CAPACITARY ESTIMATES AND THE LOCAL BEHAVIOR OF SOLUTIONS OF NONLINEAR SUBELLIPTIC EQUATIONS

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Dedicated to James Serrin

Abstract. We establish sharp capacitary estimates for Carnot-Carathéodory rings associated to a system of vector fields of Hörmander type. Such estimates are instrumental to the study of the local behavior of singular solutions of a wide class of nonlinear subelliptic equations. One of the main results is a generalization of fundamental estimates obtained independently by Sanchez-Calle and Nagel, Stein and Wainger.

1. Introduction. Since Hörmander's famous hypoellipticity paper [H] the study of second order pde's arising from noncommuting vector fields has undergone considerable progress. In this context an important class of models is that of stratified, nilpotent Lie groups with their associated sub-Laplacians. The homogeneous structure of such groups allows the development of a harmonic analysis which, in turn, plays a central role in the regularity theory of general Hörmander type operators. This program was developed in the works [FS], [F1], [F2], [RS], following a circle of ideas outlined by E. Stein [St1] in his address at the Nice International Congress in 1970.

Among the stratified Lie groups the Heisenberg group $\mathbb{H}^n = (\mathbb{C}^n \times \mathbb{R}, \cdot)$ plays a prominent role. For the latter, Folland [F1] discovered a remarkably simple fundamental solution of the real part of the Kohn-Laplacian given by

$$\Gamma(u,v) = C\rho(u^{-1}\cdot v)^{2-Q}. \tag{1.1}$$

Here, for $u = (z,t) \in \mathbb{H}^n$ we have denoted by $\rho(u) = (|z|^4 + t^2)^{\frac{1}{4}}$ the norm function on $\mathbb{H}^n$, whereas $Q = 2n+2$ is the homogeneous dimension associated to the anisotropic dilations $\delta_\lambda(z,t) = (\lambda z, \lambda^2 t)$. If we introduce the distance $d(u,v) = \rho(u^{-1}\cdot v)$, and corresponding balls $B_d(u,r) = \{v \in \mathbb{H}^n \mid d(u,v) < r\}$, then (1.1) can be recast in the form

$$\Gamma(u,v) = c\frac{d(u,v)^2}{|B_d(u,d(u,v))|}. \tag{1.2}$$

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A deep result of Nagel, Stein, and Wainger [NSW] and of Sanchez-Calle [SC] states that the behavior near the singularity of the fundamental solution of any Hörmander operator is quantitatively given by the right-hand side of (1.2). Precisely, consider a family $X_1, \dots, X_m$ of C^∞ vector fields in $\mathbb{R}^n$ (or in a connected n-dimensional manifold), $n \geq 3$, satisfying Hörmander's finite rank condition [H]

$$\operatorname{rank} \operatorname{Lie}[X_1, \dots, X_m] \equiv n.$$

Let $\Gamma(x, y)$ denote the fundamental solution of

$$\mathcal{L}u = \sum_{j=1}^{m} X_j^* X_j u = 0, \tag{1.3}$$

where X_j^* is the formal adjoint of X_j. Then, it was proved in [NSW], [SC]: For every $U \subset\subset \mathbb{R}^n$ there exist C, $R_0 > 0$ such that for $x \in U$ and $0 < d(x, y) \leq R_0$ one has

$$C \frac{d(x,y)^2}{|B_d(x, d(x,y))|} \leq \Gamma(x,y) \leq C^{-1} \frac{d(x,y)^2}{|B_d(x, d(x,y))|}. \tag{1.4}$$

In (1.4) $d(x, y)$ denotes the Carnot-Carathéodory distance associated to X_1, $\dots, X_m$, and $B_d(x, r) = \{y \in \mathbb{R}^n \mid d(x, y) < r\}$ the corresponding ball.

It is interesting to compare (1.4) with a famous result due to Littman, Stampacchia and Weinberger [LSW]. These authors proved that the fundamental solution $G(x, y)$ of a uniformly elliptic operator (with bounded measurable coefficients) satisfies the estimate

$$G(x,y) \approx \operatorname{cap}(B(x,r), B(x,R))^{-1},$$

where cap denotes the Newtonian capacity, $R > 0$ is fixed and $r = |x - y|$ is small compared to R. Since

$$\operatorname{cap}(B(x,r), B(x,R)) \approx r^{n-2},$$

one infers that

$$G(x,y) \approx |x-y|^{2-n}. \tag{1.5}$$

The estimate (1.4) constitutes the subelliptic analogue of (1.5). The above also underlines the crucial role of capacitary estimates of a spherical ring in determining the local behavior of singular solutions.

In this paper we study the pointwise behavior of singular solutions of a general class of nonlinear subelliptic equations. The prototypes of such equations naturally arise in questions concerning the geometry of CR manifolds or in the theory of quasi-regular mappings on stratified Lie groups. Consider a family $X_1, \dots, X_m$ of C^∞ vector fields satisfying the above mentioned finite rank condition. Denoting by $Xu = (X_1 u, \dots, X_m u)$ the gradient of a function u, we introduce the Sobolev functional

$$J_p(u) = \int |Xu|^p \, dx = \int \left[\sum_{j=1}^{m} (X_j u)^2 \right]^{\frac{p}{2}} dx, \quad 1 < p < \infty.$$

The Euler-Lagrange equation associated to J_p is given by

$$\sum_{j=1}^{m} X_j^* (|Xu|^{p-2} X_j u) = 0, \tag{1.6}$$

which reduces to (1.3) when $p = 2$. Equation (1.6) is the subelliptic analogue of the so-called p-Laplacian. The latter is obtained when $m = n$ and $X_j = \frac{\partial}{\partial x_j}$.

The case in which $X_1, \dots, X_m$ are left-invariant vector fields on a stratified Lie group and $p = Q$, the homogeneous dimension of the group, has a special geometrical meaning in connection with the theory of quasi-conformal or quasi-regular mappings.

Originated with Mostow's celebrated work on rigidity [M], the theory of quasi-conformal mappings on nilpotent Lie groups has been recently developed by several authors. We recall, in particular, the papers [KR1], [KR2], [KR3], [P1], [P2], [HR], [HH]. From the point of view of applications it is important to study equations more general than (1.6). There are several reasons for this. First, in the study of topological properties of quasi-regular mappings the theory of general structure quasi-linear equations plays a pervasive and fundamental role. This is well-known for the Euclidean setting (see, e.g. the classical monograph by Reshetnyak [R], and the more recent one by Heinonen, Kilpelainen and Martio [HKM]), and still holds true for stratified nilpotent Lie groups (see, for instance, the papers [HH] and [HR]). Secondly, as it was pointed out by Pansu in [P1], the Euclidean proofs of the regularity of 1-quasiconformal mappings (the so-called Liouville theorem) of Gehring [G] and Reshetnyak [R], "... rely on nonlinear elliptic regularity theory. In the nilpotent case, the corresponding equations are hypoellipitc and the necessary regularity is not yet available." We mention that the relevant equation here is (1.6), with $p = Q$ and for solutions u such that $M^{-1} \geq |Xu| \geq M > 0$. It should be emphasized that the regularity of the (horizontal) gradient constitutes, in this framework, an extremely hard problem. For nilpotent stratified Lie groups of step two the optimal smoothness has been recently achieved by one of us in [Ca]. Our paper represents some needed progress in the direction of a general regularity theory. The recent works by Mostow and Margulis [MM], and by Gromov [Gr], constitute further justification for developing the theory in the more general setting of Carnot-Carathéodory spaces, rather than only in stratified nilpotent Lie groups.

In this paper we consider equations of the type

$$\sum_{j=1}^{m} X_j^* A_j(x,u,Xu) = f(x,u,Xu). \tag{1.7}$$

Here, the functions $A = (A_1, \dots, A_m)\colon \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ and $f\colon \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ are measurable and satisfy the following structural conditions. There exist $1 < p < \infty$, $c_1 \geq 0$ and measurable functions f_1, f_2, f_3, g_2, g_3, and h_3 such that for a.e. $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $\zeta \in \mathbb{R}^m$:

$$\begin{cases} |A(x,u,\zeta)| \leq c_1 |\zeta|^{p-1} + g_2|u|^{p-1} + g_3, \\ |f(x,u,\zeta)| \leq f_1|\zeta|^{p-1} + f_2|u|^{p-1} + f_3, \\ A(x,u,\zeta)\cdot\zeta \geq |\zeta|^p - f_2|u|^p - h_3. \end{cases} \tag{1.8}$$

The relevant integrability assumptions on the functions f_i, $i = 1,2,3$, g_2, g_3 and h_3 will be introduced in section 3. It should be noted that the choice $A_j(x,u,\zeta) = |\zeta|^{p-2}\zeta_j$ makes (1.6) a particular case of (1.7).

One of the main results in this paper is a precise quantitative description of the local behavior of singular solutions of (1.7), see Theorem 7.1 below. Roughly speaking, the latter states that if u is a solution of (1.7) in a punctured ball $B_d(x_0,R) \setminus \{x_0\}$, then either x_0 is a removable singularity, or for some $C > 0$

$$\begin{aligned} &C \operatorname{cap}_p (B_d(x_0, d(x,x_0)); B_d(x_0,R))^{\frac{1}{1-p}} \leq u(x) \\ &\qquad \leq C^{-1} \operatorname{cap}_p \cdot (B_d(x_0, d(x,x_0)); B_d(x_0,R))^{\frac{1}{1-p}}. \end{aligned} \tag{1.9}$$

In the above inequality we have denoted by $\operatorname{cap}_p(E;\Omega)$ the subelliptic p-capacity of the condenser $(E;\Omega)$ defined via the functional J_p.

As a corollary of (1.9) and of the optimal capacity estimates of a metric ring, established in Theorems 4.1 and 4.2, we obtain the following generalization of results in [NSW], [SC]:

$$C\left(\frac{d(x,y)^p}{|B_d(x,d(x,y))|}\right)^{\frac{1}{p-1}} \leq \Gamma_p(x,y) \leq C^{-1}\left(\frac{d(x,y)^p}{|B_d(x,d(x,y))|}\right)^{\frac{1}{p-1}} \tag{1.10}$$

where $\Gamma_p(x,y)$ denotes the fundamental solution of (1.7).

Our approach is based on that of Serrin's ingenious papers [S1] and [S2], where the elliptic case $m = n$ and $X_j = \frac{\partial}{\partial x_j}$, $j = 1, \dots, n$ was extensively studied.

Although the present work much owes to Serrin's original ideas, it should be emphasized that the subelliptic geometry introduces new substantial difficulties in the problem. Only recently we have been able to establish two basic results:

An optimal embedding for the Sobolev spaces associated to the functional J_p; the Harnack inequality for positive solutions of (1.7), see [CDG1], [CDG3]. Another important aspect of the analysis in [S1], [S2] is the existence of sharp capacitary estimates of Euclidean rings. In the subelliptic setting these estimates are proved in Theorems 4.1 and 4.2. We would like to point out that one of the starting motivations for us was provided by Pansu's observation in [P1] that "... a big part of the analytic theory of quasiconformal mappings in the Euclidean space can be carried out on Carnot groups. However, it seems to be harder to obtain capacity estimates." Theorems 4.1 and 4.2 generalize results of Pansu [P2], Korányi and Reimann [KR1] and Heinonen and Holopainen [HH]. We remark that (1.9) in the conformally invariant case $p = Q$ for equations of a simpler form on stratified Lie groups has also been established in [HH].

We were led to conjecture that (1.10) or, more generally, (1.9) should hold by the discovery of a remarkable explicit fundamental solution of (1.6) for groups of Heisenberg type. The construction of such a fundamental solution is presented in Section 2. Once known, the latter is used to compute, with explicit geometric constants, the p-capacity of a spherical ring (see Theorem 2.2). Even for the Heisenberg group $\mathbb{H}^n$ the existence of these fundamental solutions is a nontrivial fact. Groups of Heisenberg type were introduced by Kaplan [K] as direct generalizations of $\mathbb{H}^n$. There are infinitely many isomorphism classes of such groups and they include the nilpotent component in the Iwasawa decomposition of simple groups of rank one. The relatively simple structure of Heisenberg type groups provides an ideal framework for testing conjectures and for constructing interesting examples in analysis and geometry. We hope that the results in Section 2 will motivate the reader to undertake the more technical part of the paper.

Before closing we would like to briefly discuss an interesting aspect of the capacitary estimates in Section 4. A fundamental result in [NSW] describes the volume of the Carnot-Caratheodory balls introduced above: For any $U \subset\subset \mathbb{R}^n$ there exist $C, R_0 > 0$ such that if $x \in U$ and $0 < r \leq R_0$,

$$C\Lambda(x,r) \leq |B(x,r)| \leq C^{-1}\Lambda(x,r).$$

Here, $\Lambda(x,r)$ is a polynomial in r whose coefficients are positive (continuous) functions of x. Furthermore, the following estimate holds:

$$Cr^Q \leq \Lambda(x,r) \leq C^{-1}r^{Q(x)},$$

for all $x \in U$ and $0 < r \leq R_0$, where

$$Q \geq Q(x) \geq n.$$

The number Q is called the local homogeneous dimension, relative to $U \subset\subset \mathbb{R}^n$, of the family $X_1, \ldots, X_m$. The justification for such a name comes from the role played by Q in the Sobolev embedding Theorem 3.1, or in the isoperimetric inequality Theorem 3.2.

In view of the close connection between capacitary estimates and Sobolev embeddings it came as a surprise to us that in the former it is not the local homogeneous dimension Q, but rather the number $Q(x)$ which constitutes the threshold. By this we mean that the subelliptic p-capacity of a metric ring centered at x changes drastically depending on whether $1 < p < Q(x)$, $p = Q(x)$ or $p > Q(x)$. This unsettling behavior is not observed in the analysis of stratified Lie groups, since one has in that case $Q(x) \equiv Q$, the homogeneous dimension of the group associated to the dilations.

Our work can be seen as a generalization to the subelliptic context of the quasi-linear theory developed by Serrin, Ladyzenskaja, Uraltseva and many others. In this respect, this paper is a sequel to the works [CDG1], [CDG3], [CDG2], [D1], and [D2]. We mention that a general theory of Sobolev and isoperimetric inequalities for systems of (nonsmooth) vector fields has been recently developed in [GN]. Combining the ideas in the present paper with the results in [GN] it is now possible to study the local behavior of solutions to quasilinear subelliptic equations in the general context of Carnot-Carathéodory spaces as in [GN] or [Gr].

Finally, we would like to acknowledge our indebtedness to the results in [NSW].

2. Fundamental solutions in groups of Heisenberg type. This section serves to motivate the subsequent developments. We construct explicit fundamental solutions for the model equation (1.6) in the context of Kaplan, or H-type groups. Aside from its aesthetical appeal the existence of such singular solutions is a remarkable fact which has several implications. On the one hand it demonstrates the sharpness of the results in this paper (see the remarks at the end of this section). On the other hand, in the conformally invariant case $p = Q$ such exact solutions play an important role in the theory of quasiregular mappings (see [KR3], [HH]).

Groups of H-type arise as a generalization of the Heisenberg group $\mathbb{H}^n$. Since their introduction by Kaplan [K] in 1980 they have provided an interesting framework in which to construct examples both in analysis and geometry. A group of Heisenberg type is best described in terms of its Lie algebra g. We assume that we are given a positive definite inner product $\langle\ ,\ \rangle$ on g and an orthogonal decomposition $g = \eta \oplus \xi$ (stratification of Step 2). Here, ξ is the center of g and for every unit $z \in \xi$ the mapping $J_z\colon\ \eta \to \eta$, defined by

$$\langle J_z(v), v' \rangle = \langle z, [v, v'] \rangle \tag{2.1}$$

is orthogonal. We observe explicitly that

$$\langle J_z(v), v\rangle = 0, \qquad \langle J_z(v), J_z(v)\rangle = |z|^2|v|^2 \tag{2.2}$$

for every $v \in \eta$, $z \in \xi$.

There exist infinitely many isomorphism classes of groups of H-type which include, in particular, the nilpotent component in the Iwasawa decomposition of simple groups of rank one. For any integer n there is a group of H-type having center of dimension n, see [K]. When $n = 1$ one obtains the Heisenberg groups.

Let $m = \dim \eta$, $k = \dim \xi$. The homogeneous dimension of G is $Q = m + 2k$. Let now $\{X_j\}_{j=1,\ldots,m}$ be an orthogonal basis for η and denote in the same way the corresponding left-invariant vector fields on G. Consider a Riemannian metric on G defined so that $\langle X_i, X_j\rangle = \delta_{ij}$. Since the exponential map of a group of Heisenberg type is an analytic diffeomorphism, we can define analytic mappings $v\colon G \to \eta$ and $z\colon G \to \xi$ by $x = \exp[v(x) + z(x)]$, for every $x \in G$. Denote by $\{\delta_\lambda\}_{\lambda>0}$ the group of dilations on G defined by $\delta_\lambda(x) = \exp[\lambda v(x) + \lambda^2 z(x)]$. We observe that, since for each $j = 1, \ldots, m$ the translations generated by X_j are isometries, then $X_j^* = -X_j$. This means that for every $\phi, \psi \in C_0^\infty(G)$ we have

$$\int_G \phi X_j \psi \, d\mu = -\int_G \psi X_j \phi \, d\mu.$$

Hereafter, $d\mu$ will denote a bi-invariant Haar measure on G obtained by lifting the Lebesgue measure on g via the exponential mapping. Following [K] we introduce the norm function

$$\rho(x) = \left[|v(x)|^4 + 16|z(x)|^2\right]^{\frac{1}{4}}, \qquad x \in G.$$

Let $e \in G$ denote the identity. The following lemma provides a remarkable description of the "radial" solutions of $\mathcal{L}_p u = \sum_{j=1}^m X_j^*(|Xu|^{p-2}X_j u) = 0$ in $G \setminus \{e\}$.

LEMMA 2.1. *Let $p > 1$, $f \in C^2(\mathbb{R}^+)$ and set $u(x) = f(\rho(x))$. Then, one has*

$$\mathcal{L}_p u = (p-1)|X\rho|^p |f'(\rho)|^{p-2}\left[f''(\rho) + \frac{Q-1}{p-1}\frac{f'(\rho)}{\rho}\right]$$

at every point $x \in G \setminus \{e\}$ where $f'(\rho(x)) \neq 0$.

Proof. For the reader's convenience we develop in the sequel the computations in [K]. Given $j = 1, \ldots, m$ we set for $t > 0$

$$\rho_j(t) = \left[|v(x \exp(tX_j))|^4 + 16|z(x \exp(tX_j))|^2\right]^{\frac{1}{4}}$$

and note that $\rho_j(0) = \rho(x)$ for $j = 1, \dots, m$. The Baker-Campbell-Hausdorff formula [V] gives at once

$$v(x \exp(tX_j)) = v(x) + tX_j, \qquad z(x \exp(tX_j)) = z(x) + \frac{t}{2}[v, X_j].$$

As a consequence we have

$$X_j(|v(x)|^2) = \frac{d}{dt}|v(x \exp(tX_j))|^2 \,|_{t=0} = 2\langle v(x), X_j\rangle, \tag{2.3}$$

$$X_j(|z(x)|^2) = \frac{d}{dt}|z(x \exp(tX_j))|^2 \,|_{t=0} = \langle z(x), [v(x), X_j]\rangle = \langle J_{z(x)}(v(x)), X_j\rangle, \tag{2.4}$$

where in the last equality in (2.4) we have used (2.1). From now on, in order to simplify the notation we will write v and z instead of $v(x)$, $z(x)$. Using the definition of $\rho(x)$ and (2.3), (2.4) we obtain

$$X_j\rho = \frac{1}{4}\rho^{-3}X_j(\rho^4) = \rho^{-3}[|v|^2\langle v, X_j\rangle + 4\langle J_z(v), X_j\rangle]. \tag{2.5}$$

This gives

$$\begin{aligned} |X\rho|^2 &= \sum_{j=1}^m (X_j\rho)^2 \\ &= \rho^{-6}\sum_{j=1}^m \left\{ |v|^4\langle v, X_j\rangle^2 + 16\langle J_z(v), X_j\rangle^2 + 8|v|^2\langle v, X_j\rangle \, \langle J_z(v), X_j\rangle \right\}. \end{aligned} \tag{2.6}$$

Since $\{X_j\}_{j=1,\dots,m}$ is an orthonormal basis, by (2.2) we infer

$$\sum_{j=1}^m \langle v, X_j\rangle \, \langle J_z(v), X_j\rangle = \langle J_z(v), v\rangle = 0.$$

Using (2.2) once more we conclude

$$|X\rho|^2 = \rho^{-6}[|v|^6 + 16|J_z(v)|^2] = \frac{|v|^2}{\rho^2}. \tag{2.7}$$

Observe that (2.7) implies that $|X\rho|$ is δ_λ-homogeneous of degree zero. This is not surprising since ρ is δ_λ-homogeneous of degree one.

We next choose an orthonormal basis $\{z_i\}_{i=1,\ldots,k}$ of ξ and observe that by (2.1), (2.2)

$$\sum_{j=1}^m |[v, X_j]|^2 = \sum_{i=1}^k \sum_{j=1}^m \langle z_i, [v, X_j]\rangle^2 = \sum_{i=1}^k \sum_{j=1}^m \langle J_{z_i}(v), X_j\rangle^2 = \sum_{i=1}^k |J_{z_i}(v)|^2$$
$$= \sum_{i=1}^k |z_i|^2|v|^2 = k|v|^2. \tag{2.8}$$

Using (2.8) we obtain

$$\sum_{j=1}^m X_j^2(\rho^4) = \sum_{j=1}^m \frac{d^2\rho_j}{dt^2}\bigg|_{t=0} = 4\sum_{j=1}^m \left[|v|^2 + 2\langle v, X_j\rangle^2 + 2[v, X_j]^2\right]$$
$$= 4(m + 2k + 2)|v|^2 = 4(Q+2)|v|^2. \tag{2.9}$$

We deduce from (2.7), (2.9)

$$\sum_{j=1}^m X_j^2\rho = \frac{1}{4}\rho^{-3}\left[\sum_{j=1}^m X_j^2(\rho^4) - 12\rho^2|X\rho|^2\right]$$
$$= \rho^{-3}[(Q+2)|v|^2 - 3|v|^2] = \frac{Q-1}{\rho}|X\rho|^2. \tag{2.10}$$

After these preliminaries we return to the proof of Lemma 2.1. To make the following arguments rigorous we should replace the function ρ with its regularization $\rho_\varepsilon = \left((|v|^2 + \varepsilon^2)^2 + 16|z|^2\right)^{\frac{1}{4}}$ and let $\varepsilon \to 0$ in the end. For the sake of simplicity we will, however, proceed formally. Recalling that $u = f \circ \rho$ we have

$$\mathcal{L}_p u = f'|f'|^{p-2}|X\rho|^{p-2}\sum_{j=1}^m X_j^2\rho + |X\rho|^{p-2}\sum_{j=1}^m X_j\rho X_j(f'|f'|^{p-2})$$
$$+ f'|f'|^{p-2}\sum_{j=1}^m X_j\rho X_j(|X\rho|^{p-2}) = (I) + (II) + (III). \tag{2.11}$$

By (2.7) and (2.10) we obtain easily

$$(I) + (II) = (p-1)|f'|^{p-2}|X\rho|^p\left[f'' + \frac{Q-1}{p-1}\frac{f'}{\rho}\right]. \tag{2.12}$$

We now turn our attention to (III) and prove that the latter vanishes. Here, even more than before, the remarkable structure of H-type groups plays a crucial role. We have by (2.5), (2.7)

$$\begin{aligned}
&\sum_{j=1}^{m} X_j\rho X_j(|X\rho|^{p-2}) \\
&\qquad = \frac{p-2}{2}|X\rho|^{p-4}\Big[-2\rho^{-3}|v|^2|X\rho|^2 + \rho^{-2}\sum_{j=1}^{m} X_j\rho X_j(|v|^2)\Big] \\
&\qquad\quad = \frac{p-2}{2}|X\rho|^{p-4}\Big\{-2\rho^{-1}|X\rho|^4 \\
&\qquad\qquad\qquad + 2\rho^{-5}\sum_{j=1}^{m}\langle v, X_j\rangle[|v|^2\langle v, X_j\rangle + 4\langle J_z(v), X_j\rangle]\Big\}.
\end{aligned}$$

From (2.2), (2.7) we conclude $(III) = 0$. In virtue of (2.11), and (2.12) the proof of the lemma is complete. □

Lemma 2.1 allows us to compute explicit fundamental solutions of (1.6) on H-type groups. We need to introduce some notation.

For $r > 0$ let $B_r = \{x \in G \mid \rho(x) < r\}$ and define for $1 < p < \infty$

$$|B_r|_p = \int_{B_r} |X\rho|^p \, d\mu.$$

We denote ω_p the number $|B_1|_p$. Since $d\mu \circ \delta_\lambda = \lambda^Q d\mu$, a rescaling and (2.7) give

$$|B_r|_p = \omega_p r^Q. \tag{2.13}$$

Next, we define

$$|\partial B_r|_p = \frac{d}{dr}|B_r|_p = Q\omega_p r^{Q-1}.$$

On the other hand, Federer's co-area formula [C] yields

$$|B_r|_p = \int_0^r \int_{\partial B_\tau} \frac{|X\rho|^p}{|\operatorname{grad}\rho|} \, dH_{N-1}\, d\tau$$

where $\operatorname{grad}\rho$ denotes the Riemannian gradient of ρ, $N = m + k$, and dH_{N-1} is the $(N-1)$-dimensional Riemannian density on ∂B_r. From the latter equality we obtain

$$|\partial B_r|_p = \int_{\partial B_r} \frac{|X\rho|^p}{|\operatorname{grad}\rho|} \, dH_{N-1} = Q\omega_p r^{Q-1}.$$

We define $\sigma_p = Q\omega_p$ and introduce the constants

$$C_p = \begin{cases} \dfrac{p-1}{p-Q}\sigma_p^{-1/(p-1)} & \text{when } p \neq Q, \\ \sigma_p^{-1/(p-1)} & \text{when } p = Q. \end{cases} \tag{2.14}$$

THEOREM 2.1. *Let $1 < p < \infty$ and C_p be as in (2.14). The function*

$$\Gamma_p(x) = \begin{cases} C_p \rho^{(p-Q)/(p-1)} & \textit{when } p \neq Q, \\ C_p \log \rho & \textit{when } p = Q, \end{cases}$$

is a fundamental solution of (1.6) with singularity at the identity element $e \in G$. A fundamental solution with singularity at any other point of G is obtained by Γ_p by left-translation.

Proof. From Lemma 2.1 it is clear that Γ_p is an analytic solution of (1.6) in $G \setminus \{e\}$. To prove the theorem we only need to show that for every $\phi \in C_0^\infty(G)$

$$\int_G |X\Gamma_p|^{p-2}\langle X\Gamma_p, X\phi\rangle \, d\mu = -\phi(e).$$

Using the polar coordinates in [FS] it is immediately recognized that $\Gamma_p \in L^1_{\mathrm{loc}}(G)$ only if $p > \frac{2Q}{Q+1}$.

Nonetheless, for $x \neq e$ we have $|X\Gamma_p(x)| \approx \rho^{(1-Q)/(p-1)}(x)$, and therefore

$$|X\Gamma_p|^{p-1} \in L^1_{\mathrm{loc}}(G).$$

Consider now $\phi \in C_0^\infty(G)$ and choose $R > 0$ such that $\operatorname{supp}\phi \subset B_R$. For $0 < \varepsilon < R$ we have

$$\begin{aligned} &\int_{B_R \setminus B_\varepsilon} |X\Gamma_p|^{p-2}\langle X\Gamma_p, X\phi\rangle \, d\mu \\ &\qquad = -\sum_{j=1}^m \int_{B_R \setminus B_\varepsilon} \operatorname{div}\left(|X\Gamma_p|^{p-2}(X_j\Gamma_p)X_j\right)\phi \, d\mu \\ &\qquad\quad + \sum_{j=1}^m \int_{\partial(B_R \setminus B_\varepsilon)} \phi |X\Gamma_p|^{p-2}(X_j\Gamma_p)\langle X_j, \eta\rangle \, dH_{N-1}, \end{aligned} \tag{2.15}$$

where we have denoted by $\eta(= \pm \operatorname{grad}\rho/|\operatorname{grad}\rho|^{-1})$ the unit normal.

Since $X_j^* = -X_j$ and $X_j(|X\Gamma_p|^{p-2}X_j\Gamma_p) = 0$ in $B_R \setminus B_\varepsilon$ the first integral in the right-hand side of (2.15) vanishes. On the other hand

$$
\begin{aligned}
-\sum_{j=1}^m \int_{\partial(B_R\setminus B_\varepsilon)} &\phi|X\Gamma_p|^{p-2}(X_j\Gamma_p)\langle X_j,\eta\rangle\, dH_{N-1} \\
&= \sum_{j=1}^m \int_{\partial B_\varepsilon} \phi|X\Gamma_p|^{p-2}(X_j\Gamma_p)\frac{X_j\rho}{|\operatorname{grad}\rho|}\, dH_{N-1} \\
&= C_p|C_p|^{p-2}\frac{p-Q}{p-1}\left(\frac{|p-Q|}{p-1}\right)^{p-2}\frac{1}{\varepsilon^{Q-1}}\int_{\partial B_\varepsilon}\phi\frac{|X\rho|^p}{|\operatorname{grad}\rho|}\, dH_{N-1} \\
&= -\frac{1}{|\partial B_\varepsilon|_p}\int_{\partial B_\varepsilon}\phi\frac{|X\rho|^p}{|\operatorname{grad}\rho|}\, dH_{N-1} \to -\phi(e)
\end{aligned}
$$

as $\varepsilon \to 0$. This completes the proof of the theorem in the case $p \neq Q$. The case $p = Q$ is treated similarly. □

Remark 1. The case $p = Q$ of Theorem 2.1 has also been recently established in the interesting paper [HH]. As a remarkable application the authors prove that every quasiregular map on a group of H-type is either constant or sense-preserving, discrete and open.

The fundamental solutions in Theorem 2.1 can be used to find explicit formulas for the p-capacity of a spherical ring in Heisenberg type groups (we refer to Section 4 for the definition of capacity). By virtue of Lemma 2.1 the function

$$
u(x) = \begin{cases} \dfrac{\rho(x)^{(p-Q)/(p-1)} - R^{(p-Q)/(p-1)}}{r^{(p-Q)/(p-1)} - R^{(p-Q)/(p-1)}}, & \text{when } p \neq Q, \\ \dfrac{\log[\rho(x)/R]}{\log(r/R)} & \text{when } p = Q, \end{cases}
$$

is a weak (and a classical, smooth) solution to the Dirichlet problem

$$
\begin{cases} \displaystyle\sum_{j=1}^m X_j(|Xu|^{p-2}X_ju) = 0 \text{ in } B_R \setminus B_r, \\ u\,|_{\partial B_r} = 1 \text{ and } u\,|_{\partial B_R} = 0. \end{cases} \tag{2.16}
$$

By the energy minimizing property of solutions of (2.16) and Definition 4.1 one sees easily that given a ring $B_R \setminus B_r$, then for any $r < \varepsilon < R$

$$
\operatorname{cap}_p(B_r, B_R) = -\int_{\partial B_\varepsilon} |Xu|^{p-2}X_ju\langle X_j,\eta\rangle\, dH_{N-1}. \tag{2.17}
$$

Calculations similar to those in the proof of Theorem 2.1 allow us to compute the right-hand side of (2.17) and establish the following:

THEOREM 2.2. *Let* $0 < r < R$, $1 < p < \infty$. *Then, we have*

$$\operatorname{cap}_p(B_r, B_R) = \begin{cases} \left(\dfrac{Q-p}{p-1}\right)^{p-1} \sigma_p[1-(r/R)^{(Q-p)/(p-1)}]^{1-p} r^{Q-p}, & 1 < p < Q, \\ \sigma_Q \left[\log \dfrac{R}{r}\right]^{1-Q}, & p = Q, \\ \left(\dfrac{p-Q}{p-1}\right)^{p-1} \sigma_p[R^{(p-Q)/(p-1)} - r^{(p-Q)/(p-q)}]^{1-p}, & p > Q. \end{cases}$$

We close this section with some remarks on the relevance of Theorems 2.1, 2.2 for the results in this paper. The former can be restated in the case $p \neq Q$ in the following suggestive fashion

$$\Gamma_p(x, y) = c_p \left(\frac{d(x, y)^p}{|B(x, d(x, y))|} \right)^{1/(p-1)}, \tag{2.18}$$

where we have let $d(x, y) = \rho(x^{-1}y)$, $x, y \in G$. It was (2.18) that led us to conjecture that a similar result should hold for any nonlinear operator modeled on (1.6) in the vein of what was proved in [NSW] for the case $p = 2$. This question will be taken up in Section 7, where more general results will be established.

In an analogous way, Theorem 2.2 yields when $1 < p < Q$ the estimate

$$\operatorname{cap}_p(B_r, B_R) \approx \frac{|B_r|}{r^p}. \tag{2.19}$$

It will be seen in Section 4 that (2.19) locally holds far beyond the setting of H-type groups.

3. Preliminaries. In this section we introduce the definitions and results that will be needed subsequently. Let $X_1, \ldots, X_m$ be a family of smooth vector fields in $\mathbb{R}^n$ satisfying the above cited Hörmander's condition [H]. Also, we label with $Y_1, \ldots, Y_\ell$ the collection of the vectors X_j's and of those commutators which are needed to generate $\mathbb{R}^n$. Following [NSW] a "degree" is assigned to each Y_i, namely the order of the commutator needed to obtain the vector itself starting from the original set $X_1, \ldots, X_m$. If $I = (i_1, \ldots, i_n)$, $1 \le i_j \le \ell$ is a n-tuple of integers then we define $d(I) = \sum_{j=1}^n \deg(Y_{i_j})$, and $a_I(x) = \det(Y_{i_1}, \ldots, Y_{i_n})$. Throughout the paper we denote by $Xu = (X_1 u, \ldots, X_m u)$ the subelliptic gradient of a function u. A piecewise C^1 curve $\gamma\colon [0, T] \to \mathbb{R}^n$ is called subunitary if for every $y \in \mathbb{R}^n$ and $t \in (0, T)$ for which $\dot\gamma(t)$ exists one has

$$\langle \dot\gamma(t), y \rangle^2 \le \sum_{j=1}^m \langle X_j(\gamma(t)), y \rangle^2.$$

Given $x, y \in \mathbb{R}^n$ the Carnot-Carathéodory distance between x and y is defined as follows:

$$\begin{aligned} d(x,y) \ = \ & \inf\{T > 0 |\ \text{There exists } \gamma\colon\ [0,T] \to \mathbb{R}^n \text{subunitary,} \\ & \text{with } \ \gamma(0) = x,\ \ \gamma(T) = y\}. \end{aligned}$$

For $x \in \mathbb{R}^n$ and $R > 0$, denote $B_d(x,R) = \{y \in \mathbb{R}^n \mid d(x,y) < R\}$. A fundamental result due to Nagel, Stein and Wainger [NSW] is the existence for any $U \subset\subset \mathbb{R}^n$ of constants $C > 0$, $0 < R_0 < 1$, and of a polynomial function

$$\Lambda(x,r) = \sum_I |a_I(x)| r^{d(I)}, \tag{3.1}$$

where $I = (i_1, \ldots, i_n)$ ranges in the set of n-tuples with $1 \le i_j \le \ell$, such that for every $x \in U$ and $0 < r \le R_0$

$$C\Lambda(x,r) \le |B_d(x,r)| \le C^{-1}\Lambda(x,r). \tag{3.2}$$

An important consequence of (3.2) is the doubling condition: namely, there exists a constant $C_d > 0$ such that for x and r as above

$$|B_d(x,2r)| \le C_d |B_d(x,r)|. \tag{3.3}$$

We denote by

$$Q = \frac{\log(C_d)}{\log 2} = \sup\{d(I) \mid \ |a_I(x)| \neq 0 \text{ and } x \in U\} \tag{3.4}$$

the homogeneous dimension relative to the $X_1, \ldots, X_m$ in U, and by

$$Q(x) = \inf\{d(I) \mid \ |a_I(x)| \neq 0\}, \tag{3.5}$$

the homogeneous dimension at x relative to $X_1, \ldots, X_m$. Clearly $n \le Q(x) \le Q$. It is easy to see that there exists a constant C depending only on U and $X_1, \ldots, X_m$ for which

$$\lambda^{Q(x)} |B_d(x,r)| \le |B_d(x,\lambda r)| \le C\lambda^Q |B_d(x,r)|, \tag{3.6}$$

for any $\lambda > 1$. For future reference we also point out the following trivial estimate:

$$|a_{Q(x)}(x)| r^{Q(x)} \le |B_d(x,r)| \le C r^{Q(x)}. \tag{3.7}$$

A particularly interesting case is that of stratified Lie groups in which case $Q(x) = Q$ in every point $x \in U$ (see Section 8). Closely related to (3.2) are the important estimates of the fundamental solution $\Gamma(x, y)$ of the operator $\mathcal{L} = \sum_{j=1}^{m} X_j^* X_j$ established by Nagel, Stein, Wainger [NSW] and by Sanchez-Calle [SC],

$$C \frac{d(x,y)^2}{|B_d(x, d(x,y))|} \leq \Gamma(x,y) \leq C^{-1} \frac{d(x,y)^2}{|B_d(x, d(x,y))|}, \tag{3.8}$$

$$|X\Gamma(x,y)| \leq C^{-1} \frac{d(x,y)}{|B_d(x, d(x,y))|}, \tag{3.9}$$

where $x \in U \subset\subset \mathbb{R}^n$, $0 < d(x,y) \leq R_0$ and both C and R_0 depend on U.

Next, we introduce a different family of "balls" via the fundamental solution $\Gamma(x, y)$. There are several advantages in working with the latter. The first is that, unlike the metric balls, they are smooth sets. Secondly, they are better fitted to the geometry of the operator $\mathcal{L} = \sum_{j=1}^{m} X_j^* X_j$, in the sense that they support ad hoc cut-off functions (see Lemma 3.1) below. Another remarkable aspect is the existence of mean value formulas.

Consider the polynomial $\Lambda(x, r)$ in (3.1) and set

$$E(x,r) = \frac{\Lambda(x,r)}{r^2}.$$

Since the function $r \to E(x, r)$ is strictly increasing it admits an inverse $F(x, r)$. We define the $\mathcal{L}$-balls

$$B(x,r) = \left\{ y \in \mathbb{R}^n \mid \Gamma(x,y) > \frac{1}{E(x,r)} \right\}. \tag{3.10}$$

The following facts were proved in [CGL]:

$$B_d(x, r/a) \subset B(x,r) \subset B_d(x, ra), \tag{3.11}$$

$$C\, d(x_0, y) \leq F(x_0, \Gamma(x_0,y)^{-1})) \leq C^{-1} d(x_0,y), \tag{3.12}$$

for $x \in U \subset\subset \mathbb{R}^n$, and $r < R_0$ with $C, a, R_0 > 0$ depending on U.

Throughout the paper we will work with the sets (3.10) rather than with the metric balls B_d. It is important to observe that in virtue of (3.11), (3.12) the estimate (3.6) continues to hold. Namely, for every $U \subset\subset \mathbb{R}^n$ there exist $C, R_0 > 0$ such that

$$C\lambda^{Q(x)} |B(x,r)| \leq |B(x, \lambda r)| \leq C^{-1} \lambda^{Q} |B(x,r)|, \tag{3.13}$$

for any $\lambda > 1$.

Remark 2. Let U, x and R_0 be as above, k a positive integer with $2^{k+1}r < R_0$. There exists a positive constant C such that if $x \in B(x, 2^{k+1}r) \setminus B(x, 2^k r)$ one has

$$C2^k r \le d(x,y) \le C^{-1}2^{k+1}r.$$

This important observation follows from (3.11), (3.12), and will be used several times in the paper without explicit notice.

As a consequence of (3.11), (3.12) and of the estimates (3.2), (3.8) and (3.9) we have

$$\left| XF(x, \Gamma(x,y)^{-1}) \right| \le C, \tag{3.14}$$

with $d(x,y) \le R_0$, $x \in U \subset\subset \mathbb{R}^n$ and C depending on U. The estimate (3.14) is crucial in the construction of a class of test functions. More precisely one has from [CGL]

LEMMA 3.1. *Let $U \subset\subset \mathbb{R}^n$. There exists $R_0 > 0$ such that given a $\mathcal{L}$-ball $B(x,t) \subset\subset U$, with $t \le R_0$ and $0 < s < t$, one can find a function $\varphi \in C_0^\infty(B(x,t))$ satisfying $0 \le \varphi \le 1$, $\varphi = 1$ in $B(x,s)$ and $|X\varphi| \le C(t-s)^{-1}$. Here, $C > 0$ is a constant independent of s and t.*

In the same paper the following representation formula was found:

LEMMA 3.2. *Given $U \subset\subset \mathbb{R}^n$ there exists a positive constant R_0 such that for every $x \in \mathbb{R}^n$, $0 < r < R_0$, $u \in C_0^\infty(B(x,r))$ and $\xi \in B(x,r)$*

$$u(\xi) = -\int_{B(x,r)} \langle Xu(y), X\Gamma(\xi,y)\rangle \, dy.$$

Now let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Given $1 \le p < \infty$ we will denote by $S^{1,p}(\Omega)$ the completion of $\mathrm{Lip}\,(\Omega)$, the space of Lipschitz functions in Ω, with respect to the norm

$$\|u\|_{S^{1,p}(\Omega)} = \left(\int_\Omega (|u|^p + |Xu|^p)\, dx \right)^{\frac{1}{p}}. \tag{3.15}$$

The Sobolev space $\overset{\circ}{S}{}^{1,p}(\Omega)$ is defined as the completion of $\mathrm{Lip}_0\,(\Omega)$, the space of compactly supported Lipschitz functions in Ω, with respect to the same norm. We let $S^{1,p}_{\mathrm{loc}}(\Omega)$ denote the space of those $u \in L^p(\Omega)$ such that $\varphi u \in S^{1,p}(\Omega)$ for any $\varphi \in \mathrm{Lip}_0\,(\Omega)$. In the next sections we will make frequent use of the following embedding theorem (see [D1], [CDG1], [CDG3]).

THEOREM 3.1. *Let $U \subset \mathbb{R}^n$ be a bounded open set and denote by Q the homogeneous dimension relative to U. Given $1 \le p < Q$ there exist $C > 0$ and $R_0 > 0$, such that for any $x \in U$, $R \le R_0$, we have*

$$\left(\frac{1}{|B(x,R)|}\int_{B(x,R)} |u|^{kp}\,dx\right)^{\frac{1}{kp}} \le CR\left(\frac{1}{|B(x,R)|}\int_{B(x,R)} |Xu|^p\,dx\right)^{\frac{1}{p}}$$

for any $u \in \overset{\circ}{S}{}^{1,p}(B(x,R))$. Here $1 \le k \le \frac{Q}{Q-p}$.

Theorem 3.1 holds for metric balls $B_d(x,R)$ as well. It seems worthwhile to recall here that the geometric case ($p = 1$) of the Sobolev embedding is equivalent to a remarkable isoperimetric inequality proved in [CDG3] and also independently in [FGW]. The role played in this inequality by the homogeneous dimension Q is analogous to the one played by the topological dimension in the euclidean setting.

THEOREM 3.2. *Let $U \subset \mathbb{R}^n$ be a bounded open set and denote by Q the homogeneous dimension relative to U. For every C^1 open set $E \subset \overline{E} \subset B(x,R)$ one has*

$$|E|^{(Q-1)/Q} \le CR|B(x,R)|^{-1/Q}P_X(E,B(x,R)),$$

where $P_X(E;B(x,R))$ denotes the X-perimeter of E in $B(x,R)$ (see [CDG3], Definition 4).

Let $p > 1$. At this point we can state the relevant integrability requirements on the functions f_i, g_i and h_i in the structural assumptions (1.9) for the equation (1.7):

$$\begin{aligned} &\text{(i)} && g_2, g_3 \in L^r_{loc}(U), \text{ with } r > Q(p-1)^{-1};\\ &\text{(ii)} && f_2, f_3, h_3 \in L^s_{loc}(U), \text{ with } s > Q/p;\\ &\text{(iii)} && f_1 \in L^t_{loc}(U), \text{ with } t > Q. \end{aligned} \tag{3.16}$$

Assumptions (i)–(iii) above allow us to write, for some $0 < \varepsilon < 1$, $s = Q(p-\varepsilon)^{-1}$, $t = Q(1-\varepsilon)^{-1}$, and $r = Q(p-1-\varepsilon)^{-1}$.

Given the assumptions (3.16) a function $u \in S^{1,p}_{\mathrm{loc}}(\Omega)$ is called a weak solution to (1.7) if for any $\varphi \in \overset{\circ}{S}{}^{1,p}(\Omega)$, such that $\mathrm{supp}(\varphi) \subset\subset \Omega$, we have

$$\sum_{j=1}^m \int_\Omega A_j(x,u,Xu)X_j\varphi\,dx = \int_\Omega f(x,u,Xu)\varphi\,dx. \tag{3.17}$$

Under the above integrability assumptions the following uniform Harnack inequality for positive weak solutions of (1.7) was proved in [CDG1], THEOREM 3.3. *Let $1 < p \leq Q$ and $u \in S^{1,p}_{\mathrm{loc}}(\Omega)$ be a nonnegative solution to (1.7). Then, there exist $C > 0$ and $R_0 > 0$ such that for any $B_R = B(x,R) \subset \Omega$, with $B(x,4R) \subset \Omega$, and $R \leq R_0$*

$$\operatorname{essup}_{B_R} u \leq C\big(\operatorname{essinf}_{B_R} u + k(R)\big).$$

Here,

$$k(R) = \begin{cases} \left(|B_R|^{\frac{\varepsilon}{Q}}\|f_3\|_{L^s(B_R)} + \|g_3\|_{L^r(B_R)}\right)^{\frac{1}{p-1}} + \left(|B_R|^{\frac{\varepsilon}{Q}}\|h_3\|_{L^s(B_R)}\right)^{\frac{1}{p}}, & 1<p<Q, \\ \left(|B_R|^{\frac{\varepsilon}{Q}}\|f_3\|_{L^s(B_R)} + |B_R|^{\frac{\varepsilon}{Q}}\|g_3\|_{L^r(B_R)}\right)^{\frac{1}{Q-1}} + \left(|B_R|^{\frac{\varepsilon}{Q}}\|h_3\|_{L^s(B_R)}\right)^{\frac{1}{Q}}, & p = Q. \end{cases}$$

with r, s as in (3.16).

It is not difficult to extend Theorem 3.3 to rings whose inner and outer radii have proportional length. More precisely, if u and B_R are as in Theorem 3.3, and $B(x,4R) \subset \Omega$, then

$$\operatorname{essup}_{B_R \setminus B_{R/2}} u \leq C\left(\operatorname{essinf}_{B_R \setminus B_{R/2}} u + k(R)\right). \tag{3.18}$$

Here $k(R)$ is defined as above but the Lebesgue norms are computed on $B_R \setminus B_{R/2}$. In order to prove (3.18) it suffices to show that we can cover the ring with a fixed number of balls, not depending on R. Such covering of Wiener type is a consequence of (3.3) and (3.11) (see [CW], Theorem 1.2, chapter 3).

One of the main steps in the proof of Theorem 3.3 is the following Caccioppoli type inequality. If u is a weak solution of (1.7) in Ω and $B(x,s) \subset B(x,t) \subset \Omega$, then there exists a positive constant C depending on Ω and (S), such that

$$\int_{B(x,s)} |Xu|^p \, dx \leq C(t-s)^{-p} \int_{B(x,t)} \big[|u| + k(t)\big]^p \, dx. \tag{3.19}$$

As for the Harnack inequality, a version of (3.19) holds for rings. The argument is based on a suitable choice of the test function in Lemma 3.1 and on the inclusions (3.11). More precisely, let $U \subset\subset \mathbb{R}^n$. Then, there exist $C, R_0 > 0$ such that for any $x \in U$ and $R < R_0$ we have with $B_R = B(x,R)$

$$\int_{B_{2R} \setminus B_R} |Xu|^p \, dx \leq CR^{-p} \int_{B_{3R} \setminus B_{R/2}} \big[|u| + k(3R)\big]^p \, dx. \tag{3.20}$$

The next lemma will often be used in Section 7.

LEMMA 3.3. *Let Ω be a bounded, open set in $\mathbb{R}^n$, $x_0 \in \Omega$, and $u \in S^{1,p}_{\mathrm{loc}}(\Omega \setminus \{x_0\})$ a weak solution of (1.7) in $\Omega \setminus \{x_0\}$. Then, there exist constants K_1 and K_2 such* *that for every* $\phi \in \overset{\circ}{S}^{1,p}(\Omega)$ *identically equal to one in some neighborhood of* x_0 *the following holds:*

$$\text{(a)} \qquad \int_\Omega \sum_{j=1}^m A_j(x,u,Xu)X_j\phi\, dx - \int_\Omega f(x,u,Xu)\phi\, dx = K_1,$$

$$\text{(b)} \qquad \int_\Omega \sum_{j=1}^m A_j(x,u,Xu)X_j\phi\, dx - \int_\Omega f(x,u,Xu)(\phi-1)\, dx = K_2.$$

The simple proof of Lemma 3.3 rests on the definition of weak solution; we leave the details to the reader. For later purposes we include a numerical lemma from [S1].

LEMMA 3.4. *Let* $z > 0$ *and suppose* $z^\alpha \leq \sum_{i=1}^N a_i z^{\beta_i}$ *with* $a_i > 0$ *and* $0 \leq \beta_i < \alpha$. *Then* $z \leq \sum_{i=1}^N a_i^{\gamma_i}$, *where* $\gamma_i^{-1} = \alpha - \beta_i$, *and* $C > 0$ *only depends upon* N, α *and* β_i.

4. Capacitary estimates. The aim of this section is to establish sharp estimates for the subelliptic p-capacity of a ring. Throughout, we work with C^∞ vector fields $X_1, \ldots, X_m$ satisfying the finite rank condition. In the setting of stratified Lie groups Pansu [P1], [P2] proved that the capacity of such a condenser is positive, and remarked the difficulty involved in finding better estimates. Korányi and Reimann [KR1] computed explicitly the Q-capacity of a metric ring in the Heisenberg group. Their computation made use of a suitable choice of "polar" coordinates in $\mathbb{H}^n$. More recently, Heinonen and Holopainen [HH] have proved sharp estimates for the Q-capacity of a ring in the setting of Carnot groups. Our results generalize all previous ones. We emphasize an interesting feature of Theorems 4.1 and 4.2 that cannot be observed in the setting of stratified groups: The dependence of the estimates on the center of the ring. This is a consequence of the fact that in the general case $Q(x) \neq Q$ when $x \in \Omega$ (see (3.4), (3.5)). The results in this section will play an important role in the subsequent developments.

Definition 4.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set, and $K \subset \Omega$ a compact set. For $1 \leq p < \infty$ we define the p-capacity of the condenser (K, Ω) as

$$\operatorname{cap}_p(K,\Omega) = \inf\left\{ \int_\Omega |Xu|^p\, dx \middle|\ u \in C_0^\infty(\Omega), u = 1 \text{ on } K\right\}. \tag{4.1}$$

THEOREM 4.1. (Estimates from below) *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set,* $x_0 \in \Omega$ *and* $Q(x_0)$ *be the homogeneous dimension in* x_0 *associated to the vector fields* $X_1, \ldots, X_m$. *Then, there exists* $0 < R_0 = R_0(\Omega)$ *such that for any* $0 < r < R < R_0$

we have

$$\operatorname{cap}_p\left(B(x_0,r),B(x_0,R)\right) \geq \begin{cases} C_1\dfrac{|B(x_0,r)|}{r^p}, & \text{if } 1<p<Q(x_0),\\ C_2\left(\log\dfrac{R}{r}\right)^{1-Q(x_0)}, & \text{if } p=Q(x_0),\\ C_3\left|(2R)^{\frac{p-Q(x_0)}{p-1}}-r^{\frac{p-Q(x_0)}{p-1}}\right|^{(1-p)}, & \text{if } p>Q(x_0),\end{cases}$$

where

$$\begin{aligned} C_1 &= C\left[2^{\frac{Q(x_0)-p}{p-1}}\Big/\left(2^{\frac{Q(x_0)-p}{p-1}}-1\right)\right]^{(1-p)},\\ C_2 &= C\left(\frac{|B(x_0,r)|}{r^{Q(x_0)}}\right),\\ C_3 &= C\frac{|B(x_0,r)|}{r^{Q(x_0)}}\left(2^{\frac{p-Q(x_0)}{p-1}}-1\right)^{(p-1)}, \end{aligned}$$

with $C>0$ depending only on Ω and $X_1,\dots,X_m$.

Remark 3. In virtue of (3.7), C_2 and C_3 are bounded from below by a strictly positive constant whose value depends on x_0, Ω, the vectors $X_1,\dots,X_m$ but not on r.

THEOREM 4.2. (Estimates from above) *Let Ω, x_0 and $Q(x_0)$ be as in Theorem 4.1. Then, there exists $R_0=R_0(\Omega)>0$ such that for any $0<r<R<R_0$ we have*

$$\operatorname{cap}_p\left(B(x_0,r),B(x_0,R)\right) \leq \begin{cases} C_1\dfrac{|B(x_0,r)|}{r^p} & \text{if } 1<p<Q(x_0),\\ C_2\left(\log\dfrac{R}{r}\right)^{1-Q(x_0)} & \text{if } p=Q(x_0),\\ C_3\left|(2R)^{\frac{p-Q(x_0)}{p-1}}-r^{\frac{p-Q(x_0)}{p-1}}\right|^{1-p} & \text{if } p>Q(x_0).\end{cases}$$

Here C_1 and C_2 are positive constants depending on Ω and $X_1,\dots,X_m$. The last constant has the form

$$C_3 = C\big(2^{\frac{p-Q(x_0)}{p-1}}-1\big)^{-1},$$

where $C>0$ depends only on Ω and $X_1,\dots,X_m$.

Proof of Theorem 4.1. Let $u\in C_0^\infty(B(x_0,R))$ be such that $u=1$ on $B(x_0,r)$. By Lemma 3.2 we have

$$1 \;=\; u(x_0) = \left|\int_{B(x_0,R)}\langle Xu(y),X\Gamma(x_0,y)\rangle\,dy\right|$$

$$\leq \left(\int_{B(x_0,R)} |Xu(y)|^p \, dy \right)^{\frac{1}{p}} \left(\int_{B(x_0,R)\setminus B(x_0,r)} |X\Gamma(x_0,y)|^{p'} \, dy \right)^{\frac{1}{p'}},$$

where we have let $p' = \frac{p}{p-1}$. We choose $k_0 \in \mathbf{N}$ such that $2^{k_0+1}r > R \geq 2^{k_0}r$. Then, by (3.9) and (3.3) we have

$$\begin{aligned} \text{(4.2)} \qquad & \int_{B(x_0,R)\setminus B(x_0,r)} |X\Gamma(x_0,y)|^{p'} \, dy \\ & \leq C \sum_{k=0}^{k_0} \int_{B(x_0,2^{k+1}r)\setminus B(x_0,2^k r)} \frac{d(x_0,y)^{p'}}{|B(x_0,d(x_0,y))|^{p'}} \, dy \\ & \leq C \sum_{k=0}^{k_0} \frac{(2^k r)^{p'}}{|B(x_0,2^k r)|^{(p'-1)}} \qquad \text{(by (3.6))} \\ & \leq C \frac{r^{p'}}{|B(x_0,r)|^{(p'-1)}} \sum_{k=0}^{k_0} 2^{k(p'-Q(x_0)(p'-1))}. \end{aligned}$$

Now, if $1 < p < Q(x_0)$, then $p' < Q(x_0)(p'-1)$ and we have

$$\text{(4.3)} \qquad 1 \leq C \left(\frac{2^{\frac{Q(x_0)-p}{p-1}}}{2^{\frac{Q(x_0)-p}{p-1}} - 1} \right)^{p-1} \frac{r^p}{|B(x_0,r)|} \int_{B(x_0,R)} |Xu(y)|^p \, dy.$$

If, instead, $p = Q(x_0)$, then $Q(x_0)(p'-1) = p'$ and (4.2) yields

$$\text{(4.4)} \quad 1 \leq C \, k_0^{(p-1)} \left(\int_{B(x_0,R)} |Xu(y)|^p \, dy \right) \frac{r^p}{|B(x_0,r)|} = C \frac{r^p}{|B(x_0,r)|} \left(\log \frac{R}{r} \right)^{(p-1)}.$$

Finally, if $p > Q(x_0)$, then $p' > Q(x_0)(p'-1)$ and we have

$$\begin{aligned} \text{(4.5)} \qquad 1 \ \leq \ & C \frac{r^p}{|B(x_0,r)|} \left((R/2)^{\frac{p-Q(x_0)}{p-1}} - 1 \right)^{(p-1)} \left(2^{\frac{p-Q(x_0)}{p-1}} - 1 \right)^{(1-p)} \\ & \cdot \int_{B(x_0,R)} |Xu(x)|^p \, dy. \end{aligned}$$

Taking the infimum on all competing u's in (4.2)-(4.4) we obtain the desired estimates. □

Proof of Theorem 4.2. We consider the function $E(x_0,\cdot)$ and its inverse $F(x_0,\cdot)$, introduced in Section 3. For $i = 0, 1$, we define

$$H_i(y) = \begin{cases} 0 & \text{if } y \notin B(x_0, R), \\ \dfrac{F(x_0, \Gamma(x_0, y)^{-1})^{\frac{p-Q_i}{p-1}} - R^{\frac{p-Q_i}{p-1}}}{r^{\frac{p-Q_i}{p-1}} - R^{\frac{p-Q_i}{p-1}}} & \text{if } y \in B(x_0, R) \setminus B(x_0, r), \\ 1 & \text{if } y \in B(x_0, r), \end{cases}$$

where $Q_0 = Q$, $Q_1 = Q(x_0)$, and

$$H_2(y) = \begin{cases} 0 & \text{if } y \notin B(x_0, R), \\ \left[\log\left(\dfrac{R}{r}\right)\right]^{-1} \log\left(\dfrac{R}{F(x_0, \Gamma(x_0, y)^{-1})}\right) & \text{if } y \in B(x_0, R) \setminus B(x_0, r), \\ 1 & \text{if } y \in B(x_0, r). \end{cases}$$

By the chain rule for $S^{1,p}_{\text{loc}}$ and the important estimate (3.14) we obtain

$$|XH_i(y)| \le C \left| r^{\frac{p-Q_i}{p-1}} - R^{\frac{p-Q_i}{p-1}} \right|^{-1} \cdot F(x_0, \Gamma(x_0, y)^{-1})^{\frac{1-Q_i}{p-1}}, \tag{4.6}$$

for $i = 0, 1$ and

$$|XH_2(y)| \le C \left| \log\left(\frac{R}{r}\right) \right|^{-1} F(x_0, \Gamma(x_0, y)^{-1})^{-1}. \tag{4.7}$$

Recalling (3.12), we thus conclude from (4.6) and (4.7)

$$|XH_i(y)| \ \le\ C \left| r^{\frac{p-Q_i}{p-1}} - R^{\frac{p-Q_i}{p-1}} \right|^{-1} d(x_0, y)^{\frac{1-Q_i}{p-1}}, \tag{4.8}$$

$$|XH_2(y)| \ \le\ C \left| \log\left(\frac{R}{r}\right) \right|^{-1} d(x_0, y)^{-1}. \tag{4.9}$$

Since H_i, $H_2 \in \overset{\circ}{S}{}^{1,p}(B(x_0, R))$, $i = 0, 1$, and $H_i \equiv H_2 \equiv 1$ in $B(x_0, r)$, we infer for $p \ne Q(x_0)$ and $1 < p$

$$\text{cap}_p(B(x_0, r), B(x_0, R)) \ \le\ \int_{B(x_0,R)\setminus B(x_0,r)} |XH_i(y)|^p \, dy \tag{4.10}$$

$$\leq \sum_{k=0}^{k_0} \int_{(B(x_0,2^{k+1}r)\setminus B(x_0,2^k r)} |XH_i(y)|^p \, dy$$

$$\leq C\left|r^{\frac{p-Q_i}{p-1}} - R^{\frac{p-Q_i}{p-1}}\right|^{-p} \sum_{k=0}^{k_0} (2^k r)^{\frac{1-Q_i}{p-1}p} |B(x_0, 2^k r)|,$$

where k_0 is as in the previous theorem.

At this point we need to make a distinction. If $1 < p < Q(x_0)$ then we select $i = 0$ and observe that from (3.2) there exists a positive constant C_0 depending on Ω and $X_1, \ldots, X_m$ such that

$$|B(x_0, 2^k r)| \leq C_0 2^{kQ} |B(x_0, r)|. \tag{4.11}$$

Substituting (4.11) in (4.12) we have

$$\begin{aligned} \operatorname{cap}_p\left(B(x_0, r), B(x_0, R)\right) &\leq C\left|r^{\frac{p-Q}{p-1}} - R^{\frac{p-Q}{p-1}}\right|^{-p} |B(x_0, r)| r^{\frac{1-Q}{p-1}p} \sum_{k=0}^{k_0} (2^k)^{\frac{1-Q}{p-1}p+Q} \\ &\leq C\left(\frac{1}{1 - 2^{\frac{p-Q}{p-1}}}\right)\left|1 - \left(\frac{R}{r}\right)^{\frac{p-Q}{p-1}}\right|^{-p} \frac{|B(x_0, r)|}{r^p} \\ &\leq \left[C\left(\frac{1}{1 - 2^{\frac{p-Q}{p-1}}}\right)\right] \frac{|B(x_0, r)|}{r^p}. \end{aligned}$$

This completes the proof in the range $1 < p < Q(x_0)$. When $p > Q(x_0)$ we observe that as a consequence of (3.2) and (3.7)

$$|B(x_0, 2^k r)| \leq C r^{Q(x_0)} 2^{kQ(x_0)}. \tag{4.12}$$

At this point we select $i = 1$ and from (4.10) we deduce

$$\begin{aligned} \operatorname{cap}_p\left(B(x_0, r), B(x_0, R)\right) &\leq C r^{Q(x_0)} \left|r^{\frac{p-Q(x_0)}{p-1}} - R^{\frac{p-Q(x_0)}{p-1}}\right|^{-p} \\ &\quad \cdot r^{\frac{1-Q(x_0)}{p-1}p} \left(\sum_{k=0}^{k_0} (2^k)^{\frac{p-Q(x_0)}{p-1}}\right) \\ &\leq C\left(2^{\frac{p-Q(x_0)}{p-1}} - 1\right)^{-1} r^{Q(x_0)} \\ &\quad \cdot \left[(2R)^{\frac{p-Q(x_0)}{p-1}} - r^{\frac{p-Q(x_0)}{p-1}}\right]^{1-p} r^{\frac{p-pQ(x_0)}{p-1} - \frac{p-Q(x_0)}{p-1}} \end{aligned}$$

$$\leq\ C(2^{\frac{p-Q(x_0)}{p-1}}-1)^{-1}\left[(2R)^{\frac{p-Q(x_0)}{p-1}}-r^{\frac{p-Q(x_0)}{p-1}}\right]^{1-p}.$$

The latter concludes the proof in the case $p > Q(x_0)$. When $p = Q(x_0)$, a similar argument, involving $H_2(y)$, applies. □

5. Maximum principle. The maximum principle for weak solution of (1.7) will play a crucial role in the sequel. We follow closely the arguments in [S1] and [S2], the only significant difference being the fact that the Sobolev embedding Theorem 3.1 holds only in suitably small domains.

THEOREM 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, Q be the homogeneous dimension relative to Ω, and $D \subset \Omega$. If $1 < p \leq Q$, and $u \in S^{1,p}_{\rm loc}(\Omega)$ is a weak solution of (1.7) in D such that $u \leq M$ on the boundary of D, then there exist positive constants C_1, C_2 and D_0, depending only on Ω and the structure conditions* (3.16) *such that if $|D| \leq D_0$, then*

$$\sup_{x\in D} u(x) \leq M + C_1|D|^{\varepsilon/[Q(p-1)]}|M| + C_2 k,$$

where $k = (|D|^{\varepsilon/Q}\|f_3\|_s)^{\frac{1}{p-1}} + (|D|^{\varepsilon/Q}\|h_3\|_s)^{\frac{1}{p}}$.

Proof. In the course of the proof we will assume $k > 0$, the case $k = 0$ can be treated by an approximation argument.

Let us assume initially $M = 0$. For every $\varepsilon' > 0$ we have $(u-\varepsilon')^+ \in \overset{\circ}{S}{}^{1,p}(D)$. Let $\overline{u} = \max(\varepsilon', u) + k - \varepsilon' \in S^{1,p}_{\rm loc}(D)$, then clearly $\overline{u} = k$ in a neighborhood of ∂D. Define for $q \geq 1$, $\ell > k$,

$$F(\overline{u}) \ = \ \begin{cases} \overline{u}^q, & \text{if } k \leq \overline{u} \leq \ell, \\ q\ell^{q-1}\overline{u} - (q-1)\ell^q, & \text{if } \ell \leq \overline{u}, \end{cases}$$

$$G(u) \ = \ F(\overline{u})F'(\overline{u})^{p-1} - q^{p-1}k^{\beta} \in \overset{\circ}{S}{}^{1,p}(D),$$

where $\beta = pq - p + 1$. We observe that in the set where $u < \varepsilon'$, $G(u) = 0$, while in the set $\{u \geq \varepsilon'\}$,

$$\begin{aligned} f(x,u,Xu) \ &\leq \ f_1|X\overline{u}|^{p-1} + \overline{f}_2|\overline{u}|^{p-1}, \\ A(x,u,Xu)\cdot Xu \ &\geq \ |X\overline{u}|^p - \overline{f}_2|\overline{u}|^p, \end{aligned} \tag{5.1}$$

with $\overline{f}_2 = f_2 + k^{1-p}f_3 + k^{-p}h_3$, and f_2, f_3 and h_3 as in (3.16). We observe explicitly that with this choice we have

$$\|\overline{f}_2\|_{L^s(D)} \leq \|f_2\|_{L^s(D)} + 2|D|^{-\varepsilon/Q}.$$

Now use $G(u)$ as a test function in (3.17). In the case $1 < p < Q$ the estimates (5.1) imply

$$\int_D |Xv|^p\, dx \le \frac{q}{\beta}\left[(1+\beta)q^{p-1}\int_D \overline{f}_2 |v|^p\, dx + \int_D f_1 v|Xv|^{p-1}\, dx\right] \tag{5.2}$$

with $v = F(\overline{u})$. We now estimate the two integrals on the right-hand side of (5.2). Hölder inequality gives

$$\begin{aligned}\int_D f_1 v|Xv|^{p-1}\, dx \;\le\; &\left(\int_D f_1^{\frac{Q}{1-\varepsilon}}\, dx\right)^{\frac{1-\varepsilon}{Q}}\left(\int_D v^p\, dx\right)^{\frac{\varepsilon}{p}}\\ &\cdot\left(\int_D |Xv|^p\, dx\right)^{\frac{p-1}{p}}\left(\int_D v^{\frac{Qp}{Q-p}}\, dx\right)^{(1-\varepsilon)\frac{Q-p}{Qp}}.\end{aligned} \tag{5.3}$$

We note that $v - k^q \in \overset{\circ}{S}{}^{1,p}(D)$. If $\operatorname{diam}(D)$ is sufficiently small we can apply Theorem 3.1 to a small ball containing D, after having extended $v - k^q$ with zero outside D. In the case when $\operatorname{diam}(D)$ is large we proceed as follows. By Theorem 1.2, Chapter 3 in [CW] it is possible to choose a covering of Ω with a family of balls $\{B_j\}_{j=1}^N$ having a fixed radius $r < R_0$, where R_0 is as in Theorem 3.1. Furthermore, the B_j's can be chosen so that $\frac{1}{3}B_i \cap \frac{1}{3}B_j = 0$ for $i \neq j$. We stress that the number of balls N solely depends on the bounded open set Ω. Let $\{\phi_j\}_{j=1}^N$ be a partition of unity subordinated to the covering $\{B_j\}_{j=1}^N$. We have

$$\begin{aligned}\int_D v^{\frac{Qp}{Q-p}}\, dx \;&\le\; \int_D |v-k^q|^{\frac{Qp}{Q-p}}\, dx + |D|k^{q\frac{Qp}{Q-p}}\\ &\le\; C\sum_{j=1}^N \int_{B_j\cap D} |v-k^q|^{\frac{Qp}{Q-p}}\phi_j^{\frac{Qp}{Q-p}}\, dx + |D|k^{q\frac{Qp}{Q-p}}.\end{aligned}$$

Theorem 3.1 yields

$$\begin{aligned}\int_D v^{\frac{Qp}{Q-p}}\, dx \;\le\; &C\sum_{j=1}^N\left(\int_{B_j\cap D}|Xv|^p\phi_j^p\, dx + \int_{B_j\cap D}|v-k^q|^p|X\phi_j|^p\, dx\right)^{\frac{Q}{Q-p}}\\ &+ |D|k^{q\frac{Qp}{Q-p}}.\end{aligned} \tag{5.4}$$

In conclusion there exists a positive constant $C = C(\Omega)$ such that

$$\int_D v^{\frac{Qp}{Q-p}}\, dx \le C\left(\int_D |Xv|^p + v^p\, dx\right)^{Q/(Q-p)} + C|D|k^{qQ/(Q-p)}. \tag{5.5}$$

From (5.3) and (5.5) we obtain

$$
\begin{aligned}
(5.6)\quad \int_D f_1 v |Xv|^{p-1}\,dx \;\le\; & \|f_1\|_t \left(\int_D v^p\,dx\right)^{\varepsilon/p} \left(\int_D |Xv|^p\,dx\right)^{(p-1)/p} \\
& \cdot \left[\left(\int_D v^p + |Xv|^p\,dx\right)^{(1-\varepsilon)/p} + C|D|^{(Q-p)(1-\varepsilon)/Qp} k^{q(1-\varepsilon)}\right].
\end{aligned}
$$

A similar argument leads to an estimate of the first integral on the right-hand side of (5.2), specifically

$$
\begin{aligned}
(5.7)\quad \int_D \overline{f}_2 |v|^p\,dx \;\le\; & \|\overline{f}_2\|_s \left(\int_D v^p\,dx\right)^{\varepsilon/p} \\
& \cdot \left[\left(\int_D v^p + |Xv|^p\,dx\right)^{(p-\varepsilon)/p} + C_{p,Q}|D|^{(Q-p)(p-\varepsilon)/Qp} k^{q(p-\varepsilon)}\right].
\end{aligned}
$$

Inequalities (5.2), (5.6) and (5.7) yield

$$
\begin{aligned}
(5.8)\quad \int_D |Xv|^p\,dx \;\le\; & \|\overline{f}_2\|_s \frac{q}{\beta} \|v\|_p^{\varepsilon} \\
& \times \left[(1+\beta) q^{p-1} (\|v\|_p^{p-\varepsilon} + \|Xv\|_p^{p-\varepsilon} + |D|^{(Q-p)(p-\varepsilon)/Qp} k^{q(p-\varepsilon)})\right. \\
& + \|f_1\|_t \|Xv\|_p^{p-1} \|v\|_p^{\varepsilon} \\
& \left. \times (\|v\|_p^{1-\varepsilon} + \|Xv\|_p^{1-\varepsilon} + |D|^{(Q-p)(1-\varepsilon)/Qp} k^{q(1-\varepsilon)})\right].
\end{aligned}
$$

All the L^p norms are taken over the set D. Since $v \ge k^q$, then $k^q |D|^{(Q-p)/Qp} \le |D|^{-1/Q} \|v\|_p$. Using this observation and the numerical Lemma 3.3, we obtain from (5.8) the following Caccioppoli type inequality for v

$$
\begin{aligned}
(5.9)\quad \left(\int_D |Xv|^p\,dx\right)^{1/p} \;\le\; & \Big[\|f_1\|_t (|D|^{-1/Q} + 1) + (\|\overline{f}_2\|_s)^{1/p} \\
& \cdot (|D|^{-(1-\varepsilon/p)/Q} + 1)\Big] q^{\frac{p}{\varepsilon}} \left(\int_D v^p\,dx\right)^{1/p}.
\end{aligned}
$$

Although v does not vanish on ∂D, the function $v - k^q$ does. Repeating the partition of unity argument that led to (5.4) we can use Theorem 3.1 in (5.9) and obtain

$$
\begin{aligned}
(5.10)\quad \left(\int_D v^{Qp/(Q-p)}\,dx\right)^{(Q-p)/Qp} \;\le\; & C(1 + |D|^{-1/Q}) q^{p/\varepsilon} \left(\int_D v^p\,dx\right)^{1/p} \\
& + C k^q |D|^{(Q-p)/Qp}.
\end{aligned}
$$

In the case $p = Q$ we follow Serrin's argument substituting p with $Q(1 + \varepsilon/2Q)^{-1}$ in the preceding computations. In this way (5.10) can still be proved (see also [CDG1] Theorem 3.4 for more details).

Now the Moser iteration process can be applied without having to shrink the domain at each step. One concludes

$$\sup_D \overline{u} \leq C\left(|D|^{-1/p}\|\overline{u}\|_p\right).$$

Recalling the definition of $\overline{u}$, the latter inequality yields

$$\sup_D u \leq C\left(|D|^{-1/p}\|u\|_{L^p(\{u\geq 0\})} + k\right) \tag{5.11}$$

once we let $\varepsilon' \to 0$. Next we consider the case $M > 0$. We observe that $u' = u - M$ is a weak solution of

$$X_j^* A_j'(x, u', Xu') = f'(x, u', Xu'), \tag{5.12}$$

the functions A' and f' satisfy conditions similar to those in (1.9), (3.16) with coefficients f_1, f_2, f_3', g_2, g_3', and h_3 where

$$\begin{aligned} f_3' &= f_3 + C_1 f_2 M^{p-1}, \\ g_3' &= g_3 + C_2 g_2 M^{p-1}, \end{aligned} \tag{5.13}$$

and C_1, C_2 are positive constants depending only on p. The new constant k' associated to (5.12), (5.13) satisfies the following estimate

$$k' \leq k + C|D|^{\frac{\varepsilon}{Q(p-1)}}|M|, \tag{5.14}$$

where C and ε depend on (3.16). Now we can apply (5.11) to u' and obtain

$$\sup_D u \leq C\left(|D|^{-1/p}\|u\|_{L^p(\{u\geq M\})} + k + C_4|D|^{\varepsilon/Q(p-1)}|M|\right) + M. \tag{5.15}$$

Let us remark that (5.15) holds without restrictions on the measure of D. In order to conclude the proof, we need to estimate the term $\|u\|_{L^p(\{u\geq M\})}$. To this end we assume once more $M = 0$ and consider

$$\overline{u} = \max(\varepsilon', u) + K\delta^{-1} - \varepsilon', \tag{5.16}$$

for some small $\delta > 0$, to be fixed at a later time. This function solves an equation similar to (1.7) with reduced coefficient $\overline{f}_2 = f_2 + (k\delta^{-1})^{1-p} f_3 + (k\delta^{-1})^{-p} h_3$.

Modifying the definition of F and G properly and proceeding as above, we arrive at the Caccioppoli inequality (5.10) for $\overline{u}$. Setting $q = 1$ in the latter one obtains

$$\int_D |X\overline{u}|^p\,dx \le C_0^p \int_D \overline{u}^p\,dx, \tag{5.17}$$

where $C_0 = \left[\|f_1\|_t(|D|^{-1/Q} + 1) + (\|\overline{f}_2\|_s)^{1/p}(|D|^{-(1-\varepsilon/p)/Q} + 1)\right]$. Since $\|\overline{f}_2\|_s \le \|f_2\|_s + \delta^{p-1}|D|^{-\varepsilon/Q}$, then for a suitable choice of $|D|$ and δ we can make $|D|^{1/Q}C_0$ as small as needed. The Sobolev embedding Theorem 3.1 now gives

$$\left(\int_D \overline{u}^{Qp/(Q-p)}\,dx\right)^{(Q-p)Qp} \le C\left(\int_D |X\overline{u}|^p + \overline{u}^p\,dx\right)^{1/p} + C_{p,Q}|D|^{(Q-p)/Qp}k\delta^{-1}. \tag{5.18}$$

The Hölder inequality, (5.17) and (5.18) yield

$$\begin{aligned}\left(\int_D \overline{u}^p\,dx\right)^{1/p} &\le |D|^{1/Q}\left(\int_D \overline{u}^{Qp/(Q-p)}\,dx\right)^{(Q-p)/Qp}\\ &\le |D|^{1/Q}C_0C\left(\int_D \overline{u}^p\,dx\right)^{1/p} + C|D|^{1/Q}\left(\int_D \overline{u}^p\,dx\right)^{1/p}\\ &\quad + C|D|^{1/p}k\delta^{-1}.\end{aligned}$$

Choosing C_0 sufficiently small we have the desired estimate in the case $M = 0$. The theorem now follows from an argument similar to the one in (5.12)–(5.14). □

6. Removable singularities. In this section the classical result on removability of singularities in [S1] for solutions of quasilinear elliptic equations is extended to the subelliptic setting.

THEOREM 6.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and Q be the homogeneous dimension relative to Ω. Let $E \subset \Omega$ be a compact set and for $1 < p \le Q$, $u \in S^{1,p}_{\mathrm{loc}}(\Omega \setminus E)$ be a weak solution of (1.7) in $\Omega \setminus E$. Suppose that for $p \le s \le Q$, $\mathrm{cap}_s(E, \Omega) = 0$. If $u \in L^r(\Omega)$ for some $r > \frac{s(p-1)}{s-p}$, when $s > p$, or $u \in L^\infty(\Omega)$ when $s = p$, then u can be extended to become a solution in all of Ω.*

In the next section we will use the following corollary.

COROLLARY 6.1. *Suppose that $E = \{x_0\}$ in Theorem 6.1 and denote by $Q(x_0)$ the homogeneous dimension at x_0. Let $p \le s \le Q(x_0)$. If $u \in S^{1,p}_{\mathrm{loc}}(\Omega \setminus \{x_0\})$ is a solution to (1.7) in $\Omega \setminus \{x_0\}$ and $u \in L^r(\Omega)$ with r as in Theorem 6.1, then the singularity at x_0 is removable.*

The next propositions are basic ingredients for the proof of Theorem 6.1. They follow from the definition of capacity and the Poincare inequality based on Theorem 3.1.

PROPOSITION 6.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $1 < s \le Q$, and $E \subset \Omega$ a relatively closed subdomain of zero s-capacity. There exists a sequence $\pi_i \in \overset{\circ}{S}{}^{1,s}(\Omega \setminus E)$ such that $\pi_i \to 1$ almost everywhere in Ω and $\int_\Omega |X\pi_i|^s\,dx \to 0$ as i approaches infinity.*

PROPOSITION 6.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and $1 < p < \infty$. If $E \subset \Omega$ is a relatively closed subdomain of zero p-capacity, then $|E| = 0$.*

Proof of Theorem 6.1. By Theorem 3.3 it suffices to show that u is a solution in the neighborhood of any point x in E. We will only consider a ball $B(x, 2r)$ where r is properly small (so as to use the Sobolev embedding Theorem 3.1). Let $\overline{u} = |u| + k$, with

$$k = k(2r) = \left(|B(x,2r)|^{\frac{\varepsilon}{Q}}\|f_3\|_s + \|g_3\|_r\right)^{\frac{1}{p-1}} + \left(|B(x,2r)|^{\frac{\varepsilon}{Q}}\|h_3\|_s\right)^{\frac{1}{p}},$$

then the structure conditions (1.8) can be restated as

$$\begin{aligned} f(x,u,Xu) &\le f_1|X\overline{u}|^{p-1} + \overline{f}_2|\overline{u}|^{p-1}, \\ A(x,u,Xu) &\le c_1|X\overline{u}|^{p-1} + \overline{g}_2|\overline{u}|^{p-1}, \\ A(x,u,Xu)\cdot Xu &\ge |X\overline{u}|^p - \overline{f}_2|\overline{u}|^p, \end{aligned} \tag{6.1}$$

with

$$\overline{f}_2 = f_2 + k^{1-p}f_3 + k^{-p}h_3 \in L^s_{\mathrm{loc}}(B(x,2r)),$$

and

$$\overline{g}_2 = g_2 + k^{1-p}g_3 \in L^r_{\mathrm{loc}}(B(x,2r)).$$

Define for $\beta_0 = \varepsilon(p-1)$, $p_0 = p + \beta_0 - 1$, $q \ge q_0 = p_0/p$, and $\ell > k$

$$\begin{aligned} F(\overline{u}) &= \begin{cases} \overline{u}^q, & \text{if } k \le \overline{u} \le \ell, \\ q_0[q\ell^{q-q_0}\overline{u}^{q_0} - (q-q_0)\ell^q], & \text{if } \ell \le \overline{u}, \end{cases} \\ G(u) &= \operatorname{sign}(u)\{F(\overline{u})F'(\overline{u})^{p-1} - q^{p-1}k^{\beta}\} \in S^{1,p}_{\mathrm{loc}}(B(x,2r)\setminus E). \end{aligned}$$

Let $\eta \in C_0^\infty(B(x,2r))$ and π_i be as in Proposition 6.1. Since u is locally bounded outside E we are allowed to use $\gamma = (\eta\pi_i)^p G(u)$ as a test function in

(3.17). The structure conditions (6.1), and the Sobolev embedding Theorem 3.1 lead to the Caccioppoli inequality

$$\int_{B(x,2r)\setminus E} |Xv|^p(\eta\pi_i)^p\,dx \leq Cq^{\frac{p}{\varepsilon}}\left[\int_{B(x,2r)\setminus E} |v|^p(\eta\pi_i)^p\,dx + \int_{B(x,2r)\setminus E} |v|^p|X(\eta\pi_i)|^p\,dx\right] \tag{6.2}$$

with $v = F(\overline{u})$. We now observe that $v <$ const. $\overline{u}^{q_0} \in L^{\frac{sp}{s-p}}(B(x,2r)\setminus E)$, because of the integrability assumption on u. By Hölder inequality we have

$$\int_{B(x,2r)\setminus E} v^p\eta^p|X\pi_i|^p\,dx \leq \left(\int_{B(x,2r)\setminus E} (v\eta)^{\frac{sp}{s-p}}\right)^{\frac{s-p}{s}}\left(\int_{B(x,2r)\setminus E} |X\pi_i|^s\right)^{\frac{p}{s}}.$$

By Proposition 6.1 the right-hand side tends to zero as $i \to \infty$. We thus obtain from (6.2)

$$\int_{B(x,2r)\setminus E} |Xv|^p\eta^p\,dx \leq Cq^{\frac{p}{\varepsilon}}\left[\int_{B(x,2r)\setminus E} |v|^p\eta^p\,dx + \int_{B(x,2r)\setminus E} |v|^p|X\eta|^p\,dx\right]. \tag{6.3}$$

We now let $\ell \to \infty$ in (6.3). By Moser's iteration process, starting with $q = q_0$, one obtains

$$\sup_{B(x,r)\setminus E} \overline{u} \leq C\left(\int_{B(x,2r)\setminus E} |\overline{u}|^{p_0}\right)^{\frac{1}{p_0}}.$$

Since $p_0 < s(p-1)/(s-p)$, we infer $u \in L^\infty(B(x,r)\setminus E)$. In particular, from the Caccioppoli inequality (6.3) we have $Xu \in L^p(B(x,r)\setminus E)$. Being E a set of s-capacity zero, and $s \geq p$, it is easy to show that there exists a sequence of functions $u_j \in C_0^\infty(B(x,r)\setminus E)$ such that $u_j \to u$ in $S^{1,p}(B(x,r)\setminus E)$. Proposition 6.2 finally implies that $u \in S^{1,p}(B(x,r))$. To conclude the proof we need to show that u is solution of (1.7) in $B(x,r)$. Since u is a weak solution in $B(x,r)\setminus E$, then

$$\int_{B(x,r)} A_j(x,u,Xu)X_j\varphi\,dx = \int_{B(x,r)} \varphi f(x,u,Xu)\,dx,$$

for every test function $\varphi \in S_0^{1,p}(B(x,r)\setminus E)$. Let $\psi \in C_0^\infty(B(x,r))$, and π_i as in Proposition 6.1. Then the proof follows from Lebesgue dominated convergence theorem once we set $\varphi = \psi\pi_i$ and let $i \to \infty$. □

Remark 4. If $y \in \Omega$ is a nonremovable singularity for a weak solution u to (1.7) which is bounded from below, then as a consequence of the Harnack inequality Theorem 3.3, the maximum principle Theorem 5.1 and Theorem 6.1

above we have

$$\operatorname{Lim}_{x\to y} u(x) = \infty.$$

7. Local behavior of singular solutions. This section is devoted to establishing the main results about the local behavior of solutions to (1.7) having a singularity at one point.

THEOREM 7.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $x_0 \in \Omega$ and $Q(x_0)$ denote the homogeneous dimension at x_0. If $u \in S^{1,p}_{\mathrm{loc}}(\Omega \setminus \{x_0\})$ is a nonnegative (or bounded from below) solution to (1.7) in $\Omega \setminus \{x_0\}$ with $1 < p < Q(x_0)$, then one of the following holds:*

(i) *The singularity at x_0 is removable,*

(ii) *There exist positive constants A, B and R such that for $d(x,x_0) < \frac{R}{2}$ we have*

$$\begin{aligned} A\left(\operatorname{cap}_p(B(x_0,d(x,x_0)); B(x_0,R)\right)^{-\frac{1}{p-1}} \\ \le u(x) \\ \le B\left(\operatorname{cap}_p(B(x_0,d(x,x_0)); B(x_0,R))\right)^{-\frac{1}{p-1}}. \end{aligned}$$

The constants R, A and B depend on u, Ω, (3.16) *and p. The constants B and R depend also on C_1 in Theorem 4.1.*

Proof. Take $R > 0$ to be subsequently chosen. Without loss of generality we can assume that $u \le 0$ on $\partial B(x_0,R)$. Suppose x_0 is not a removable singularity. By Remark 4, $\operatorname{Lim}_{x\to x_0} u(x) = \infty$. Therefore, there exists $0 < \sigma_0 < R$ such that $u(x) \ge 1$ in $B(x_0,\sigma_0)$. Let $0 < \sigma \le \sigma_0$ and denote $m(\sigma) = \min_{\partial B(x_0,\sigma)} u$. In $B(x_0,R) \setminus B(x_0,\sigma)$ we define

$$v(x) = \begin{cases} 0 & \text{if } u \le 0, \\ u & \text{if } 0 < u < m(\sigma), \\ m(\sigma) & \text{if } u \ge m(\sigma). \end{cases}$$

Extend v by letting $v \equiv m(\sigma)$ in $B(x_0,\sigma)$, then $v \in \overset{\circ}{S}{}^{1,p}(B(x_0,R))$. By (b) of Lemma 3.3 we have

$$\begin{aligned} (7.1)\qquad & \sum_{j=1}^{m} \int_{B(x_0,R)\setminus B(x_0,\sigma)} A_j(x,u,Xu)X_j v\, dx \\ & - \int_{B(x_0,R)\setminus B(x_0,\sigma)} f(x,u,Xu)(v - m(\sigma))\, dx = K_2 m(\sigma). \end{aligned}$$

Observing that $Xv \equiv 0$ on the set where $v \not\equiv u$, we can rewrite (7.1) as follows

$$\sum_{j=1}^{m} \int_{B(x_0,R)} A_j(x,v,Xv)X_j v\, dx \le K_2 m(\sigma) + \mathcal{F}(\sigma)\|v - m(\sigma)\|_{L^\infty(B(x_0,R))}, \tag{7.2}$$

where we have let

$$\mathcal{F}(\sigma) = \int_{B(x_0,R)\setminus B(x_0,\sigma)} f(x,u,Xu)\, dx.$$

Using the assumption (1.8) we infer

$$\begin{aligned}\sum_{j=1}^{m} \int_{B(x_0,R)} A_j(x,v,Xv)X_j v\, dx \ \ge\ & \int_{B(x_0,R)} |Xv|^p\, dx \\ & - \int_{B(x_0,R)} f_2 v^p\, dx - \int_{B(x_0,R)} h_3\, dx.\end{aligned} \tag{7.3}$$

By 3.16 on f_2 and Sobolev's embedding Theorem 3.1 (see also estimate (3.10) in [CDG1]) we infer from (7.3) for some $C_1 = C_1(Q,p) > 0$

$$\begin{aligned}\sum_{j=1}^{m} \int_{B(x_0,R)} A_j(x,v,Xv)X_j v\, dx \ \ge\ & \int_{B(x_0,R)} |Xv|^p\, dx - C_1\|f_2\|_{L^s(B(x_0,R))} \\ & \cdot \int_{B(x_0,R)} |Xv|^p\, dx - \|h_3\|_{L^s(B(x_0,R))} \\ \ge\ & \frac{1}{2}\int_{B(x_0,R)} |Xv|^p\, dx - C_2\end{aligned} \tag{7.4}$$

provided that $R > 0$ is chosen sufficiently small. Since $\frac{v}{m(\sigma)}$ is an admissible function for the definition of $\operatorname{cap}_p(B(x_0,\sigma); B(x_0,R))$, from (7.2) and (7.4) we conclude

$$m(\sigma)^p \operatorname{cap}_p(B(x_0,\sigma); B(x_0,R)) \le C_1[1 + K_2 m(\sigma) + \mathcal{F}(\sigma) m(\sigma)],$$

where we have used $\|v - m(\sigma)\|_{L^\infty(B(x_0,R))} \le m(\sigma)$. Since $m(\sigma) \ge 1$ we obtain

$$m(\sigma)^{p-1} \le C_4[1 + \mathcal{F}(\sigma)] \operatorname{cap}_p(B(x_0,\sigma); B(x_0,R))^{-1}. \tag{7.5}$$

At this point we consider the ring $B(x_0,R) \setminus B(x_0,\sigma)$. By (3.18) there exists $C_5 > 0$, depending only on Ω and the structure of the equation, such that

$$\sup_{B(x_0,3\sigma)\setminus B(x_0,\frac{\sigma}{2})} u \le C_5\Big(\inf_{B(x_0,3\sigma)\setminus B(x_0,\frac{\sigma}{2})} u + k(\sigma)\Big).$$

This implies

$$\sup_{B(x_0,3\sigma)\setminus B(x_0,\frac{\sigma}{2})} u \le C_6 m(\sigma). \tag{7.6}$$

We now proceed to estimate $\mathcal{F}(\sigma)$. We begin with estimating the integral of f on the ring $B(x_0, 2\sigma) \setminus B(x_0, \sigma)$. By (1.8) and (7.6) we have

$$\begin{aligned} \int_{B(x_0,2\sigma)\setminus B(x_0,\sigma)} & f(x,u,Xu)\,dx \\ &\le \int_{B(x_0,2\sigma)\setminus B(x_0,\sigma)} \left[f_1|Xu|^{p-1} + f_2|u|^{p-1} + f_3\right]\,dx, \\ &\le \int_{B(x_0,2\sigma)\setminus B(x_0,\sigma)} f_1|Xu|^{p-1}\,dx + C_1 m(\sigma)^{p-1}|B(x_0,\sigma)|^{\frac{Q-p+\varepsilon}{Q}}, \end{aligned} \tag{7.7}$$

where we have let $C_7 = C_6^{p-1}(\int_{B(x_0,R)}(f_2+f_3)^s\,dx)^{\frac{1}{s}}$ with $s = \frac{Q}{p-\varepsilon}$. To estimate the integral containing $|Xu|$ we use Hölder inequality, and then (3.20), obtaining

$$\begin{aligned} \int_{B(x_0,2\sigma)\setminus B(x_0,\sigma)} f_1|Xv|^{p-1}\,dx &\le C_8\|f_1\|_{L^t(B(x_0,R))}|B(x_0,\sigma)|^{\frac{Q-p+p\varepsilon}{pQ}} \\ &\quad\cdot \left[\left(\int_{B(x_0,3\sigma)\setminus B(x_0,\frac{\sigma}{2})}|u|^p\,dx\right)^{\frac{1}{p}} + |B(x_0,\sigma)|^{\frac{1}{p}}k\right]^{p-1} \\ &\quad\cdot\ \sigma^{1-p} \end{aligned} \tag{7.8}$$

where k is as in Theorem 3.3. By (7.6) we infer from (7.8)

$$\int_{B(x_0,2\sigma)\setminus B(x_0,\sigma)} f_1|Xu|^{p-1}\,dx \le C_9|B(x_0,\sigma)|^{\frac{pQ-p+p\varepsilon}{pQ}}\sigma^{1-p}m(\sigma)^{p-1}. \tag{7.9}$$

Inserting (7.9) in (7.7) we conclude

$$\begin{aligned} \int_{B(x_0,2\sigma)\setminus B(x_0,\sigma)} f(x,u,Xu)\,dx \ &\le\ C_{10}m(\sigma)^{p-1}|B(x_0,\sigma)|^{\frac{Q+\varepsilon}{Q}} \\ &\quad\cdot\ \left[|B(x_0,\sigma)|^{-\frac{p}{Q}} + \sigma^{1-p}|B(x_0,\sigma)|^{-\frac{1}{Q}}\right]. \end{aligned} \tag{7.10}$$

At this point we emphasize that the term in brackets in the right-hand side of (7.10) lacks any scale invariance. This represents a serious threat to the possibility of carrying on Serrin's idea, were it not for the sharp capacity estimates of

Theorem 4.1. From the latter and (7.5) it follows

$$m(\sigma)^{p-1} \le C_{11}[1 + \mathcal{F}(\sigma)]\frac{\sigma^p}{|B(x_0,\sigma)|}, \tag{7.11}$$

where now the constant C_{11} depends on p, C_4, and C_1 from the statement of Theorem 4.1.

Observing that

$$\mathcal{F}(\sigma) - \mathcal{F}(2\sigma) = \int_{B(x_0,2\sigma)\setminus B(x_0,\sigma)} f(x,u,Xu)\,dx,$$

we arrive to the crucial estimate

$$\mathcal{F}(\sigma) - \mathcal{F}(2\sigma) \le C_{12}[1 + \mathcal{F}(\sigma)]\big|B(x_0,\sigma)\big|^{\frac{\varepsilon}{Q}} \le C_{13}[1 + \mathcal{F}(\sigma)]\sigma^{\frac{\varepsilon n}{Q}}. \tag{7.12}$$

Here we have made repeated use of the fundamental result (3.2). An elementary iteration argument gives from (7.12) for $\sigma \le \sigma_0$ and some small $\eta > 0$

$$\mathcal{F}(\sigma) \le C_{14} \tag{7.13}$$

and

$$\int_{B(x_0,\sigma)} |f(x,u,Xu)|\,dx \le C_{14}\sigma^{\eta}. \tag{7.14}$$

The estimate from above in the statement of the theorem now follows from (7.5), (7.13) and the Harnack inequality. $\square$

The preceding argument holds in the case $p = Q(x_0)$ as well.

We turn our attention to the estimates from below. The following result plays an important role.

LEMMA 7.1. *If* $1 < p < Q(x_0)$, *then the constant* K_1 *in part* (a) *of Lemma 3.3 is strictly positive.*

Proof. We recall the function v introduced before (7.1). Arguing by contradiction we assume that $K_1 \le 0$. Then,

$$\begin{aligned} 0 &\ge \sum_{j=1}^{m}\int_{B(x_0,R)} A_j(x,u,Xu)X_jv\,dx - \int_{B(x_0,R)} f(x,u,Xu)v\,dx \\ &= \sum_{j=1}^{m}\int_{B(x_0,R)} A_j(x,v,Xv)X_jv\,dx - \int_{V_1} v\,f(x,v,Xv)\,dx \\ &\quad - \int_{V_2} v\,f(x,u,Xu)\,dx = I + II + III, \end{aligned} \tag{7.15}$$

where we have let $V_1 = [B(x_0,R) \setminus B(x_0,\sigma)] \cap \{x \in \Omega \mid u(x) < m(\sigma)\}$, $V_2 = B(x_0,\sigma) \cup \{x \in \Omega \mid u(x) \geq m(\sigma)\}$. As in (7.4) one obtains

$$I \geq \frac{1}{2}\int_{B(x_0,R)} |Xv|^p \, dx - C_1, \tag{7.16}$$

for some $C_1 > 0$ depending only on Q, p and the structural assumptions. On the set V_1 we have $u \equiv v$ and, therefore,

$$|II| \leq \int_{V_1} |v| \, |f(x,v,Xv)| \, dx \leq \int_{V_1} f_1 v |Xv|^{p-1} + f_2 v^p + f_3 v.$$

To estimate the latter we use Hölder inequality and Sobolev Theorem 3.1, obtaining for R small enough

$$|II| \leq \frac{1}{3}\left[\int_{B(x_0,R)} |Xv|^p \, dx + C_2\right] \tag{7.17}$$

with $C_2 > 0$ depending on the structural assumptions. From (7.15)–(7.17) one has (recalling that $v \equiv m(\sigma)$ on V_2)

$$\int_{B(x_0,R)} |Xv|^p \, dx \leq C_3\left[m(\sigma)\int_{V_2} |f(x,u,Xu)| \, dx + C_4\right]. \tag{7.18}$$

Since $v/m(\sigma)$ is admissible for $\text{cap}_p\,(B(x_0,\sigma); B(x_0,R))$ we conclude

$$m(\sigma)^p \leq C_4 \,\text{cap}_p\,(B(x_0,\sigma); B(x_0,R))^{-1}\left[m(\sigma)\int_{V_2} |f(x,u,Xu)| \, dx + C_4\right]. \tag{7.19}$$

At this point we claim that there exists a positive constant A, depending on x_0, Ω, p and the structure assumptions (3.16) such that for every $\sigma < \sigma_0$,

$$m(\sigma) \leq A\left(\frac{|B(x_0,\sigma)|}{\sigma^p}\right)^{\frac{1-\delta}{1-p}}, \tag{7.20}$$

where $\delta = \frac{\varepsilon}{Q(x_0)-p+\varepsilon}$, with ε as in (S). We observe that once the claim is proved, then (7.20) implies that the solution v belongs to $L^t(B(x_0,R))$ with $t = \frac{Q(x_0)(p-1)}{Q(x_0)-p}(1+\delta)$. This can be recognized by means of a decomposition of the ball $B(x_0,R)$ in dyadic rings. The argument is similar to those used in the proof of Theorems 4.1, 4.2. As a consequence of Corollary 6.1 we infer that v has a removable singularity at x_0. This contradiction stems from the assumption $K_1 \leq 0$.

We are thus left with proving the claim. To this regard suppose that for $\sigma < \sigma_0$ we have

$$m(\sigma) > \left(\frac{|B(x_0,\sigma)|}{\sigma^p}\right)^{\frac{1-\delta}{1-p}}, \tag{7.21}$$

since, if the opposite inequality holds there is nothing to prove. Let $x \in B(x_0, \frac{\sigma_0}{2}) \setminus B(x_0, \theta\sigma^{1-\delta})$, where $\theta > 1$ is to be determined later. Setting $r = F(x_0, \Gamma(x_0,x)^{-1})$, by the Harnack inequality there exists a positive constant C_5 such that

$$u(x) \le \sup_{\partial B(x_0,r)} u \le C_5 \inf_{\partial B(x_0,r)} u = C_5 m(r).$$

From this estimate, (7.11) and (7.13) one obtains

$$u(x) \le C_5 m(r) \le C_6 \left(\frac{r^p}{|B(x_0,r)|}\right)^{\frac{1}{p-1}}.$$

In view of (3.7) the previous inequality yields

$$u(x) \le C_7 r^{-\frac{Q(x_0)-p}{p-1}}. \tag{7.22}$$

From (7.20), (7.21) we infer

$$u(x) \le C_8 \theta^{-\frac{Q(x_0)-p}{p-1}} m(\sigma),$$

where (3.6) has been used. Choosing θ appropriately large we conclude

$$u(x) < m(\sigma)$$

for every $x \in B(x_0, \frac{\sigma_0}{2}) \setminus B(x_0, \theta\sigma^{1-\delta})$. The latter inequality, along with the fact $m(\sigma) \to +\infty$ as $\sigma \to 0$, implies that $V_2 \subset B(x_0, \theta\sigma^{1-\delta})$, provided that σ is small enough. Thereby, we obtain from (7.19)

$$\begin{aligned} m(\sigma)^p \ \le\ & C_3 \operatorname{cap}_p (B(x_0,\sigma); B(x_0,R))^{-1} \\ & \cdot \left[m(\sigma) \int_{B(x_0,\theta\sigma^{1-\delta})} |f(x,u,Xu)|\, dx + C_4 \right]. \end{aligned} \tag{7.23}$$

Using (7.14) in the right-hand side of (7.23) we conclude

$$m(\sigma)^p \le C_9 [m(\sigma)\sigma^{\eta(1-\delta)} + C_4] \operatorname{cap}_p (B(x_0,\sigma), B(x_0,R))^{-1}.$$

Estimate (3.6) and Theorem 4.1 yield

$$m(\sigma)^p \leq C_{10}\left[m(\sigma)\sigma^{p-Q(x_0)+\eta(1-\delta)} + \sigma^{p-Q(x_0)}\right]. \tag{7.24}$$

Applying Lemma 3.4 in (7.24), and recalling the definition of δ, we conclude

$$m(\sigma) \leq C_{11}\left[\sigma^{\frac{p-Q(x_0)+\varepsilon(1-\delta)}{p-1}} + \sigma^{\frac{p-Q(x_0)}{p}}\right] = C_{11}\left[\sigma^{\frac{p-Q(x_0)}{p-1}(1-\delta)} + \sigma^{\frac{p-Q(x_0)}{p}}\right].$$

Since $\frac{Q(x_0)-p}{p-1}(1-\delta) > \frac{Q(x_0)-p}{p}$ (at least for ε small), the latter inequality and (3.6) imply the claim (7.19). This concludes the proof of Lemma 7.1. □

Remark 5. We explicitly observe that the restriction $p < Q(x_0)$ in the statement of the theorem is only due to the previous lemma. If the positivity of K_1 is assumed as a hypothesis, then the following can be easily extended to include the endpoint case $p = Q(x_0)$.

We can now complete the proof of Theorem 7.1 by showing that the estimate from below holds for u as in the statement. By (6.14) and Lemma 7.1 one can find $\sigma_2 \in (0, \sigma_0)$ such that for $0 < \sigma < \sigma_2$ we have

$$\int_{B(x_0,\sigma)} |f(x, u, Xu)|\, dx \leq \frac{1}{2}K_1. \tag{7.25}$$

Now fix $1 > \eta > 0$ and choose $\theta = \theta(\eta) \in \overset{\circ}{S}{}^{1,p}(B(x_0, \sigma_2))$, with $\theta \equiv 1$ on $B(x_0, \sigma)$, such that

$$\int_{B(x_0,\sigma_2)} |X\theta|^p\, dx \leq \frac{\operatorname{cap}_p(B(x_0, \sigma); B(x_0, \sigma_2))}{1-\eta}. \tag{7.26}$$

Since $\max_{\partial B(x_0,\sigma)} \theta = 1$, Lemma 3.3 (a) yields

$$K_1 = \sum_{J=1}^{m} \int_{B(x_0,R)} A_j(x, u, Xu) X_J\theta\, dx - \int_{B(x_0,R)} \theta f(x, u, Xu)\, dx. \tag{7.27}$$

Hölder's inequality, (7.25) and (7.26) give

$$\begin{aligned} &C(1-\eta)^{\frac{1}{p-1}} \operatorname{cap}_p(B(x_0, \sigma); B(x_0, \sigma_2))^{-\frac{1}{p-1}} \\ &\qquad \leq \int_{B(x_0,\sigma_2)\setminus B(x_0,\sigma)} |A(x, u, Xu)|^{p'}\, dx, \end{aligned} \tag{7.28}$$

where $C > 0$ depends on K_1 and P, but not on η.

Let $M(\sigma) = \max_{\partial B(x_0,\sigma)} u$ and

$$v(x) = \begin{cases} u^+(x) & x \in B(x_0,R) \setminus \overline{B(x_0,\sigma)}, \\ \min(M(\sigma),u(x)) & x \in B(x_0,\sigma) \setminus \{x_0\}, \\ M(\sigma) & x = x_0. \end{cases}$$

We observe that $v \in \overset{\circ}{S}{}^{1,p}(B(x_0,R))$. This can be seen by noting that $\tilde{v} = \min(M(\sigma),u^+) \in \overset{\circ}{S}{}^{1,p}(B(x_0,R))$ (recall that u is negative on $\partial B(x_0,R)$), and that $(u - M(\sigma))^+ \chi_{B(x_0,\sigma)^C} = v - \tilde{v} \in \overset{\circ}{S}{}^{1,p}(B(x_0,R) \cap B(x_0,\sigma)^C) \subset \overset{\circ}{S}{}^{1,p}(B(x_0,R))$.

With this choice of the function v, along with the maximum principle Theorem 5.1 and Sobolev's embedding Theorem 3.1 the argument in [S2] can be repeated almost word by word, concluding from (7.28) that

$$C(1-\eta)^{\frac{1}{p-1}} \operatorname{cap}_p(B(x_0,\sigma); B(x_0,\sigma_2))^{-\frac{1}{p-1}} \le M(\sigma).$$

Letting $\eta \to 0$ and using the Harnack inequality Theorem 3.3, we obtain the left-hand side inequality in the statement of Theorem 7.1. This completes the proof. □

Remark 6. The range of p can be extended to include $Q(x_0) = p$ in Theorem 6.1, if we restrict ourselves to the estimate from above. More precisely we have the following: There exist positive constants B and R depending on Ω, p, (3.16), u and C_2 from Theorem 4.1 such that if $0 < d(x,x_0) < R/2$ then

$$u(x) \le B \operatorname{cap}_{Q(x_0)}(B(x_0,d(x,x_0)), B(x_0,R))^{\frac{-1}{Q(x_0)-1}}.$$

This is showed by means of a slight variation of the first part of the proof of Theorem 7.1. We exploit the fact that the number K_2 in (7.1) needs not be positive in this argument.

Theorem 7.1 and the capacity estimates in Section 4 can be used to extend to the nonlinear contest the famous estimates (3.8) of Sanchez-Calle [SC] and Nagel, Stein and Wainger [NSW].

We start with the following:

LEMMA 7.2. *Let Ω, x_0, $Q(x_0)$ and u be as in the statement of Theorem 7.1 If the singularity of u in x_0 is not removable, we have for any $\phi \in C_0^\infty(\Omega)$*

$$\sum_{j=1}^m \int_\Omega A_j(x,u,Xu) X_j\phi(x)\,dx - \int_\Omega f(x,u,Xu)\phi(x)\,dx = K_1\phi(x_0),$$

with K_1 as in Lemma 3.3 (a).

We postpone the proof of Lemma 7.2 to the end of the section.

Definition 7.1. Let Ω, p and x_0 as before. Let $\Gamma_p \in S^{1,p}_{\mathrm{loc}}(\Omega \setminus \{x_0\})$ be a nonnegative solution of (1.7) in $\Omega \setminus \{x_0\}$ having a nonremovable singularity in x_0. We say that Γ_p is a fundamental solution in Ω with pole at x_0 for the operator

$$\mathcal{L}_p u = \sum_{j=1}^{m} X_j^* A_j(x, u, Xu) - f(x, u, Xu),$$

if for any $\phi \in C_0^\infty(\Omega)$

$$\sum_{j=1}^{m} \int_\Omega A_j(x, \Gamma_p, X\Gamma_p) X_j \phi \, dx - \int_\Omega f(x, \Gamma_p, X\Gamma_p)\phi \, dx = \phi(x_0).$$

THEOREM 7.2. *Let Ω, x_0, $Q(x_0)$ be as in Theorem 7.1, and let Γ_p be a fundamental solution with pole at $x_0 \in \Omega$ for the operator $\mathcal{L}_p$, $1 < p \le Q(x_0)$. Then there exist positive constants C and R_0 such that for any $0 < r < R_0$ and $x \in B(x_0, r)$*

$$C\left(\frac{d(x, x_0)^p}{|B(x_0, d(x, x_0))|}\right)^{\frac{1}{p-1}} \le \Gamma_p(x) \le C^{-1}\left(\frac{d(x, x_0)^p}{|B(x_0, d(x, x_0))|}\right)^{\frac{1}{p-1}},$$

when $1 < p < Q(x_0)$, whereas

$$C \log\left(\left(\frac{R_0}{d(x, x_0)}\right) \le \Gamma_p(x) \le C^{-1} \log\left(\left(\frac{R_0}{d(x, x_0)}\right),$$

when $p = Q(x_0)$. The constants C and R_0 depend on Ω, the structure conditions (3.16)*, and C_1 (or C_2, depending on the value of p) from the statement of Theorem 4.1, and Theorem 4.2.*

Proof. The proof follows at once from Theorem 7.1 in the case $1 < p < Q(x_0)$. If $p = Q(x_0)$ we observe that by Lemma 7.2, Γ_p is a singular solution for (1.7) with a nonremovable singularity at x_0, for which $K_1 = 1$. This observation allows us to carry the proof of Theorem 7.1 without having to go through Lemma 7.1. The desired estimates on the fundamental solutions now follow easily. □

Proof of Lemma 7.2 We divide the proof in two steps:

(1) There exist positive constants C, and R_0 such that if $0 < r < R_0$ and $x \in B(x_0, r)$, then if $p < Q(x_0)$

$$\int_{B(x_0,r)} |A(x, u, Xu)| \, dx \le Cr, \tag{7.29}$$

while, if $p = Q(x_0)$

$$\int_{B(x_0,r)} |A(x,u,Xu)|\,dx \le Cr\left(\log\frac{2R_0}{r}\right)^{Q(x_0)-1}, \tag{7.30}$$

where $|A|^2 = \sum_{j=1}^m A_j^2$. In order to prove (7.29) and (7.30) recall that from the structure assumption (3.16) we have

$$\begin{aligned}\int_{B(x_0,2r)\setminus B(x_0,r)} |A(x,u,Xu)|\,dx \;\le\; & C_1 \int_{B(x_0,2r)\setminus B(x_0,r)} |Xu|^{p-1}\,dx \\ &+ \int_{B(x_0,2r)\setminus B(x_0,r)} g_2|u|^{p-1}\,dx \\ &+ \int_{B(x_0,2r)\setminus B(x_0,r)} g_3\,dx.\end{aligned} \tag{7.31}$$

By Caccioppoli inequality, Hölder inequality and the doubling condition one obtains

$$\begin{aligned}\int_{B(x_0,2r)\setminus B(x_0,r)} |Xu|^{p-1}\,dx \;\le\; & C|B(x_0,r)|^{\frac{1}{p}} r^{1-p} \\ & \cdot\left(\int_{B(x_0,3r)\setminus B(x_0,r/2)} |u|^p\,dx + C|B(x_0,r)|\right)^{\frac{p-1}{p}}.\end{aligned} \tag{7.32}$$

The Harnack inequality and (3.18) yield

$$\sup_{B(x_0,3r)\setminus B(x_0,r/2)} |u| \le Cm(r), \tag{7.33}$$

where $m(r)$ is as in the proof of Theorem 7.1. From (7.32), the estimates from above in Theorem 7.1, Theorem 4.1 and (7.33) we infer

$$\int_{B(x_0,2r)\setminus B(x_0,r)} |Xu|^{p-1}\,dx \le Cr, \tag{7.34}$$

in the case $p < Q(x_0)$. Recalling (3.5) we also have if $p = Q(x_0)$

$$\begin{aligned}\int_{B(x_0,2r)\setminus B(x_0,r)} |Xu|^{p-1}\,dx \;\le\; & Cr\left(\frac{|B(x_0,r)|}{r^{Q(x_0)}}\right)\left(\log\frac{1}{r}\right)^{p-1} \\ \le\; & Cr\left(\log\frac{1}{r}\right)^{p-1}.\end{aligned} \tag{7.35}$$

Next we estimate the remaining terms in (7.31). By Hölder inequality, Harnack inequality, and the doubling condition one obtains

$$\int_{B(x_0,2r)\setminus B(x_0,r)} g_2|u|^{p-1}\,dx \ \le\ C|B(x_0,r)|^{1-\frac{p-1}{Q(x_0)}}m(r)^{p-1},$$
$$\int_{B(x_0,2r)\setminus B(x_0,r)} g_3\,dx \ \le\ C|B(x_0,r)|^{1-\frac{p-1}{Q(x_0)}}.$$

After a decomposition argument similar to the one used in Theorem 7.1 the inequalities above, together with Theorem 7.1 and the capacity estimates, yield (7.29) and (7.30).

(2) Let $\phi \in C_0^\infty(B(x_0,R_0))$ and $\eta \in C_0^\infty(B(x_0,2r))$ be such that $\eta = 1$ on $B(x_0,r)$ and $|X\eta| \le C/r$ (for the existence of such a function see Lemma 3.1). Define $\theta = (1-\eta)\phi + \eta\phi(x_0)$. Lemma 3.3 (a) gives

$$\begin{aligned}(7.36)\qquad \int_{B(x_0,R_0)} A_j(x,u,Xu)X_j\theta \ -\ \int_{B(x_0,R_0)} f(x,u,Xu)\theta &= K_1\phi(x_0)\\
&= \int_{B(x_0,R_0)\setminus B(x_0,r)} A_j(x,u,Xu)(1-\eta)X_j\phi\\
&\quad - \int_{B(x_0,2r)\setminus B(x_0,r)} A_j(x,u,Xu)\phi X_j\eta\\
&\quad - \int_{B(x_0,R_0)\setminus B(x_0,r)} (1-\eta)\phi f\\
&\quad + \int_{B(x_0,2r)\setminus B(x_0,r)} A_j(x,u,Xu)\phi(x_0)X_j\eta\\
&\quad - \int_{B(x_0,2r)} f\phi(x_0)\eta.\end{aligned}$$

The result now follows from Step 1, (7.14), (7.36) and the estimate

$$\sup_{B(x_0,2r)\setminus B(x_0,r)} |\phi(x)-\phi(x_0)| \le \sup_{B(x_0,R_0)} |X\phi(x)|r,$$

when we let $r \to 0$. ◻

8. Stratified Lie groups. The results of the previous sections acquire a special appeal in the setting of stratified Lie groups. By this is meant (see [F2] and [St2]) a simply connected nilpotent Lie group together with a stratification $g = \oplus_{j=1}^{\ell} V_j$ of its Lie algebra g, that is a decomposition of g as a vector space sum $g = V_1 \oplus \ldots \oplus V_\ell$ such that $[V_1, V_j] = V_{j+1}$ for $1 \le j \le \ell$ and $[V_1, V_\ell] = 0$.

There is a natural family of dilations on such a Lie algebra, namely

$$\delta_\lambda(X_1 + X_2 + \ldots + X_\ell) = \lambda X_1 + \lambda^2 X_2 + \ldots + \lambda^\ell X_\ell,$$

with $X_i \in V_i$. Since the exponential map exp: $g \to G$ is a diffeomorphism, then the dilations δ_λ can be lifted to a one parameter group of automorphisms of G, still denoted by δ_λ. Accordingly, we define $Q = \sum_{j=1}^{\ell} j(\dim(V_j))$, to be the homogeneous dimension of the Lie group G. If $X_1, \ldots, X_m \in V_1$ is a vector basis, then the corresponding left-invariant vector fields on G (that we still call $X_1, \ldots, X_m$) will satisfy the Hörmander condition for hypoellipticity [H]. The homogeneous dimension in G related to the family $X_1, \ldots, X_m$ is constant in G and coincides with the number Q defined above. Moreover, the Nagel, Stein and Wainger's polynomial $\Lambda(x, R)$ described in (3.1) is in this case a monomial, so $Q = Q(x_0)$ for every $x_0 \in G$. This last observation allows us to simplify the statement of Theorems 4.1, 4.2, 7.1 and 7.2. In the following, we write $A \approx B$ if there exists a positive constant C such that $CA < B < C^{-1}A$.

THEOREM 8.1. *Let $x_0 \in G$ and $r < R$. We have*

$$\operatorname{cap}_p(B(x_0, r), B(x_0, R)) \cong \begin{cases} r^{(Q-p)} & \text{when } 1 < p < Q; \\ \left(\log \dfrac{R}{r}\right)^{(1-Q)} & \text{when } p = Q; \\ \left[R^{\frac{p-Q}{p-1}} - r^{\frac{p-Q}{p-1}}\right]^{(1-p)} & \text{when } p < Q. \end{cases}$$

The constant of proportionality depends only on p, G and $X_1, \ldots, X_m$.

THEOREM 8.2. *Let $\Omega \subset G$ be a bounded open set, $x_0 \in \Omega$ and $u \in S^{1,p}_{\text{loc}}(\Omega \setminus \{x_0\})$ be a solution of (1.7) in $\Omega \setminus \{x_0\}$ with $1 < p < Q$. Then one of the following holds:*

(i) *The singularity at x_0 is removable.*

(ii) *There exists $R_0 > 0$ such that for $x \in B(x_0, R_0/2)$,*

$$u(x) \approx \operatorname{cap}_p(B(x_0, d(x, x_0)), B(x_0, R_0))^{\frac{-1}{p-1}}.$$

The constant of proportionality depends on u, p, (3.16), G and $X_1, \ldots, X_m$.

THEOREM 8.3. *Let $\Omega \subset G$ be a bounded open set, $x_0 \in \Omega$ and $\Gamma_p(x, x_0)$ be the fundamental solution in Ω for the operator $\mathcal{L}_p$ with pole at x_0. Then there exists $R_0 > 0$ depending only on Ω and* (3.16) *such that if $x \in B(x_0, R_0)$*

$$\Gamma_p(x, x_0) \approx \left(\frac{d(x, x_0)^p}{|B(x_0, d(x, x_0))|}\right)^{\frac{1}{p-1}} \approx d(x, x_0)^{\frac{p-Q}{p-1}},$$

when $1 < p < Q$*, whereas*

$$\Gamma_p(x, x_0) \approx \log\left(\frac{R_0}{d(x, x_0)}\right),$$

when $p = Q$*. The constant of proportionality depends only on* p*, (3.16),* G *and* $X_1, \ldots, X_m$.

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