2}\sum_{i=1}^{m} b_{im}^k x_i(g_0)\phi_{t_k} \neq 0,$$

where in the last equation we have used (4.6). This proves that rank $J_F(g_0) = k+1 = N-m+1$, and therefore the claim follows for groups of step $r=2$.

If instead $\mathcal{S} \subset \boldsymbol{G} = \mathfrak{E}$ is a hypersurface in the Engel group, with $g_0 = (x_0, y_0, t_0, s_0) \in \mathcal{S} \setminus \Sigma$, then we can assume for instance that we have at g_0

$$X_2\phi(g_0) = \phi_y(g_0) + \frac{x_0}{2}\phi_t(g_0) + \frac{x_0^2}{12}\phi_s(g_0) \neq 0. \tag{4.7}$$

We consider the function $F = (\phi, \Psi_1, \Psi_2) : \mathbb{R}^4 \to \mathbb{R}^3$, where Ψ_i, $i = 1, 2$ are as in (4.2). Its Jacobian matrix is given by

$$J_F(g_0) = \begin{pmatrix} \phi_x & \phi_y & \phi_t & \phi_s \\ \frac{y_0}{2} & -\frac{x_0}{2} & 1 & 0 \\ \frac{6t_0 + x_0 y_0}{12} & -\frac{x_0^2}{12} & 0 & 1 \end{pmatrix}.$$

One readily sees that the 3×3 minor

$$\tilde{J}_F(g_0) = \begin{pmatrix} \phi_y & \phi_t & \phi_s \\ -\frac{x_0}{2} & 1 & 0 \\ -\frac{x_0^2}{12} & 0 & 1 \end{pmatrix}$$

has determinant given by $X_2\phi(g_0)$. From (4.7) we conclude that rank $\tilde{J}_F(g_0) = 3 = N - m + 1$, and again the claim follows. □

5. Horizontal Levi-Civita connection

Let $\boldsymbol{G}$ be a Carnot group of step r. Henceforth in this paper we will assume that $\boldsymbol{G}$ is endowed with a left-invariant Riemannian metric $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = g_{ij} u^i v^j$, where $\boldsymbol{u}, \boldsymbol{v} \in T\boldsymbol{G}$, with respect to which the left-invariant vector fields defined in (2.15)

$$\{X_1, \ldots, X_m, T_1, \ldots, T_k, \ldots, X_{r,1}, \ldots, X_{r,m_r}\}$$

constitute an orthonormal frame for $T\boldsymbol{G}$. No other inner product will be used on $T\boldsymbol{G}$, thereby when we write $\langle \cdot, \cdot \rangle$ there will be no risk of confusion. We denote with by ∇ the corresponding Levi-Civita connection on $\boldsymbol{G}$. Recall that ∇ is torsion free,

$$\nabla_X Y - \nabla_Y X = [X, Y], \tag{5.1}$$

and that it is metric preserving, i.e., $\nabla g = 0$ or, equivalently,

$$X\langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle. \tag{5.2}$$

Permuting cyclically the roles of X, Y, Z in (5.2), one obtains the basic Koszul identity, see e.g. [87, (1.13), p. 28],

$$2\langle\nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle - \big\langle Y, [X, Z]\big\rangle - \big\langle X, [Y, Z]\big\rangle + \big\langle Z, [X, Y]\big\rangle. \tag{5.3}$$

Using (5.3) it is easy to check that

$$\nabla_{X_i} X_i = 0, \quad i = 1, \dots, m, \qquad \dots, \qquad \nabla_{X_{j,m_j}} X_{j,m_j} = 0, \quad j = 1, \dots, r. \tag{5.4}$$

In addition to (5.4), we can easily verify from (5.3) and the grading of the Lie algebra, that

$$\langle\nabla_{X_i} X_j, X_\ell\rangle = 0, \quad i, j, l = 1, \dots, m. \tag{5.5}$$

The remaining covariant derivatives and the Christoffel symbols can be determined from the group constants. For instance, we have the following proposition.

Proposition 5.1. *Let $\mathbf{G}$ be a Carnot group of step r, then*

$$\nabla_{X_i} X_j = \frac{1}{2}\sum_{s=1}^{k} b_{ij}^s T_s, \quad i, j = 1, \dots, m. \tag{5.6}$$

$$\nabla_{T_p} T_s = 0, \quad p, s = 1, \dots, k. \tag{5.7}$$

$$\nabla_{X_i} T_s = -\frac{1}{2}\sum_{j=1}^{m} b_{ij}^s X_j + \frac{1}{2}\sum_{p=1}^{m_3}\big\langle [X_i, T_s], X_{3,p}\big\rangle X_{3,p}, \quad i = 1, \dots, m,\ s = 1, \dots, k. \tag{5.8}$$

In particular, when $\mathbf{G} = \mathbb{H}^n$ one has for $i, j = 1, \dots, n$,

$$\nabla_{X_i} X_{n+j} = \frac{\delta_{ij}}{2} T, \qquad \nabla_{X_i} T = \nabla_T X_i = -\frac{1}{2} X_{n+i}, \qquad \nabla_{X_{n+i}} T = \nabla_T X_{n+i} = \frac{1}{2} X_i.$$

Proof. Using (5.3), for any vector field

$$Z = \sum_{\ell=1}^{m} a_\ell X_\ell + \sum_{s=1}^{k} b_s T_s + \sum_{h=3}^{r}\sum_{p=1}^{m_h} c_p X_{h,p},$$

we obtain

$$\langle\nabla_{X_i} X_j, Z\rangle = \sum_{\ell=1}^{m} a_\ell \langle\nabla_{X_i} X_j, X_\ell\rangle + \sum_{s=1}^{k} b_s \langle\nabla_{X_i} X_j, T_s\rangle + \sum_{h=3}^{r}\sum_{p=1}^{m_h} c_p \langle\nabla_{X_i} X_j, X_{h,p}\rangle.$$

Now (5.5) gives $\langle \nabla_{X_i} X_j, X_\ell\rangle = 0$, whereas using (5.3) again, we find

$$2\langle \nabla_{X_i} X_j, T_s\rangle = -\langle T_s, [X_j, X_i]\rangle = \sum_{p=1}^{k} b_{ij}^{p}\delta_{sp} = b_{ij}^{s}.$$

Similarly, for $h \in \{3, \dots, r\}$ we have

$$2\langle \nabla_{X_i} X_j, X_{h,p}\rangle = \sum_{s=1}^{k} b_{ij}^{s}\langle X_{h,p}, T_s\rangle = 0.$$

From these equations we obtain

$$\langle \nabla_{X_i} X_j, Z\rangle = \left\langle \frac{1}{2}\sum_{s=1}^{k} b_{ij}^{s} T_s, Z\right\rangle.$$

From the arbitrariness of Z we conclude that (5.11) holds. In a similar way, one obtains (5.12), and (5.13). We leave the details to the reader. □

Next, we want to introduce a connection on the horizontal bundle. We do this by projecting onto $H\boldsymbol{G}$ the Levi-Civita connection ∇.

Definition 5.2. If X is a vector field on $\boldsymbol{G}$, and Y is a horizontal vector field on $\boldsymbol{G}$, then we define the (Levi-Civita) *horizontal connection* on $H\boldsymbol{G}$ as follows:

$$\nabla_X^H Y \stackrel{\text{def}}{=} \sum_{i=1}^{m} \langle \nabla_X Y, X_i\rangle X_i. \tag{5.9}$$

Let us notice that ∇^H satisfies the metric compatibility condition

$$X\langle Y, Z\rangle = \langle \nabla_X^H Y, Z\rangle + \langle Y, \nabla_X^H Z\rangle, \tag{5.10}$$

for every triple of vector fields X, Y, Z on $\boldsymbol{G}$, such that Y and Z are horizontal. This follows from the corresponding compatibility condition (5.2) satisfied by the Levi-Civita connection ∇, and from the definition of ∇^H. From Proposition 5.1 and Definition 5.2 we obtain the following.

Proposition 5.3. *Let $\boldsymbol{G}$ be a Carnot group of step r, then*

$$\nabla_{X_i}^H X_j = 0, \quad i, j = 1, \dots, m, \tag{5.11}$$

$$\nabla_{T_p}^H T_s = 0, \quad p, s = 1, \dots, k, \tag{5.12}$$

$$\nabla_{X_i}^H T_s = -\frac{1}{2}\sum_{j=1}^{m} b_{ij}^{s} X_j, \quad i = 1, \dots, m,\ s = 1, \dots, k. \tag{5.13}$$

Remark 5.4. We mention that the horizontal Levi-Civita connection $\nabla^H_X Y$ is intimately connected with the notion of non-holonomic connection introduced by Cartan in his address at the 1928 International Congress of Mathematicians in Bologna [12]. In this respect we refer the reader to the interesting re-visitation of Cartan's address by Koiller, Rodrigues and Pitanga, see [63,64], where the authors generalize some of the ideas in [12] and also introduce a non-holonomic connection (see their Definition 1.1 in [63]) which, for a Carnot group, gives precisely our Definition 5.2. A general framework has been recently set forth by Hladky and Pauls in [56] for what they call *vertically rigid spaces*. These are sub-Riemannian manifolds which include, in particular, Carnot groups. When specialized to Carnot groups, the adapted connection in [56] coincides with the horizontal connection in Definition 5.2.

Hereafter, for a given vector field X we indicate with $X^H = \sum_{i=1}^m \langle X, X_i\rangle X_i$ the projection of X on the horizontal bundle $H\boldsymbol{G}$.

Proposition 5.5. *Given horizontal vector fields X and Y, one has*

$$\nabla^H_X Y - \nabla^H_Y X = [X, Y]^H \stackrel{\text{def}}{=} \sum_{i=1}^m \big\langle [X, Y], X_i\big\rangle X_i.$$

Proof. From Definition 5.2 and the torsion freeness (5.1) of the Levi-Civita connection we obtain

$$\begin{aligned}\nabla^H_X Y - \nabla^H_Y X &= \sum_{i=1}^m \langle \nabla_X Y - \nabla_Y X, X_i\rangle X_i \\ &= \sum_{i=1}^m \big\langle [X, Y], X_i\big\rangle X_i = [X, Y]^H. \qquad \square\end{aligned}$$

If we define the *horizontal torsion* as follows:

$$T^H(X, Y) = \nabla^H_X Y - \nabla^H_Y X - [X, Y]^H,$$

then Proposition 5.5 asserts that the horizontal connection is torsion free, and this is why we call it the horizontal Levi-Civita connection. Permuting cyclically the roles of X, Y, Z in (5.10), and using Proposition 5.5, we obtain the following *horizontal Koszul identity* for ∇^H.

Proposition 5.6. *Let X, Y, Z be horizontal vector fields on $\boldsymbol{G}$, then*

$$\begin{aligned}2\big\langle \nabla^H_X Y, Z\big\rangle &= X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle \\ &\quad - \big\langle Y, [X, Z]^H\big\rangle - \big\langle X, [Y, Z]^H\big\rangle + \big\langle Z, [X, Y]^H\big\rangle. \end{aligned} \tag{5.14}$$

Proposition 5.6 shows in particular that ∇^H is completely determined by the Riemannian inner product in $\boldsymbol{G}$ and by the horizontal bundle $H\boldsymbol{G}$. Given a function $u \in C^1(\boldsymbol{G})$, its Riemannian gradient with respect to the inner product $\langle\cdot,\cdot\rangle$ is given by

$$\nabla u = X_1uX_1 + \cdots + X_muX_m + T_1uT_1 + \cdots + T_kuT_k + \cdots + X_{r,1}uX_{r,1} + \cdots + X_{r,m_r}uX_{r,m_r}. \tag{5.15}$$

If we let $G = \det(g_{ij})$, then as a consequence of (2.19), and of the fact that $G \equiv 1$ (see [41], or [23]), we obtain for the divergence of X_i (see [55, p. 387])

$$\operatorname{div} X_i = \frac{1}{\sqrt{G}} \sum_{k=1}^{N} \frac{\partial}{\partial \xi_k}\big(\sqrt{G}(X_i)_k\big) = \operatorname{div}_E X_i + \sum_{k=1}^{N} (X_i)_k \frac{\partial}{\partial \xi_k}(\log \sqrt{G}\,) = 0, \tag{5.16}$$

for every $i = 1, \ldots, m$. The horizontal gradient of u is obtained by projecting ∇u on the subbundle $\boldsymbol{HG}$ (see Definition 5.2). The resulting horizontal vector field on $\boldsymbol{G}$ is nothing but the horizontal connection acting on u

$$\nabla^H u = \langle \nabla u, X_1\rangle X_1 + \cdots + \langle \nabla u, X_m\rangle X_m = X_1uX_1 + \cdots + X_muX_m. \tag{5.17}$$

If $\zeta = \zeta_1 X_1 + \cdots + \zeta_m X_m \in C^1(\boldsymbol{G}, \boldsymbol{HG})$, then the horizontal divergence of ζ is given by

$$\operatorname{div}_H \zeta = X_1\zeta_1 + \cdots + X_m\zeta_m. \tag{5.18}$$

The horizontal Laplacian (also known as sub-Laplacian) of a function $u \in C^2(\boldsymbol{G})$ is given by

$$\Delta_H u = \operatorname{div}_H \nabla^H u = \sum_{i=1}^{m} X_i^2. \tag{5.19}$$

Except for the Abelian case when the step $r = 1$ and Δ_H is just the standard Laplacian $\Delta = \sum_{i=1}^m \partial^2/\partial x_i^2$, such operator fails to be elliptic at every point of $\boldsymbol{G}$. We notice that $\Delta_H u = \operatorname{trace}(\nabla_H^2 u)$, where we have denoted by $\nabla_H^2 u$ the $m \times m$ matrix-valued function on $\boldsymbol{G}$ defined by

$$\nabla_H^2 u = u_{,ij} = \frac{X_iX_ju + X_jX_iu}{2}, \quad i, j = 1, \ldots, m. \tag{5.20}$$

The following proposition contains a useful property of Carnot groups.

Proposition 5.7. *Let* $\boldsymbol{G}$ *be a Carnot group, then*

$$X_i x_j = \delta_{ij}, \qquad \Delta_H x_j = 0, \quad i, j = 1, \ldots, m. \tag{5.21}$$

As a consequence, we find

$$\big|\nabla^H\big(|x|^2\big)\big|^2 = 4|x|^2. \tag{5.22}$$

One also has

$$X_i t_s = \frac{1}{2}\big\langle [\xi_1, e_i], \epsilon_s\big\rangle = \frac{1}{2}\sum_{j=1}^{m} x_j b_{ji}^s, \qquad X_jX_it_s = \frac{1}{2} b_{ji}^s. \tag{5.23}$$

In particular, we obtain $\nabla_H^2(t_s) = 0$*, and therefore* $\Delta_H t_s = 0$*,* $s = 1, \ldots, k$*.*

6. Horizontal Gauss map and tangent space to a hypersurface

In this section we introduce two basic geometric concepts for an hypersurface in a Carnot group $\boldsymbol{G}$ which are adapted to the horizontal subbundle of $\boldsymbol{G}$. We consider the Riemannian manifold $M = \boldsymbol{G}$ with the metric tensor with respect to which $X_1, \dots, X_m, \dots, X_{r,m_r}$ is an orthonormal basis, the corresponding Levi-Civita connection ∇ on $\boldsymbol{G}$, and the horizontal Levi-Civita connection ∇^H introduced in Definition 5.2. Let $\mathcal{S} \subset \boldsymbol{G}$ be a C^k oriented hypersurface, with $k \geqslant 2$. We will denote by $\boldsymbol{N}$ the non-unit Riemannian normal to $\mathcal{S}$, and will indicate with $\boldsymbol{\nu} = \boldsymbol{N}/|\boldsymbol{N}|$ the Riemannian Gauss map of $\mathcal{S}$. It will be convenient to introduce the following notation:

$$p_j = \langle \boldsymbol{N}, X_j \rangle, \quad i = j, \dots, m, \qquad W = \sqrt{p_1^2 + \cdots + p_m^2}. \tag{6.1}$$

We now set

$$\overline{p}_j = \frac{p_j}{W}, \quad \text{so that} \quad \overline{p}_1^2 + \cdots + \overline{p}_m^2 \equiv 1 \quad \text{on } \mathcal{S} \setminus \Sigma_{\mathcal{S}}. \tag{6.2}$$

We also define

$$\begin{aligned} \omega_s &= \langle \boldsymbol{N}, T_s \rangle, \qquad \overline{\omega}_s = \frac{\omega_s}{W}, \quad s = 1, \dots, k, \\ \omega_{j,s} &= \langle \boldsymbol{N}, X_{j,s} \rangle, \qquad \overline{\omega}_{j,s} = \frac{\omega_{j,s}}{W}, \quad j = 1, \dots, r,\ s = 1, \dots, m_j. \end{aligned} \tag{6.3}$$

If $g_0 \in \mathcal{S}$ is characteristic, then we have $p_j(g_0) = 0$, $j = 1, \dots, m$, and therefore we have the alternative characterization of $\Sigma_{\mathcal{S}}$ as the zero set of the continuous function W

$$\Sigma_{\mathcal{S}} = \{ g \in \mathcal{S} \mid W(g) = 0 \}, \tag{6.4}$$

which shows that $\Sigma_{\mathcal{S}}$ is a closed subset of $\mathcal{S}$. The next definition plays a basic role in the sequel.

Definition 6.1. We define the *horizontal normal* $\boldsymbol{N}^H : \mathcal{S} \to H\boldsymbol{G}$ by the formula

$$\boldsymbol{N}^H = \sum_{j=1}^m \langle \boldsymbol{N}, X_j \rangle X_j = \sum_{j=1}^m p_j X_j. \tag{6.5}$$

The *horizontal Gauss map* $\boldsymbol{\nu}^H$ is defined by

$$\boldsymbol{\nu}^H = \frac{\boldsymbol{N}^H}{|\boldsymbol{N}^H|} = \sum_{j=1}^m \overline{p}_j X_j, \quad \text{on } \mathcal{S} \setminus \Sigma_{\mathcal{S}}. \tag{6.6}$$

We note that $\boldsymbol{N}^H$ is the projection of the Riemannian normal $\boldsymbol{N}$ on the horizontal subbundle $H\boldsymbol{G} \subset T\boldsymbol{G}$. Such projection vanishes only at characteristic points, and this is why the horizontal Gauss map is not defined on $\Sigma_{\mathcal{S}}$. A trivial consequence of the definition which, however, will be important in the sequel is

$$|\boldsymbol{\nu}^H|^2 = \overline{p}_1^2 + \cdots + \overline{p}_m^2 \equiv 1, \quad \text{in } \mathcal{S} \setminus \Sigma_{\mathcal{S}}, \tag{6.7}$$

which is of course a re-formulation of the second equation in (6.2). One also has

$$\langle \boldsymbol{\nu}^H, \boldsymbol{N}^H\rangle = |\boldsymbol{N}^H|, \qquad \boldsymbol{N}^H - \langle \boldsymbol{N}^H, \boldsymbol{\nu}^H\rangle\boldsymbol{\nu}^H = 0. \tag{6.8}$$

We note explicitly that, with these quantities in place, the Riemannian (non-unit) normal to $\mathcal{S}$ is given at every $g \in \mathcal{S} \setminus \Sigma_{\mathcal{S}}$ by

$$\begin{aligned} \boldsymbol{N} &= \boldsymbol{N}^H + \omega_1 T_1 + \cdots + \omega_k T_k + \cdots + \omega_{r,m_r} X_{r,m_r} \\ &= W\{\overline{p}_1 X_1 + \cdots + \overline{p}_m X_m + \overline{\omega}_1 T_1 + \cdots + \overline{\omega}_k T_k + \cdots + \overline{\omega}_{r,m_r} X_{r,m_r}\} \\ &= W\{\boldsymbol{\nu}^H + \overline{\omega}_1 T_1 + \cdots + \overline{\omega}_k T_k + \cdots + \overline{\omega}_{r,m_r} X_{r,m_r}\}. \end{aligned} \tag{6.9}$$

Since $\langle \boldsymbol{\nu}^H, T_s\rangle = \langle \boldsymbol{\nu}^H, X_{j,m_j}\rangle = 0$ for $s = 1, \ldots, k$, and $j = 3, \ldots, r$, it is obvious from (6.9) that

$$\langle \boldsymbol{N}, \boldsymbol{\nu}^H\rangle = \langle \boldsymbol{N}^H, \boldsymbol{\nu}^H\rangle = W, \quad \text{hence } \cos(\boldsymbol{\nu}^H \angle \boldsymbol{N}) = \frac{W}{|\boldsymbol{N}|}. \tag{6.10}$$

Because of (6.10), the function W is also called the *angle function*.

Remark 6.2. To help the reader's comprehension, we sometimes give proofs or examples in the special case when $\boldsymbol{G} = \mathbb{H}^1$, the first Heisenberg group. Furthermore, Sections 13–15 are devoted to this special setting. It will thus be convenient to simplify the notation introduced above as follows. For surfaces $\mathcal{S} \subset \mathbb{H}^1$ we will let

$$\begin{gathered} p = p_1, \qquad q = p_2, \qquad \omega = \omega_1, \qquad W = \sqrt{p^2 + q^2}, \\ \overline{p} = \overline{p}_1, \qquad \overline{q} = \overline{p}_2, \qquad \overline{\omega} = \overline{\omega}_1. \end{gathered} \tag{6.11}$$

Consequently, in this setting the normal $\boldsymbol{N}$ and the horizontal Gauss map $\boldsymbol{\nu}^H$ will always be respectively written as

$$\boldsymbol{N} = pX_1 + qX_2 + \omega T = \boldsymbol{N}^H + \omega T, \qquad \boldsymbol{\nu}^H = \overline{p}X_1 + \overline{q}X_2, \tag{6.12}$$

so that (6.9) becomes

$$\boldsymbol{N} = W\{\boldsymbol{\nu}^H + \overline{\omega}T\}.$$

The horizontal vector field defined on $\mathcal{S} \setminus \Sigma_{\mathcal{S}}$ by

$$(\boldsymbol{\nu}^H)^\perp = \overline{q}X_1 - \overline{p}X_2, \tag{6.13}$$

is perpendicular to $\boldsymbol{\nu}^H$, but it is also orthogonal to the Riemannian normal $\boldsymbol{N}$ to $\mathcal{S}$.

Definition 6.3. At a point $g \in \mathcal{S} \setminus \Sigma_{\mathcal{S}}$ the *horizontal tangent space* is defined as follows:

$$HT_g\mathcal{S} \stackrel{\text{def}}{=} \{\boldsymbol{v} \in H_g \mid \langle \boldsymbol{v}, \boldsymbol{\nu}^H\rangle_g = 0\}.$$

The *horizontal tangent bundle* of $\mathcal{S}$ is defined by

$$HT\mathcal{S} = \bigcup_{g\in\mathcal{S}\setminus\Sigma_{\mathcal{S}}} HT_g\mathcal{S}.$$

One can check that $HT\mathcal{S}$ has the structure of a vector bundle. It is clear that, since $\dim H_g = m$, then $\dim HT_g\mathcal{S} = m-1$, and one has in fact

$$H_g = HT_g\mathcal{S} \oplus \operatorname{span}\{\boldsymbol{\nu}^H(g)\}. \tag{6.14}$$

For instance, when $\boldsymbol{G} = \mathbb{H}^1$, then if for a C^2 surface $\mathcal{S}\subset\mathbb{H}^1$ we consider the unit vector field on $\mathcal{S}$ given by (6.13), then it is clear that at every point $g\in\mathcal{S}\setminus\Sigma$, one has

$$HT_g\mathcal{S} = \operatorname{span}\{(\boldsymbol{\nu}^H)^\perp(g)\}. \tag{6.15}$$

If we consider $\Gamma = \mathcal{S}\cap H_g$, then from Proposition 4.6 we know that Γ is submersed manifold of dimension one (a curve). Its Riemannian tangent space in g can be identified in a canonical way with HT_g. We also observe that an orthonormal basis for the Riemannian tangent space $T_g\mathcal{S}$ of $\mathcal{S}$ at g is given by

$$T_g\mathcal{S} = \operatorname{span}\left\{(\boldsymbol{\nu}^H)^\perp, \frac{\omega}{|\boldsymbol{N}|}\boldsymbol{\nu}^H - \frac{W}{|\boldsymbol{N}|}T\right\}. \tag{6.16}$$

Proposition 6.4. *Let $g\in\mathcal{S}\setminus\Sigma_{\mathcal{S}}$, then one has*

$$HT_g\mathcal{S} = T_g\mathcal{S}\cap H_g.$$

Proof. We begin by observing that, since by hypothesis $g\notin\Sigma_{\mathcal{S}}$, then $H_g\not\subset T_g\mathcal{S}$, and therefore $\boldsymbol{N}^H\neq 0$. Now, from (6.9) and the fact that $X_1,\ldots,X_m,T_1,\ldots,X_{r,m_r}$ constitute an orthonormal basis of $T_g\mathcal{S}$ at every $g\in\boldsymbol{G}$, one sees from (6.9) that $\boldsymbol{N}-\boldsymbol{N}^H\perp H_g$. Therefore, if $\boldsymbol{v}\in HT_g\mathcal{S}$, then we have

$$\langle\boldsymbol{v},\boldsymbol{N}\rangle = \langle\boldsymbol{v},\boldsymbol{N}-\boldsymbol{N}^H\rangle + \langle\boldsymbol{v},\boldsymbol{N}^H\rangle = 0, \tag{6.17}$$

which shows $\boldsymbol{v}\in T_g\mathcal{S}$. We thus have the inclusion $HT_g\mathcal{S}\subset T_g\mathcal{S}\cap H_g$. To establish the opposite inclusion, let $\boldsymbol{v}\in T_g\mathcal{S}\cap H_g$. We thus have that the left-hand side of (6.17) is zero, and since $\langle\boldsymbol{v},\boldsymbol{N}-\boldsymbol{N}^H\rangle = 0$ because of the fact that $\boldsymbol{v}\in H_g$, we conclude that it must be $\langle\boldsymbol{v},\boldsymbol{N}^H\rangle = 0$, hence $\boldsymbol{v}\in HT_g\mathcal{S}$. □

7. Horizontal connection on a hypersurface

We recall the classical definition of the Levi-Civita connection of a n-dimensional immersed submanifold $N = N^n$ of an m-dimensional Riemannian manifold $M = M^m$. Denoting with $i: N\hookrightarrow M$ the immersion, and having endowed N with the induced Riemannian metric i^*g, let $i_*: TN\to TM$ be the differential of i. We identify T_pN with the subspace $(i_*)_p(T_pN)$ of T_pM, and denote by $T_pN^\perp$ its orthogonal complement. $TN^\perp = \bigcup_{p\in N} T_pN^\perp$ has the structure of a $(m-n)$-dimensional vector bundle, traditionally referred to as the normal bundle of N.

We can thus write $TM \mid N \cong TN \oplus TN^\perp$, and for every $\boldsymbol{u} \in T_pM$, we indicate with $\boldsymbol{u}^\top$ its T_pN component, and with $\boldsymbol{u}^\perp$ its $T_pN^\perp$ component. Since $p \to (\nabla^M_X Y)^\top(p)$ satisfies all the assumptions of a Levi-Civita connection on N, by the uniqueness of the latter we obtain

$$\nabla^N_X Y = \left(\nabla^M_X Y\right)^\top. \tag{7.1}$$

Before proceeding we need to say a few words concerning (7.1). First of all, since X, Y are only initially defined on the submanifold N, we need to give a meaning to right-hand side. Using a partition of unity argument, we can extend X, Y to smooth vector fields $\overline{X}$, $\overline{Y}$ on M, and therefore interpret the right-hand side as follows

$$\left(\nabla^M_X Y\right)^\top = \left(\nabla^M_{\overline{X}} \overline{Y}\right)^\top. \tag{7.2}$$

This immediately raises the question of whether (7.2) is a good definition, in other words, whether it is independent of the particular extensions of X, Y that we have picked. Since the value of $\nabla^M_X Y$ at $p \in N$ depends only on X_p, it is clear that (7.2) is independent of the extension of X. On the other hand, $(\nabla^M_X Y)_p$ depends only on the values of Y along any curve on M whose initial tangent vector is X_p. By picking a curve which lies entirely on N, we see that (7.2) is also independent of the extension of Y. This fact, can be also recognized by the following observations, which also establish the torsion freeness of the connection $(\nabla^M_X Y)^\top$. Denoting with i_* the differential of the immersion, we have $i_*(X) = \overline{X}$, $i_*(Y) = \overline{Y}$, and therefore, see Theorem 7.9 in [7], $i_*[X, Y] = [\overline{X}, \overline{Y}]$. This implies, in particular, that $[X, Y] = [\overline{X}, \overline{Y}]^\top$. From the torsion freeness of ∇^M, we thus conclude that

$$\left(\nabla^M_{\overline{X}} \overline{Y}\right)^\top - \left(\nabla^M_{\overline{Y}} \overline{X}\right)^\top = [\overline{X}, \overline{Y}]^\top = [X, Y]. \tag{7.3}$$

We notice that $[X, Y]_p$ only depends on the values of X, Y in a neighborhood of p in N, and therefore (7.3) shows at once that (7.2) is a good definition, and that $(\nabla^M_X Y)^\top$ is torsion free. The remaining properties of a Levi-Civita connection are checked easily.

Inspired by the Riemannian situation we now introduce a notion of horizontal connection on a hypersurface $\mathcal{S} \subset \boldsymbol{G}$ by projecting the horizontal Levi-Civita connection ∇^H in the ambient Lie group $\boldsymbol{G}$ onto the horizontal tangent space $HT\mathcal{S}$.

Definition 7.1. Let $\mathcal{S} \subset \boldsymbol{G}$ be a non-characteristic, C^k hypersurface, $k \geqslant 2$, then we define the *horizontal connection* on $\mathcal{S}$ as follows. Let ∇^H denote the horizontal Levi-Civita connection introduced in Definition 5.2. For every $X, Y \in C^1(\mathcal{S}; HT\mathcal{S})$ we define

$$\nabla^{H,\mathcal{S}}_X Y = \nabla^H_{\overline{X}} \overline{Y} - \left\langle \nabla^H_{\overline{X}} \overline{Y}, \boldsymbol{\nu}^H \right\rangle \boldsymbol{\nu}^H,$$

where $\overline{X}$, $\overline{Y}$ are any two horizontal vector fields on $\boldsymbol{G}$ such that $\overline{X} = X$, $\overline{Y} = Y$ on $\mathcal{S}$.

Arguing as above one can check that Definition 7.1 is well posed, i.e., it is independent of the extensions $\overline{X}, \overline{Y}$ of the vector fields X, Y. From Proposition 5.5 we immediately obtain.

Proposition 7.2. *For every $X, Y \in C^1(\mathcal{S}; HT\mathcal{S})$ one has*

$$\nabla^{H,\mathcal{S}}_X Y - \nabla^{H,\mathcal{S}}_Y X = [X, Y]^H - \left\langle [X, Y]^H, \boldsymbol{\nu}^H \right\rangle \boldsymbol{\nu}^H.$$

It is clear from this proposition that the horizontal connection $\nabla^{H,\mathcal{S}}$ on $\mathcal{S}$ is not necessarily torsion free. This depends on the fact that it is not true in general that, if $X, Y \in C^1(\mathcal{S}; HT\mathcal{S})$, then $[X, Y]^H \in C^1(\mathcal{S}; HT\mathcal{S})$. In the special case of the first Heisenberg group this fact is true, and we have the following result.

Proposition 7.3. *Given a C^k non-characteristic surface $\mathcal{S} \subset \mathbb{H}^1$, $k \geqslant 2$, one has $[X, Y]^H \in HT\mathcal{S}$ for every $X, Y \in C^1(\mathcal{S}; HT\mathcal{S})$, and therefore the horizontal connection on $\mathcal{S}$ is torsion free.*

Proof. According to (6.15), for every $g \in \mathcal{S}$ we have $HT_g\mathcal{S} = \operatorname{span}\{\boldsymbol{e}_1(g)\}$, where $\boldsymbol{e}_1 = (\boldsymbol{\nu}^H)^\perp$. Therefore, if we take two vector fields $X, Y \in C^1(\mathcal{S}; HT\mathcal{S})$, then we can write $X = a\boldsymbol{e}_1$, $Y = b\boldsymbol{e}_1$, for appropriate C^{k-1} functions a and b. We thus have

$$[X, Y] = [a\boldsymbol{e}_1, b\boldsymbol{e}_1] = \big\{a\boldsymbol{e}_1(b) - b\boldsymbol{e}_1(a)\big\}\boldsymbol{e}_1.$$

This shows that $[X, Y] \in C(\mathcal{S}; HT\mathcal{S})$, and therefore Proposition 7.2 gives

$$\nabla_X^{H,\mathcal{S}} Y - \nabla_Y^{H,\mathcal{S}} X = [X, Y].$$

This gives the desired conclusion. □

Definition 7.4. Let $\mathcal{S}$ be as in Definition 7.1. Consider a function $u \in C^1(\mathcal{S})$. We define the *tangential horizontal gradient* of u as follows

$$\nabla^{H,\mathcal{S}} u \stackrel{\text{def}}{=} \nabla^H \bar{u} - \big\langle \nabla^H \bar{u}, \boldsymbol{\nu}^H \big\rangle \boldsymbol{\nu}^H,$$

where $\bar{u} \in C^1(\boldsymbol{G})$ is such that $\bar{u} = u$ on $\mathcal{S}$.

We note that $\nabla^{H,\mathcal{S}} u = \sum_{i=1}^m \nabla_i^{H,\mathcal{S}} u X_i$, where

$$\nabla_i^{H,\mathcal{S}} u = X_i \bar{u} - \big\langle \nabla^H \bar{u}, \boldsymbol{\nu}^H \big\rangle \nu_i^H = X_i \bar{u} - \overline{p}_i \overline{p}_j X_j \bar{u}.$$

Observe also that $\nabla^{H,\mathcal{S}} u \in HT\mathcal{S}$. One has in fact from (6.7) and Definition 7.4

$$\big\langle \nabla^{H,\mathcal{S}} u, \boldsymbol{\nu}^H \big\rangle \equiv 0 \quad \text{in } \mathcal{S} \setminus \Sigma, \tag{7.4}$$

and therefore

$$\big|\nabla^{H,\mathcal{S}} u\big|^2 = \big|\nabla^H u\big|^2 - \big\langle \nabla^H u, \boldsymbol{\nu}^H \big\rangle^2. \tag{7.5}$$

8. Perimeter measure and horizontal first fundamental form

In a Carnot group $\boldsymbol{G}$, given an open set $\Omega \subset \boldsymbol{G}$, we let

$$\mathcal{F}(\Omega) = \left\{ \zeta = \sum_{i=1}^m \zeta_i X_i \in C_0^1(\Omega, H\boldsymbol{G}) \;\middle|\; |\zeta|_\infty = \sup_\Omega \left(\sum_{i=1}^m \zeta_i^2 \right)^{1/2} \leqslant 1 \right\}.$$

For a function $u \in L^1_{\mathrm{loc}}(\Omega)$, the H-variation of u with respect to Ω is defined by

$$\mathrm{Var}_H(u;\Omega) = \sup_{\zeta \in \mathcal{F}(\Omega)} \int_{\boldsymbol{G}} u \,\mathrm{div}_H \zeta \, dg.$$

We say that $u \in L^1(\Omega)$ has bounded H-variation in Ω if $\mathrm{Var}_H(u;\Omega) < \infty$. The space $BV_H(\Omega)$ of functions with bounded H-variation in Ω, endowed with the norm

$$\|u\|_{BV_H(\Omega)} = \|u\|_{L^1(\Omega)} + \mathrm{Var}_H(u;\Omega),$$

is a Banach space.

Definition 8.1. Let $E \subset \boldsymbol{G}$ be a measurable set, Ω be an open set. The H-perimeter of E with respect to Ω is defined by

$$P_H(E;\Omega) = \mathrm{Var}_H(\chi_E;\Omega),$$

where χ_E denotes the indicator function of E. We say that E is a H-Caccioppoli set if $\chi_E \in BV_H(\Omega)$ for every $\Omega \Subset \boldsymbol{G}$.

The above definitions are taken from [9], see also [48]. Following classical arguments [38,97], one obtains from the Riesz representation theorem.

Theorem 8.2. *Given an open set $\Omega \subset \boldsymbol{G}$, let $E \subset \boldsymbol{G}$ be a H-Caccioppoli set in Ω. There exist a Radon measure $\|\partial^H E\|$ in Ω, and a $\|\partial^H E\|$-measurable function $\boldsymbol{\nu}^H_E : \Omega \to H\boldsymbol{G}$, such that*

$$\left|\boldsymbol{\nu}^H_E(g)\right| = 1 \quad \text{for } \left\|\partial^H E\right\|\text{-a.e. } g \in \Omega,$$

and for which one has for every $\zeta \in C^1_0(\Omega; H\boldsymbol{G})$

$$\int_E \mathrm{div}_H \zeta \, dg = \int_\Omega \left\langle \zeta, \boldsymbol{\nu}^H_E \right\rangle d\left\|\partial^H E\right\| = \int_\Omega \left\langle \zeta, d\left[\partial^H E\right]\right\rangle.$$

Let $E \subset \boldsymbol{G}$ be a C^1 domain, with Riemannian outer unit normal $\boldsymbol{\nu}$. If $\zeta \in C^1_0(\Omega; H\boldsymbol{G})$, we have

$$\int_E \mathrm{div}_H \zeta \, dg = \int_{\partial E \cap \Omega} \sum_{i=1}^m \zeta_i \langle X_i, \boldsymbol{\nu}\rangle \, dH_{N-1}.$$

From this observation, and from Theorem 8.2, we conclude the following result.

Proposition 8.3. *Let $E \subset \boldsymbol{G}$ be a C^1 domain. For every open set $\Omega \subset \boldsymbol{G}$, and any $\zeta \in C^1_0(\Omega; H\boldsymbol{G})$, one has*

$$\int_\Omega \left\langle \zeta, \boldsymbol{\nu}^H_E \right\rangle d\left\|\partial^H E\right\| = \int_{\partial E \cap \Omega} \left\langle \zeta, \frac{\boldsymbol{N}^H}{|\boldsymbol{N}|} \right\rangle dH_{N-1},$$

where $\boldsymbol{N}^H$ *is defined in* (6.5). *Moreover,*

$$d\left\|\partial^H E\right\| = \left|\boldsymbol{N}^H\right| d(H_{N-1}\lfloor\partial E), \tag{8.1}$$

and one has

$$\left\|\partial^H E\right\|(\Omega) = P_H(E;\Omega) = \int\limits_{\partial E\cap\Omega} \frac{|\boldsymbol{N}^H|}{|\boldsymbol{N}|}\, dH_{N-1} = \int\limits_{\partial E\cap\Omega} \frac{W}{|\boldsymbol{N}|}\, dH_{N-1}, \tag{8.2}$$

where W *is the angle function defined in* (6.1).

Definition 8.4. Given an oriented C^2 hypersurface $\mathcal{S}\subset\boldsymbol{G}$, we will denote by

$$d\sigma_H = \frac{|\boldsymbol{N}^H|}{|\boldsymbol{N}|}\, dH_{N-1}\lfloor\mathcal{S} = \frac{W}{|\boldsymbol{N}|}\, dH_{N-1}\lfloor\mathcal{S}, \tag{8.3}$$

the H-perimeter measure supported on $\mathcal{S}$ (see (8.2) and (6.10)).

For a detailed local study of such measure the reader should see [24,25,69,70]. An interesting interpretation of the H-perimeter measure is that the latter is obtained by blowing-up the Riemannian regularizations of the sub-Riemannian metric of the group $\boldsymbol{G}$. In a different context, this idea was first exploited systematically by Korányi [65] in his computations of the sub-Riemannian geodesics in $\mathbb{H}^n$. For simplicity, and to illustrate the main idea, we will state the relevant result in the case when $\boldsymbol{G}=\mathbb{H}^n$.

Theorem 8.5. *Consider in the Heisenberg group* $\mathbb{H}^n$ *the left-invariant Riemannian metric tensor* $\{g^{\epsilon}_{ij}\}_{i,j=1,\dots,2n+1}$ *with respect to which* $\{X_1,\dots,X_{2n},\sqrt{\epsilon}T\}$ *constitutes an orthonormal frame of* $T\mathbb{H}^n$. *Let* $\mathcal{S}\subset\mathbb{H}^n$ *be a* C^2 *hypersurface, with* $\Sigma_{\mathcal{S}}=\emptyset$, *and denote by* $I^{\mathcal{S}}_{\epsilon}(\cdot,\cdot)$ *the first fundamental form in the Riemannian metric on* $\mathcal{S}$ *induced by* $\{g^{\epsilon}_{ij}\}_{i,j=1,\dots,2n+1}$. *Denote by* σ^{ϵ} *the corresponding surface area on* $\mathcal{S}$, *then for any bounded open chart* $U\subset\mathcal{S}$ *one has*

$$\sigma_H(U) = \lim_{\epsilon\to 0} \frac{\sigma^{\epsilon}(U)}{\sqrt{\det(g^{\epsilon}_{ij})}}.$$

Proof. For simplicity, we present the proof in the case $n=1$. Let $T_\epsilon=\sqrt{\epsilon}T$, and consider in $\mathbb{H}^1$ the one-parameter family of left-invariant Riemannian metrics $\{(g^{\epsilon}_{ij})_{i,j=1,2,3}\}_{\epsilon>0}$ with respect to which $\{X_1,X_2,T_\epsilon\}$ constitute an orthonormal basis of $T\mathbb{H}^1$. Similarly to (3.8), we find

$$\left(g^{\epsilon}_{ij}\right) = \begin{pmatrix} 1+\frac{y^2}{4\epsilon} & -\frac{xy}{4\epsilon} & \frac{y}{2\epsilon} \\ -\frac{xy}{4\epsilon} & 1+\frac{x^2}{4\epsilon} & -\frac{x}{2\epsilon} \\ \frac{y}{2\epsilon} & -\frac{x}{2\epsilon} & \frac{1}{\epsilon} \end{pmatrix}. \tag{8.4}$$

One easily verifies that

$$G^{\epsilon} = \det\left(g^{\epsilon}_{ij}\right) = \epsilon^{-1},$$

and that letting $((g^\epsilon)^{ij}) = (g^\epsilon_{ij})^{-1}$, then

$$\big((g^\epsilon)^{ij}\big) = \begin{pmatrix} 1 & 0 & -\frac{y}{2} \\ 0 & 1 & \frac{x}{2} \\ -\frac{y}{2} & \frac{x}{2} & \epsilon + \frac{x^2+y^2}{4} \end{pmatrix}. \tag{8.5}$$

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set such that U is represented by $\theta : \Omega \to U$, with $\theta \in C^2(\Omega)$. We have $\theta(u, v) = x(u, v)X_1 + y(u, v)X_2 + t(u, v)T$, see (14.15). We now use some of the computations from Section 14. We rewrite (14.17) as follows

$$\begin{cases} \theta_u = x_u X_1 + y_u X_2 + \frac{1}{\sqrt{\epsilon}}\big(t_u + \frac{yx_u - xy_u}{2}\big)T_\epsilon, \\ \theta_v = x_v X_1 + y_v X_2 + \frac{1}{\sqrt{\epsilon}}\big(t_v + \frac{yx_v - xy_v}{2}\big)T_\epsilon. \end{cases} \tag{8.6}$$

Denoting by $\wedge_\epsilon$ the wedge product with respect to the orthonormal frame $\{X_1, X_2, T_\epsilon\}$, similarly to (14.19) we obtain for the non-unit Riemannian normal to $\mathcal{S}$ with respect to $I_\epsilon(\cdot,\cdot)$

$$\begin{aligned} \boldsymbol{N}^\epsilon = \theta_u \wedge_\epsilon \theta_v &= \frac{1}{\sqrt{\epsilon}}\left(y_u t_v - y_v t_u - \frac{y}{2}(x_u y_v - x_v y_u)\right)X_1 \\ &\quad + \frac{1}{\sqrt{\epsilon}}\left(x_v t_u - x_u t_v + \frac{x}{2}(x_u y_v - x_v y_u)\right)X_2 + (x_u y_v - x_v y_u)T_\epsilon \\ &= \frac{1}{\sqrt{\epsilon}}pX_1 + \frac{1}{\sqrt{\epsilon}}qX_2 + \omega T_\epsilon, \end{aligned} \tag{8.7}$$

where in the last equality we have used (14.20). From (8.7) we conclude that

$$\frac{\sigma_\epsilon(U)}{\sqrt{g^\epsilon}} = \sqrt{\epsilon}\int_U d\sigma_\epsilon = \sqrt{\epsilon}\int_\Omega \sqrt{I_\epsilon\big(\boldsymbol{N}^\epsilon, \boldsymbol{N}^\epsilon\big)}\, du \wedge dv = \int_\Omega \sqrt{p^2 + q^2 + \epsilon\omega^2}\, du \wedge dv. \tag{8.8}$$

Letting $\epsilon \to 0$ in (8.8), we conclude

$$\lim_{\epsilon\to 0} \frac{\sigma^\epsilon(U)}{\sqrt{g^\epsilon}} = \int_\Omega W\, du \wedge dv = \int_U \frac{W}{|\boldsymbol{N}|}\, d\sigma = \sigma_H(U),$$

where in the last equality we have used (8.3). This completes the proof. □

We close this section by collecting two basic properties of the H-perimeter. The former is a trivial consequence of the left-invariance on the vector fields $X_1, \ldots, X_m$, and of the definition of H-perimeter.

Proposition 8.6. *For any H-Caccioppoli set E in a Carnot group $\boldsymbol{G}$, and any open set $\Omega \subset \boldsymbol{G}$, one has*

$$P_H\big(L_{g_0}(E); L_{g_0}(\Omega)\big) = P_H(E; \Omega), \quad g_0 \in \boldsymbol{G}, \tag{8.9}$$

where $L_{g_0} g = g_0 g$ is the left-translation on the group. In particular,

$$P_H\big(L_{g_0}(E); \boldsymbol{G}\big) = P_H(E; \boldsymbol{G}), \quad g_0 \in \boldsymbol{G}. \tag{8.10}$$

Proposition 8.7. *In a Carnot group $\boldsymbol{G}$ one has for every H-Caccioppoli set $E \subset \boldsymbol{G}$, any open set $\Omega \subset \boldsymbol{G}$, and every $\lambda > 0$*

$$P_H(\delta_\lambda E; \delta_\lambda \Omega) = \lambda^{Q-1} P_H(E; \Omega). \tag{8.11}$$

In particular,

$$P_H(\delta_\lambda E; \boldsymbol{G}) = \lambda^{Q-1} P_H(E; \Omega). \tag{8.12}$$

Proof. We observe that if $\zeta \in C_0^1(\boldsymbol{G}, H\boldsymbol{G})$, then $\zeta \circ \delta_{1/\lambda} \in C_0^1(\delta_\lambda \Omega; H\boldsymbol{G})$. Furthermore, $\zeta \in \mathcal{F}(\Omega)$ if and only if $\zeta \circ \delta_{1/\lambda} \in \mathcal{F}(\delta\Omega)$. The divergence theorem, and a rescaling now give

$$\int_E \operatorname{div}_H \zeta \, dg = \int_E \sum_{j=1}^m X_j \zeta_j \, dg = \lambda^{-Q} \int_{\delta_\lambda E} \sum_{j=1}^m X_j \zeta_j (\delta_{1/\lambda} g) \, dg. \tag{8.13}$$

Since

$$X_j(\zeta_j \circ \delta_{1/\lambda}) = \lambda^{-1} (X_j \zeta_j) \circ \delta_{1/\lambda},$$

we conclude from (8.13)

$$\int_E \operatorname{div}_H \zeta \, dg = \lambda^{-(Q-1)} \int_{\delta_\lambda E} \sum_{j=1}^m X_j (\zeta_j \circ \delta_{1/\lambda}) \, dg.$$

Taking the supremum on all $\zeta \in \mathcal{F}(\Omega)$ in the latter equation, we reach the desired conclusion. □

Combining Propositions 8.6 and 8.7 we obtain the following result.

Corollary 8.8. *Let $\mathcal{S} \subset \boldsymbol{G}$ be a C^2 hypersurface with finite H-perimeter, then for every $g_0 \in \boldsymbol{G}$, and every $\lambda > 0$, one has*

$$\sigma_H\big(L_{g_0}(\mathcal{S})\big) = \sigma_H(\mathcal{S}),$$
$$\sigma_H(\delta_\lambda \mathcal{S}) = \lambda^{Q-1} \sigma_H(\mathcal{S}).$$

9. Horizontal second fundamental form and mean curvature

We open this section by computing the first variation of the H-perimeter for deformations of a hypersurface $\mathcal{S}$ along the Riemannian normal $\boldsymbol{N}$ to $\mathcal{S}$. This will provide a first motivation for the introduction of the notion of H-mean curvature.

Theorem 9.1. *Let $\mathcal{U} \subset \boldsymbol{G}$ be a bounded open set and consider $\phi \in C^2(\mathcal{U})$ with $|\nabla\phi| \geqslant \alpha > 0$ in $\mathcal{U}$, and for small $\lambda \in [-\lambda_0, \lambda_0]$ consider the one-parameter family of $\mathcal{S}^\lambda = \partial\mathcal{U}_\lambda$, where we have let $\mathcal{U}_\lambda = \{g \in \mathcal{U} \mid \phi(g) < \lambda\}$. Assume that each of the $\mathcal{S}^\lambda$ be a C^2 non-characteristic hypersurface. Let $\mathcal{S} = \mathcal{S}^0$ and define a function $\mathcal{H} : \mathcal{S} \to \mathbb{R}$ by letting*

$$\mathcal{H} \stackrel{\text{def}}{=} \sum_{i=1}^{m} X_i \overline{p}_i, \tag{9.1}$$

where the $\overline{p}_i$ are the components of the horizontal Gauss map introduced in (6.2). *We then have*

$$\frac{d}{d\lambda} P_H\big(\mathcal{S}^\lambda\big)|_{\lambda=0} \stackrel{\text{def}}{=} \frac{d}{d\lambda} P_H(\mathcal{U}_\lambda; \boldsymbol{G})|_{\lambda=0} = \int_{\mathcal{S}} \frac{\mathcal{H}}{|\boldsymbol{N}|}\, dH_{N-1}.$$

In particular, $\mathcal{S}$ is a critical point of the H-perimeter with respect to the deformations $\mathcal{S} \to \mathcal{S}^\lambda$ if and only of $\mathcal{H} \equiv 0$.

Proof. Using Federer's coarea formula [39] we can write

$$\int_{\mathcal{U}_\lambda} \big|\nabla^H \phi\big|\, dg = \int_{-\infty}^{\lambda} \int_{\partial\mathcal{U}_\tau} \frac{W}{|\boldsymbol{N}|}\, dH_{N-1}\, d\tau = \int_{-\infty}^{\lambda} P_H(\mathcal{U}_\tau; \boldsymbol{G})\, d\tau, \tag{9.2}$$

where the second equality is a consequence of (8.2). The identity (9.2) gives

$$P_H(\mathcal{U}_\lambda; \boldsymbol{G}) = \frac{d}{d\lambda} \int_{\mathcal{U}_\lambda} \big|\nabla^H \phi\big|\, dg. \tag{9.3}$$

Using the summation convention over repeated indices, and integration by parts, we now compute:

$$\begin{aligned}
\int_{\mathcal{U}_\lambda} \big|\nabla^H \phi\big|\, dg &= \int_{\mathcal{U}_\lambda} X_i \phi \nu_i^H\, dg \\
&= \int_{\partial\mathcal{U}_\lambda} \phi \langle X_i, \boldsymbol{\nu}\rangle \nu_i^H\, dH_{N-1} - \int_{\mathcal{U}_\lambda} \phi X_i \nu_i^H\, dg \\
&= \int_{\partial\mathcal{U}_\lambda} \phi \big|\boldsymbol{N}^H\big|\, dH_{N-1} - \int_{\mathcal{U}_\lambda} \phi X_i \nu_i^H\, dg \\
&= \lambda P_H(\mathcal{U}_\lambda; \boldsymbol{G}) - \int_{\mathcal{U}_\lambda} \phi X_i \nu_i^H\, dg,
\end{aligned} \tag{9.4}$$

where we have used (8.2). From (9.3), (9.4), and the coarea formula again, we find

$$P_H(\mathcal{U}_\lambda; \boldsymbol{G}) = \frac{d}{d\lambda}\{\lambda P_H(\mathcal{U}_\lambda; \boldsymbol{G})\} - \lambda \int_{\partial \mathcal{U}_\lambda} \frac{X_i \bar{p}_i}{|\boldsymbol{N}|} \, dH_{N-1}. \tag{9.5}$$

Equation (9.5) easily implies the desired conclusion. □

Remark 9.2. As we will see in this section, the critical points of the H-perimeter are precisely the so-called H-minimal hypersurfaces. We will return to this question in Section 14, where we will develop more precise intrinsic first and second variation formulas of the H-perimeter in the setting of the Heisenberg group.

We are now ready to introduce the central notions of sub-Riemannian, or horizontal second fundamental form, and of H-mean curvature. According to Theorem 9.1, hypersurfaces for which the function $\mathcal{H}$ in (9.1) vanishes identically on $\mathcal{S}$ are critical points of the H-perimeter functional with respect to deformations of the surface in the direction of the Riemannian normal $\boldsymbol{N}$ to $\mathcal{S}$. This suggests a notion of horizontal mean curvature of $\mathcal{S}$ based on Eq. (9.1). Such notion was proposed by one of us back in 1997, see [46], and it produces precisely the function in (9.1). We will in fact introduce a more intrinsic notion which is based on that of horizontal second fundamental form, and then recognize that such definition coincides with (9.1). This closely parallels the classical development of the subject. We recall the classical definition of the mean curvature H of a n-dimensional immersed submanifold $N = N^n$ of an m-dimensional Riemannian manifold $M = M^m$. Denoting with $i : N \hookrightarrow M$ the immersion, we recall that the Levi-Civita connection of N is given by Eq. (7.1). The second fundamental form of N is defined by

$$II_N(X, Y) = \left(\nabla^M_X Y\right)^\perp,$$

where ∇^M is the Levi-Civita connection of M. Since for vector fields on N one has

$$II_N(X, Y) - II_N(Y, X) = \left(\nabla^M_X Y - \nabla^M_Y X\right)^\perp = [X, Y]^\perp = 0,$$

II_N defines a symmetric tensor field on N of type $(0, 2)$, which takes values in $TN^\perp$. The mean curvature of N at a point $p \in N$ is defined by

$$H = -\frac{1}{n}\operatorname{trace}(II_N) = -\frac{1}{n}\sum_{i=1}^n II_N(\boldsymbol{e}_i, \boldsymbol{e}_i),$$

where $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ is an orthonormal basis of T_pN.

We now consider the Riemannian manifold $M = \boldsymbol{G}$ with the metric tensor with respect to which $X_1, \dots, X_m, \dots, X_{r,m_r}$ is an orthonormal basis, and the corresponding Levi-Civita connection ∇ on $\boldsymbol{G}$. Let ∇^H denote the horizontal Levi-Civita connection introduced in Definition 5.2. Let $\mathcal{S} \subset \boldsymbol{G}$ be a C^2 hypersurface. Inspired by the Riemannian situation we introduce a notion of horizontal second fundamental on $\mathcal{S}$ as follows.

Definition 9.3. Let $\mathcal{S} \subset \boldsymbol{G}$ be a C^2 hypersurface, with $\Sigma_{\mathcal{S}} = \emptyset$, then we define a tensor field of type $(0,2)$ on $HT\mathcal{S}$, as follows: for every $X, Y \in C^1(\mathcal{S}; HT\mathcal{S})$

$$II^{H,\mathcal{S}}(X,Y) = \langle \nabla_X^H Y, \boldsymbol{\nu}^H \rangle \boldsymbol{\nu}^H. \tag{9.6}$$

We call $II^{H,\mathcal{S}}(\cdot,\cdot)$ the *horizontal second fundamental form* of $\mathcal{S}$. We also define $\mathcal{A}^{H,\mathcal{S}} : HT\mathcal{S} \to HT\mathcal{S}$ by letting for every $g \in \mathcal{S}$ and $\boldsymbol{u}, \boldsymbol{v} \in HT_g$

$$\langle \mathcal{A}^{H,\mathcal{S}} \boldsymbol{u}, \boldsymbol{v} \rangle = -\langle II^{H,\mathcal{S}}(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{\nu}^H \rangle = -\langle \nabla_X^H Y, \boldsymbol{\nu}^H \rangle, \tag{9.7}$$

where $X, Y \in C^1(\mathcal{S}, HT\mathcal{S})$ are such that $X_g = \boldsymbol{u}$, $Y_g = \boldsymbol{v}$. We call the endomorphism $\mathcal{A}^{H,\mathcal{S}} : HT_g\mathcal{S} \to HT_g\mathcal{S}$ the *horizontal shape operator*. If $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{m-1}$ denotes a local orthonormal frame for $HT\mathcal{S}$, then the matrix of the horizontal shape operator with respect to the basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{m-1}$ is given by the $(m-1) \times (m-1)$ matrix $[-\langle \nabla_{\boldsymbol{e}_i}^H \boldsymbol{e}_j, \boldsymbol{\nu}^H \rangle]_{i,j=1,\ldots,m-1}$.

Using the horizontal Koszul identity (5.14), one easily verifies that

$$\langle \nabla_{\boldsymbol{e}_i}^H \boldsymbol{e}_j, \boldsymbol{\nu}^H \rangle = -\langle \nabla_{\boldsymbol{e}_i}^H \boldsymbol{\nu}^H, \boldsymbol{e}_j \rangle.$$

Using Proposition 5.5 in Definition 9.3 we immediately recognize that

$$II^{H,\mathcal{S}}(X,Y) - II^{H,\mathcal{S}}(Y,X) = \langle [X,Y]^H, \boldsymbol{\nu}^H \rangle \boldsymbol{\nu}^H, \tag{9.8}$$

and therefore, unlike its Riemannian counterpart, the horizontal second fundamental form of $\mathcal{S}$ is not necessarily symmetric. This depends on the fact, already observed, that if $X, Y \in C^1(\mathcal{S}; HT\mathcal{S})$, then it is not necessarily true that $[X,Y]^H \in C(\mathcal{S}; HT\mathcal{S})$. The next proposition gives a necessary and sufficient condition for the symmetry of $II^{H,\mathcal{S}}$.

Proposition 9.4. *The horizontal second fundamental form $II^{H,\mathcal{S}}(\cdot,\cdot)$ is a $(0,2)$ symmetric tensor field on $HT\mathcal{S}$ if and only if for any local orthonormal basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{m-1}$ of $HT\mathcal{S}$, one has*

$$[\boldsymbol{e}_i, \boldsymbol{e}_j]^H \in HT\mathcal{S}, \quad i,j = 1, \ldots, m-1.$$

Proof. Let $X = \sum_{i=1}^{m-1} a_i \boldsymbol{e}_i$, $Y = \sum_{j=1}^{m-1} b_j \boldsymbol{e}_j$, then

$$[X,Y] = \sum_{j=1}^{m-1} \left\{ \sum_{i=1}^{m-1} \big(a_i \boldsymbol{e}_i(b_j) - b_i \boldsymbol{e}_i(a_j) \big) \right\} \boldsymbol{e}_j + \sum_{i,j=1}^{m-1} a_i b_j [\boldsymbol{e}_i, \boldsymbol{e}_j].$$

This identity gives

$$\begin{aligned} \langle [X,Y]^H, \boldsymbol{\nu}^H \rangle &= \sum_{i,j=1}^{m-1} \big[a_i \boldsymbol{e}_i(b_j) - b_i \boldsymbol{e}_i(a_j) \big] \langle \boldsymbol{e}_j, \boldsymbol{\nu}^H \rangle + \sum_{i,j=1}^{m-1} a_i b_j \langle [\boldsymbol{e}_i, \boldsymbol{e}_j]^H, \boldsymbol{\nu}^H \rangle \\ &= \sum_{i,j=1}^{m-1} a_i b_j \langle [\boldsymbol{e}_i, \boldsymbol{e}_j]^H, \boldsymbol{\nu}^H \rangle = 0, \end{aligned}$$

provided that $[\boldsymbol{e}_i, \boldsymbol{e}_j]^H \in HT\mathcal{S}$. Therefore, under the assumption $[\boldsymbol{e}_i, \boldsymbol{e}_j]^H \in HT\mathcal{S}$, we finally obtain from Definition 9.3

$$II^{H,\mathcal{S}}(X,Y) - II^{H,\mathcal{S}}(Y,X) = \langle \nabla_X^H Y - \nabla_Y^H X, \boldsymbol{\nu}^H \rangle \boldsymbol{\nu}^H = \langle [X,Y]^H, \boldsymbol{\nu}^H \rangle \boldsymbol{\nu}^H = 0,$$

which proves the symmetry of $II^{H,\mathcal{S}}$. Vice-versa, suppose that $II^{H,\mathcal{S}}$ be symmetric, then applying the latter identity with $X = \boldsymbol{e}_i$, $Y = \boldsymbol{e}_j$, we reach the conclusion that $\langle [\boldsymbol{e}_i, \boldsymbol{e}_j]^H, \boldsymbol{\nu}^H \rangle = 0$. □

Corollary 9.5. *The horizontal second fundamental form of a C^2 non-characteristic surface $\mathcal{S} \subset \mathbb{H}^1$ is symmetric.*

Proof. In this situation the assumption of Proposition 9.4 is trivially satisfied since a basis of $HT\mathcal{S}$ is given by the single vector field $\boldsymbol{e}_1 = (\boldsymbol{\nu}^H)^\perp$, and therefore $[\boldsymbol{e}_1, \boldsymbol{e}_1]^H = 0 \in HT\mathcal{S}$, see also Proposition 7.3. □

Another situation in which the assumption of Proposition 9.4 is fulfilled is that when $\mathcal{S}$ is a cylindrical hypersurface over the first layer of the Lie algebra.

Proposition 9.6. *Suppose that the hypersurface $\mathcal{S}$ is a* vertical cylinder, *i.e., it can be represented in the form*

$$\mathcal{S} = \{ g \in \boldsymbol{G} \mid \mathfrak{h}(x_1(g), \ldots, x_m(g)) = 0 \}, \tag{9.9}$$

where $\mathfrak{h} \in C^2(\mathbb{R}^m)$, and there exist an open set $\omega \subset \mathbb{R}^m$ and $\alpha > 0$ such that $|\nabla \mathfrak{h}| \geqslant \alpha$ in ω. Under these assumptions, we have $\Sigma_{\mathcal{S}} = \emptyset$, and the horizontal second fundamental form is symmetric.

Proof. The function $\phi(g) = \mathfrak{h}(x_1(g), \ldots, x_m(g))$ is a defining function of $\mathcal{S}$. Using the global exponential coordinates, we obtain from Lemma 2.1

$$X_i \phi(g) = \frac{\partial \mathfrak{h}}{\partial x_i},$$

hence $\nabla^H \phi = \nabla_x \mathfrak{h}$, which proves in particular that $\Sigma_{\mathcal{S}} = \emptyset$, and that $\boldsymbol{\nu}^H = \frac{\nabla \mathfrak{h}}{|\nabla \mathfrak{h}|} = \boldsymbol{\nu}$. We next observe that for every $g_0 \in \boldsymbol{G}$, the left-translated surface $\tilde{\mathcal{S}} = L_{g_0}(\mathcal{S})$ is again a vertical cylinder, with defining function $\tilde{\mathfrak{h}}(x) = \mathfrak{h}(x(g_0) + x(g)) = 0$. As a consequence of this observation, if $g_0 \in \mathcal{S}$, then by left-translation we can assume without restriction that $g_0 = e$. We thus immediately see that the horizontal plane $H_e = \exp(V_1)$ is given by $t_1 = \cdots = t_k = \cdots = x_{r,m_r} = 0$. Furthermore, by an orthogonal transformation in the horizontal layer of the Lie algebra, we can assume that the tangent space of $\mathcal{S}$ in e be given by the hyperplane $x_m = 0$. Since thanks to Proposition 6.4 the horizontal tangent space at e is given by $H_e \cap T_e\mathcal{S}$, from the previous considerations we see that $HT_e\mathcal{S} = \operatorname{span}\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{m-1}\}$, where $\boldsymbol{e}_i = (\partial/\partial x_i)_e$. Since $[\partial/\partial x_i, \partial/\partial x_j] = 0$, $i, j = 1, \ldots, m-1$, we conclude that $[\boldsymbol{e}_i, \boldsymbol{e}_j]^H = 0$. In view of Proposition 9.4 we conclude that $II^{H,\mathcal{S}}$ is symmetric, thus completing the proof. □

From the proof of Proposition 9.6 one easily obtains the following corollary.

Corollary 9.7. *Let $\mathcal{S}$ be a vertical cylinder as in* (9.9), *then the H-mean curvature at $g \in \mathcal{S}$ is given by the formula*

$$\mathcal{H}(g) = (m-1)H\big(x(g)\big), \tag{9.10}$$

where $H(x(g))$ represents the Riemannian mean curvature of the projection $\pi_{V_1}(\mathcal{S})$ of $\mathcal{S}$ onto the horizontal layer V_1. In particular, $\mathcal{S}$ is H-minimal if and only if $\pi_{V_1}(\mathcal{S})$ is a classical minimal surface in $V_1 \simeq \mathbb{R}^m$.

Definition 9.8. We define the *horizontal principal curvatures* as the real eigenvalues $\kappa_1, \dots, \kappa_{m-1}$ of the symmetrized operator

$$\mathcal{A}^{H,\mathcal{S}}_{\mathrm{sym}} = \frac{1}{2}\big\{\mathcal{A}^{H,\mathcal{S}} + \big(\mathcal{A}^{H,\mathcal{S}}\big)^t\big\}.$$

The H-mean curvature of $\mathcal{S}$ at a non-characteristic point $g_0 \in \mathcal{S}$ is defined as

$$\mathcal{H} = -\mathrm{trace}\,\mathcal{A}^{H,\mathcal{S}}_{\mathrm{sym}} = \sum_{i=1}^{m-1} \kappa_i = \sum_{i=1}^{m-1} \big\langle \nabla^H_{\boldsymbol{e}_i} \boldsymbol{e}_i, \boldsymbol{\nu}^H \big\rangle.$$

If g_0 is characteristic, then we let

$$\mathcal{H}(g_0) = \lim_{g \to g_0,\, g \in \mathcal{S} \setminus \Sigma_{\mathcal{S}}} \mathcal{H}(g),$$

provided that such limit exists, finite or infinite. We do not define the H-mean curvature at those points $g_0 \in \Sigma_{\mathcal{S}}$ at which the limit does not exist. Finally, we call $\vec{\mathcal{H}} = \mathcal{H}\boldsymbol{\nu}^H$ the *H-mean curvature vector*.

Proposition 9.9. *The H-mean curvature in Definition* 9.8 *coincides with the function defined in* (9.1). *In fact, the following intrinsic identity holds*:

$$\mathcal{H} = \sum_{i=1}^{m} \nabla^{H,\mathcal{S}}_i \big\langle \boldsymbol{\nu}^H, X_i \big\rangle = \sum_{i=1}^{m} \nabla^{H,\mathcal{S}}_i \overline{p}_i. \tag{9.11}$$

Proof. In what follows to simplify the exposition we continue to indicate with $\overline{p}_1, \dots, \overline{p}_m$ an m-tuple of C^1 extensions to the whole of $\boldsymbol{G}$ of the coefficients of the horizontal Gauss map with respect to the basis $X_1, \dots, X_m$. We begin by observing that using the horizontal Koszul identity (5.14), one easily recognizes that

$$\big\langle \nabla^H_{\boldsymbol{e}_i} \boldsymbol{e}_j, \boldsymbol{\nu}^H \big\rangle = -\big\langle \boldsymbol{e}_i, \big[\boldsymbol{e}_i, \boldsymbol{\nu}^H\big]^H \big\rangle.$$

From Definition 9.8 we thus obtain

$$\mathcal{H} = -\sum_{i=1}^{m-1} \big\langle \nabla^H_{\boldsymbol{e}_i} \boldsymbol{e}_i, \boldsymbol{\nu}^H \big\rangle = \sum_{i=1}^{m-1} \big\langle \boldsymbol{e}_i, \big[\boldsymbol{e}_i, \boldsymbol{\nu}^H\big]^H \big\rangle. \tag{9.12}$$

Recalling (6.6), we find

$$\left[\boldsymbol{e}_i, \boldsymbol{\nu}^H\right] = \sum_{j=1}^m \boldsymbol{e}_i(\overline{p}_j) X_j + \sum_{j=1}^m [\boldsymbol{e}_i, X_j].$$

To proceed in the calculations, we write

$$\boldsymbol{e}_i = \sum_{\ell=1}^m a_i^\ell X_\ell, \quad i = 1, \ldots, m-1,$$

with $\{a_i^\ell\}$ satisfying the orthogonality conditions

$$\sum_{\ell=1}^m a_i^\ell \overline{p}_\ell = 0, \qquad \sum_{\ell=1}^m a_i^\ell a_j^\ell = \delta_{ij}, \quad i, j = 1, \ldots, m-1. \tag{9.13}$$

We thus obtain

$$\left[\boldsymbol{e}_i, \boldsymbol{\nu}^H\right] = \sum_{j=1}^m \boldsymbol{e}_i(\overline{p}_j) X_j - \sum_{\ell=1}^m \sum_{j=1}^m X_j(a_i^\ell) X_\ell + \sum_{s=1}^k \left(\sum_{\ell=1}^m \sum_{j=1}^m a_i^\ell b_{\ell j}^s \right) T_s.$$

The latter identity gives

$$\left[\boldsymbol{e}_i, \boldsymbol{\nu}^H\right]^H = \sum_{j=1}^m \boldsymbol{e}_i(\overline{p}_j) X_j - \sum_{\ell=1}^m \sum_{j=1}^m X_j\left(a_i^\ell\right) X_\ell,$$

and therefore,

$$\begin{aligned}
\sum_{i=1}^{m-1} \left\langle \left[\boldsymbol{e}_i, \boldsymbol{\nu}^H\right]^H, \boldsymbol{e}_i \right\rangle &= \sum_{j=1}^m \sum_{i=1}^{m-1} \boldsymbol{e}_i(\overline{p}_j) a_i^j - \sum_{\ell,j=1}^m \sum_{i=1}^{m-1} a_i^\ell X_j\left(a_i^\ell\right) \\
&= \sum_{j=1}^m \sum_{i=1}^{m-1} \boldsymbol{e}_i(\overline{p}_j) a_i^j \\
&= \sum_{\ell,j=1}^m X_\ell(\overline{p}_j) \sum_{i=1}^{m-1} a_i^\ell a_i^j,
\end{aligned} \tag{9.14}$$

where in the second to the last equality we have used (9.13). We now observe that, since $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{m-1}, \boldsymbol{\nu}^H\}$ is an orthonormal basis of $H_g T\mathcal{S}$, we have

$$\begin{aligned}
\sum_{i=1}^{m-1} a_i^\ell a_i^j &= \sum_{i=1}^{m-1} \langle X_\ell, \boldsymbol{e}_i \rangle \langle X_j, \boldsymbol{e}_j \rangle \\
&= \langle X_\ell, X_j \rangle - \left\langle X_\ell, \boldsymbol{\nu}^H \right\rangle \left\langle X_j, \boldsymbol{\nu}^H \right\rangle = \delta_{\ell j} - \overline{p}_\ell \overline{p}_j.
\end{aligned}$$

Substituting this identity in (9.14), and recalling (9.12), we finally have

$$\mathcal{H}=\sum_{i=1}^{m-1}\langle[\boldsymbol{e}_i,\boldsymbol{\nu}^H]^H,\boldsymbol{e}_i\rangle=\sum_{j=1}^{m}X_j(\overline{p}_j)-\sum_{\ell,j=1}^{m}\overline{p}_jX_\ell(\overline{p}_j)=\sum_{j=1}^{m}X_j(\overline{p}_j).$$

This concludes the proof. □

It is clear from Definition 9.8 that $\mathcal{H}\in C(\mathcal{S}\setminus\Sigma)$.

Definition 9.10. An oriented C^k hypersurface $\mathcal{S}\subset\boldsymbol{G}$, $k\geqslant 2$, is said to have *constant H-mean curvature* if $\mathcal{H}\equiv$ const on $\mathcal{S}$. We say that $\mathcal{S}$ is *H-minimal* if its H-mean curvature $\mathcal{H}$ vanishes everywhere as a continuous function on $\mathcal{S}$.

Remark 9.11. Consider the product group $\hat{\boldsymbol{G}}=\boldsymbol{G}\times\mathbb{R}$ with the canonical group law $(g,s)\circ(g',s')=(gg',s+s')$, induced by the one on $\boldsymbol{G}$. The stratification of the Lie algebra for $\hat{\boldsymbol{G}}$ is then given by $\hat{V}_1\oplus\cdots\oplus\hat{V}_r$ where $\hat{V}_1=V_1\times\mathbb{R}$, $\hat{V}_j=V_j\times\{0\}$ for $j=2,3,\ldots,r$. If $\{e_1,\ldots,e_m\}$ is a basis for V_1, we let $\hat{e}_j=(e_j,0)$ for $j=1,\ldots,m$, $\hat{e}_{m+1}=(0,\ldots,0,1)$. A basis for $\hat{V}_1$ is then given by $\{\hat{e}_1,\ldots,\hat{e}_{m+1}\}$. This identifies a subbundle $H\hat{\boldsymbol{G}}$. We can naturally identify $\boldsymbol{G}$ with the hypersurface $\mathcal{S}=\boldsymbol{G}\times\{0\}\subset\hat{\boldsymbol{G}}$ with global defining function $\phi(g,s)=s$. Now observe that $\nabla_i^{H,\mathcal{S}}\phi=X_i\phi$, $i=1,\ldots,m$ and $\nabla_{m+1}^{H,\mathcal{S}}\phi=1$ and therefore, $\boldsymbol{\nu}^H=\hat{e}_{m+1}$. As a consequence, we have $\nabla_i^{H,\mathcal{S}}\nu_i^H=0$ for $i=1,\ldots,m+1$. In view of Proposition 9.9 we conclude that the H-mean curvature of $\boldsymbol{G}$ in $\hat{\boldsymbol{G}}$ is zero.

To state the next proposition we consider for a function $u:\boldsymbol{G}\to\mathbb{R}$ the *symmetrized horizontal Hessian* of u at $g\in\boldsymbol{G}$. This is the $m\times m$ matrix with entries

$$u_{,ij}\overset{\text{def}}{=}\frac{1}{2}\{X_iX_ju+X_jX_iu\},\quad i,j=1,\ldots,m.\tag{9.15}$$

Setting $\nabla_H^2u=[u_{,ij}]$, the mapping $g\to\nabla_H^2u(g)$ defines a 2-covariant tensor on the subbundle $H\boldsymbol{G}$. We recall that the horizontal Laplacian associated with the basis $\{e_1,\ldots,e_m\}$ of V_1 is given by $\Delta_Hu=\operatorname{tr}\nabla_H^2u$. We also consider the following nonlinear operator:

$$\Delta_{H,\infty}u\overset{\text{def}}{=}\sum_{i,j=1}^{m}u_{,ij}X_iuX_ju=\frac{1}{2}\langle\nabla^H(|\nabla^Hu|^2),\nabla^Hu\rangle,\tag{9.16}$$

which, by analogy with its by now classical Euclidean ancestor, we call the *horizontal ∞-Laplacian*. The reason for introducing the operator $\Delta_{H,\infty}$ is in the following result which is often useful in computing the H-mean curvature. We consider a C^2 hypersurface $\mathcal{S}\subset\boldsymbol{G}$, and for a given $g_0\in\mathcal{S}\setminus\Sigma_{\mathcal{S}}$, suppose that there exist a neighborhood $\mathcal{U}$ of g_0, and $\phi\in C^2(\mathcal{U})$, such that $\mathcal{S}\cap\mathcal{U}=\mathcal{S}\cap\partial\{g\in\mathcal{U}\mid\phi(g)<0\}$. We observe that the hypothesis that $g_0\notin\Sigma_{\mathcal{S}}$ implies that $\nabla^H\phi(g_0)\neq 0$, and therefore, by possibly restricting $\mathcal{U}$ we can assume that $\nabla^H\phi(g)\neq 0$ for every $g\in\mathcal{S}\cap\mathcal{U}$. We thus have

$$\boldsymbol{N}^H(g)=\nabla^H\phi(g),\quad\text{for every } g\in\mathcal{S}\cap\mathcal{U},\tag{9.17}$$

and therefore

$$\boldsymbol{\nu}^H = \frac{\nabla^H \phi}{|\nabla^H \phi|} \quad \text{for every } g \in \mathcal{S} \cap \mathcal{U}. \tag{9.18}$$

Proposition 9.12. *At every point of $\mathcal{S} \cap \mathcal{U}$ one has*

$$|\nabla_H \phi|^3 \mathcal{H} = \{|\nabla_H \phi|^2 \Delta_H \phi - \Delta_{H,\infty} \phi\}.$$

Proof. We use the summation convention over repeated indices. Invoking Proposition 9.9 and (9.18), we obtain at every point in $\mathcal{S} \setminus \Sigma$

$$\mathcal{H} = \nabla_i^{H,\mathcal{S}} \overline{p}_i = X_i(\nu_{H,i}) = X_i\left(\frac{X_i \phi}{|\nabla^H \phi|}\right) = \frac{\Delta_H \phi}{|\nabla^H \phi|} - \frac{\Delta_{H,\infty} \phi}{|\nabla^H \phi|^3}. \qquad \Box$$

It is interesting to consider a nonlinear operator which interpolates in an appropriate sense between the H-mean curvature operator in Definition 9.8, and the operator $\Delta_{H,\infty}$. Consider the one-parameter family of quasilinear operators defined by

$$\Delta_{H,p} u = \operatorname{div}_H\left(\left|\nabla^H u\right|^{p-2} \nabla^H u\right) = 0, \quad 1 < p < \infty. \tag{9.19}$$

Supposing that $|\nabla^H u| \neq 0$, we formally rewrite in the more suggestive fashion

$$\Delta_{H,p} u = (p-2)\left|\nabla^H u\right|^{p-4}\left\{\frac{1}{p-2}\left|\nabla^H u\right|^2 \Delta_H u + \Delta_{H,\infty} u\right\}. \tag{9.20}$$

If u_p is a solution to $\Delta_{H,p} u_p = 0$, then Eq. (9.20) gives

$$\frac{1}{p-2}\left|\nabla^H u_p\right|^2 \Delta_H u_p + \Delta_{H,\infty} u_p = 0.$$

If we assume that $u_p \to u_\infty$ as $p \to \infty$, and that $|\nabla^H u_p|^2 \Delta_H u_p$ is bounded independently of p large, then by letting $p \to \infty$ we formally find that u_∞ must be a solution to $\Delta_{H,\infty} u_\infty = 0$. On the other hand, if we know instead that $u_p \to u_1$ as $p \to 1$, then we discover from (9.20) and Proposition 9.12 that

$$\Delta_{H,p} u_p \underset{p \to 1}{\longrightarrow} -\mathcal{H}(u_1),$$

where $\mathcal{H}(u_1)$ is the H-mean curvature of the level sets of u_1! This suggests that one should study the behavior as $p \to 1$ of the one-parameter family of quasilinear operators $\Delta_{H,p}$.

9.1. Comparison with S. Pauls' notion of horizontal mean curvature

In the first Heisenberg group $\mathbb{H}^1$ another notion of horizontal mean curvature was introduced by S. Pauls in [79]. Such notion is based on the procedure of Riemannian ϵ-regularization defined in the proof of Theorem 8.5. Using (8.5) one sees that, given a function $\phi \in C^1(\mathbb{H}^n)$, its gradient with respect to the metric (g^ϵ_{ij}) is given by

$$\nabla_\epsilon \phi = X_1\phi X_1 + X_2\phi X_2 + T_\epsilon\phi T_\epsilon. \tag{9.21}$$

Let us note in passing that (9.21) gives

$$|\nabla_\epsilon \phi|^2 = |\nabla_H \phi|^2 + \epsilon (T\phi)^2, \quad \text{and} \quad \Delta_\epsilon \phi = \Delta_H \phi + \epsilon T^2 \phi, \tag{9.22}$$

where we have denoted by Δ_ϵ the Laplace–Beltrami operator with respect to the metric (g^ϵ_{ij}).

In [79] the author defined the horizontal mean curvature of $\mathcal{S}$ at a point $g_0 \in \mathcal{S} \setminus \Sigma_{\mathcal{S}}$ as follows:

$$\mathcal{H}_P(g_0) \stackrel{\text{def}}{=} \lim_{\epsilon \to 0} H^\epsilon_{\mathcal{R}}(g_0), \tag{9.23}$$

where $H^\epsilon_{\mathcal{R}}$ indicates the mean curvature of $\mathcal{S}$ in the Riemannian metric (8.4). We now recognize that such notion coincides with the one introduced in Definition 9.8.

Proposition 9.13. *The horizontal mean curvature defined by* (9.23) *coincides with the one in Definition* 9.8.

Proof. Let $\mathcal{S} \subset \mathbb{H}^1$ be a C^2 surface. The Riemannian Gauss map of $\mathcal{S}$ with respect to the metric (8.4) is given by $\boldsymbol{\nu}^\epsilon = \frac{\boldsymbol{N}^\epsilon}{|\boldsymbol{N}^\epsilon|_\epsilon}$, where we have denoted by $|\boldsymbol{N}^\epsilon|_\epsilon$ the length of $\boldsymbol{N}^\epsilon$ in such metric. Let us notice that $|\boldsymbol{N}^\epsilon|_\epsilon = \frac{1}{\sqrt{\epsilon}}\sqrt{W^2 + \epsilon\omega^2}$. Recalling (8.7) we see that

$$\boldsymbol{\nu}_\epsilon = \alpha^\epsilon \{\overline{p} X_1 + \overline{q} X_2 + \sqrt{\epsilon}\, \overline{\omega} T_\epsilon\}, \tag{9.24}$$

where we have let $\alpha^\epsilon = W/\sqrt{W^2 + \epsilon\omega^2}$. From (9.24) and (13.2) below, we recognize that the expression of the Gauss map in the Cartesian coordinates (x, y, t) is given by

$$\boldsymbol{\nu}_\epsilon = \left(\alpha^\epsilon \overline{p}, \alpha^\epsilon \overline{q}, \epsilon\alpha^\epsilon \overline{\omega} + \frac{\alpha^\epsilon}{2}(x\overline{q} - y\overline{p})\right).$$

Using Proposition 9.9, (5.16), and the fact that $\det(g^\epsilon_{ij}) = \epsilon^{-1}$, we then see that at any $g_0 \in \mathcal{S}$

$$\begin{aligned} H^\epsilon_{\mathcal{R}}(g_0) &= \operatorname{div}_\epsilon \boldsymbol{\nu}_\epsilon = \partial_x\big(\alpha^\epsilon \overline{p}\big) + \partial_y\big(\alpha^\epsilon \overline{q}\big) + \partial_t\left(\epsilon\alpha^\epsilon\overline{\omega} + \frac{\alpha^\epsilon}{2}(x\overline{q} - y\overline{p})\right) \\ &= X_1\big(\alpha^\epsilon \overline{p}\big) + X_2\big(\alpha^\epsilon \overline{q}\big) + \epsilon T\big(\alpha^\epsilon \overline{\omega}\big) \\ &= \alpha^\epsilon \mathcal{H} + \overline{p} X_1\big(\alpha^\epsilon\big) + \overline{q} X_2\big(\alpha^\epsilon\big) + \epsilon\big(\alpha^\epsilon T\overline{\omega} + \overline{\omega} T\big(\alpha^\epsilon\big)\big), \end{aligned}$$

where we have used the fact that $\mathcal{H} = X_1\overline{p} + X_2\overline{q}$, see Definition 9.8 and Proposition 9.9. We now claim that at any $g_0 \in \mathcal{S} \setminus \Sigma$ we have $\alpha^\epsilon \to 1$ as $\epsilon \to 0$, and that furthermore the following cancelation relations hold:

$$X_1\big(\alpha^\epsilon\big) \to 0, \qquad X_1\big(\alpha^\epsilon\big) \to 0, \qquad T\big(\alpha^\epsilon\big) \to 0, \quad \text{as } \epsilon \to 0. \tag{9.25}$$

We only prove the first relation of (9.25), leaving the analogous details of the remaining two to the reader. We have

$$X_1\alpha^\epsilon = \frac{(W^2+\epsilon\omega^2)X_1W - W(WX_1W+\epsilon\omega X_1\omega)}{(W^2+\epsilon\omega^2)^{3/2}}$$
$$= \epsilon\omega\frac{\omega X_1W - WX_1\omega}{(W^2+\epsilon\omega^2)^{3/2}} \to 0.$$

From (9.25) we conclude that $\mathcal{H}_P(g_0) = \lim_{\epsilon\to 0} H^\epsilon_{\mathcal{R}}(g_0) = \mathcal{H}(g_0)$, at every $g_0 \in \mathcal{S}\setminus\Sigma$. □

9.2. Comparison with the notion of pseudo-hermitian mean curvature of Cheng, Hwang, Malchiodi and Yang

In [15] the authors have introduced the following notion of pseudo-Hermitian curvature for a surface $\mathcal{S}\subset M$, where (M, J, Θ) is a three-dimensional oriented CR manifold, with CR structure J, and global contact form Θ. At every point $g\in\mathcal{S}\setminus\Sigma_{\mathcal{S}}$, they consider the one-dimensional space $HT_g\mathcal{S}$. They fix a unit vector field $\boldsymbol{e}_1\in HT\mathcal{S}$ with respect to the metric $G=\frac{1}{2}d\Theta(\cdot, J\cdot)$ associated with the Levi form, and then define $\boldsymbol{e}_2 = J(\boldsymbol{e}_1)$. They call $\boldsymbol{e}_2$ the Legendrian normal or Gauss map. They denote by $\nabla^{\text{p.h.}}$ the pseudo-Hermitian connection associated with (J,Θ). There exists a function $H^{\text{p.h.}}$ such that

$$\nabla^{\text{p.h.}}_{\boldsymbol{e}_1}\boldsymbol{e}_1 = H^{\text{p.h.}}\boldsymbol{e}_2. \tag{9.26}$$

Such function $H_{\text{p.h.}}$ is called the pseudo-Hermitian mean curvature of $\mathcal{S}$, see (2.1) in [15].

Proposition 9.14. *Let M be the Heisenberg group $\mathbb{H}^1$, then the function $H^{\text{p.h.}}$ coincides* (*up to a choice of the orientation*) *with the horizontal H-mean curvature in Definition* 9.8.

Proof. One can check that in the Heisenberg group the pseudo-Hermitian connection $\nabla^{\text{p.h.}}$ is nothing but the horizontal Levi-Civita connection ∇^H introduced in Section 5. Since a basis for $HT\mathcal{S}$ is given by $\boldsymbol{e}_1 = (\boldsymbol{\nu}^H)^\perp$, see Corollary 9.5, and we clearly have $\boldsymbol{e}_2 = \boldsymbol{\nu}^H$, from the horizontal Koszul identity (5.14) we obtain for every vector field $X = a\boldsymbol{e}_1 + b\boldsymbol{e}_2 + cT$

$$2\big\langle\nabla^H_{\boldsymbol{e}_1}\boldsymbol{e}_1, X\big\rangle = 2\boldsymbol{e}_1\langle\boldsymbol{e}_1, X\rangle - X\langle\boldsymbol{e}_1,\boldsymbol{e}_1\rangle - 2\big\langle\boldsymbol{e}_1, [\boldsymbol{e}_1, X]^H\big\rangle + \big\langle X, [\boldsymbol{e}_1,\boldsymbol{e}_1]^H\big\rangle$$
$$= 2\boldsymbol{e}_1(a) - 2\big\langle\boldsymbol{e}_1, [\boldsymbol{e}_1, X]^H\big\rangle. \tag{9.27}$$

We now have

$$[\boldsymbol{e}_1, X] = \boldsymbol{e}_1(a)\boldsymbol{e}_1 + \boldsymbol{e}_1(b)\boldsymbol{e}_2 + b[\boldsymbol{e}_1,\boldsymbol{e}_2] + \boldsymbol{e}_1(c)T + c[\boldsymbol{e}_1, T].$$

The commutators in the right-hand side of the latter equation have been computed in Section 13 below, where the vector fields $\boldsymbol{e}_1$, and $\boldsymbol{e}_2$ are respectively denoted by Z and Y. From Lemmas 13.8 and 13.9 we find

$$[\boldsymbol{e}_1, X] = \boldsymbol{e}_1(a)\boldsymbol{e}_1 + \boldsymbol{e}_1(b)\boldsymbol{e}_2 + b\big\{T + \mathcal{H}\boldsymbol{e}_1 + \big(\bar{q}\boldsymbol{e}_2(\bar{p}) - \bar{p}\boldsymbol{e}_2(\bar{q})\big)\boldsymbol{e}_2\big\}$$
$$+ \boldsymbol{e}_1(c)T + c(\bar{q}T\bar{p} - \bar{p}T\bar{q})\boldsymbol{e}_2.$$

From the latter expression we obtain

$$[\boldsymbol{e}_1, X]^H = \boldsymbol{e}_1(a)\boldsymbol{e}_1 + \boldsymbol{e}_1(b)\boldsymbol{e}_2 + b\big\{\mathcal{H}\boldsymbol{e}_1 + \big(\bar{q}\boldsymbol{e}_2(\bar{p}) - \bar{p}\boldsymbol{e}_2(\bar{q})\big)\boldsymbol{e}_2\big\} + c(\bar{q}T\bar{p} - \bar{p}T\bar{q})\boldsymbol{e}_2,$$

and therefore (9.27) gives

$$\big\langle \nabla^H_{\boldsymbol{e}_1}\boldsymbol{e}_1, X\big\rangle = \boldsymbol{e}_1(a) - \boldsymbol{e}_1(a) - b\mathcal{H} = \langle -\mathcal{H}\boldsymbol{e}_2, X\rangle.$$

From the arbitrariness of X we conclude

$$\nabla^H_{\boldsymbol{e}_1}\boldsymbol{e}_1 = -\mathcal{H}\boldsymbol{e}_2. \tag{9.28}$$

This proves the proposition. □

10. Sub-Riemannian calculus on hypersurfaces

In this section we establish some basic integration by parts formulas involving the tangential horizontal gradient on a hypersurface, and the horizontal mean curvature of the latter. Such formulas are reminiscent of the classical one, and in fact they encompass the latter. However, an important difference is that the ordinary volume form on the hypersurface $\mathcal{S}$ is replaced by the H-perimeter measure $d\sigma_H$. Furthermore, they contain additional terms which are due to the non-trivial commutation relations, which is reflected in the lack of torsion freeness of the horizontal connection on $\mathcal{S}$. Such term prevents the corresponding horizontal Laplace–Beltrami operator from being formally self-adjoint in $L^2(\mathcal{S}, d\sigma_H)$ in general. Since the framework we work in does not lend itself to a preferred choice of coordinates, for ease of computation we have developed an approach, based on Federer's co-area formula, which is coordinate-free. In fact, our proof slightly simplifies several of the classical formulas for hypersurfaces in $\mathbb{R}^n$ which are derived by writing $\mathcal{S}$ as a graph, see e.g. [52,71].

Theorem 10.1 *(First sub-Riemannian integration by parts formula). Consider a C^2 oriented hypersurface in a Carnot group $\mathcal{S} \subset \boldsymbol{G}$. If $u \in C_0^1(\mathcal{S} \setminus \Sigma_{\mathcal{S}})$, then we have*

$$\int_{\mathcal{S}} \nabla_i^{H,\mathcal{S}} u \, d\sigma_H = \int_{\mathcal{S}} u\big\{\mathcal{H}\nu_i^H - \boldsymbol{c}_i^{H,\mathcal{S}}\big\}\, d\sigma_H, \quad i = 1, \ldots, m, \tag{10.1}$$

where the C^1 functions $\boldsymbol{c}_i^{H,\mathcal{S}}$ on $\mathcal{S} \setminus \Sigma$ are defined by

$$\boldsymbol{c}_i^{H,\mathcal{S}} = \sum_{s=1}^{k}\left(\sum_{j=1}^{m} b_{ij}^s \bar{p}_j\right)\overline{\omega}_s, \tag{10.2}$$

with b_{ij}^s denoting the horizontal group constants defined in (2.14). *Moreover, the horizontal vector field $\boldsymbol{c}^{H,\mathcal{S}} = \sum_{i=1}^m \boldsymbol{c}_i^{H,\mathcal{S}} X_i$ is perpendicular to the horizontal Gauss map $\boldsymbol{\nu}^H$, i.e., one has*

$$\big\langle \boldsymbol{c}^{H,\mathcal{S}}, \boldsymbol{\nu}^H\big\rangle = 0. \tag{10.3}$$

As a consequence, we have $\boldsymbol{c}^{H,\mathcal{S}} \in C^1(\mathcal{S} \setminus \Sigma_{\mathcal{S}}, HT\mathcal{S})$.

Proof. Since the question is local, to prove the theorem we will assume, without loss of generality, that $\mathcal{S}$ is the level set of a C^2 defining function ϕ, and that $\mathcal{S}$ is oriented in such a way that $\boldsymbol{N} = \nabla\phi$. Furthermore, thanks to the assumption of the support of u, we can also assume that $\mathcal{S}$ be non-characteristic. Using a partition of unity we can always reduce ourselves to this situation. For every $\rho \in \mathbb{R}$, we define $\mathcal{U}_\rho = \{g \in \boldsymbol{G} \mid \phi(g) < \rho\}$. By the non-characteristic assumption on $\mathcal{S}$ we can assume that, if $\mathcal{S} = \partial\mathcal{U}_{\rho_0}$, then for every ρ sufficiently close to ρ_0 the characteristic locus of $\partial\mathcal{U}_\rho$ is empty. Federer's coarea formula gives, see [39],

$$\int_{\mathcal{U}_\rho} \nabla_i^{H,\mathcal{S}} u W\, dg = \int_{-\infty}^{\rho} \int_{\partial\mathcal{U}_\tau} \nabla_i^{H,\mathcal{S}} u \frac{W}{|\boldsymbol{N}|}\, dH_{N-1}\, d\tau = \int_{-\infty}^{\rho} \int_{\partial\mathcal{U}_\tau} \nabla_i^{H,\mathcal{S}} u\, d\sigma_H\, d\rho. \tag{10.4}$$

Recalling (8.3) and (9.17), we obtain from (10.4)

$$\int_{\partial\mathcal{U}_\rho} \nabla_i^{H,\mathcal{S}} u\, d\sigma_H = \frac{d}{d\rho} \int_{\mathcal{U}_\rho} \nabla_i^{H,\mathcal{S}} u W\, dg. \tag{10.5}$$

This crucial observation allows us to reduce the computation of the surface integral to that of an integral over the solid region $\mathcal{U}_\rho$. Recalling Definition 7.4, we have

$$\int_{\mathcal{U}_\rho} \nabla_i^{H,\mathcal{S}} u W\, dg = \int_{\mathcal{U}_\rho} X_i u W\, dg - \int_{\mathcal{U}_\rho} X_j u \overline{p}_j \overline{p}_i W\, dg, \tag{10.6}$$

where we have adopted the summation convention over repeated indices. Integrating by parts in the first integral in the right-hand side of (10.6) we find

$$\begin{aligned} \int_{\mathcal{U}_\rho} X_i u W\, dg &= \int_{\partial\mathcal{U}_\rho} u \langle \boldsymbol{N}, X_i \rangle \frac{W}{|\boldsymbol{N}|}\, dH_{N-1} - \int_{\mathcal{U}_\rho} u X_i W\, dg \\ &= \int_{\partial\mathcal{U}_\rho} u \overline{p}_i W\, d\sigma_H - \int_{\mathcal{U}_\rho} u \frac{\overline{p}_j X_i p_j}{W} W\, dg, \end{aligned} \tag{10.7}$$

where we have used $\operatorname{div} X_i = 0$, see (5.16). We next integrate by parts in the second integral in the right-hand side of (10.6), obtaining as in (10.7)

$$\begin{aligned} \int_{\mathcal{U}_\rho} X_j u \overline{p}_j \overline{p}_i W\, dg &= \int_{\partial\mathcal{U}_\rho} u \overline{p}_j \overline{p}_j \overline{p}_i W\, d\sigma_H - \int_{\mathcal{U}_\rho} u X_j (\overline{p}_j \overline{p}_i W)\, dg \\ &= \int_{\partial\mathcal{U}_\rho} u \overline{p}_i W\, d\sigma_H - \int_{\mathcal{U}_\rho} u (X_j \overline{p}_j) \overline{p}_i W\, dg - \int_{\mathcal{U}_\rho} u \frac{\overline{p}_j X_j p_i}{W} W\, dg. \end{aligned} \tag{10.8}$$

Inserting (10.7), (10.8) into (10.6), we see that the boundary integrals disappear and we finally obtain

$$\int_{\mathcal{U}_\rho} \nabla_i^{H,\mathcal{S}} u W\, dg = \int_{\mathcal{U}_\rho} u X_j(\overline{p}_j)\overline{p}_i W\, dg - \int_{\mathcal{U}_\rho} u \boldsymbol{c}_i^{\mathcal{S}} W\, dg, \tag{10.9}$$

where we have let

$$\boldsymbol{c}_i^{\mathcal{S}} = \frac{\overline{p}_j}{W}\{X_i p_j - X_j p_i\}.$$

Formula (10.9) is the crucial point in the proof. Proceeding now as in (10.4), and applying (10.5), we conclude for every $\rho \in \mathbb{R}$ in a sufficiently small neighborhood of a given $\rho_0 \in \mathbb{R}$

$$\int_{\partial\mathcal{U}_\rho} \nabla_i^{H,\mathcal{S}} u\, d\sigma_H = \int_{\partial\mathcal{U}_\rho} u X_j(\overline{p}_j)\overline{p}_i\, d\sigma_H - \int_{\partial\mathcal{U}_\rho} u \boldsymbol{c}_i^{\mathcal{S}}\, d\sigma_H. \tag{10.10}$$

We now have

$$\boldsymbol{c}_i^{\mathcal{S}} = \frac{\overline{p}_j}{W}\{X_i X_j \phi - X_j X_i \phi\} = \sum_{s=1}^{k}\left(\sum_{j=1}^{m} b_{ij}^s \overline{p}_j\right)\overline{\omega}_s.$$

Recalling (9.11) in Proposition 9.9, we conclude that (10.1) holds. Finally, (10.3) follows from the skew-symmetry of the matrix $\{b_{ij}^s\}_{i,j=1,\dots,m}$ defined by (2.14) which gives $\sum_{j=1}^m b_{ij}^s \overline{p}_i \overline{p}_j = 0$ for every $s = 1, \dots, k$. Hence,

$$\langle \boldsymbol{c}^{H,\mathcal{S}}, \boldsymbol{\nu}^H \rangle = \sum_{s=1}^{k}\left(\sum_{j=1}^{m} b_{ij}^s \overline{p}_i \overline{p}_j\right)\overline{\omega}_s = 0.$$

This completes the proof. □

Remark 10.2. We emphasize that in the Abelian case $\boldsymbol{G} = \mathbb{R}^m$, we have $X_i = \partial/\partial x_i$, $i = 1, \dots, m$, and so $[X_i, X_j] = 0$ and thereby $\boldsymbol{c}_i^{\mathcal{S}} \equiv 0$. In this case formula (10.15) recaptures the classical integration by parts formula on a hypersurface, see for instance [71,91].

Remark 10.3. We note explicitly that when $\boldsymbol{G} = \mathbb{H}^n$, the Heisenberg group, then the horizontal vector field $\boldsymbol{c}^{H,\mathcal{S}}$ is given by

$$\boldsymbol{c}^{H,\mathcal{S}} = \overline{\omega} J(\boldsymbol{\nu}^H), \tag{10.11}$$

where $J : H\mathbb{H}^n \to H\mathbb{H}^n$ is the symplectic transformation which, in the orthonormal basis $\{X_1, \dots, X_{2n}\}$ of $H\mathbb{H}^n$, is represented by the block matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

We thus obtain from (10.11)

$$\boldsymbol{c}^{H,\mathcal{S}} = \overline{\omega}\big(\boldsymbol{\nu}^H\big)^{\perp} = \overline{\omega}(\overline{p}_{n+1}X_1 + \cdots + \overline{p}_{2n}X_n - \overline{p}_1X_{n+1} - \cdots - \overline{p}_nX_{2n}). \tag{10.12}$$

Therefore, for $\mathbb{H}^n$ formula (10.1) reads

$$\int_{\mathcal{S}} \nabla^{H,\mathcal{S}} u \, d\sigma_H = \int_{\mathcal{S}} u\big\{\mathcal{H}\boldsymbol{\nu}^H - \overline{\omega}\big(\boldsymbol{\nu}^H\big)^{\perp}\big\} \, d\sigma_H. \tag{10.13}$$

In particular, when $n = 1$ then in the notation of Section 13, see also Remark 6.2, we have $\boldsymbol{c}^{H,\mathcal{S}} = \overline{\omega}Z$, and we can write (10.13) as follows:

$$\int_{\mathcal{S}} \nabla^{H,\mathcal{S}} u \, d\sigma_H = \int_{\mathcal{S}} u\{\mathcal{H}Y - \overline{\omega}Z\} \, d\sigma_H. \tag{10.14}$$

We have the following notable consequences of Theorem 10.1.

Theorem 10.4. *Let $\mathcal{S} \subset \boldsymbol{G}$ be a C^2 oriented hypersurface, with characteristic set $\Sigma_{\mathcal{S}}$. If $\zeta \in C_0^1(\mathcal{S} \setminus \Sigma_{\mathcal{S}}, HT\mathcal{S})$, then we have*

$$\int_{\mathcal{S}} \big\{\operatorname{div}_{H,\mathcal{S}} \zeta + \big\langle \boldsymbol{c}^{\mathcal{S}}, \zeta\big\rangle\big\} \, d\sigma_H = \int_{\mathcal{S}} \mathcal{H}\big\langle \zeta, \boldsymbol{\nu}^H\big\rangle \, d\sigma_H, \tag{10.15}$$

where we have let

$$\operatorname{div}_{H,\mathcal{S}} \zeta = \sum_{i=1}^{m} \nabla_i^{H,\mathcal{S}} \zeta_i.$$

Theorem 10.5. *In a Carnot group $\boldsymbol{G}$ suppose that the hypersurface $\mathcal{S}$ is a vertical cylinder as in Proposition* 9.6. *If $u \in C_0^1(\mathcal{S})$ we have*

$$\int_{\mathcal{S}} \nabla_i^{H,\mathcal{S}} u \, d\sigma_H = \int_{\mathcal{S}} u\mathcal{H}\nu_i^H \, d\sigma_H. \tag{10.16}$$

Proof. First of all we notice that the assumption on $\mathcal{S}$ guarantees that the characteristic set $\Sigma_{\mathcal{S}}$ is empty, see Proposition 9.6. It is thereby legitimate to assume $u \in C_0^1(\mathcal{S})$, instead of $u \in C_0^1(\mathcal{S} \setminus \Sigma_{\mathcal{S}})$. Next, we observe that since the defining function of $\mathcal{S}$ depends only on the horizontal variables, then the normal has no component along V_2, and therefore $\omega_s = 0$ for $s = 1, \ldots, k$. This implies $\boldsymbol{c}^{H,\mathcal{S}} \equiv 0$, see (10.2). The conclusion thus follows from (10.15). □

We next establish another integration by parts formula which involves differentiation along a special combination of the vector fields $\boldsymbol{\nu}^H$ and T_s, $s = 1, \ldots, k$, where T_s constitute the orthonormal basis of the first vertical layer defined in (2.16). Such result plays a central role in the last two sections of this paper.

Theorem 10.6 *(Second sub-Riemannian integration by parts formula). Let $\mathcal{S}$ be a C^2 oriented hypersurface in a Carnot group $\boldsymbol{G}$. For every $f, \zeta \in C_0^1(\mathcal{U} \setminus \Sigma_{\mathcal{S}})$, where $\mathcal{U} \subset \boldsymbol{G}$ is an open neighborhood of $\mathcal{S}$, then one has for $s = 1, \ldots, k$*

$$\int_{\mathcal{S}} f\big(T_s\zeta - \overline{\omega}_s\langle\nabla\zeta, \boldsymbol{\nu}^H\rangle\big)\, d\sigma_H$$
$$= -\int_{\mathcal{S}} \zeta\big(T_s f - \overline{\omega}_s\langle\nabla f, \boldsymbol{\nu}^H\rangle\big)\, d\sigma_H + \int_{\mathcal{S}} f\zeta\left\{\overline{\omega}_s\mathcal{H} + \sum_{\ell=1}^{m_3}\sum_{i=1}^{m} b_{is}^{\ell}\,\overline{p}_i\overline{\omega}_{3,\ell}\right\} d\sigma_H, \tag{10.17}$$

where $\overline{\omega}_s$ are defined in (6.3), *and for $i = 1, \ldots, m$, $s = 1, \ldots, k$ and $\ell = 1, \ldots, m_3$, we have let $b_{is}^{\ell} = \langle[e_i, \epsilon_s], e_{3,\ell}\rangle$. In particular, for a hypersurface $\mathcal{S} \subset \mathbb{H}^n$, we have $k = 1$, and therefore letting $\overline{\omega}_1 = \overline{\omega}$ and setting $Yf \stackrel{\text{def}}{=} \langle\nabla f, \boldsymbol{\nu}^H\rangle$, see also Section* 13, *we obtain,*

$$\int_{\mathcal{S}} f(T - \overline{\omega}Y)\zeta\, d\sigma_H = -\int_{\mathcal{S}} \zeta(T - \overline{\omega}Y)f\, d\sigma_H + \int_{\mathcal{S}} f\zeta\overline{\omega}\mathcal{H}\, d\sigma_H. \tag{10.18}$$

Proof. We use the same idea of the proof of Theorem 10.1, except that this time we consider

$$\begin{aligned}
&\int_{\mathcal{U}_\rho} \big[T_s f - \big\langle\nabla(\overline{\omega}_s f), \boldsymbol{\nu}^H\big\rangle\big] W\, dg \\
&\quad= \int_{\partial\mathcal{U}_\rho} \langle T_s, \boldsymbol{N}\rangle f \frac{W}{|\boldsymbol{N}|}\, d\sigma - \int_{\mathcal{U}_\rho} f\, \operatorname{div}(WT_s)\, dg \\
&\qquad - \int_{\partial\mathcal{U}_\rho} \big\langle\boldsymbol{\nu}^H, \boldsymbol{N}\big\rangle\overline{\omega}_s f \frac{W}{|\boldsymbol{N}|}\, d\sigma + \int_{\mathcal{U}_\rho} \overline{\omega}_s f\, \operatorname{div}\big(W\boldsymbol{\nu}^H\big)\, dg \\
&\quad= \int_{\partial\mathcal{U}_\rho} \omega_s f\, d\sigma_H - \int_{\mathcal{U}_\rho} f \frac{T_s W}{W} W\, dg - \int_{\partial\mathcal{U}_\rho} W\overline{\omega}_s f\, d\sigma_H \\
&\qquad + \int_{\mathcal{U}_\rho} \overline{\omega}_s f W \operatorname{div}\big(\boldsymbol{\nu}^H\big)\, dg + \int_{\mathcal{U}_\rho} \overline{\omega}_s f \frac{\langle\nabla W, \boldsymbol{\nu}^H\rangle}{W} W\, dg,
\end{aligned}$$

where we have used the identity $\langle\boldsymbol{\nu}^H, \boldsymbol{N}\rangle = W$, see (6.10). Since $\overline{\omega}_s W = \omega_s$, the two boundary terms drop and we are left with

$$\begin{aligned}
&\int_{\mathcal{U}_\rho} \big[T_s f - \big\langle\nabla(\overline{\omega}_s f), \boldsymbol{\nu}^H\big\rangle\big] W\, dg \\
&\quad= -\int_{\mathcal{U}_\rho} f \frac{T_s W}{W} W\, dg + \int_{\mathcal{U}_\rho} \overline{\omega}_s f W \operatorname{div}\big(\boldsymbol{\nu}^H\big)\, dg + \int_{\mathcal{U}_\rho} \overline{\omega}_s f \frac{\langle\nabla W, \boldsymbol{\nu}^H\rangle}{W} W\, dg.
\end{aligned}$$

Using the coarea formula as in the proof of Theorem 10.1, and differentiating the resulting integrals, we obtain from the latter identity

$$\int_{\mathcal{S}} \left[T_s f - \left\langle \nabla(\overline{\omega}_s f), \boldsymbol{\nu}^H \right\rangle\right] d\sigma_H$$
$$= - \int_{\mathcal{S}} f \frac{T_s W}{W} d\sigma_H + \int_{\mathcal{S}} \overline{\omega}_s f \operatorname{div}\left(\boldsymbol{\nu}^H\right) d\sigma_H + \int_{\mathcal{S}} \overline{\omega}_s f \frac{\langle \nabla W, \boldsymbol{\nu}^H \rangle}{W} d\sigma_H. \qquad (10.19)$$

Since Definition 9.8 and Proposition 9.9 give

$$\mathcal{H} = \operatorname{div}_{H,\mathcal{S}}\left(\boldsymbol{\nu}^H\right) = \sum_{i=1}^m X_i \overline{p}_i = \operatorname{div}_H\left(\boldsymbol{\nu}^H\right),$$

we can re-write (10.19) as follows:

$$\int_{\mathcal{S}} \left[T_s f - \overline{\omega}_s \left\langle \nabla f, \boldsymbol{\nu}^H \right\rangle\right] d\sigma_H$$
$$= \int_{\mathcal{S}} f \left\langle \nabla \overline{\omega}_s, \boldsymbol{\nu}^H \right\rangle d\sigma_H + \int_{\mathcal{S}} \overline{\omega}_s f \mathcal{H} \, d\sigma_H - \int_{\mathcal{S}} f \left(\frac{T_s W}{W} - \overline{\omega}_s \frac{\langle \nabla W, \boldsymbol{\nu}^H \rangle}{W} \right) d\sigma_H. \qquad (10.20)$$

We now observe that (6.3) gives

$$\left\langle \nabla \overline{\omega}_s, \boldsymbol{\nu}^H \right\rangle = \frac{\langle \nabla \omega_s, \boldsymbol{\nu}^H \rangle}{W} - \overline{\omega}_s \frac{\langle \nabla W, \boldsymbol{\nu}^H \rangle}{W}. \qquad (10.21)$$

On the other hand, we have

$$\frac{\langle \nabla \omega_s, \boldsymbol{\nu}^H \rangle}{W} = \frac{T_s W}{W} + \sum_{\ell=1}^{m_3} \sum_{i=1}^m b_{is}^\ell \overline{p}_i \overline{\omega}_{3,\ell}. \qquad (10.22)$$

To prove (10.22), suppose, as we may, that $\mathcal{S}$ is locally described as the zero set of a C^2 function ϕ, and that $\boldsymbol{N} = \nabla \phi = \sum_{i=1}^m p_i X_i + \sum_{s=1}^k \omega_s T_s + \sum_{j=3}^r \sum_{\ell=1}^{m_j} \omega_{j,\ell} X_{j,\ell}$. We thus have

$$\begin{aligned}
\left\langle \nabla \omega_s, \boldsymbol{\nu}^H \right\rangle &= \left\langle \nabla (T_s \phi), \boldsymbol{\nu}^H \right\rangle = \sum_{i=1}^m \overline{p}_i X_i (T_s \phi) \\
&= \sum_{i=1}^m \overline{p}_i T_s (X_i \phi) + W \sum_{i=1}^m \sum_{\ell=1}^{m_3} b_{is}^\ell \overline{p}_i \overline{\omega}_{3,\ell} \\
&= \sum_{i=1}^m \overline{p}_i T_s (\overline{p}_i W) + W \sum_{\ell=1}^{m_3} \sum_{i=1}^m b_{is}^\ell \overline{p}_i \overline{\omega}_{3,\ell}
\end{aligned}$$

$$= \left(\sum_{i=1}^{m} \overline{p}_i^2\right) T_s W + \left(\sum_{i=1}^{m} \overline{p}_i T_s \overline{p}_i\right) W + W \sum_{\ell=1}^{m_3} \sum_{i=1}^{m} b_{is}^{\ell} \overline{p}_i \overline{\omega}_{3,\ell}$$

$$= T_s W + W \sum_{\ell=1}^{m_3} \sum_{i=1}^{m} b_{is}^{\ell} \overline{p}_i \overline{\omega}_{3,\ell},$$

where we have used the commutation relations $[X_i, T_s] = \sum_{\ell=1}^{m_3} b_{is}^{\ell} X_{3,\ell}$. This proves (10.22). Inserting (10.22) in (10.21), and the resulting equation in (10.20), we reach the conclusion

$$\int_{\mathcal{S}} \left(T_s f - \overline{\omega}_s \langle \nabla f, \boldsymbol{\nu}^H \rangle\right) d\sigma_H = \int_{\mathcal{S}} f \overline{\omega}_s \mathcal{H}\, d\sigma_H + \int_{\mathcal{S}} f \sum_{\ell=1}^{m_3} \sum_{i=1}^{m} b_{is}^{\ell} \overline{p}_i \overline{\omega}_{3,\ell}\, d\sigma_H.$$

If we replace f by $f\zeta$ in the latter integral identity we obtain the sought for integration by parts formula. □

11. Tangential horizontal Laplacian

In this section we introduce a tangential partial differential operator, $\Delta_{H,\mathcal{S}}$ (and a modified version of the latter), which constitutes the sub-Riemannian counterpart of the classical Laplace–Beltrami operator on a hypersurface. In fact, as we will see, it reduces to the latter when the group $\boldsymbol{G}$ is Abelian. It has however one aspect which distinguishes it from its classical predecessor, and this is lack of self-adjointness in $L^2(\mathcal{S}, d\sigma_H)$. This phenomenon is caused by the presence of the "drift" term $\boldsymbol{c}^{H,\mathcal{S}}$ in the integration by parts formula in Theorem 10.1. In the next section we will show that the horizontal mean curvature flow recently proposed by Bonk and Capogna [6] satisfies a nonlinear pde which involves the operator $\Delta_{H,\mathcal{S}}$, see Theorem 12.1.

Definition 11.1. Given a function $u \in C^2(\mathcal{S})$, the *tangential horizontal Laplacian* of u on $\mathcal{S}$ is defined as follows at points of $\mathcal{S} \setminus \Sigma_{\mathcal{S}}$

$$\Delta_{H,\mathcal{S}} u \stackrel{\text{def}}{=} \sum_{i=1}^{m} \nabla_i^{H,\mathcal{S}} \nabla_i^{H,\mathcal{S}} u. \tag{11.1}$$

We also introduce the *modified tangential horizontal Laplacian* on $\mathcal{S}$

$$\hat{\Delta}_{H,\mathcal{S}} u \stackrel{\text{def}}{=} \Delta_{H,\mathcal{S}} u + \langle \boldsymbol{c}^{\mathcal{S}}, \nabla^{H,\mathcal{S}} u \rangle, \tag{11.2}$$

where $\boldsymbol{c}^{H,\mathcal{S}}$ is given by (10.2).

Remark 11.2. One should keep in mind that when $\mathcal{S}$ is a vertical cylinder given by (9.9), then the operators $\Delta_{H,\mathcal{S}}$ and $\hat{\Delta}_{H,\mathcal{S}}$ coincide

$$\hat{\Delta}_{H,\mathcal{S}} = \Delta_{H,\mathcal{S}}.$$

In such case it is easy to show from Theorem 10.1 that $\Delta_{H,\mathcal{S}}$ is formally self-adjoint in $L^2(\mathcal{S}, d\sigma_H)$.

One basic raison d'être for the operator $\hat{\Delta}_{H,\mathcal{S}}$ is in the following sub-Riemannian Stokes' theorem which follows from Theorem 10.1.

Corollary 11.3. *Let $u \in C_0^2(\mathcal{S} \setminus \Sigma_{\mathcal{S}})$, then we have*

$$\int_{\mathcal{S}} \hat{\Delta}_{H,\mathcal{S}} u \, d\sigma_H = 0. \tag{11.3}$$

Proof. It suffices to take $\nabla_i^{H,\mathcal{S}} u$ instead of u in Theorem 10.1, and then add the resulting identities in $i = 1, \dots, m$. Keeping in mind the definition (11.2), formula (10.15) gives,

$$\int_{\mathcal{S}} \hat{\Delta}_{H,\mathcal{S}} u \, d\sigma_H = \int_{\mathcal{S}} \mathcal{H} \langle \nabla^{H,\mathcal{S}} u, \boldsymbol{\nu}^H \rangle \, d\sigma_H = 0,$$

since by (9.6) one has $\langle \nabla^{H,\mathcal{S}} u, \boldsymbol{\nu}^H \rangle = 0$ on $\mathcal{S} \setminus \Sigma_{\mathcal{S}}$. $\square$

Corollary 11.4. *Let $u \in C^1(\mathcal{S})$, then for every $\zeta \in C_0^2(\mathcal{S} \setminus \Sigma_{\mathcal{S}})$ we have*

$$\int_{\mathcal{S}} \langle \nabla^{H,\mathcal{S}} u, \nabla^{H,\mathcal{S}} \zeta \rangle \, d\sigma_H = -\int_{\mathcal{S}} u \hat{\Delta}_{H,\mathcal{S}} \zeta \, d\sigma_H. \tag{11.4}$$

Proof. We take $u \nabla_i^{H,\mathcal{S}} \zeta$, instead of u, in Theorem 10.1. $\square$

Remark 11.5. In connection with Remark 9.11 we see that for the product group $\hat{\boldsymbol{G}} = \boldsymbol{G} \times \mathbb{R}$ one has $\hat{\Delta}_{H,\mathcal{S}} = \Delta_{H,\mathcal{S}} = \Delta_H$ on $\mathcal{S} = \boldsymbol{G} \times \{0\}$.

The following formulas are verified by direct computation from the definition.

Lemma 11.6. *Let $u, v \in C^2(\mathcal{O})$, $F \in C^2(\mathbb{R})$, then we have on $\mathcal{O} \setminus \Sigma$*

$$\hat{\Delta}_{H,\mathcal{S}}(uv) = u \hat{\Delta}_{H,\mathcal{S}} v + v \hat{\Delta}_{H,\mathcal{S}} u + 2 \langle \nabla^{H,\mathcal{S}} u, \nabla^{H,\mathcal{S}} v \rangle, \tag{11.5}$$

$$\hat{\Delta}_{H,\mathcal{S}}(F \circ u) = (F'' \circ u) |\nabla^{H,\mathcal{S}} u|^2 + (F' \circ u) \hat{\Delta}_{H,\mathcal{S}} u. \tag{11.6}$$

The next result provides a useful mean for computing the operators $\Delta_{H,\mathcal{S}}$ and $\hat{\Delta}_{H,\mathcal{S}}$ on $\mathcal{S}$, using the vector fields $X_1, \dots, X_m$ in the ambient group $\boldsymbol{G}$.

Proposition 11.7. *Let $u \in C^2(\mathcal{S})$, then we have on $\mathcal{S} \setminus \Sigma$*

$$\Delta_{H,\mathcal{S}} u = \Delta_H \bar{u} - \langle \nabla_H^2 \bar{u} \boldsymbol{\nu}^H, \boldsymbol{\nu}^H \rangle - \langle \nabla^H \bar{u}, \boldsymbol{\nu}^H \rangle \mathcal{H}, \tag{11.7}$$

$$\hat{\Delta}_{H,\mathcal{S}} u = \Delta_H \bar{u} + \langle \boldsymbol{c}^{H,\mathcal{S}}, \nabla^H \bar{u} \rangle - \langle \nabla_H^2 \bar{u} \boldsymbol{\nu}^H, \boldsymbol{\nu}^H \rangle - \langle \nabla^H \bar{u}, \boldsymbol{\nu}^H \rangle \mathcal{H}, \tag{11.8}$$

where $\bar{u}$ denotes any extension of u. In the above formulas, the notation $\nabla_H^2 \bar{u}$ indicates the horizontal Hessian of $\bar{u}$ introduced in (9.15).

Proof. We begin with Definition 7.4 which gives

$$\nabla^{H,\mathcal{S}} u = \nabla^H \bar{u} - \langle \nabla^H \bar{u}, \boldsymbol{\nu}^H \rangle \boldsymbol{\nu}^H,$$

where $\bar{u}$ is any extension of u. Applying (11.1), and using the summation convention over repeated indices, we find

$$\nabla_i^{H,\mathcal{S}} \nabla_i^{H,\mathcal{S}} u = \nabla_i^{H,\mathcal{S}} (X_i \bar{u}) - \nabla_i^{H,\mathcal{S}} \big(\langle \nabla^H \bar{u}, \boldsymbol{\nu}^H \rangle \big) \nu_i^H - \langle \nabla^H \bar{u}, \boldsymbol{\nu}^H \rangle \nabla_i^{H,\mathcal{S}} \nu_i^H. \tag{11.9}$$

We now compute the terms in the right-hand side of (11.9).

$$\begin{aligned} \nabla_i^{H,\mathcal{S}} (X_i \bar{u}) &= X_i X_i \bar{u} - \langle \nabla^H (X_i \bar{u}), \boldsymbol{\nu}^H \rangle \nu_i^H \\ &= \Delta_H \bar{u} - X_j X_i \bar{u} \nu_i^H \nu_j^H \\ &= \Delta_H \bar{u} - \langle \nabla_H^2 \bar{u} \boldsymbol{\nu}^H, \boldsymbol{\nu}^H \rangle. \end{aligned} \tag{11.10}$$

Next, Eq. (6.7) gives

$$\begin{aligned} \nabla_i^{H,\mathcal{S}} \big(\langle \nabla^H \bar{u}, \boldsymbol{\nu}^H \rangle \big) \nu_i^H &= X_i \big(X_j \bar{u} \nu_j^H \big) \nu_i^H - \langle \nabla^H \big(\langle \nabla^H \bar{u}, \boldsymbol{\nu}^H \rangle \big), \boldsymbol{\nu}^H \rangle \nu_i^H \nu_i^H \\ &= 0. \end{aligned} \tag{11.11}$$

Finally, we find from Proposition 9.9

$$\langle \nabla^H \bar{u}, \boldsymbol{\nu}^H \rangle \nabla_i^{H,\mathcal{S}} \nu_i^H = \langle \nabla^H \bar{u}, \boldsymbol{\nu}^H \rangle \mathcal{H}. \tag{11.12}$$

We now substitute (11.10)–(11.12) in (11.9). To reach the desired conclusion we only need to observe that thanks to (10.2) one has

$$\langle \boldsymbol{c}^{H,\mathcal{S}}, \nabla^{H,\mathcal{S}} \bar{u} \rangle = \langle \boldsymbol{c}^{H,\mathcal{S}}, \nabla^H \bar{u} \rangle - \langle \nabla^H \bar{u}, \boldsymbol{\nu}^H \rangle \langle \boldsymbol{c}^{H,\mathcal{S}}, \boldsymbol{\nu}^H \rangle = \langle \boldsymbol{c}^{H,\mathcal{S}}, \nabla^H \bar{u} \rangle. \qquad \square$$

The first elementary example of solutions of the tangential operators $\Delta_{H,\mathcal{S}}$ and $\hat{\Delta}_{H,\mathcal{S}}$ is provided by the following consequence of Proposition 11.7.

Proposition 11.8. *If the function u is constant on $\mathcal{S}$, then*

$$\Delta_{H,\mathcal{S}} u = \hat{\Delta}_{H,\mathcal{S}} u = 0.$$

Proof. First of all, let us notice that, since $\Delta_{H,\mathcal{S}} u$ and $\hat{\Delta}_{H,\mathcal{S}} u$ only depend on the values of u on $\mathcal{S}$, we can without restriction assume that $\bar{u} \equiv 1$ in $\boldsymbol{G}$. Under such hypothesis the conclusion now follows trivially from Proposition 11.7. $\square$

Another interesting consequence of Proposition 11.7 and of the grading structure of a Carnot group is the following.

Proposition 11.9. *Let $\mathcal{S} \subset \boldsymbol{G}$ be a H-minimal hypersurface, then if $x(g) = (x_1(g), \dots, x_m(g))$ denote the projection onto the horizontal layer of the exponential coordinates of $g \in \boldsymbol{G}$ (see* (2.12)*), one has*

$$\Delta_{H,\mathcal{S}}(x_i) = 0, \quad i = 1, \dots, m.$$

Proof. From Proposition 5.7 we have $\Delta_H(x_i) = 0$, and also $\nabla_H^2(x_i) = 0$. The desired conclusion thus follows immediately from (11.7). □

We next analyze a situation of special interest, namely when $\boldsymbol{G}$ is a Carnot group of step $r = 2$, and one has a hypersurface $\mathcal{S}$ given as a graph over the first layer of the Lie algebra. In such case, identifying via the exponential map $g = \exp \xi(g)$ with $\xi(g) \cong (x(g), t(g))$, we can find an open set $\Omega \subset V_1$, and a C^2 function $h : \Omega \to \mathbb{R}$, such that for some $s \in \{1, \dots, k\}$, $\mathcal{S}$ can be written as

$$\mathcal{S} = \bigl\{ \bigl(x(g), t(g)\bigr) \in \boldsymbol{G} \bigm| x(g) \in \Omega,\ t_s(g) = h\bigl(x(g)\bigr) \bigr\}. \tag{11.13}$$

For instance, in the special case of the Heisenberg group $\mathbb{H}^n$ we would be considering a graph over $\mathbb{R}^{2n}$, i.e., $\mathcal{S} = \{(x, y, t) \in \mathbb{H}^n \mid (z, y) \in \Omega \subset \mathbb{R}^{2n},\ t = h(x, y)\}$.

Theorem 11.10. *Let $\boldsymbol{G}$ be a Carnot group of step $r = 2$, and $\mathcal{S} \subset \boldsymbol{G}$ be a H-minimal hypersurface of the type* (11.13)*, then outside the characteristic set $\Sigma_{\mathcal{S}}$ the coordinate functions $x_1, \dots, x_m, t_1, \dots, t_k$ are solutions of the tangential sub-Laplacian on $\mathcal{S}$.*

Proof. For the horizontal coordinates $x_1, \dots, x_m$ the conclusion follows from Proposition 11.9. We now recall (5.23) in Proposition 5.7

$$X_i t_s = \frac{1}{2} \bigl\langle [\xi_1, e_i], \epsilon_s \bigr\rangle, \qquad \Delta_H t_s = 0, \quad i = 1, \dots, m,\ s = 1, \dots, k. \tag{11.14}$$

The first equation in (11.14) can be written

$$X_i t_s = \frac{1}{2} \sum_{j=1}^{m} x_j \bigl\langle [e_j, e_i], \epsilon_s \bigr\rangle.$$

Thanks to (5.21) this gives

$$X_j X_i t_s = \frac{1}{2} \bigl\langle [e_j, e_i], \epsilon_s \bigr\rangle = -\frac{1}{2} \bigl\langle [e_i, e_j], \epsilon_s \bigr\rangle = -X_i X_j t_s,$$

and therefore for every $s = 1, \dots, k$, one has

$$\nabla_H^2(t_s) = 0. \tag{11.15}$$

Now Proposition 11.7 gives for any $l \in \{1, \dots, k\}$, with $l \neq s$

$$\Delta_{H,\mathcal{S}} t_l = \Delta_H t_l - \bigl\langle \nabla_H^2(t_l) \boldsymbol{\nu}^H, \boldsymbol{\nu}^H \bigr\rangle - \bigl\langle \nabla^H h, \boldsymbol{\nu}^H \bigr\rangle \mathcal{H} = 0,$$

when $\mathcal{S}$ is H-minimal, thanks to (11.14) and (11.15). We are left with proving that, if $\mathcal{S}$ is H-minimal then $\Delta_{H,\mathcal{S}}(t_s)=0$. Since on $\mathcal{S}$ we have $t_s=h(x)$, we need to show that $\Delta_{H,\mathcal{S}}h=0$ on $\mathcal{S}$. With this objective in mind we begin by expressing the H-mean curvature of $\mathcal{S}$ in terms of the function h. We consider the function $\phi(g)=t_s-h(x)$ defining $\mathcal{S}$. According to Proposition 9.12, one has on $\mathcal{S}\setminus\Sigma_{\mathcal{S}}$,

$$\mathcal{H}=\frac{1}{|\nabla^H\phi|^3}\left\{\left|\nabla^H\phi\right|^2\Delta_H\phi-\Delta_{H,\infty}\phi\right\}. \tag{11.16}$$

On the other hand, we have from Proposition 11.7

$$\begin{aligned}\Delta_{H,\mathcal{S}}h&=\Delta_H h-\left\langle\nabla_H^2 h\boldsymbol{\nu}^H,\boldsymbol{\nu}^H\right\rangle-\left\langle\nabla^H h,\boldsymbol{\nu}^H\right\rangle\mathcal{H}\\&=\frac{1}{|\nabla^H\phi|^2}\left\{\left|\nabla^H\phi\right|^2\Delta_H h-\left\langle\nabla_H^2 h\nabla^H\phi,\nabla^H\phi\right\rangle\right\}-\left\langle\nabla^H h,\boldsymbol{\nu}^H\right\rangle\mathcal{H}\\&=\frac{1}{|\nabla^H\phi|^2}\left\{\left|\nabla^H\phi\right|^2\Delta_H(h-t_s)+\Delta_H t_s-\left\langle\nabla_H^2(h-t_s)\nabla^H\phi,\nabla^H\phi\right\rangle\right.\\&\quad\left.-\left\langle\nabla_H^2(t_s)\nabla^H\phi,\nabla^H\phi\right\rangle\right\}-\left\langle\nabla^H h,\boldsymbol{\nu}^H\right\rangle\mathcal{H}.\end{aligned} \tag{11.17}$$

If in (11.17) we use (11.15) and the second equation in (11.14), we obtain

$$\Delta_{H,\mathcal{S}}h=-\frac{1}{|\nabla^H\phi|^2}\left\{\left|\nabla^H\phi\right|^2\Delta_H\phi-\Delta_{H,\infty}\phi\right\}-\left\langle\nabla^H h,\boldsymbol{\nu}^H\right\rangle\mathcal{H}. \tag{11.18}$$

We now compare (11.18) with (11.16) to reach the following interesting conclusion:

$$\Delta_{H,\mathcal{S}}h=-\left\{\left|\nabla^H\phi\right|+\left\langle\nabla^H h,\nu_H\right\rangle\right\}\mathcal{H}. \tag{11.19}$$

It is now clear from (11.19) that if $\mathcal{H}\equiv 0$, then $\Delta_{H,\mathcal{S}}h=0$, and this completes the proof. □

Corollary 11.11. *In the Heisenberg group let*

$$\mathcal{S}=\left\{(x,y,t)\in\mathbb{H}^n\;\middle|\;(x,y)\in\Omega,\ t=h(x,y)\right\},$$

where $\Omega\subset\mathbb{R}^{2n}$ is an open set, and $h\in C^2(\Omega)$. Is $\mathcal{S}$ is H-minimal, then the coordinate functions $x_1,\dots,x_n,y_1,\dots,y_n,t$ are solutions of $\Delta_{H,\mathcal{S}}$ on $\mathcal{S}$.

Corollary 11.12. *Let $\boldsymbol{G}$ be a Carnot group, and consider the exponential horizontal coordinates $x_1(g),\dots,x_m(g)$ in $\boldsymbol{G}$, then*

$$\Delta_{H,\mathcal{S}}(x_i)=-\left\langle\boldsymbol{\nu}^H,X_i\right\rangle\mathcal{H}=-\overline{p}_i\mathcal{H},\quad i=1,\dots,m.$$

Consider the exponential coordinates $t_1(g),\dots,t_k(g)$ in the first vertical layer V_2, then

$$\Delta_{H,\mathcal{S}}(t_s)=-\frac{1}{2}\sum_{i,j=1}^m b_{ij}^s x_i\overline{p}_j\mathcal{H},\quad s=1,\dots,k.$$

In particular, when $\boldsymbol{G} = \mathbb{H}^1$, then

$$\Delta_{H,\mathcal{S}}(t) = -\frac{1}{2}(x\overline{p} + y\overline{q}).$$

We close this section with introducing the notions of p-Dirichlet integral and of p-harmonic function on an hypersurface. Such notions play a central role in the development of geometric subelliptic pde's on hypersurfaces in Carnot groups.

Definition 11.13. Suppose that $\mathcal{S} \subset \boldsymbol{G}$ be a C^2 hypersurface, with $\Sigma_{\mathcal{S}} = \emptyset$. Given $1 < p < \infty$ we define the *p-Dirichlet integral* of a function $u \in C_0^1(\mathcal{S})$ as

$$\mathcal{E}_{H,\mathcal{S}}(u) = \frac{1}{p} \int_{\mathcal{S}} \left|\nabla^{H,\mathcal{S}} u\right|^p d\sigma_H.$$

Suppose that $u \in L^p(\mathcal{S}, d\sigma_H)$, and that moreover $\nabla_i^{H,\mathcal{S}} u \in L^p(\mathcal{S}, d\sigma_H)$, for $i = 1, \dots, m$. We say that u is *p-subharmonic* (*-superharmonic*) in $\mathcal{S}$ if for every $\zeta \in C_0^1(\mathcal{S})$, $\zeta \geqslant 0$, one has

$$\int_{\mathcal{S}} \left|\nabla^{H,\mathcal{S}} u\right|^{p-2} \left\langle \nabla^{H,\mathcal{S}} u, \nabla^{H,\mathcal{S}} \zeta \right\rangle d\sigma_H \leqslant 0 \quad (\geqslant 0).$$

We say that u is *p-harmonic* in $\mathcal{S}$ if u is simultaneously p-subharmonic and p-superharmonic. When $p = 2$ we simply say that u is *subharmonic, superharmonic* or *harmonic* in $\mathcal{S}$.

According to Corollary 11.4 we can adopt the following alternative notion of subharmonicity.

Definition 11.14. A function $u \in L^1_{\text{loc}}(\mathcal{S}, d\sigma_H)$ is called subharmonic in $\mathcal{S}$ if

$$0 \leqslant \int_{\mathcal{S}} u \hat{\Delta}_{H,\mathcal{S}} \zeta \, d\sigma_H, \quad \text{for every } \zeta \in C_0^2(\mathcal{S}),\ \zeta \geqslant 0. \tag{11.20}$$

12. Flow by horizontal mean curvature

In connection with Proposition 11.7, we recall the Riemannian counterpart of (11.7):

$$\Delta_M u = \Delta u - \left\langle \nabla^2 u \boldsymbol{\nu}, \boldsymbol{\nu} \right\rangle - (n-1)\langle \nabla_M u, \boldsymbol{\nu} \rangle H, \tag{12.1}$$

where Δ_M and ∇_M respectively represent the Laplace–Beltrami operator and the intrinsic gradient on an $(n-1)$-dimensional Riemannian manifold M. Formula (12.1) plays a crucial role, for instance, in the derivation of the equation for flow by mean curvature, see for instance [37]. If one considers a family of smooth embeddings $F(\cdot, t) : M \to \mathbb{R}^n$, then with $M_t = F(M, t)$, the equation of flow by mean curvature is given by

$$\frac{\partial F}{\partial t}(p, t) = -(n-1)H\boldsymbol{\nu}. \tag{12.2}$$

If we write $\boldsymbol{x} = F(p,t)$, then using (12.1) and (12.2) we obtain the nonlinear partial differential equation

$$\frac{\partial \boldsymbol{x}}{\partial t} = \Delta_{M_t} \boldsymbol{x}, \tag{12.3}$$

which is satisfied by the components $(x_1, \dots, x_n)$ of $\boldsymbol{x}$. This can be readily recognized as follows. Equation (12.1) gives for each component x_i

$$\begin{aligned} \Delta_{M_t}(x_i) &= \Delta(x_i) - \langle \nabla^2(x_i)\boldsymbol{\nu}, \boldsymbol{\nu} \rangle - (n-1)\langle \nabla(x_i), \boldsymbol{\nu} \rangle H \\ &= -(n-1)\langle e_i, \boldsymbol{\nu} \rangle H = -(n-1)\boldsymbol{\nu}_i H. \end{aligned}$$

In other words, we have $\Delta_{M_t}\boldsymbol{x} = -(n-1)H\boldsymbol{\nu}$. This equation, combined with (12.2), proves (12.7).

We next want to prove a sub-Riemannian analogue of (12.3) for the mean curvature flow in the Heisenberg group recently proposed by Bonk and Capogna in [6]. We consider a smooth hypersurface in a Carnot group $\mathcal{S} \subset \boldsymbol{G}$, and a family of smooth embeddings $F : \mathcal{S} \times (0,T) \to \boldsymbol{G}$. We will denote by $S^\lambda = F(\mathcal{S}, \lambda)$. The reader should note that we are using the unconventional parameter $\lambda \in (0,T)$ to indicate time. The reason is due to the fact that, to keep a homogeneous notation with the Heisenberg group, we have already reserved the letter $t = (t_1, \dots, t_k)$ to indicate the exponential coordinates in the first vertical layer V_2 of the Lie algebra of $\boldsymbol{G}$, see (2.12). In [6] the authors have introduced the following definition of *horizontal mean curvature flow* when the group $\boldsymbol{G}$ is $\mathbb{H}^n$. At any point $F(g,\lambda) \in \mathcal{S}^\lambda \setminus \Sigma^\lambda$ (Σ^λ denotes the characteristic set of $\mathcal{S}^\lambda$), they require that

$$\left\langle \frac{\partial F}{\partial \lambda}, \boldsymbol{N} \right\rangle = -\mathcal{H}\langle \boldsymbol{\nu}^H, \boldsymbol{N} \rangle. \tag{12.4}$$

We notice that it is important to project the flow along the normal direction since the vector equation $\frac{\partial F}{\partial \lambda} = -\mathcal{H}\boldsymbol{\nu}^H$ is meaningless: the right-hand side evolves in the horizontal bundle $H\boldsymbol{G}$, whereas the left-hand side has components which move outside of it. Also, as noted in [6], "any tangential component of the velocity field only gives rise to a re-parametrization of the surface with no effect on the geometric evolution." At characteristic points the equation (12.4) is not defined and the way the authors circumvent this obstacle is by restricting to $\mathcal{S}$ the Riemannian ϵ-regularization of the sub-Riemannian metric of $\mathcal{S}$ introduced in (8.4). We refer the reader to [6] for the relevant details. We want to next prove the following result which underscores the interest of the operator $\Delta_{H,\mathcal{S}}$ introduced in the previous section. It should be thought of as the sub-Riemannian analogue of (12.3).

Theorem 12.1. *Let $F : \mathcal{S} \times (0,T) \to \boldsymbol{G}$ be a C^2 solution of the horizontal mean curvature flow* (12.4)*, then at any non-characteristic point $F(g,\lambda) \in \mathcal{S}^\lambda$ one has*

$$\left\langle \frac{\partial F}{\partial \lambda}, \boldsymbol{N} \right\rangle = \langle \Delta_{H,\mathcal{S}^\lambda} F, \boldsymbol{N} \rangle, \tag{12.5}$$

where the latter equation must be interpreted component-wise.

Proof. To make our proof as transparent as possible we discuss in detail the case of the first Heisenberg group $\mathbb{H}^1$. The details of the more general case, as well as some applications of (12.5), will appear elsewhere. We consider $F(g,\lambda) = (x(g,\lambda), y(g,\lambda), t(g,\lambda))$ and notice that we have from (13.2) below,

$$\begin{aligned}\Delta_{H,\mathcal{S}^\lambda} F &= \big(\Delta_{H,\mathcal{S}^\lambda}(x), \Delta_{H,\mathcal{S}^\lambda}(y), \Delta_{H,\mathcal{S}^\lambda}(t)\big)\\ &= \Delta_{H,\mathcal{S}^\lambda}(x) X_1 + \Delta_{H,\mathcal{S}^\lambda}(y) X_2 + \left(\Delta_{H,\mathcal{S}^\lambda}(t) + \frac{y\Delta_{H,\mathcal{S}^\lambda}(x) - x\Delta_{H,\mathcal{S}^\lambda}(y)}{2}\right) T. \qquad (12.6)\end{aligned}$$

At this point we use (12.6) and the fact that

$$\boldsymbol{N} = \left(\overline{p} X_1 + \overline{q} X_2 + \frac{\langle \boldsymbol{N}, T\rangle}{W} T\right) W,$$

to discover that

$$\begin{aligned}\langle \Delta_{H,\mathcal{S}^\lambda} F, \boldsymbol{N}\rangle = W\bigg\{&\overline{p}\Delta_{H,\mathcal{S}^\lambda}(x) + \overline{q}\Delta_{H,\mathcal{S}^\lambda}(y)\\ &+ \frac{\langle \boldsymbol{N}, T\rangle}{W}\left(\Delta_{H,\mathcal{S}^\lambda}(t) + \frac{y\Delta_{H,\mathcal{S}^\lambda}(x) - x\Delta_{H,\mathcal{S}^\lambda}(y)}{2}\right)\bigg\}. \qquad (12.7)\end{aligned}$$

We now use Corollary 11.12, which in the present situation gives,

$$\Delta_{H,\mathcal{S}^\lambda}(x) = -\overline{p}\mathcal{H}, \qquad \Delta_{H,\mathcal{S}^\lambda}(y) = -\overline{q}\mathcal{H}, \qquad \Delta_{H,\mathcal{S}^\lambda}(t) = -\frac{x\overline{q} - y\overline{p}}{2}\mathcal{H}. \qquad (12.8)$$

Substituting (12.8) in (12.7) we obtain the remarkable conclusion

$$\begin{aligned}\langle \Delta_{H,\mathcal{S}^\lambda} F, \boldsymbol{N}\rangle &= W\left\{-\overline{p}^2\mathcal{H} - \overline{q}^2\mathcal{H} + \frac{\langle \boldsymbol{N}, T\rangle}{W}\left(-\frac{x\overline{q} - y\overline{p}}{2}\mathcal{H} + \frac{x\overline{q} - y\overline{p}}{2}\mathcal{H}\right)\right\}\\ &= -W\mathcal{H}. \qquad (12.9)\end{aligned}$$

On the other hand, (6.10) gives

$$\langle \boldsymbol{\nu}^H, \boldsymbol{N}\rangle \mathcal{H} = W\mathcal{H}.$$

Combining the latter equation with (12.9) we reach the conclusion

$$\langle \Delta_{H,\mathcal{S}^\lambda} F, \boldsymbol{N}\rangle = -\mathcal{H}\langle \boldsymbol{\nu}^H, \boldsymbol{N}\rangle. \qquad (12.10)$$

Finally, from (12.10) and (12.4) we obtain (12.5). □

13. Some geometric identities in the Heisenberg group

In this section we collect several geometric identities in the Heisenberg group $\mathbb{H}^1$ which, besides their intrinsic interest, play an important role in the development of the first and second variation formulas in Section 14. We note preliminarily that

$$X_1 \wedge X_2 = T, \qquad X_2 \wedge T = X_1, \qquad X_1 \wedge T = -X_2, \tag{13.1}$$

where the wedge products are computed with respect to the left-invariant Riemannian metric with respect to which $\{X_1, X_2, T\}$ constitute an orthonormal basis. We also observe that the passage from the orthonormal basis $\{X_1, X_2, T\}$ to the standard rectangular coordinates of $\mathbb{R}^3$ is given by the formula

$$aX_1 + bX_2 + cT = \left(a, b, c + \frac{bx - ay}{2}\right). \tag{13.2}$$

Throughout this section $\mathcal{S} \subset \mathbb{H}^1$ denotes an oriented C^2 surface, with non-unit normal $\boldsymbol{N}$, and Riemannian Gauss map $\boldsymbol{\nu}$, we consider the functions p_1, p_2 and W on $\mathcal{S}$ defined in (6.1). As we have mentioned in Remark 6.2, for computational ease it will be convenient to adopt in this and the next two sections the slightly different notation $p = p_1, q = p_2$, i.e.,

$$p = \langle \boldsymbol{N}, X_1\rangle, \qquad q = \langle \boldsymbol{N}, X_2\rangle, \qquad W = \sqrt{p^2 + q^2}. \tag{13.3}$$

The horizontal Gauss map defined in (6.6) is now given on $\mathcal{S} \setminus \Sigma_{\mathcal{S}}$ by

$$\boldsymbol{\nu}^H = \bar{p}X_1 + \bar{q}X_2, \tag{13.4}$$

where we have let

$$\bar{p} = \frac{p}{W}, \qquad \bar{q} = \frac{q}{W}, \quad \text{so that} \quad \bar{p}^2 + \bar{q}^2 \equiv 1 \quad \text{on } \mathcal{S} \setminus \Sigma_{\mathcal{S}}. \tag{13.5}$$

We also introduce the notation

$$\omega = \langle \boldsymbol{N}, T\rangle, \qquad \bar{\omega} = \frac{\omega}{W}. \tag{13.6}$$

We notice explicitly that if $\mathcal{S} \in C^k$, $k \geqslant 2$, then $p, q, \omega, \bar{p}, \bar{q}, \bar{\omega} \in C^{k-1}(\mathcal{S} \setminus \Sigma_{\mathcal{S}})$. We also note that, thanks to Proposition 9.9, the H-mean curvature of $\mathcal{S}$ is presently given by the formula

$$\mathcal{H} = X_1\bar{p} + X_2\bar{q}. \tag{13.7}$$

Along with the horizontal Gauss map $\boldsymbol{\nu}^H$ we consider the vector field

$$(\boldsymbol{\nu}^H)^\perp = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \boldsymbol{\nu}^H = \bar{q}X_1 - \bar{p}X_2, \tag{13.8}$$

which, as already noticed in Section 9, constitutes a basis of $HT\mathcal{S}$. It will be convenient to keep a different notation for the action of the vector fields $\boldsymbol{\nu}^H$, $(\boldsymbol{\nu}^H)^\perp$ on a function $\zeta \in C_0^1(\mathcal{S} \setminus \Sigma_{\mathcal{S}})$.

We thus set

$$Y\zeta \overset{\text{def}}{=} \langle \nabla\zeta, \boldsymbol{\nu}^H \rangle = \overline{p} X_1\zeta + \overline{q} X_2\zeta, \tag{13.9}$$

$$Z\zeta \overset{\text{def}}{=} \langle \nabla\zeta, (\boldsymbol{\nu}^H)^{\perp} \rangle = \overline{q} X_1\zeta - \overline{p} X_2\zeta. \tag{13.10}$$

We mention that in the right-hand sides of (13.9), (13.10) the vector fields X_1, X_2 act on an extension $\overline{\zeta}$ of ζ. However, for the sake of simplifying the notation we have used, and will continue to do so below, the same notation for both functions. It is worth observing that $\{Z, Y, T\}$ constitutes an orthonormal frame on $\mathcal{S}$. One has in fact

$$Z \wedge Y = T, \qquad Y \wedge T = Z, \qquad T \wedge Z = Y. \tag{13.11}$$

Moreover, the (Riemannian) divergence in $\mathbb{H}^1$ of these vector fields is given by

$$\operatorname{div} Y = X_1\overline{p} + X_2\overline{q} = \mathcal{H}, \qquad \operatorname{div} Z = X_1\overline{q} - X_2\overline{p}. \tag{13.12}$$

Using Cramer's rule one easily obtains from (13.9) and (13.10),

$$X_1 = \overline{p}Y + \overline{q}Z, \qquad X_2 = \overline{q}Y - \overline{p}Z. \tag{13.13}$$

One also has

$$\nabla_1^{H,\mathcal{S}} = \overline{q}Z, \qquad \nabla_2^{H,\mathcal{S}} = -\overline{p}Z. \tag{13.14}$$

To prove (13.14) we proceed as follows:

$$\begin{aligned}
\nabla_1^{H,\mathcal{S}}\zeta &= X_1\zeta - \langle \nabla^H\zeta, \boldsymbol{\nu}^H \rangle \boldsymbol{\nu}_1^H = X_1\zeta - (\overline{p}X_1\zeta + \overline{q}X_2\zeta)\overline{p} \\
&= X_1\zeta - \overline{p}^2 X_1\zeta - \overline{p}\overline{q}X_2\zeta = \overline{q}^2 X_1\zeta - \overline{p}\overline{q}X_2\zeta = \overline{q}(\overline{q}X_1\zeta - \overline{p}X_2\zeta) \\
&= \overline{q}Z\zeta, \\
\nabla_2^{H,\mathcal{S}}\zeta &= X_2\zeta - \langle \nabla^H\zeta, \boldsymbol{\nu}^H \rangle \boldsymbol{\nu}_2^H = X_2\zeta - (\overline{p}X_1\zeta + \overline{q}X_2\zeta)\overline{q} \\
&= X_2\zeta - \overline{p}\overline{q}X_1\zeta - \overline{q}^2 X_2\zeta = \overline{p}^2 X_2\zeta - \overline{p}\overline{q}X_1\zeta = -\overline{p}(\overline{q}X_1\zeta - \overline{p}X_2\zeta) \\
&= -\overline{p}Z\zeta.
\end{aligned}$$

These formulas give

$$\nabla^{H,\mathcal{S}}\zeta = \overline{q}Z\zeta X_1 - \overline{p}Z\zeta X_2. \tag{13.15}$$

From (13.15) and (13.5) we obtain

$$\left|\nabla^{H,\mathcal{S}}\zeta\right|^2 = (Z\zeta)^2 = (\overline{q}X_1\zeta - \overline{p}X_2\zeta)^2. \tag{13.16}$$

We next establish some identities that will be used times and again in Sections 14 and 15.

Lemma 13.1. *One has on $\mathcal{S} \setminus \Sigma_{\mathcal{S}}$*

$$\overline{p} Z \overline{p} + \overline{q} Z \overline{q} = \overline{p} Y \overline{p} + \overline{q} Y \overline{q} = \overline{p} T \overline{p} + \overline{q} T \overline{q} = 0, \tag{13.17}$$

$$\overline{p} Z^2 \overline{p} + \overline{q} Z^2 \overline{q} = -(Z\overline{p})^2 - (Z\overline{q})^2. \tag{13.18}$$

It is useful to note the following alternative expression of the first two identities in (13.17):

$$\overline{p}\overline{q} X_1 \overline{p} - \overline{p}^2 X_2 \overline{p} + \overline{q}^2 X_1 \overline{q} - \overline{p}\overline{q} X_2 \overline{q} = 0, \tag{13.19}$$

$$\overline{p}^2 X_1 \overline{p} + \overline{p}\overline{q} X_2 \overline{p} + \overline{p}\overline{q} X_1 \overline{q} + \overline{q}^2 X_2 \overline{q} = 0. \tag{13.20}$$

Proof. The proof of (13.17) follows trivially by differentiating the identity $\overline{p}^2 + \overline{q}^2 \equiv 1$, whereas (13.18) follows by differentiating $\overline{p} Z \overline{p} + \overline{q} Z \overline{q} = 0$ with respect to Z. One has from (13.17) and (13.10)

$$0 = \overline{p} Z \overline{p} + \overline{q} Z \overline{q} = \overline{p}(\overline{q} X_1 \overline{p} - \overline{p} X_2 \overline{p}) + \overline{q}(\overline{q} X_1 \overline{q} - \overline{p} X_2 \overline{q}),$$

which proves (13.19). Similarly,

$$0 = \overline{p} Y \overline{p} + \overline{q} Y \overline{q} = \overline{p}(\overline{p} X_1 \overline{p} + \overline{q} X_2 \overline{p}) + \overline{q}(\overline{p} X_1 \overline{q} + \overline{q} X_2 \overline{q}),$$

which implies (13.20). □

Lemma 13.2. *One has on $\mathcal{S} \setminus \Sigma_{\mathcal{S}}$*

$$\langle Z, \boldsymbol{N} \rangle = 0, \qquad \langle Y, \boldsymbol{N} \rangle = W, \tag{13.21}$$

$$Y\omega = TW, \tag{13.22}$$

$$\overline{q} Y \overline{p} - \overline{p} Y \overline{q} = X_2 \overline{p} - X_1 \overline{q}, \tag{13.23}$$

$$\frac{ZW}{W} = \overline{q} Y \overline{p} - \overline{p} Y \overline{q} + \overline{\omega}, \tag{13.24}$$

and

$$\frac{Z\omega}{W} = \overline{q}\, T \overline{p} - \overline{p}\, T \overline{q}. \tag{13.25}$$

Proof. The first identity in (13.21) is obvious, while the second one is simply a reformulation of (6.10). The identity (13.22) is just a special case of (10.22). To prove (13.23), it suffices to use (13.13) and (13.17) to find

$$X_2 \overline{p} - X_1 \overline{q} = \overline{q} Y \overline{p} - \overline{p} Y \overline{q} - (\overline{p} Y \overline{p} + \overline{q} Y \overline{q}) = \overline{q} Y \overline{p} - \overline{p} Y \overline{q}.$$

As for (13.24) we have

$$\omega = T\phi = X_1 X_2 \phi - X_2 X_1 \phi = X_1(\overline{q} W) - X_2(\overline{p} W) = -(X_2 \overline{p} - X_1 \overline{q}) W + ZW,$$

from which the desired conclusion follows immediately.

Finally, we turn to the proof of (13.25). Applying T to both sides of (13.21) we obtain

$$\begin{aligned}0 &= T(Z\phi) = T(\bar{q}X_1\phi - \bar{p}X_2\phi) = T\bar{q}X_1\phi + \bar{q}\,TX_1\phi - T\bar{p}X_2\phi - \bar{p}TX_2\phi \\ &= T\bar{q}\,X_1\phi - T\bar{p}\,X_2\phi + \bar{q}X_1T\phi - \bar{p}X_2T\phi = pT\bar{q} - qT\bar{p} + Z(T\phi).\end{aligned}$$

It follows that

$$\frac{Z\omega}{W} = \frac{Z(T\phi)}{W} = \bar{q}T\bar{p} - \bar{p}T\bar{q}. \qquad \square$$

Corollary 13.3. *One has on $\mathcal{S} \setminus \Sigma_{\mathcal{S}}$*

$$\begin{aligned}\mathcal{A} \overset{\text{def}}{=} -Z\bar{\omega} &= (\bar{p}T\bar{q} - \bar{q}T\bar{p}) + \bar{\omega}(\bar{q}Y\bar{p} - \bar{p}Y\bar{q}) + \bar{\omega}^2 \\ &= \bar{p}(T\bar{q} - \bar{\omega}Y\bar{q}) - \bar{q}(T\bar{p} - \bar{\omega}Y\bar{p}) + \bar{\omega}^2.\end{aligned}$$

Proof. We have

$$Z\bar{\omega} = \frac{Z\omega}{W} - \bar{\omega}\frac{ZW}{W},$$

so the desired result follows immediately from (13.24), (13.25). $\square$

The next lemma expresses a useful orthogonality property which enters several times in the computations of Section 14.

Lemma 13.4. *Let $\mathcal{X}$, $\mathcal{Y}$ be smooth vector fields on $\mathcal{S}$, then on the set $\mathcal{S} \setminus \Sigma_{\mathcal{S}}$ one has*

$$\mathcal{X}\bar{q}\mathcal{Y}\bar{p} - \mathcal{X}\bar{p}\mathcal{Y}\bar{q} = 0.$$

In particular, letting $\mathcal{X} = Y$ or T, and $\mathcal{Y} = Z$ or Y, we find

$$\begin{aligned}Y\bar{q}Z\bar{p} - Y\bar{p}Z\bar{q} &= 0, \\ T\bar{q}Z\bar{p} - T\bar{p}Z\bar{q} &= 0, \\ T\bar{q}Y\bar{p} - T\bar{p}Y\bar{q} &= 0.\end{aligned}$$

Proof. To prove the lemma we note that

$$\mathcal{X}W = \bar{p}\mathcal{X}p + \bar{q}\mathcal{X}q, \qquad \mathcal{Y}W = \bar{p}\mathcal{Y}p + \bar{q}\mathcal{Y}q,$$

and proceed as follows:

$$\begin{aligned}&\mathcal{X}\bar{q}\mathcal{Y}\bar{p} - \mathcal{X}\bar{p}\mathcal{Y}\bar{q} \\ &\quad = \mathcal{X}\big(qW^{-1}\big)\mathcal{Y}\big(pW^{-1}\big) - \mathcal{X}\big(pW^{-1}\big)\mathcal{Y}\big(qW^{-1}\big) \\ &\quad = \frac{1}{W^2}\big\{(\mathcal{X}q - \bar{q}\mathcal{X}W)(\mathcal{Y}p - \bar{p}\mathcal{Y}W) - (\mathcal{X}p - \bar{p}\mathcal{X}W)(\mathcal{Y}q - \bar{q}\mathcal{Y}W)\big\}\end{aligned}$$

$$= \frac{-p^2\mathcal{X}q\mathcal{Y}p - pq\mathcal{X}q\mathcal{Y}q - pq\mathcal{X}p\mathcal{Y}p - q^2\mathcal{X}q\mathcal{Y}p + pq\mathcal{X}p\mathcal{Y}p + q^2\mathcal{X}p\mathcal{Y}q + p^2\mathcal{X}p\mathcal{Y}q}{W^4}$$
$$+ \frac{pq\mathcal{X}q\mathcal{Y}q}{W^4} + \frac{\mathcal{X}q\mathcal{Y}p - \mathcal{X}p\mathcal{Y}q}{W^2}$$
$$= \frac{q^2\mathcal{X}q\mathcal{Y}p - p^2\mathcal{X}p\mathcal{Y}q - q^2\mathcal{X}q\mathcal{Y}p + p^2\mathcal{X}p\mathcal{Y}q}{W^4} = 0. \qquad \square$$

In the following lemma we collect some geometric identities involving the H-mean curvature of $\mathcal{S}$ which play an essential role in the sequel.

Lemma 13.5. *One has on the set* $\mathcal{S} \setminus \Sigma_{\mathcal{S}}$

$$\overline{q}^2 X_1\overline{p} - \overline{p}\overline{q}(X_2\overline{p} + X_1\overline{q}) + \overline{p}^2 X_2\overline{q} = \mathcal{H}, \tag{13.26}$$
$$\overline{q}Z\overline{p} - \overline{p}Z\overline{q} = \mathcal{H}, \tag{13.27}$$
$$Z\overline{p} = \overline{q}\mathcal{H}, \qquad Z\overline{q} = -\overline{p}\mathcal{H}. \tag{13.28}$$

The following formula is dual to (13.26)*,* (13.27)*,*

$$\overline{p}\overline{q}X_1\overline{p} + \overline{q}^2 X_2\overline{p} - \overline{p}^2 X_1\overline{q} - \overline{p}\overline{q}X_2\overline{q} = X_2\overline{p} - X_1\overline{q}. \tag{13.29}$$

We also have the following expressions for the derivatives of the H-mean curvature along Y and T:

$$\overline{q}Y(Z\overline{p}) - \overline{p}Y(Z\overline{q}) = Y\mathcal{H}, \tag{13.30}$$
$$\overline{q}T(Z\overline{p}) - \overline{p}T(Z\overline{q}) = T\mathcal{H}. \tag{13.31}$$

Proof. In view of (13.7) one has that (13.26) is equivalent to

$$\overline{p}^2 X_2\overline{q} + \overline{q}^2 X_1\overline{p} - \overline{p}\overline{q}(X_2\overline{p} + X_1\overline{q}) = X_1\overline{p} + X_2\overline{q},$$

which is in turn equivalent to

$$\overline{q}^2 X_2\overline{q} + \overline{p}^2 X_1\overline{p} + \overline{p}\overline{q}(X_2\overline{p} + X_1\overline{q}) = 0,$$

and this is nothing but (13.20). We now use (13.26) to prove (13.27) as follows:

$$\overline{q}Z\overline{p} - \overline{p}Z\overline{q} = \overline{q}^2 X_1\overline{p} - \overline{p}\overline{q}X_2\overline{p} - \overline{p}\overline{q}X_1\overline{q} + \overline{p}^2 X_2\overline{q} = \mathcal{H}.$$

The proof of (13.28) immediately follows from the equation $\overline{p}Z\overline{p} + \overline{q}Z\overline{q} = 0$, from (13.27), and from Cramer's rule. Next, it is easy to recognize that (13.29) is equivalent to (13.19). The proof of (13.30) follows from differentiating (13.27) with respect to Y, upon using Leibniz rule and Lemma 13.4. Similarly, we establish (13.31) by differentiating (13.27) with respect to T and then using Lemma 13.4. $\square$

We now establish a result which says that one of the two horizontal principal curvatures is zero. We stress that this phenomenon, whose Riemannian counterpart is obviously not true, reflects the fact that H-minimal surfaces are ruled surfaces, see [15,49].

Proposition 13.6. *One has on $\mathcal{S} \setminus \Sigma_{\mathcal{S}}$*

$$\left|\nabla_1^{H,\mathcal{S}} \boldsymbol{\nu}_1^H\right|^2 + \left|\nabla_2^{H,\mathcal{S}} \boldsymbol{\nu}_2^H\right|^2 = (Z\overline{p})^2 + (Z\overline{q})^2 \equiv \mathcal{H}^2.$$

In particular, if $\mathcal{S}$ is H-minimal, we have

$$\left|\nabla_1^{H,\mathcal{S}} \boldsymbol{\nu}_1^H\right|^2 = \left|\nabla_2^{H,\mathcal{S}} \boldsymbol{\nu}_2^H\right|^2 = 0.$$

Proof. According to (13.16), (13.27) and (13.28), we have

$$\begin{aligned}
&\mathcal{H}^2 - \left|\nabla_1^{H,\mathcal{S}} \boldsymbol{\nu}_1^H\right|^2 - \left|\nabla_2^{H,\mathcal{S}} \boldsymbol{\nu}_2^H\right|^2 \\
&\quad = (\overline{q}Z\overline{p} - \overline{p}Z\overline{q})^2 - (Z\overline{p})^2 - (Z\overline{q})^2 \\
&\quad = \overline{q}^2(Z\overline{p})^2 + \overline{p}^2(Z\overline{q})^2 - 2\overline{p}\overline{q}Z\overline{p}Z\overline{q} - (Z\overline{p})^2 - (Z\overline{q})^2 \\
&\quad = -\big(\overline{p}^2(Z\overline{p})^2 + \overline{q}^2(Z\overline{q})^2 + 2\overline{p}\overline{q}Z\overline{p}Z\overline{q}\big) = -(\overline{p}Z\overline{p} + \overline{q}Z\overline{q})^2 = 0,
\end{aligned}$$

where the last equation follows from Lemma 13.1. □

Lemma 13.7. *One has on $\mathcal{S} \setminus \Sigma_{\mathcal{S}}$*

$$Z\overline{p}X_1 + Z\overline{q}X_2 = \mathcal{H}Z, \qquad Z\overline{q}X_1 - Z\overline{p}X_2 = -\mathcal{H}Y.$$

Proof. One easily obtains from Eqs. (13.13)

$$\begin{aligned}
Z\overline{p}X_1 + Z\overline{q}X_2 &= (\overline{p}Z\overline{p} + \overline{q}Z\overline{q})Y + (\overline{q}Z\overline{p} - \overline{p}Z\overline{q})Z \\
&= (\overline{q}Z\overline{p} - \overline{p}Z\overline{q})Z = \mathcal{H}Z,
\end{aligned}$$

where in the second to the last equality we have used (13.17), and in the last one we have used (13.27). The proof of the second identity is similar. □

The next commutator formulas will be useful in the sequel.

Lemma 13.8. *One has on $\mathcal{S} \setminus \Sigma_{\mathcal{S}}$*

$$[Z, Y] = T + \mathcal{H}Z + (\overline{q}Y\overline{p} - \overline{p}Y\overline{q})Y.$$

Proof. To compute the commutator between Z and Y we use Eqs. (13.9) and (13.10) to find

$$\begin{aligned}
[Z,Y] &= Z(Y) - Y(Z) \\
&= \overline{q}X_1(\overline{p}X_1 + \overline{q}X_2) - \overline{p}X_2(\overline{p}X_1 + \overline{q}X_2) - \overline{p}X_1(\overline{q}X_1 - \overline{p}X_2) - \overline{q}X_2(\overline{q}X_1 - \overline{p}X_2) \\
&= X_1X_2 - X_2X_1 + (\overline{q}X_1\overline{p} - \overline{p}X_1\overline{q})X_1 + (\overline{q}X_2\overline{p} - \overline{p}X_2\overline{q})X_2 \\
&= T + \big(Z\overline{p} + \overline{p}(X_2\overline{p} - X_1\overline{q})\big)X_1 + \big(Z\overline{q} + \overline{q}(X_2\overline{p} - X_1\overline{q})\big)X_2 \\
&= T + Z\overline{p}X_1 + Z\overline{q}X_2 + (X_2\overline{p} - X_1\overline{q})Y,
\end{aligned}$$

where we have repeatedly used (13.17) along with the identity $\overline{p}^2 + \overline{q}^2 = 1$. We now appeal to Lemma 13.7 and to (13.29) to reach the desired conclusion. □

Lemma 13.9. *On $\mathcal{S} \setminus \Sigma_{\mathcal{S}}$, one has*

$$[Z,T] = (\overline{q}T\overline{p} - \overline{p}T\overline{q})Y.$$

Proof. Using the trivial commutation relations $[X_i, T] = 0$, $i = 1, 2$ we obtain

$$T(Z) = T\overline{q}X_1 - T\overline{p}X_2 + Z(T).$$

From this identity, and from (13.13), we obtain

$$\begin{aligned}
[Z,T] &= T\overline{p}X_2 - T\overline{q}X_1 = T\overline{p}(\overline{q}Y - \overline{p}Z) - T\overline{q}(\overline{q}Z + \overline{p}Y) \\
&= (\overline{q}T\overline{p} - \overline{p}T\overline{q})Y - (\overline{p}T\overline{p} + \overline{q}T\overline{q})Z = (\overline{q}T\overline{p} - \overline{p}T\overline{q})Y,
\end{aligned}$$

where in the last equality we have used Lemma 13.1. □

Corollary 13.10. *One has on $\mathcal{S} \setminus \Sigma_{\mathcal{S}}$*

$$[T - \overline{\omega}Y, Z] = \overline{\omega}\big\{(T - \overline{\omega}Y) + \mathcal{H}Z\big\}.$$

In particular, if $\mathcal{S}$ is H-minimal, then

$$[T - \overline{\omega}Y, Z] = \overline{\omega}(T - \overline{\omega}Y).$$

Proof. One has from Lemmas 13.8 and 13.9

$$\begin{aligned}
[T - \overline{\omega}Y, Z] &= [T,Z] - \overline{\omega}[Y,Z] + Z\overline{\omega}Y \\
&\quad - (\overline{q}T\overline{p} - \overline{p}T\overline{q})Y + \overline{\omega}\big(T + (\overline{q}Y\overline{p} - \overline{p}Y\overline{q})\big)Y - \mathcal{A}Y \\
&= \overline{\omega}T + \big(\overline{\omega}(\overline{q}Y\overline{p} - \overline{p}Y\overline{q}) - (\overline{q}T\overline{p} - \overline{p}T\overline{q}) - \mathcal{A}\big)Y,
\end{aligned}$$

where we have used the hypothesis that $\mathcal{S}$ be H-minimal. Using Corollary 13.3 we reach the desired conclusion. □

Corollary 13.11. *If $\mathcal{S}$ is H-minimal, one has*

$$Z\mathcal{A} = \overline{\omega}\big(\overline{\omega}^2 - 3\mathcal{A}\big).$$

For an arbitrary C^2 surface we have instead on $\mathcal{S} \setminus \Sigma_{\mathcal{S}}$

$$Z\mathcal{A} = \overline{\omega}\big\{\big(\overline{\omega}^2 - 3\mathcal{A}\big) + \mathcal{H}^2\big\}.$$

Proof. From the assumption of H-minimality of $\mathcal{S}$, (13.28) and from Corollary 13.3 we obtain

$$\begin{aligned} Z\mathcal{A} &= \overline{p}Z(T\overline{q} - \overline{\omega}Y\overline{q}) - \overline{q}Z(T\overline{p} - \overline{\omega}Y\overline{p}) + 2\overline{\omega}Z\overline{\omega} \\ &= \overline{p}[Z, T - \overline{\omega}Y]\overline{q} - \overline{q}[Z, T - \overline{\omega}Y]\overline{p} - 2\overline{\omega}\mathcal{A} \\ &= -\overline{\omega}\big[\overline{p}(T\overline{q} - \overline{\omega}Y\overline{q}) - \overline{q}(T\overline{p} - \overline{\omega}Y\overline{p}) + 2\mathcal{A}\big] = \overline{\omega}\big(\overline{\omega}^2 - 3\mathcal{A}\big). \end{aligned}$$

The proof of the second part of the corollary is based on a longer computation which, in addition, exploits also Corollary 13.10, (13.17), (13.27) and (13.28). We leave the details to the interested reader. □

Corollary 13.12. *One has on $\mathcal{S} \setminus \Sigma_{\mathcal{S}}$*

$$\begin{gathered} [Z, Y]\overline{p} = T\overline{p} + Z\overline{p}\mathcal{H} + (\overline{q}Y\overline{p} - \overline{p}Y\overline{q})Y\overline{p}, \\ [Z, Y]\overline{q} = T\overline{q} + Z\overline{q}\mathcal{H} + (\overline{q}Y\overline{p} - \overline{p}Y\overline{q})Y\overline{q}, \\ [Z, T]\overline{p} = (\overline{q}T\overline{p} - \overline{p}T\overline{q})Y\overline{p}, \qquad [Z, T]\overline{q} = (\overline{q}T\overline{p} - \overline{p}T\overline{q})Y\overline{q}. \end{gathered}$$

Lemma 13.13. *On $\mathcal{S} \setminus \Sigma_{\mathcal{S}}$ one has*

$$Z(\overline{q}Y\overline{p} - \overline{p}Y\overline{q}) - (\overline{q}Y\overline{p} - \overline{p}Y\overline{q})^2 - (\overline{q}T\overline{p} - \overline{p}T\overline{q}) = Y\mathcal{H} + \mathcal{H}^2.$$

Proof. Thanks to Lemma 13.4 we have

$$\begin{aligned} &Z(\overline{q}Y\overline{p} - \overline{p}Y\overline{q}) \\ &\quad = \overline{q}Z(Y\overline{p}) - \overline{p}Z(Y\overline{q}) \\ &\quad = \overline{q}Y(Z\overline{p}) - \overline{p}Y(Z\overline{q}) + \overline{q}[Z, Y]\overline{p} - \overline{p}[Z, Y]\overline{q} \\ &\quad = Y\mathcal{H} + \overline{q}\big\{T\overline{p} + Z\overline{p}\mathcal{H} + (\overline{q}Y\overline{p} - \overline{p}Y\overline{q})Y\overline{p}\big\} - \overline{p}\big\{T\overline{q} + Z\overline{q}\mathcal{H} + (\overline{q}Y\overline{p} - \overline{p}Y\overline{q})Y\overline{q}\big\} \\ &\quad = Y\mathcal{H} + (\overline{q}T\overline{p} - \overline{p}T\overline{q}) + (\overline{q}Z\overline{p} - \overline{p}Z\overline{q})\mathcal{H} + (\overline{q}Y\overline{p} - \overline{p}Y\overline{q})^2 \\ &\quad = Y\mathcal{H} + \mathcal{H}^2 + (\overline{q}T\overline{p} - \overline{p}T\overline{q}) + (\overline{q}Y\overline{p} - \overline{p}Y\overline{q})^2, \end{aligned}$$

where we have used (13.30) and Corollary 13.12 in the third equality, and (13.27) in the second to the last equality. □

Lemma 13.14. *One has on the set $\mathcal{S} \setminus \Sigma_{\mathcal{S}}$*

$$Z(\overline{q}T\overline{p} - \overline{p}T\overline{q}) = T\mathcal{H} + (\overline{q}T\overline{p} - \overline{p}T\overline{q})(\overline{q}Y\overline{p} - \overline{p}Y\overline{q}).$$

Proof. Using Lemma 13.4 we obtain

$$\begin{aligned} Z(\bar{q}T\bar{p} - \bar{p}T\bar{q}) &= \bar{q}Z(T\bar{p}) - \bar{p}Z(T\bar{q}) \\ &= \bar{q}T(Z\bar{p}) - \bar{p}T(Z\bar{q}) + \bar{q}[Z,T]\bar{p} - \bar{p}[Z,T]\bar{q}. \end{aligned}$$

Now using Lemma 7.4 again we obtain from (13.27)

$$T\mathcal{H} = \bar{q}T(Z\bar{p}) - \bar{p}T(Z\bar{q}),$$

which, substituted in the above equation gives, along with Corollary 13.12, the desired result. □

14. First and second variation of the H-perimeter in the Heisenberg group

A fundamental tool in Riemannian geometry are the first and second variation formulas for the area functional. Consider a C^2 oriented hypersurface $\mathcal{S} \subset \mathbb{R}^n$, with Gauss map $\nu : \mathcal{S} \to \mathbb{S}^{n-1}$, and denote by $\mathcal{S}^\lambda = G_\lambda(\mathcal{S})$ the hypersurface obtained by deforming $\mathcal{S}$ in the normal direction with the one-parameter family of local diffeomorphisms $G_\lambda(x) = x + \lambda\zeta(x)\nu(x)$, where $\zeta \in C_0^\infty(\mathcal{S})$, and $\lambda \in \mathbb{R}$ is small. One has the following theorem, see for instance (10.12), (10.13) in [52], or also [8,22,71,91].

Theorem 14.1. *The first variation of the area of $\mathcal{S}$ is given by the formula*

$$\frac{d}{d\lambda} H_{n-1}\big(G_\lambda(\mathcal{S})\big)\big|_{\lambda=0} = \int_{\mathcal{S}} H\zeta \, dH_{n-1}, \tag{14.1}$$

where $H = \kappa_1 + \cdots + \kappa_{n-1}$ indicates the sum of the principal curvatures of $\mathcal{S}$. The second variation is given by

$$\frac{d^2}{d\lambda^2} H_{n-1}\big(G_\lambda(\mathcal{S})\big)\big|_{\lambda=0} = \int_{\mathcal{S}} \left\{ |\nabla\zeta|^2 + \zeta^2\left(H^2 - \sum_{i=1}^n |\nabla\nu_i|^2\right)\right\} dH_{n-1}, \tag{14.2}$$

where ∇ denotes the Levi-Civita connection on $\mathcal{S}$, and it can be shown that $\sum_{i=1}^n |\nabla\nu_i|^2$ is the sum of the squares of the principal curvatures of $\mathcal{S}$.

In this section we consider an oriented surface in the Heisenberg group $\mathbb{H}^1$, with non-unit Riemannian normal $\boldsymbol{N}$, and horizontal Gauss map $\boldsymbol{\nu}^H$, and compute the first and second variation for general deformations of $\mathcal{S}$. We observe that, in view of applications to the fundamental question of stability of H-minimal surfaces, it is important to be able to treat general deformations, versus deformations along a specific direction. More precisely, we consider deformations of $\mathcal{S}$ given by $\mathcal{S} \to \mathcal{S}^\lambda = J_\lambda(S)$, where

$$J_\lambda(\mathcal{S}) = \mathcal{S} + \lambda\mathcal{X} = \mathcal{S} + \lambda(aX_1 + bX_2 + kT), \tag{14.3}$$

$a, b, k \in C_0^1(\mathcal{S} \setminus \Sigma_{\mathcal{S}})$, and $\lambda \in \mathbb{R}$ is a small parameter. Throughout this section, and the following one, we will continue to use the notations of Section 13. We recall that $\mathbb{H}^1$ is endowed with a left-invariant Riemannian metric with respect to which $\{X_1, X_2, T\}$ constitute an orthonormal basis with inner product $\langle\cdot,\cdot\rangle$. Since no other inner product will be used, there will not be any confusion, for instance, with the standard Euclidean inner product of $\mathbb{R}^3$.

Definition 14.2. Let $\mathcal{S} \subset \mathbb{H}^1$ be an oriented C^2 surface, consider the family of vector fields $\mathcal{X} = aX_1 + bX_2 + kT$, with $a, b, k \in C_0^2(\mathcal{S} \setminus \Sigma_{\mathcal{S}})$, and the family of surfaces $\mathcal{S}^\lambda$. We define the *first variation* of the H-perimeter with respect to the deformation (14.3) as

$$\mathcal{V}_I^H(\mathcal{S}; \mathcal{X}) = \frac{d}{d\lambda} P_H(\mathcal{S}^\lambda)\big|_{\lambda=0}.$$

If $\Sigma_{\mathcal{S}} = \emptyset$, then we say that $\mathcal{S}$ is *stationary* if $\mathcal{V}_I^H(\mathcal{S}; \mathcal{X}) = 0$, for every $\mathcal{X}$.

Classical minimal surfaces are stationary points of the perimeter (the area functional for graphs). It is natural to ask what is the connection between the notion of H-minimal surface and that of H-perimeter. The answer to this question is contained in the following result. To simplify the formulas we introduce the following notation

$$F \overset{\text{def}}{=} \overline{p}a + \overline{q}b + \overline{\omega}k = \frac{\langle \mathcal{X}, \boldsymbol{N}\rangle}{\langle \boldsymbol{\nu}^H, \boldsymbol{N}\rangle}. \tag{14.4}$$

Theorem 14.3. *Let $\mathcal{S} \subset \mathbb{H}^1$ be an oriented C^2 surface, then*

$$\mathcal{V}_I^H(\mathcal{S}; \mathcal{X}) = \int_{\mathcal{S}} \mathcal{H} F \, d\sigma_H. \tag{14.5}$$

In particular, $\mathcal{S}$ is stationary if and only if it is H-minimal.

Versions of Theorem 14.3 have also been obtained independently by other people. An approach based on motion by H-mean curvature can be found in [6]. When $\mathcal{X} = a\boldsymbol{\nu}^H + kT$, then a proof based on CR-geometry can be found in [15], and [84,85]. We mention that Hladky and Pauls have recently proved in [56] (with a different approach which does not directly use the first variation) that, for a wide class of sub-Riemannian spaces, a non-characteristic C^2 hypersurface is a critical point of the H-perimeter if and only if it is H-minimal.

Definition 14.4. Given an oriented C^2 surface $\mathcal{S} \subset \mathbb{H}^1$, we define the *second variation* of the H-perimeter with respect to the deformation (14.3) as

$$\mathcal{V}_{II}^H(\mathcal{S}; \mathcal{X}) = \frac{d^2}{d\lambda^2} P_H(\mathcal{S}^\lambda)\big|_{\lambda=0}.$$

Our main result in this section is the following theorem.

Theorem 14.5. *The second variation of the H-perimeter with respect to the deformation of $\mathcal{S}$ given by* (14.3) *is expressed by the formula*

$$
\begin{aligned}
\mathcal{V}_{II}^{H}(\mathcal{S};\mathcal{X}) = \int_{\mathcal{S}} \big\{ & 2(\overline{q}Za - \overline{p}Zb)(Tk - \overline{\omega}Yk) \\
& + (Ta - \overline{\omega}Ya)\big[-2\overline{q}Zk - \overline{q}(a\overline{p} + b\overline{q}) - \overline{p}(a\overline{q} - b\overline{p})\big] \\
& + (Tb - \overline{\omega}Yb)\big[2\overline{p}Zk + \overline{p}(a\overline{p} + b\overline{q}) - \overline{q}(a\overline{q} - b\overline{p})\big] \\
& + 2(a\overline{q} - b\overline{p})(\overline{q}Za - \overline{p}Zb)\overline{\omega} + (Za + \overline{p}\overline{\omega}Zk)^2 + (Zb + \overline{q}\overline{\omega}Zk)^2 \\
& + \big(a^2 + b^2\big)\overline{\omega}^2 + 2\overline{\omega}(aZa + bZb) + 2\overline{\omega}^2(a\overline{p} + b\overline{q})Zk \\
& - \big(\overline{q}Za - \overline{p}Zb + (a\overline{q} - b\overline{p})\overline{\omega}\big)^2\big\}\, d\sigma_H. \qquad (14.6)
\end{aligned}
$$

In order to prove Theorems 14.3 and 14.5 we develop some preliminary material which constitutes the necessary geometric backbone. We begin by deriving from Theorem 10.1 two integration by parts formulas which play a fundamental role in this section and in the following one.

Lemma 14.6. *Let $\zeta \in C_0^1(\mathcal{S} \setminus \Sigma_{\mathcal{S}})$, then*

$$\int_{\mathcal{S}} Z\zeta \, d\sigma_H = -\int_{\mathcal{S}} \zeta\overline{\omega}\, d\sigma_H.$$

Proof. We begin by noting that, thanks to (13.14), (13.9) and (13.10), we can rewrite the two identities in (10.14) in the form

$$\int_{\mathcal{S}} \{\overline{q}Zu + \overline{q}u\overline{\omega}\}\, d\sigma_H = \int_{\mathcal{S}} \overline{p}u\mathcal{H}\, d\sigma_H, \qquad (14.7)$$

$$\int_{\mathcal{S}} \{\overline{p}Zu + \overline{p}u\overline{\omega}\}\, d\sigma_H = -\int_{\mathcal{S}} \overline{q}u\mathcal{H}\, d\sigma_H. \qquad (14.8)$$

Choosing $u = \overline{q}\zeta$ in (14.7), and $u = \overline{p}\zeta$ in (14.8), and adding the resulting equations, we obtain

$$\int_{\mathcal{S}} \big\{(\overline{p}Z\overline{p} + \overline{q}Z\overline{q})\zeta + \big(\overline{p}^2 + \overline{q}^2\big)Z\zeta\big\}\, d\sigma_H + \int_{\mathcal{S}} \big(\overline{p}^2 + \overline{q}^2\big)\zeta\overline{\omega}\, d\sigma_H = 0.$$

From the latter equation, and from (13.5), (13.17), we immediately reach the conclusion. □

Remark 14.7. We have above derived Lemma 14.6 from Theorem 10.1 specialized to $\mathbb{H}^1$. The two results are in fact equivalent. To see this suppose that the identity in Lemma 14.6 hold. Applying it twice, once with the choice $\zeta = \overline{q}u$, and the other with $\zeta = \overline{p}u$, with $u \in C_0^1(\mathcal{S} \setminus \Sigma)$, we obtain (14.7) and (14.8) if we use the identities $Z\overline{p} = \overline{q}\mathcal{H}$, $Z\overline{q} = -\overline{p}\mathcal{H}$, in (13.28).

Another crucial integration by parts formula which we will need is (10.18) in Theorem 10.6. For the reader's convenience we combine this formula and Lemma 14.6 into a single statement.

Lemma 14.8. *Let $f \in C^1(\mathcal{S})$, $\zeta \in C_0^1(\mathcal{S} \setminus \Sigma_{\mathcal{S}})$, then*

$$\int_{\mathcal{S}} f Z\zeta \, d\sigma_H = -\int_{\mathcal{S}} \zeta Z f \, d\sigma_H - \int_{\mathcal{S}} f \zeta \overline{\omega} \, d\sigma_H,$$

$$\int_{\mathcal{S}} f(T\zeta - \overline{\omega} Y\zeta) \, d\sigma_H = -\int_{\mathcal{S}} \zeta(Tf - \overline{\omega} Y f) \, d\sigma_H + \int_{\mathcal{S}} f\zeta\overline{\omega}\mathcal{H} \, d\sigma_H.$$

In particular, if $\mathcal{S}$ is H-minimal, we find

$$\int_{\mathcal{S}} f(T\zeta - \overline{\omega} Y\zeta) \, d\sigma_H = -\int_{\mathcal{S}} \zeta(Tf - \overline{\omega} Y f) \, d\sigma_H.$$

After these preliminaries we turn to the proofs of the main results in this section. We first recall the representation formula for the H-perimeter of $\mathcal{S}$ given in (8.3) of Definition 8.4

$$\sigma_H(\mathcal{S}) = \int_{\mathcal{S}} \frac{W}{|\boldsymbol{N}|} \, d\sigma = \int_{\mathcal{S}} \frac{\sqrt{p^2 + q^2}}{|\boldsymbol{N}|} \, d\sigma, \tag{14.9}$$

where $d\sigma$ represents the standard surface measure on $\mathcal{S}$. Next, we establish a simple lemma which provides the general expression for the first and second variation of the H-perimeter. We consider, for small values of $\lambda \in \mathbb{R}$, a deformation of $\mathcal{S}$ of the type $\mathcal{S}^\lambda = J_\lambda(\mathcal{S}) = \mathcal{S} + \lambda\mathcal{X}$, where $\mathcal{X} \in C_0^1(\mathcal{S} \setminus \Sigma_{\mathcal{S}}; \mathbb{H}^1)$. We denote by $\boldsymbol{N}^\lambda$ the non-unit Riemannian normal on $\mathcal{S}^\lambda$. Letting

$$X_1^\lambda(g) = X_1\big(J_\lambda(g)\big), \qquad X_2^\lambda(g) = X_2\big(J_\lambda(g)\big), \quad g \in \mathcal{S},$$

we consider the functions

$$p^\lambda = \langle \boldsymbol{N}^\lambda, X_1^\lambda \rangle, \qquad q^\lambda = \langle \boldsymbol{N}^\lambda, X_2^\lambda \rangle, \qquad W^\lambda = \sqrt{\big(p^\lambda\big)^2 + \big(q^\lambda\big)^2}. \tag{14.10}$$

We stress that we are assuming that $\mathcal{X}$ is compactly supported away from the characteristic set of $\mathcal{S}$, so that the angle function W for $\mathcal{S}$ never vanishes on the support of $\mathcal{X}$, see (6.4).

Lemma 14.9. *The first variation of the H-perimeter along the deformation $J_\lambda(\mathcal{S}) = \mathcal{S} + \lambda\mathcal{X}$ is given by the formula*

$$\mathcal{V}_I^H(\mathcal{S}; \mathcal{X}) = \int_{\mathcal{S}} \frac{\frac{dW^\lambda}{d\lambda}}{W^\lambda}\bigg|_{\lambda=0} d\sigma_H = \int_{\mathcal{S}} \frac{(p^\lambda \frac{dp^\lambda}{d\lambda} + q^\lambda \frac{dq^\lambda}{d\lambda})|_{\lambda=0}}{W^2} \, d\sigma_H. \tag{14.11}$$

The second variation is given by

$$\mathcal{V}_{II}^H(\mathcal{S}; \mathcal{X}) = \int_{\mathcal{S}} \frac{(p^\lambda \frac{d^2 p^\lambda}{d\lambda^2} + q^\lambda \frac{d^2 q^\lambda}{d\lambda^2})|_{\lambda=0}}{W^2} \, d\sigma_H + \int_{\mathcal{S}} \frac{((\frac{dp^\lambda}{d\lambda})^2 + (\frac{dq^\lambda}{d\lambda})^2)|_{\lambda=0}}{W^2} \, d\sigma_H$$
$$- \int_{\mathcal{S}} \frac{(p^\lambda \frac{dp^\lambda}{d\lambda} + q^\lambda \frac{dq^\lambda}{d\lambda})^2|_{\lambda=0}}{W^4} \, d\sigma_H. \tag{14.12}$$

Proof. Because of the assumptions on $\mathcal{X}$, the integrals in the right-hand sides of (14.11), (14.12) are performed on a compact set which does not intersect $\Sigma_{\mathcal{S}}$. By a partition of unity we can thus reduce the analysis to a set $\mathcal{S}\cap\mathcal{O}$, where $\mathcal{O}\subset\mathbb{H}^1$ is an open neighborhood of a point $g_0\in\mathcal{S}\setminus\Sigma_{\mathcal{S}}$. We can thus assume that there exist an open set $\Omega\subset\mathbb{R}^2_{u,v}$ and a parametrization $\theta:\Omega\to\mathbb{H}^1$ such that $\mathcal{S}\cap\mathcal{O}=\theta(\Omega)$. We suppose that the orientation of $\mathcal{S}$ is given by $\boldsymbol{N}=\theta_u\wedge\theta_v$. Recalling that $d\sigma=|\theta_u\wedge\theta_v|\,du\wedge dv$, we can, after projecting $\mathcal{S}$ onto Ω, rewrite (14.9) as follows:

$$\sigma_H(\mathcal{S}\cap\mathcal{O})=\int_\Omega W\,d\sigma=\int_\Omega\sqrt{p^2+q^2}\,du\wedge dv. \tag{14.13}$$

According to (14.13) we have

$$\sigma_H\big(\mathcal{S}^\lambda\cap\mathcal{O}\big)=\int_\Omega W^\lambda\,du\wedge dv.$$

Observing that

$$\frac{dW^\lambda}{d\lambda}=\frac{p^\lambda\frac{dp^\lambda}{d\lambda}+q^\lambda\frac{dq^\lambda}{d\lambda}}{W^\lambda},$$

and that $W^\lambda|_{\lambda=0}=W$, we find

$$\mathcal{V}_I^H(\mathcal{S};\mathcal{X})=\int_\Omega\frac{(p^\lambda\frac{dp^\lambda}{d\lambda}+q^\lambda\frac{dq^\lambda}{d\lambda})|_{\lambda=0}}{W}\,du\wedge dv=\int_{\mathcal{S}}\frac{(p^\lambda\frac{dp^\lambda}{d\lambda}+q^\lambda\frac{dq^\lambda}{d\lambda})|_{\lambda=0}}{W^2}\,d\sigma_H,$$

which gives (14.11). To obtain (14.12) we proceed analogously, observing that

$$\frac{d^2W^\lambda}{d\lambda^2}=\frac{p^\lambda\frac{d^2p^\lambda}{d\lambda^2}+q^\lambda\frac{d^2q^\lambda}{d\lambda^2}}{W^\lambda}+\frac{(\frac{dp^\lambda}{d\lambda})^2+(\frac{dq^\lambda}{d\lambda})^2}{W^\lambda}-\frac{(p^\lambda\frac{dp^\lambda}{d\lambda}+q^\lambda\frac{dq^\lambda}{d\lambda})^2}{(W^\lambda)^3}.\qquad\square$$

We now turn to the proof of the main results. Given an open set $\Omega\subset\mathbb{R}^2$, we denote by $\theta:\Omega\to\mathbb{H}^1$ a C^2 parametrization of an oriented surface

$$\mathcal{S}=\big\{\theta(u,v)=\big(x(u,v),y(u,v),t(u,v)\big)\in\mathbb{H}^1\;\big|\;(u,v)\in\Omega\big\}. \tag{14.14}$$

We assume throughout that the orientation of $\mathcal{S}$ is given by the non-unit normal $\boldsymbol{N}=\theta_u\wedge\theta_v$. Using (13.2) we see that

$$\theta(u,v)=x(u,v)X_1\big(\theta(u,v)\big)+y(u,v)X_2\big(\theta(u,v)\big)+t(u,v)T. \tag{14.15}$$

From this equation we find

$$\theta_u=x_uX_1+y_uX_2+t_uT+xX_{1,u}+yX_{2,u},$$

with a similar expression for θ_v. Keeping in mind that

$$\begin{cases} X_{1,u} = -\frac{y_u}{2}T, \; X_{2,u} = \frac{x_u}{2}T, \\ X_{1,v} = -\frac{y_v}{2}T, \; X_{2,v} = \frac{x_v}{2}T, \end{cases} \tag{14.16}$$

we obtain

$$\begin{cases} \theta_u = x_u X_1 + y_u X_2 + \big(t_u + \frac{yx_u - xy_u}{2}\big)T, \\ \theta_v = x_v X_1 + y_v X_2 + \big(t_v + \frac{yx_v - xy_v}{2}\big)T. \end{cases} \tag{14.17}$$

From (13.1) and (14.16) we find

$$\begin{cases} \theta_u \wedge X_1 = \big(t_u + \frac{yx_u - xy_u}{2}\big)X_2 - y_u T, \\ \theta_u \wedge X_2 = -\big(t_u + \frac{yx_u - xy_u}{2}\big)X_1 + x_u T, \\ \theta_u \wedge T = y_u X_1 - x_u X_2, \\ \theta_v \wedge X_1 = \big(t_v + \frac{yx_v - xy_v}{2}\big)X_2 - y_v T, \\ \theta_v \wedge X_2 = -\big(t_v + \frac{yx_v - xy_v}{2}\big)X_1 + x_v T, \\ \theta_v \wedge T = y_v X_1 - x_v X_2. \end{cases} \tag{14.18}$$

The non-unit outer Riemannian normal to $\mathcal{S}$ is thus given by

$$\begin{aligned} \boldsymbol{N} = \theta_u \wedge \theta_v &= \left\{ y_u\left(t_v + \frac{yx_v - xy_v}{2}\right) - y_v\left(t_u + \frac{yx_u - xy_u}{2}\right)\right\}X_1 \\ &\quad + \left\{ x_v\left(t_u + \frac{yx_u - xy_u}{2}\right) - x_u\left(t_v + \frac{yx_v - xy_v}{2}\right)\right\}X_2 + \{x_u y_v - x_v y_u\}T \\ &= \left(y_u t_v - y_v t_u - \frac{y}{2}(x_u y_v - x_v y_u)\right)X_1 + \left(x_v t_u - x_u t_v + \frac{x}{2}(x_u y_v - x_v y_u)\right)X_2 \\ &\quad + (x_u y_v - x_v y_u)T. \end{aligned} \tag{14.19}$$

We denote by $\boldsymbol{\nu} = \boldsymbol{N}/|\boldsymbol{N}|$ the Riemannian Gauss map of $\mathcal{S}$. Keeping in mind (13.3), we see from (14.19) that

$$\begin{cases} p = y_u t_v - y_v t_u - \frac{y}{2}(x_u y_v - x_v y_u), \\ q = x_v t_u - x_u t_v + \frac{x}{2}(x_u y_v - x_v y_u), \\ \omega = x_u y_v - x_v y_u. \end{cases} \tag{14.20}$$

We note at this moment that, given the assumption $\theta \in C^2(\Omega)$, the functions p, q, ω, W are of class $C^1(\Omega)$, and that moreover $\overline{p}, \overline{q}, \overline{\omega}$ are of class $C^1(\mathcal{S} \setminus \Sigma_{\mathcal{S}})$. In what follows, given a function ζ defined in a neighborhood of $\mathcal{S}$, we will by abuse of notation denote with $\zeta(u,v) = \zeta \circ \theta(u,v) = \zeta(x(u,v), y(u,v), t(u,v))$. The chain rule gives

$$\zeta_u = x_u \zeta_x + y_u \zeta_y + t_u \zeta_t, \qquad \zeta_v = x_v \zeta_x + y_v \zeta_y + t_v \zeta_t. \tag{14.21}$$

Using (3.2), we obtain from (14.21)

$$\begin{cases} \zeta_u = x_u X_1\zeta + y_u X_2\zeta + \left(t_u + \frac{yx_u - xy_u}{2}\right)T\zeta, \\ \zeta_v = x_v X_1\zeta + y_v X_2\zeta + \left(t_v + \frac{yx_v - xy_v}{2}\right)T\zeta. \end{cases} \tag{14.22}$$

In the sequel, it will be convenient to also have the expression of ζ_u, ζ_v with respect to the orthonormal frame $\{Z, Y, T\}$, where Y and Z are like in (13.9), (13.10). From (14.22) and (13.13), we have

$$\begin{cases} \zeta_u = (x_u\overline{p} + y_u\overline{q})Y\zeta + (x_u\overline{q} - y_u\overline{p})Z\zeta + \left(t_u + \frac{yx_u - xy_u}{2}\right)T\zeta, \\ \zeta_v = (x_v\overline{p} + y_v\overline{q})Y\zeta + (x_v\overline{q} - y_v\overline{p})Z\zeta + \left(t_v + \frac{yx_v - xy_v}{2}\right)T\zeta. \end{cases} \tag{14.23}$$

We now fix functions $a, b, k \in C_0^\infty(\mathcal{S} \setminus \Sigma_{\mathcal{S}})$, and consider the vector field

$$\mathcal{X} = aX_1 + bX_2 + kT. \tag{14.24}$$

For small values of $\lambda \in \mathbb{R}$, we let $\mathcal{S}^\lambda$ be the surface obtained by deforming $\mathcal{S}$ through the map $J_\lambda = Id + \lambda\mathcal{X}$, so that

$$J_\lambda(g) = g + \lambda(aX_1 + bX_2 + kT), \quad g \in \mathcal{S}. \tag{14.25}$$

The parametric representation of $\mathcal{S}^\lambda$ is given by

$$\theta^\lambda = \theta + \lambda\mathcal{X}, \tag{14.26}$$

so that

$$\theta_u^\lambda = \theta_u + \lambda\mathcal{X}_u, \qquad \theta_v^\lambda = \theta_v + \lambda\mathcal{X}_v, \tag{14.27}$$

and therefore the non-unit Riemannian normal to the surface $\mathcal{S}^\lambda = J_\lambda(\mathcal{S})$ is given by

$$\boldsymbol{N}^\lambda = \theta_u^\lambda \wedge \theta_v^\lambda = \boldsymbol{N} + \lambda(\theta_u \wedge \mathcal{X}_v - \theta_v \wedge \mathcal{X}_u) + \lambda^2\mathcal{X}_u \wedge \mathcal{X}_v. \tag{14.28}$$

From (14.28) we obtain

$$\left.\frac{d\boldsymbol{N}^\lambda}{d\lambda}\right|_{\lambda=0} = \theta_u \wedge \mathcal{X}_v - \theta_v \wedge \mathcal{X}_u, \qquad \left.\frac{d^2\boldsymbol{N}^\lambda}{d\lambda^2}\right|_{\lambda=0} = 2\mathcal{X}_u \wedge \mathcal{X}_v. \tag{14.29}$$

Using (14.16) we find

$$\mathcal{X}_u = a_u X_1 + b_u X_2 + \left(k_u + \frac{bx_u - ay_u}{2}\right)T, \tag{14.30}$$

$$\mathcal{X}_v = a_v X_1 + b_v X_2 + \left(k_v + \frac{bx_v - ay_v}{2}\right)T. \tag{14.31}$$

From (14.29)–(14.31) and (14.18) one has

$$\frac{dN^\lambda}{d\lambda}\bigg|_{\lambda=0}$$
$$= \left\{ \left(t_v + \frac{yx_v - xy_v}{2} \right) b_u - \left(t_u + \frac{yx_u - xy_u}{2} \right) b_v + (y_u k_v - y_v k_u) - \frac{b}{2}(x_u y_v - x_v y_u) \right\} X_1$$
$$+ \left\{ \left(t_u + \frac{yx_u - xy_u}{2} \right) a_v - \left(t_v + \frac{yx_v - xy_v}{2} \right) a_u + (x_v k_u - x_u k_v) + \frac{a}{2}(x_u y_v - x_v y_u) \right\} X_2$$
$$+ \left\{ (y_v a_u - y_u a_v) + (x_u b_v - x_v b_u) \right\} T. \tag{14.32}$$

Equations (14.29)–(14.31) also give

$$\frac{d^2 N^\lambda}{d\lambda^2}\bigg|_{\lambda=0} = 2\Bigg\{ \left[b_u \left(k_v + \frac{bx_v - ay_v}{2} \right) - b_v \left(k_u + \frac{bx_u - ay_u}{2} \right) \right] X_1$$
$$+ \left[a_v \left(k_u + \frac{bx_u - ay_u}{2} \right) - a_u \left(k_v + \frac{bx_v - ay_v}{2} \right) \right] X_2$$
$$+ (a_u b_v - a_v b_u) T \Bigg\}. \tag{14.33}$$

We now let

$$\begin{cases} X_1^\lambda = X_1(\theta^\lambda) = X_1 - \lambda \frac{b}{2} T, \\ X_2^\lambda = X_2(\theta^\lambda) = X_2 + \lambda \frac{a}{2} T, \end{cases} \tag{14.34}$$

for which we clearly have

$$\frac{dX_1^\lambda}{d\lambda} = -\frac{b}{2} T, \qquad \frac{dX_2^\lambda}{d\lambda} = \frac{a}{2} T, \qquad \frac{d^2 X_1^\lambda}{d\lambda^2} = 0, \qquad \frac{d^2 X_2^\lambda}{d\lambda^2} = 0. \tag{14.35}$$

Consider the quantities in (14.10). From (14.35) we find

$$\frac{dp^\lambda}{d\lambda}\bigg|_{\lambda=0} = \left\langle \frac{dN^\lambda}{d\lambda}\bigg|_{\lambda=0}, X_1 \right\rangle - \frac{b}{2} \langle N, T \rangle, \tag{14.36}$$

$$\frac{d^2 p^\lambda}{d\lambda^2}\bigg|_{\lambda=0} = \left\langle \frac{d^2 N^\lambda}{d\lambda^2}|_{\lambda=0}, X_1 \right\rangle - b \left\langle \frac{dN^\lambda}{d\lambda}\bigg|_{\lambda=0}, T \right\rangle. \tag{14.37}$$

Similarly, we find

$$\frac{dq^\lambda}{d\lambda}\bigg|_{\lambda=0} = \left\langle \frac{dN^\lambda}{d\lambda}\bigg|_{\lambda=0}, X_2 \right\rangle + \frac{a}{2} \langle N, T \rangle, \tag{14.38}$$

$$\frac{d^2 q^\lambda}{d\lambda^2}\bigg|_{\lambda=0} = \left\langle \frac{d^2 N^\lambda}{d\lambda^2}\bigg|_{\lambda=0}, X_2 \right\rangle + a \left\langle \frac{dN^\lambda}{d\lambda}\bigg|_{\lambda=0}, T \right\rangle. \tag{14.39}$$

Lemma 14.10. *Let p^λ and q^λ relative to the surface $\mathcal{S}^\lambda = J_\lambda(\mathcal{S})$, where J_λ is defined by* (14.25), *then*

$$\frac{dp^\lambda}{d\lambda}\bigg|_{\lambda=0} = W\big\{-(Zb + b\overline{\omega}) - \overline{q}\,\overline{\omega} Zk + \overline{p}(Tk - \overline{\omega} Yk)\big\},$$

$$\frac{dq^\lambda}{d\lambda}\bigg|_{\lambda=0} = W\big\{(Za + a\overline{\omega}) + \overline{p}\,\overline{\omega} Zk + \overline{q}(Tk - \overline{\omega} Yk)\big\}.$$

Proof. Using (14.32), (14.36) and the third equation in (14.20), we obtain

$$\frac{dp^\lambda}{d\lambda}\bigg|_{\lambda=0} = \left(t_v + \frac{yx_v - xy_v}{2}\right)b_u - \left(t_u + \frac{yx_u - xy_u}{2}\right)b_v + (y_u k_v - y_v k_u) - b\omega. \quad (14.40)$$

We now use the equations (14.22) to express the derivatives b_u, b_v, k_u, k_v in terms of derivatives Zb, Yb, Tb with respect to the orthonormal frame $\{Z, Y, T\}$. Ordering terms one finds:

$$\begin{aligned}\frac{dp^\lambda}{d\lambda}\bigg|_{\lambda=0} ={}& \left[\left(t_v + \frac{yx_v - xy_v}{2}\right)(x_u\overline{p} + y_u\overline{q}) - \left(t_u + \frac{yx_u - xy_u}{2}\right)(x_v\overline{p} + y_v\overline{q})\right]Yb \\ &+ \left[\left(t_v + \frac{yx_v - xy_v}{2}\right)(x_u\overline{q} - y_u\overline{p}) - \left(t_u + \frac{yx_u - xy_u}{2}\right)(x_v\overline{q} - y_v\overline{p})\right]Zb \\ &+ \big[y_u(x_v\overline{p} + y_v\overline{q}) - y_v(x_u\overline{p} + y_u\overline{q})\big]Yk + \big[y_u(x_v\overline{q} - y_v\overline{p}) - y_v(x_u\overline{q} - y_u\overline{p})\big]Zk \\ &+ \left[y_u t_v - y_v t_u - \frac{y}{2}(x_u y_v - x_v y_u)\right]Tk - b\omega.\end{aligned}$$

Simplifying in the latter equation, gives

$$\begin{aligned}\frac{dp^\lambda}{d\lambda}\bigg|_{\lambda=0} ={}& \left[-\left(x_v t_u - x_u t_v + \frac{x}{2}(x_u y_v - x_v y_u)\right)\overline{p} + \left(y_u t_v - y_v t_u - \frac{y}{2}(x_u y_v - x_v y_u)\right)\overline{q}\right]Yb \\ &+ \left[-\left(x_v t_u - x_u t_v + \frac{x}{2}(x_u y_v - x_v y_u)\right)\overline{q} - \left(y_u t_v - y_v t_u - \frac{y}{2}(x_u y_v - x_v y_u)\right)\overline{p}\right]Zb \\ &- (x_u y_v - x_v y_u)\overline{p}Yk - (x_u y_v - x_v y_u)\overline{q}Zk + \overline{p}WTk - b\omega,\end{aligned}$$

where we have used (14.20). At this point we notice that, in view of (14.20) again, the coefficient of Yb vanishes, and we obtain from the remaining terms the following expression:

$$\frac{dp^\lambda}{d\lambda}\bigg|_{\lambda=0} = W\big\{-Zb - b\overline{\omega} + \overline{p}Tk - \overline{\omega}(\overline{p}Yk + \overline{q}Zk)\big\},$$

which gives the first equation in the thesis of the lemma. In a similar fashion, we obtain the desired expression of $\frac{dq^\lambda}{d\lambda}|_{\lambda=0}$. □

From Lemma 14.10 we immediately obtain the following crucial result.

Lemma 14.11. *In the situation of Lemma* 14.10 *we have*

$$\left(p^\lambda \frac{dp^\lambda}{d\lambda} + q^\lambda \frac{dq^\lambda}{d\lambda}\right)\bigg|_{\lambda=0} = W^2\big\{(Tk - \overline{\omega}Yk) + (\overline{q}Za - \overline{p}Zb) + (\overline{q}a - \overline{p}b)\overline{\omega}\big\}.$$

With Lemma 14.11 in hands we are ready to give the proof of Theorem 14.3.

Proof of Theorem 14.3. Substituting the equation in Lemma 14.11 in (14.11) of Lemma 14.9, we obtain

$$\mathcal{V}_I^H(\mathcal{S};\mathcal{X}) = \int_{\mathcal{S}} \{Tk - \overline{\omega}Yk\}\, d\sigma_H + \int_{\mathcal{S}} \big\{(\overline{q}Za - \overline{p}Zb) + (\overline{q}a - \overline{p}b)\overline{\omega}\big\}\, d\sigma_H. \tag{14.41}$$

In order to extract the geometry from (14.41) we need to convert the two integrals in the right-hand side into ones which involve only the functions a, b and k, and not their covariant derivatives along the orthonormal frame $\{Z, Y, T\}$. This is where we use Lemma 14.8 for the first time. Applying (14.7), (14.8) we find

$$\int_{\mathcal{S}} \{\overline{q}Za + \overline{q}\overline{\omega}a\}\, d\sigma_H = \int_{\mathcal{S}} \overline{p}a\mathcal{H}\, d\sigma_H,$$

$$\int_{\mathcal{S}} \{\overline{p}Zb + \overline{p}\overline{\omega}b\}\, d\sigma_H = -\int_{\mathcal{S}} \overline{q}b\mathcal{H}\, d\sigma_H. \tag{14.42}$$

Furthermore, Lemma 14.8 gives

$$\int_{\mathcal{S}} \{Tk - \overline{\omega}Yk\}\, d\sigma_H = \int_{\mathcal{S}} \overline{\omega}k\mathcal{H}\, d\sigma_H. \tag{14.43}$$

Combining (14.42), (14.43) with (14.41) we obtain

$$\mathcal{V}_I^H(\mathcal{S};\mathcal{X}) = \int_{\mathcal{S}} \mathcal{H}\{\overline{p}a + \overline{q}b + \overline{\omega}k\}\, d\sigma_H. \tag{14.44}$$

Recalling the definition (14.4) of F, we reach the desired conclusion. □

With Lemma 14.10 and some elementary computations, we obtain the following result which is useful in the proof of Theorem 14.5 since it provides the integrand of the second addend in the right-hand side of (14.12).

Lemma 14.12. *In the situation of Lemma* 14.10 *we have*

$$\begin{aligned}\frac{(\frac{dp^\lambda}{d\lambda})^2|_{\lambda=0} + (\frac{dq^\lambda}{d\lambda})^2|_{\lambda=0}}{W^2} &= (Za + \overline{p}\overline{\omega}Zk)^2 + (Zb + \overline{q}\overline{\omega}Zk)^2 + (Tk - \overline{\omega}Yk)^2 + \big(a^2 + b^2\big)\overline{\omega}^2 \\ &\quad + 2(Tk - \overline{\omega}Yk)\big(\overline{q}Za - \overline{p}Zb + \overline{\omega}(a\overline{q} - b\overline{p})\big) \\ &\quad + 2\overline{\omega}(aZa + bZb) + 2\overline{\omega}^2(a\overline{p} + b\overline{q})Zk.\end{aligned} \tag{14.45}$$

We finally turn to the computation of the first addend in the right-hand side of (14.12), i.e., $(p^\lambda \frac{d^2 p^\lambda}{d\lambda^2} + q^\lambda \frac{d^2 q^\lambda}{d\lambda^2})|_{\lambda=0}$. From (14.37) and from (14.32), (14.33) we obtain

$$\begin{aligned}
&\left(p^\lambda \frac{d^2 p^\lambda}{d\lambda^2} + q^\lambda \frac{d^2 q^\lambda}{d\lambda^2}\right)\bigg|_{\lambda=0} \\
&\quad = W\left\{\left\langle \frac{d^2 N^\lambda}{d\lambda^2}\bigg|_{\lambda=0}, \boldsymbol{\nu}^H\right\rangle + (a\overline{q} - b\overline{p})\left\langle \frac{dN^\lambda}{d\lambda}\bigg|_{\lambda=0}, T\right\rangle\right\} \\
&\quad = W\big\{\big[2k_v(\overline{p}b_u - \overline{q}a_u) + 2k_u(\overline{q}a_v - \overline{p}b_v)\big] \\
&\qquad + \big[(bx_v - ay_v)(\overline{p}b_u - \overline{q}a_u) + (bx_u - ay_u)(\overline{q}a_v - \overline{p}b_v)\big] \\
&\qquad + (a\overline{q} - b\overline{p})\big[(y_v a_u - y_u a_v) + (x_u b_v - x_v b_u)\big]\big\}.
\end{aligned} \tag{14.46}$$

Now we compute the three expressions in square brackets in the right-hand side of (14.46). Using (14.23) we obtain

$$\begin{aligned}
(\overline{p}b_u - \overline{q}a_u) &= (x_u\overline{p} + y_u\overline{q})(\overline{p}Yb - \overline{q}Ya) + (x_u\overline{q} - y_u\overline{p})(\overline{p}Zb - \overline{q}Za) \\
&\quad + \left(t_u + \frac{yx_u - xy_u}{2}\right)(\overline{p}Tb - \overline{q}Ta),
\end{aligned} \tag{14.47}$$

$$\begin{aligned}
(\overline{q}a_v - \overline{p}b_v) &= -(x_v\overline{p} + y_v\overline{q})(\overline{p}Yb - \overline{q}Ya) - (x_v\overline{q} - y_v\overline{p})(\overline{p}Zb - \overline{q}Za) \\
&\quad - \left(t_v + \frac{yx_v - xy_v}{2}\right)(\overline{p}Tb - \overline{q}Ta).
\end{aligned} \tag{14.48}$$

These formulas, combined with (14.20), give

$$\begin{aligned}
&(bx_v - ay_v)(\overline{p}b_u - \overline{q}a_u) + (bx_u - ay_u)(\overline{q}a_v - \overline{p}b_v) \\
&\quad = W\big\{\overline{p}(a\overline{p} + b\overline{q})(Tb - \overline{\omega}Yb) - \overline{q}(a\overline{p} + b\overline{q})(Ta - \overline{\omega}Ya) \\
&\qquad + (b\overline{p} - a\overline{q})(\overline{p}Zb - \overline{q}Za)\overline{\omega}\big\}.
\end{aligned} \tag{14.49}$$

Again from (14.23) and (14.20), we obtain

$$y_v a_u - y_u a_v = \big[-\overline{p}(Ta - \overline{\omega}Ya) + \overline{q}\overline{\omega}Za\big]W, \tag{14.50}$$

and

$$x_u b_v - x_v b_u = \big[-\overline{q}(Tb - \overline{\omega}Yb) - \overline{p}\overline{\omega}Zb\big]W. \tag{14.51}$$

Formulas (14.50), (14.51) give

$$\begin{aligned}
&(a\overline{q} - b\overline{p})\big[(y_v a_u - y_u a_v) + (x_u b_v - x_v b_u)\big] \\
&\quad = (a\overline{q} - b\overline{p})\big[-\overline{p}(Ta - \overline{\omega}Ya) - \overline{q}(Tb - \overline{\omega}Yb) + (\overline{q}Za - \overline{p}Zb)\overline{\omega}\big]W.
\end{aligned} \tag{14.52}$$

Finally, a (long) computation, based on (14.23), (14.20), (14.47), (14.48), and the identity $\overline{p}^2 + \overline{q}^2 = 1$, give

$$
\begin{aligned}
&2k_v(\overline{p}b_u-\overline{q}a_u)+2k_u(\overline{q}a_v-\overline{p}b_v)\\
&\quad=2W\bigl\{-(\overline{p}Zb-\overline{q}Za)(Tk-\overline{\omega}Yk)+Zk\bigl[\overline{p}(Tb-\overline{\omega}Yb)-\overline{q}(Ta-\overline{\omega}Ya)\bigr]\bigr\}. \qquad (14.53)
\end{aligned}
$$

From (14.46), (14.49), (14.52) and (14.53) we finally conclude

$$
\begin{aligned}
&\frac{(p^\lambda\frac{d^2p^\lambda}{d\lambda^2}+q^\lambda\frac{d^2q^\lambda}{d\lambda^2})|_{\lambda=0}}{W^2}\\
&\quad=-2(\overline{p}Zb-\overline{q}Za)(Tk-\overline{\omega}Yk)+2Zk\bigl[\overline{p}(Tb-\overline{\omega}Yb)-\overline{q}(Ta-\overline{\omega}Ya)\bigr]\\
&\qquad+\overline{p}(a\overline{p}+b\overline{q})(Tb-\overline{\omega}Yb)-\overline{q}(a\overline{p}+b\overline{q})(Ta-\overline{\omega}Ya)+(b\overline{p}-a\overline{q})(\overline{p}Zb-\overline{q}Za)\overline{\omega}\\
&\qquad+(a\overline{q}-b\overline{p})\bigl[-\overline{p}(Ta-\overline{\omega}Ya)-\overline{q}(Tb-\overline{\omega}Yb)+(\overline{q}Za-\overline{p}Zb)\overline{\omega}\bigr]. \qquad (14.54)
\end{aligned}
$$

We have thus proved the following lemma.

Lemma 14.13. *In the situation of Lemma* 14.10 *we have*

$$
\begin{aligned}
&\frac{(p^\lambda\frac{d^2p^\lambda}{d\lambda^2}+q^\lambda\frac{d^2q^\lambda}{d\lambda^2})|_{\lambda=0}}{W^2}\\
&\quad=-2(\overline{p}Zb-\overline{q}Za)(Tk-\overline{\omega}Yk)+(Ta-\overline{\omega}Ya)\bigl[-2\overline{q}Zk-\overline{q}(a\overline{p}+b\overline{q})-\overline{p}(a\overline{q}-b\overline{p})\bigr]\\
&\qquad+(Tb-\overline{\omega}Yb)\bigl[2\overline{p}Zk+\overline{p}(a\overline{p}+b\overline{q})-\overline{q}(a\overline{q}-b\overline{p})\bigr]\\
&\qquad+2(a\overline{q}-b\overline{p})(\overline{q}Za-\overline{p}Zb)\overline{\omega}. \qquad (14.55)
\end{aligned}
$$

We can finally give the proof of Theorem 14.5.

Proof of Theorem 14.5. Combining (14.12) in Lemma 14.9 with Lemmas 14.11, 14.12 and 14.13, we obtain the desired conclusion. □

15. The stability of *H*-minimal surfaces

Unfortunately, in its present form Theorem 14.5 is not as useful as one would wish. A completely analogous situation occurs in the Riemannian case, where one still needs to carefully use intrinsic integration by parts to extract the geometry, see [8]. The main objective of this section is to give a geometric meaning to the second variation formula of Theorem 14.5. We stress that for the sake of simplicity, and because of its relevance in the applications to stability, we state it for stationary points of the H-perimeter functional (H-minimal surfaces), but a more general formula containing the H-mean curvature H, along with its covariant derivatives, can be obtained with some additional work if we use the full form of the geometric identities in Section 13. We begin with the relevant definition.

Definition 15.1. Given an oriented C^2 surface $\mathcal{S}\subset\mathbb{H}^1$, with $\Sigma_{\mathcal{S}}=\emptyset$, we say that $\mathcal{S}$ is *stable* if it is stationary (i.e., H-minimal), and if

$$
\mathcal{V}_{II}^H(\mathcal{S};\mathcal{X})\geqslant 0,\quad\text{for every } \mathcal{X}\in C_0^2(\mathcal{S},\mathbb{H}^1).
$$

If there exists $\mathcal{X}\neq 0$ such that $\mathcal{V}_{II}^H(\mathcal{S};\mathcal{X})<0$, then we say that $\mathcal{S}$ is *unstable*.

Our main result concerning the stability is contained in the following theorem.

Theorem 15.2. *Let $\mathcal{S} \subset \mathbb{H}^1$ be H-minimal, then*

$$\mathcal{V}_{II}^H(\mathcal{S};\mathcal{X}) = \int_{\mathcal{S}} \left\{ \left|\nabla^{H,\mathcal{S}} F\right|^2 + \left(2\mathcal{A} - \overline{\omega}^2\right) F^2 \right\} d\sigma_H,$$

where F is as in (14.4). *As a consequence, $\mathcal{S}$ is stable if and only if the following stability inequality of Hardy type holds on $\mathcal{S}$:*

$$\int_{\mathcal{S}} \left(\overline{\omega}^2 - 2\mathcal{A}\right) F^2 \, d\sigma_H \leqslant \int_{\mathcal{S}} \left|\nabla^{H,\mathcal{S}} F\right|^2 d\sigma_H.$$

Corollary 15.3. *Every vertical plane $\mathcal{S} = \{(x,y,t) \in \mathbb{H}^1 \mid \alpha x + \beta y = \gamma\}$, with $\alpha^2 + \beta^2 \neq 0$, is stable.*

Proof. Consider the defining function $\phi(x,y,t) = \alpha x + \beta y - \gamma$. One has $\omega = T\phi \equiv 0$, and therefore $\omega = \mathcal{A} \equiv 0$. Since every plane in $\mathbb{H}^1$ is H-minimal, we can apply Theorem 15.2, to find for every vector field $\mathcal{X} = aX_1 + bX_2 + kT \in C_0^2(\mathcal{S}, \mathbb{H}^1)$

$$\mathcal{V}_{II}^H(\mathcal{S};\mathcal{X}) = \int_{\mathcal{S}} \left|\nabla^{H,\mathcal{S}} F\right|^2 d\sigma_H \geqslant 0.$$

This proves the stability of $\mathcal{S}$. We note explicitly that in the present situation

$$F = \frac{\alpha a + \beta b}{\sqrt{\alpha^2 + \beta^2}}. \qquad \square$$

Another interesting consequence of Theorem 15.2 is the following stability inequality for intrinsic graphs. We recall that a C^2 surface $\mathcal{S} \subset \mathbb{H}^1$ is called an intrinsic X_1-graph according to [45] provided that there exist an open set $\Omega \subset \mathbb{R}^2_{(u,v)}$ and a function $\phi \in C^2(\Omega)$ such that $\mathcal{S}$ can be described by $(x,y,t) = (0,u,v) \circ \phi(u,v)e_1 = (0,u,v) \circ (\phi(u,v),0,0)$. This means that $\mathcal{S}$ admits the parametrization

$$\theta(u,v) = \left(\phi(u,v), u, v - \frac{u}{2}\phi(u,v)\right), \quad (u,v) \in \Omega.$$

If instead $\mathcal{S}$ can be parametrized by

$$\theta(u,v) = \left(u, \phi(u,v), v + \frac{u}{2}\phi(u,v)\right), \quad (u,v) \in \Omega,$$

then we say that $\mathcal{S}$ is an intrinsic X_2-graph. We only discuss the case of an intrinsic X_1-graph, leaving to the reader to provide the trivial changes necessary to treat the case of X_2-graphs.

Given a function F denote by $\mathcal{B}_\phi(F) = F_u + \phi F_v$ the linear transport equation, so that $\mathcal{B}_\phi(\phi) = \phi_u + \phi\phi_v$ indicates the nonlinear inviscid Burger operator acting on ϕ. Since from (14.20) we obtain

$$p = 1, \qquad q = -\mathcal{B}_\phi(\phi), \qquad \omega = -\phi_v, \qquad W = \sqrt{1 + \mathcal{B}_\phi(\phi)^2}, \tag{15.1}$$

we see that the Riemannian normal to an intrinsic X_1-graph is given by

$$\boldsymbol{N} = X_1 - \mathcal{B}_\phi(\phi)X_2 - \phi_v T. \tag{15.2}$$

As a consequence of the first equality in (15.1) we deduce that an intrinsic X_1-graph always has empty characteristic locus. Furthermore, again from (15.1), and from (15.2), we see that if Ω is bounded then the H-perimeter of $\mathcal{S}$ is expressed by the functional

$$\sigma_H(\mathcal{S}) = \mathcal{P}(\phi) = \int_\Omega \sqrt{1 + \mathcal{B}_\phi(\phi)^2}\, du\, dv,$$

so that

$$d\sigma_H = \sqrt{1 + \mathcal{B}_\phi(\phi)^2}\, du\, dv,$$

see also [1].

Corollary 15.4. *Let $\mathcal{S}$ be a C^2 H-minimal, intrinsic X_1-graph, then $\mathcal{S}$ is stable if and only if*

$$\int_\Omega \frac{\phi_v^2 + 2\mathcal{B}_\phi(\phi_v)}{\sqrt{1 + \mathcal{B}_\phi(\phi)^2}} F^2\, du\, dv \leqslant \int_\Omega \frac{\mathcal{B}_\phi(F)^2}{\sqrt{1 + \mathcal{B}_\phi(\phi)^2}}\, du\, dv,$$

where F is as in (14.4).

Proof. We begin by observing that, thanks to (15.1) we have

$$\begin{cases} Y = \boldsymbol{\nu}^H = \frac{1}{\sqrt{1+\mathcal{B}_\phi(\phi)^2}} X_1 - \frac{\mathcal{B}_\phi(\phi)}{\sqrt{1+\mathcal{B}_\phi(\phi)^2}} X_2, \\ Z = (\boldsymbol{\nu}^H)^\perp = -\frac{\mathcal{B}_\phi(\phi)}{\sqrt{1+\mathcal{B}_\phi(\phi)^2}} X_1 - \frac{1}{\sqrt{1+\mathcal{B}_\phi(\phi)^2}} X_2. \end{cases} \tag{15.3}$$

Given a function f on $\mathcal{S}$, by abuse of notation we continue to indicate with the same letter the function $f(u,v) = f(\theta(u,v))$. A simple use of the chain rule as in (14.21) gives

$$f_u = \phi_u X_1 f + X_2 f - \phi T f, \qquad f_v = \phi_v X_1 f + T f,$$

where in the right-hand sides of the latter equations we have written $X_1 f$ for $X_1 f \circ \theta$, and similarly for $X_2 f$, Tf. Using the latter two equations and the second equation in (15.3), we obtain

$$Zf = -\frac{\mathcal{B}_\phi(f)}{\sqrt{1 + \mathcal{B}_\phi(\phi)^2}}. \tag{15.4}$$

We now use (15.4) to compute $\mathcal{A} = -Z\overline{\omega}$. From the latter two equations in (15.1), one has

$$\begin{aligned}\mathcal{A} &= Z\left(\frac{\phi_v}{\sqrt{1+\mathcal{B}_\phi(\phi)^2}}\right)\\ &= \frac{Z(\phi_v)}{\sqrt{1+\mathcal{B}_\phi(\phi)^2}} - \phi_v \frac{Z(\sqrt{1+\mathcal{B}_\phi(\phi)^2})}{1+\mathcal{B}_\phi(\phi)^2}\\ &= -\frac{\mathcal{B}_\phi(\phi_v)}{1+\mathcal{B}_\phi(\phi)^2} + \phi_v \frac{\mathcal{B}_\phi(\phi)\mathcal{B}_\phi(\mathcal{B}_\phi(\phi))}{(1+\mathcal{B}_\phi(\phi)^2)^2}.\end{aligned}$$

One can now recognize that the H-mean curvature of $\mathcal{S}$ is given by

$$\mathcal{B}_\phi\left(\frac{\mathcal{B}_\phi(\phi)}{\sqrt{1+\mathcal{B}_\phi(\phi)^2}}\right) = -\mathcal{H}, \tag{15.5}$$

see [50], and also [3]. Using (15.5), after some simple computations, we obtain that the condition that $\mathcal{S}$ be H-minimal is expressed by

$$\mathcal{B}_\phi\big(\mathcal{B}_\phi(\phi)\big) = 0.$$

Substituting this equation in the above formula for $\mathcal{A}$ we conclude that

$$\mathcal{A} = -\frac{\mathcal{B}_\phi(\phi_v)}{1+\mathcal{B}_\phi(\phi)^2}.$$

Again from (15.1) we finally obtain

$$\overline{\omega}^2 - 2\mathcal{A} = \frac{\phi_v^2 + 2\mathcal{B}_\phi(\phi_v)}{1+\mathcal{B}_\phi(\phi)^2}.$$

To reach the desired conclusion we are left with using the latter equation in the stability inequality in Theorem 15.2, in combination with the expression of $d\sigma_H$ and with (15.4). □

In [26] it was conjectured that the only C^2 stable intrinsic graphs in $\mathbb{H}^1$ are the vertical planes. Using also the results in [26], in [3] the authors have provided a positive answer to this conjecture. We next turn to the proof of Theorem 15.2.

Proof of Theorem 15.2. Since we want to extract a more geometrically meaningful formula from the general expression in Theorem 14.5, we will now make several reductions. First, expanding the three squares, and regrouping terms using repeatedly $\overline{p}^2 + \overline{q}^2 = 1$, we find for the integrand in the right-hand side of (14.6)

$$\begin{aligned}\mathcal{I}ntegrand &= 2(\overline{q}Za - \overline{p}Zb)(Tk - \overline{\omega}Yk) + (Ta - \overline{\omega}Ya)\big[-2\overline{q}Zk - \overline{q}(a\overline{p} + b\overline{q}) - \overline{p}(a\overline{q} - b\overline{p})\big]\\ &\quad + (Tb - \overline{\omega}Yb)\big[2\overline{p}Zk + \overline{p}(a\overline{p} + b\overline{q}) - \overline{q}(a\overline{q} - b\overline{p})\big] + (\overline{p}Za + \overline{q}Zb)^2\end{aligned}$$

$$+ \overline{\omega}^2(Zk)^2 + 2(\overline{p}Za + \overline{q}Zb)\overline{\omega}Zk + 2\overline{\omega}^2(a\overline{p} + b\overline{q})Zk + 2\overline{\omega}(aZa + bZb) + \big(a^2\overline{p}^2 + b^2\overline{q}^2 + 2\overline{p}\overline{q}ab\big)\overline{\omega}^2.$$

We easily obtain from the latter equation

$$\begin{aligned}
\mathcal{V}_{II}^H(\mathcal{S};\mathcal{X}) &= 2\int_{\mathcal{S}} (\overline{q}Za - \overline{p}Zb)(Tk - \overline{\omega}Yk)\,d\sigma_H \\
&\quad + 2\int_{\mathcal{S}} \big[-\overline{q}(Ta - \overline{\omega}Ya) + \overline{p}(Tb - \overline{\omega}Yb)\big]Zk\,d\sigma_H \\
&\quad + \int_{\mathcal{S}} \big\{(Ta - \overline{\omega}Ya)\big[-\overline{q}(a\overline{p} + b\overline{q}) - \overline{p}(a\overline{q} - b\overline{p})\big] \\
&\quad + (Tb - \overline{\omega}Yb)\big[\overline{p}(a\overline{p} + b\overline{q}) - \overline{q}(a\overline{q} - b\overline{p})\big] + (\overline{p}Za + \overline{q}Zb + \overline{\omega}Zk)^2 \\
&\quad + 2\overline{\omega}^2(a\overline{p} + b\overline{q})Zk + 2\overline{\omega}(aZa + bZb) + (a\overline{p} + b\overline{q})^2\overline{\omega}^2\big\}\,d\sigma_H.
\end{aligned} \tag{15.6}$$

Our final objective is to remove all derivatives from the functions a, b and k from the right-hand side of (15.6). This is somewhat delicate and involves some effort. The final product will be achieved by a repeated use of the basic integration by parts Lemma 14.8 and of the geometric identities in Section 13. We begin by observing that, thanks to (13.28), the H-minimality of $\mathcal{S}$ implies that

$$\overline{q}Za - \overline{p}Zb = Z(\overline{q}a - \overline{p}b).$$

Using this observation, Lemma 14.8, and Corollary 13.10, we obtain

$$\begin{aligned}
&\int_{\mathcal{S}} 2(\overline{q}Za - \overline{p}Zb)(Tk - \overline{\omega}Yk)\,d\sigma_H \\
&\quad = -2\int_{\mathcal{S}} (\overline{q}a - \overline{p}b)Z(T - \overline{\omega}Y)k\,d\sigma_H - 2\int_{\mathcal{S}} \overline{\omega}(\overline{q}a - \overline{p}b)(T - \overline{\omega}Y)k\,d\sigma_H \\
&\quad = -2\int_{\mathcal{S}} (\overline{q}a - \overline{p}b)(T - \overline{\omega}Y)Zk\,d\sigma_H + 2\int_{\mathcal{S}} (\overline{q}a - \overline{p}b)[T - \overline{\omega}Y, Z]k\,d\sigma_H \\
&\qquad - 2\int_{\mathcal{S}} \overline{\omega}(\overline{q}a - \overline{p}b)(Tk - \overline{\omega}Yk)\,d\sigma_H \\
&\quad = 2\int_{\mathcal{S}} Zk(T - \overline{\omega}Y)(\overline{q}a - \overline{p}b)\,d\sigma_H - 2\int_{\mathcal{S}} \overline{\omega}k(T - \overline{\omega}Y)(\overline{q}a - \overline{p}b)\,d\sigma_H \\
&\qquad + 2\int_{\mathcal{S}} \overline{\omega}k(T - \overline{\omega}Y)(\overline{q}a - \overline{p}b)\,d\sigma_H
\end{aligned}$$

$$= 2\int_{\mathcal{S}} \big[\overline{q}(Ta - \overline{\omega} Ya) - \overline{p}(Tb - \overline{\omega} Yb)\big] Zk\, d\sigma_H$$
$$+ 2\int_{\mathcal{S}} aZk(T\overline{q} - \overline{\omega} Y\overline{q})\, d\sigma_H - 2\int_{\mathcal{S}} bZk(T\overline{p} - \overline{\omega} Y\overline{p})\, d\sigma_H. \tag{15.7}$$

Substituting (15.7) in (15.6), we find

$$\begin{aligned}\mathcal{V}^H_{II}(\mathcal{S};\mathcal{X}) = {} & 2\int_{\mathcal{S}} aZk(T\overline{q} - \overline{\omega} Y\overline{q})\, d\sigma_H - 2\int_{\mathcal{S}} bZk(T\overline{p} - \overline{\omega} Y\overline{p})\, d\sigma_H \\ & + \int_{\mathcal{S}} \big\{(Ta - \overline{\omega} Ya)\big[-\overline{q}(a\overline{p} + b\overline{q}) - \overline{p}(a\overline{q} - b\overline{p})\big] \\ & + (Tb - \overline{\omega} Yb)\big[\overline{p}(a\overline{p} + b\overline{q}) - \overline{q}(a\overline{q} - b\overline{p})\big] \\ & + (\overline{p} Za + \overline{q} Zb + \overline{\omega} Zk)^2 + 2\overline{\omega}^2(a\overline{p} + b\overline{q})Zk + 2\overline{\omega}(aZa + bZb) \\ & + (a\overline{p} + b\overline{q})^2\overline{\omega}^2\big\}\, d\sigma_H. \end{aligned} \tag{15.8}$$

It should be clear to the reader that we have made some interesting progress, since we have eliminated terms containing products of derivatives of a, b and k. However, we are still far from our final goal. We next use the H-minimality of $\mathcal{S}$, and (13.28) again, to see that

$$\begin{aligned}(ZF)^2 &= \big((\overline{p} Za + \overline{q} Zb + \overline{\omega} Zk) - \mathcal{A}k\big)^2 \\ &= (\overline{p} Za + \overline{q} Zb + \overline{\omega} Zk)^2 + \mathcal{A}^2k^2 - 2\mathcal{A}k(\overline{p} Za + \overline{q} Zb) - \overline{\omega}\mathcal{A}Z\big(k^2\big),\end{aligned}$$

where F is as in (14.4) and $\mathcal{A}$ is the function defined in Corollary 13.3. This identity, the fact that $\overline{p} Za + \overline{q} Zb = Z(a\overline{p} + b\overline{q})$ (the H-minimality of $\mathcal{S}$), Corollary 13.11 and Lemma 14.8, give

$$\begin{aligned}&\int_{\mathcal{S}} (\overline{p} Za + \overline{q} Zb + \overline{\omega} Zk)^2\, d\sigma_H \\ &\quad = \int_{\mathcal{S}} (ZF)^2\, d\sigma_H + 2\int_{\mathcal{S}} \mathcal{A}kZ(a\overline{p} + b\overline{q})\, d\sigma_H - \int_{\mathcal{S}} \big(\overline{\omega}^2 - 2\mathcal{A}\big)\overline{\omega}^2k^2\, d\sigma_H \\ &\quad = \int_{\mathcal{S}} (ZF)^2\, d\sigma_H + \int_{\mathcal{S}} \big(2\mathcal{A} - \overline{\omega}^2\big)\big(\overline{\omega}^2k^2 + 2\overline{\omega}k(a\overline{p} + b\overline{q})\big)\, d\sigma_H \\ &\qquad - 2\int_{\mathcal{S}} \mathcal{A}(a\overline{p} + b\overline{q})Zk\, d\sigma_H.\end{aligned} \tag{15.9}$$

Substitution of (15.9) into (15.8) gives

$$\begin{aligned}\mathcal{V}^H_{II}(\mathcal{S};\mathcal{X}) = {} & \int_{\mathcal{S}} (ZF)^2\, d\sigma_H + \int_{\mathcal{S}} \big(2\mathcal{A} - \overline{\omega}^2\big)\big(\overline{\omega}^2k^2 + 2\overline{\omega}k(a\overline{p} + b\overline{q})\big)\, d\sigma_H \\ & + 2\int_{\mathcal{S}} aZk\big[(T\overline{q} - \overline{\omega} Y\overline{q}) + \overline{p}\overline{\omega}^2 - \overline{p}\mathcal{A}\big]\, d\sigma_H\end{aligned}$$

$$
\begin{aligned}
&+2\int_{\mathcal{S}} bZk\big[-(T\overline{p}-\overline{\omega}Y\overline{p})+\overline{q}\overline{\omega}^2-\overline{q}\mathcal{A}\big]\,d\sigma_H \\
&+\int_{\mathcal{S}}\big\{(Ta-\overline{\omega}Ya)\big[-\overline{q}(a\overline{p}+b\overline{q})-\overline{p}(a\overline{q}-b\overline{p})\big] \\
&+(Tb-\overline{\omega}Yb)\big[\overline{p}(a\overline{p}+b\overline{q})-\overline{q}(a\overline{q}-b\overline{p})\big]\big\}\,d\sigma_H \\
&+\int_{\mathcal{S}}\overline{\omega}Z\big(a^2+b^2\big)\,d\sigma_H+\int_{\mathcal{S}}(a\overline{p}+b\overline{q})^2\overline{\omega}^2\,d\sigma_H. \qquad (15.10)
\end{aligned}
$$

We now claim

$$
(T\overline{q}-\overline{\omega}Y\overline{q})+\overline{p}\overline{\omega}^2-\overline{p}\mathcal{A}=-(T\overline{p}-\overline{\omega}Y\overline{p})+\overline{q}\overline{\omega}^2-\overline{q}\mathcal{A}=0.
$$

We only check the first of the two equations, leaving it to the reader to provide the details of the second one. Corollary 13.3 and the identity $\overline{p}^2+\overline{q}^2=1$ give

$$
\begin{aligned}
\overline{p}\mathcal{A}&=\overline{p}^2(T\overline{q}-\overline{\omega}Y\overline{q})-\overline{p}\overline{q}(T\overline{p}-\overline{\omega}Y\overline{p})+\overline{p}\overline{\omega}^2 \\
&=(T\overline{q}-\overline{\omega}Y\overline{q})-\overline{q}^2(T\overline{q}-\overline{\omega}Y\overline{q})-\overline{p}\overline{q}(T\overline{p}-\overline{\omega}Y\overline{p})+\overline{p}\overline{\omega}^2 \\
&=(T\overline{q}-\overline{\omega}Y\overline{q})+\overline{p}\overline{\omega}^2-\overline{q}\big[\overline{q}(T\overline{q}-\overline{\omega}Y\overline{q})+\overline{p}(T\overline{p}-\overline{\omega}Y\overline{p})\big] \\
&=(T\overline{q}-\overline{\omega}Y\overline{q})+\overline{p}\overline{\omega}^2,
\end{aligned}
$$

which proves the first identity. Using the claim in (15.10), with a further application of Lemma 14.8, we obtain

$$
\begin{aligned}
\mathcal{V}_{II}^H(\mathcal{S};\mathcal{X})=&\int_{\mathcal{S}}(ZF)^2\,d\sigma_H+\int_{\mathcal{S}}\big(2\mathcal{A}-\overline{\omega}^2\big)\big(\overline{\omega}^2k^2+2\overline{\omega}k(a\overline{p}+b\overline{q})\big)\,d\sigma_H \\
&+\int_{\mathcal{S}}\big\{(Ta-\overline{\omega}Ya)\big[-\overline{q}(a\overline{p}+b\overline{q})-\overline{p}(a\overline{q}-b\overline{p})\big] \\
&+(Tb-\overline{\omega}Yb)\big[\overline{p}(a\overline{p}+b\overline{q})-\overline{q}(a\overline{q}-b\overline{p})\big]\big\}\,d\sigma_H \\
&+\int_{\mathcal{S}}\big(\mathcal{A}-\overline{\omega}^2\big)\big(a^2+b^2\big)\,d\sigma_H+\int_{\mathcal{S}}(a\overline{p}+b\overline{q})^2\overline{\omega}^2\,d\sigma_H. \qquad (15.11)
\end{aligned}
$$

Completing the square in the second integral in the right-hand side of (15.11), we find

$$
\begin{aligned}
\mathcal{V}_{II}^H(\mathcal{S};\mathcal{X})=&\int_{\mathcal{S}}(ZF)^2\,d\sigma_H+\int_{\mathcal{S}}\big(2\mathcal{A}-\overline{\omega}^2\big)F^2\,d\sigma_H+\int_{\mathcal{S}}\big\{(Ta-\overline{\omega}Ya)\big[-2a\overline{p}\overline{q}+b\big(\overline{p}^2-\overline{q}^2\big)\big] \\
&+(Tb-\overline{\omega}Yb)\big[2b\overline{p}\overline{q}+a(\overline{p}^2-\overline{q}^2)\big]\big\}\,d\sigma_H \\
&+\int_{\mathcal{S}}\big(\mathcal{A}-\overline{\omega}^2\big)\big(a^2+b^2\big)\,d\sigma_H-2\int_{\mathcal{S}}\big(\mathcal{A}-\overline{\omega}^2\big)(a\overline{p}+b\overline{q})^2\,d\sigma_H. \qquad (15.12)
\end{aligned}
$$

The proof of the theorem will be completed if we can establish the following crucial claim:

$$\int_{\mathcal{S}} \big\{(Ta-\overline{\omega}Ya)\big[-2a\overline{p}\overline{q}+b\big(\overline{p}^2-\overline{q}^2\big)\big]+(Tb-\overline{\omega}Yb)\big[2b\overline{p}\overline{q}+a\big(\overline{p}^2-\overline{q}^2\big)\big]\big\}\,d\sigma_H$$
$$+\int_{\mathcal{S}}\big(\mathcal{A}-\overline{\omega}^2\big)\big(a^2+b^2\big)\,d\sigma_H-2\int_{\mathcal{S}}\big(\mathcal{A}-\overline{\omega}^2\big)(a\overline{p}+b\overline{q})^2\,d\sigma_H=0. \tag{15.13}$$

To prove (15.13) we proceed as follows. First, Lemma 14.8 gives

$$-2\int_{\mathcal{S}} a\overline{p}\overline{q}(Ta-\overline{\omega}Ya)\,d\sigma_H=\int_{\mathcal{S}} a^2(T-\overline{\omega}Y)(\overline{p}\overline{q})\,d\sigma_H.$$

Therefore, the coefficient of a^2 in the left-hand side of (15.13) is given by

$$(T-\overline{\omega}Y)(\overline{p}\overline{q})+\big(\mathcal{A}-\overline{\omega}^2\big)-2\big(\mathcal{A}-\overline{\omega}^2\big)\overline{p}^2=(T-\overline{\omega}Y)(\overline{p}\overline{q})+\big(\mathcal{A}-\overline{\omega}^2\big)\big(\overline{q}^2-\overline{p}^2\big)=0,$$

where we have used the identity $\overline{p}^2+\overline{q}^2=1$. Similarly, we have

$$2\int_{\mathcal{S}} b\overline{p}\overline{q}(Tb-\overline{\omega}Yb)\,d\sigma_H=-\int_{\mathcal{S}} b^2(T-\overline{\omega}Y)(\overline{p}\overline{q})\,d\sigma_H,$$

hence the coefficient of b^2 in the left-hand side of (15.13) is given by

$$-(T-\overline{\omega}Y)(\overline{p}\overline{q})+\big(\mathcal{A}-\overline{\omega}^2\big)-2\big(\mathcal{A}-\overline{\omega}^2\big)\overline{q}^2=-\big[(T-\overline{\omega}Y)(\overline{p}\overline{q})+\big(\mathcal{A}-\overline{\omega}^2\big)\big(\overline{q}^2-\overline{p}^2\big)\big]=0.$$

Finally, we have

$$\int_{\mathcal{S}}\big\{b\big(\overline{p}^2-\overline{q}^2\big)(Ta-\overline{\omega}Ya)+a\big(\overline{p}^2-\overline{q}^2\big)(Tb-\overline{\omega}Yb)\big\}\,d\sigma_H$$
$$=-\int_{\mathcal{S}} ab(T-\overline{\omega}Y)\big(\overline{p}^2-\overline{q}^2\big)\,d\sigma_H.$$

We thus see that the coefficient of ab in the left-hand side of (15.13) is given by

$$4\big(\overline{\omega}^2-\mathcal{A}\big)\overline{p}\overline{q}+(T-\overline{\omega}Y)\big(\overline{q}^2-\overline{p}^2\big)=0.$$

From these considerations, the claim (15.13) follows. We have thus completed the proof. □

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