INSTABILITY OF GRAPHICAL STRIPS AND A POSITIVE ANSWER TO THE BERNSTEIN PROBLEM IN THE HEISENBERG GROUP $\mathbb{H}^1$

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1. **Introduction**

One of the most celebrated problems in geometry and calculus of variations is the Bernstein problem, which asserts that a C^2 minimal graph in $\mathbb{R}^3$ must necessarily be an affine plane. Following an old tradition, here minimal means of vanishing mean curvature. Bernstein [Be] established this property in 1915. Almost fifty years later a new insight of Fleming [Fle] sparked a major development in the geometric measure theory which, through the celebrated works [DG3], [Al], [Sim], [BDG] culminated in the following solution of the Bernstein problem.

Theorem 1.1. *Let $\mathcal{S} = \{(x, u(x)) \in \mathbb{R}^{n+1} | x \in \mathbb{R}^n, x_{n+1} = u(x)\}$ be a C^2 minimal graph in $\mathbb{R}^{n+1}$, i.e., let $u \in C^2(\mathbb{R}^n)$ be a solution of the minimal surface equation*

$$div \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) \; = \; 0 \; , \tag{1.1}$$

in the whole space. If $n \leq 7$, then there exist $a \in \mathbb{R}^n$, $\beta \in \mathbb{R}$ such that $u(x) =< a, x > +\beta$, i.e., $\mathcal{S}$ must be an affine hyperplane. If instead $n \geq 8$, then there exist non affine (real analytic) functions on $\mathbb{R}^n$ which solve (1.1)*.*

The purpose of this paper is to study, in the simplest model of a sub-Riemannian space, the three-dimensional Heisenberg group $\mathbb{H}^1$, the structure of C^2 minimal graphs with empty characteristic locus and which, on every compact set, minimize the horizontal perimeter. As a corollary of our results we obtain a positive answer to a sub-Riemannian analogue of the Bernstein problem. From the perspective of geometry the relevance of $\mathbb{H}^1$ lies in the fact that this Lie group constitutes the simplest prototype of a class of graded nilpotent Lie groups which arise as "tangent spaces" in the Gromov-Hausdorff limit of Riemannian spaces, see [Be], [Gro1],[Gro2], [Mon], [CDPT]. Furthermore, $\mathbb{H}^n$ is an interesting model of a metric space with a non-trivial geometry.

The Bernstein problem in $\mathbb{H}^1$ has recently received increasing attention, see [GP], [CHMY], [CH], [RR1], [RR2], [HP], [ASV], [DGN2], [GS], [CHY], [BSV]. While we refer to the discussion below, to section 2, and to the cited references for a detailed description of the relevant geometric setting, it seems appropriate here to provide the reader with some historical perspective and a brief overview of the main contributions of the present paper.

When approaching the sub-Riemannian Bernstein problem one is confronted with the fact that there exists smooth entire minimal graphs which are not affine. For instance, in $\mathbb{H}^1$ the surface $\mathcal{S}$ defined by $t = xy/2$ is minimal, in the sense that its H-mean curvature, defined in

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(2.10) below, vanishes identically. This is of course in sharp contrast with Theorem 1.1. What lurks in the dark here are two aspects: (i) On the one hand, the so-called characteristic locus of the surface $\mathcal{S}$, i.e., the collection of points at which the fiber of the subbundle generated by the horizontal distribution coincides with the tangent space of $\mathcal{S}$: on this set the horizontal Gauss map becomes singular (for instance, the surface $t = xy/2$ has non-empty characteristic locus); (ii) The drastically different nature of the relevant minimal surface equation, with respect to the classical case. The classical minimal surface equation (1.1) is a quasi-linear elliptic equation, whereas in $\mathbb{H}^1$, away from the characteristic locus, one has a degenerate quasi-linear hyperbolic equation (with one vanishing eigenvalue). Recall that a quasi-linear second order equation $au_{xx} + 2bu_{xy} + cu_{yy} = d$, where a, b, c, d depend on x, y, u, u_x, u_y, is called elliptic (on the solution u) if $ac - b^2 > 0$, hyperbolic is $ac - b^2 < 0$. For instance, for a C^2 graph $\mathcal{S} \subset \mathbb{R}^3$ of the type $t = u(x, y)$, (1.1) is equivalent to

$$(1 + u_y^2)u_{xx} - 2u_xu_yu_{xy} + (1 + u_x^2)u_{yy} \ = \ 0 \ ,$$

for which one has $ac-b^2 = 1+u_x^2+u_y^2 > 0$. On the other hand the condition that $\mathcal{S}$ be H-minimal becomes, away from the characteristic locus $\Sigma(\mathcal{S}) = \{(x, y, t) \in \mathcal{S} \mid u_y - \frac{x}{2} = 0\ ,\ u_x + \frac{y}{2} = 0\}$,

$$\left(u_y - \frac{x}{2}\right)^2 u_{xx} - 2\left(u_x + \frac{y}{2}u_y\right)\left(u_y - \frac{x}{2}\right)u_{xy} + \left(u_x + \frac{y}{2}\right)^2 u_{yy} \ = \ 0 \ .$$

Since in this case $ac - b^2 \equiv 0$, we conclude that the equation is degenerate hyperbolic. It is interesting to observe that since, away from the characteristic locus, we can always locally parameterize $\mathcal{S}$ in one of the two forms (1.7), (1.8) below, then by (4.19) below, in terms of the function $\phi(u, v)$ the H-minimality of $\mathcal{S}$ is expressed by the pde

$$\phi_{uu} + 2\phi\phi_{uv} + \phi^2\phi_{vv} \ = \ -\ \phi_v(\phi\phi_v + 2\phi_u) \ ,$$

which again is degenerate hyperbolic.

The above example $t = xy/2$ is not isolated since a basic result first proved in [CHMY], and also independently with a different proof in [GP], shows in particular that every H-minimal entire graph over the horizontal plane $t = 0$ must have non-empty characteristic locus, see Theorem 5.5 below. These considerations suggest that, in order to have a reasonably behaved horizontal Gauss map, in the sub-Riemannian Bernstein problem one should impose the restriction that the horizontal bundle $H\mathbb{H}^1$ be transversal to the tangent bundle $T\mathcal{S}$ at every point of the surface, i.e., $\mathcal{S}$ should have empty characteristic locus. This, in turn, immediately imposes a restriction on the type of planes which are appropriate in the sub-Riemannian Bernstein problem. Since every plane $ax + by + ct = \gamma$, for which $c \neq 0$, possesses the isolated characteristic point $(-2b/c, 2a/c, \gamma/c))$, it is clear that we want to confine the attention to the non-characteristic vertical planes

$$\tilde{P}_0 \ = \ \{(x, y, t) \in \mathbb{H}^1 \mid ax + by = \gamma\} \ . \tag{1.2}$$

The appropriateness of these planes is also confirmed by the fundamental Rademacher-Stepanov type theorem of Pansu [Pa]. Specialized to the present setting, the latter states that if $F : \mathbb{H}^1 \to \mathbb{R}$ is a Lipschitz map with respect to the Carnot-Carathéodory distance associated with the subbundle $H\mathbb{H}^1$, then F is Pansu differentiable at a.e. point $g = (x, y, t) \in \mathbb{H}^1$ (w.r.t. Lebesgue measure), and the Pansu differential is given by $F_*(x, y, t) = ax+by$, for some $a, b \in \mathbb{R}$. This result underscores the special role of the vertical planes (1.2) in sub-Riemannian geometric measure theory. A closely related remarkable fact, discovered in [FSS1], is that the blow-up à la De Giorgi of a set with locally finite H-perimeter at a point of its reduced boundary is again a plane such as (1.2). It is time to introduce a basic definition.

Definition 1.2. *We say that a surface $\mathcal{S} \subset \mathbb{H}^1$ of class C^2 is an* entire graph *if there exists a plane P, having equation $ax+by+ct = \gamma$, such that for every point $g_0 \in P$ the straight line passing* *through* g_0 *and parallel to the Euclidean normal* $\boldsymbol{N}_e = (a,b,c)$ *to* P, $L(g_0) = \{g_0 + s\boldsymbol{N}_e \mid s \in \mathbb{R}\}$, *intersects the surface* $\mathcal{S}$ *in exactly one point.*

The above considerations suggest the natural conjecture that if $\mathcal{S} \subset \mathbb{H}^1$ is an entire H-minimal graph, with empty characteristic locus, then $\mathcal{S}$ should be a vertical plane $\tilde{P}_0$ such as (1.2). Since, as we have mentioned, H-minimal is intended in the sense that the horizontal mean curvature $\mathcal{H}$ vanishes identically as a continuous function on $\mathcal{S}$, it is worth stressing that, thanks to the first variation formula in Theorem 3.2 below, a C^2 surface with empty characteristic locus is H-minimal if and only if it is a critical point of the H-perimeter given by (3.5) below. However, for $\mathbb{H}^1$ the situation is very different than in Euclidean space. In fact, in [GP] the second and the fourth named authors discovered the following counterexample to such a plausible sub-Riemannian version of the Bernstein problem. The real analytic surface

$$S \;=\; \{(x,y,t) \in \mathbb{H}^1 \mid x = y \tan\tanh(t)\} \tag{1.3}$$

is an entire H-minimal graph, with empty characteristic locus, over the coordinate (y,t)-plane in $\mathbb{H}^1$. This example seems to cast a dim light over the sub-Riemannian Bernstein problem.

There is however a deeper aspect of the problem which in the classical case is confined to the background, but which, due to the diverse nature in the sub-Riemannian setting of the relevant area functional, the horizontal perimeter, might be playing an important role. What could be happening, in fact, is that H-minimal surfaces such as (1.3) are unstable, in the sense that they are only critical points, but not local minimizers of the H-perimeter. This phenomenon, which goes back to the classical findings of Bernoulli, see e.g. [Ca], has of course no counterpart in the Bernstein problem in flat space, since, thanks to the convexity of the area functional $A(\mathcal{S}) = \int_\Omega \sqrt{1+|Du|^2}dx$, stability is automatic for a solution to (1.1), see e.g. [CM]. On the other hand, stability had played an important role in the work on complete minimal surfaces in 3-manifolds by Fischer-Colbrie and Schoen [FCS], who generalized Bernstein's theorem and proved, in particular, that the only complete stable oriented minimal surfaces in $\mathbb{R}^3$ are the planes. This latter result was also independently obtained in [DCP].

For the functional which expresses the horizontal perimeter the convexity fails. To see this negative phenomenon consider for instance the situation in which $\mathcal{S}$ is parameterized by (1.7), then it was proved in [ASV] that the H-perimeter of $\mathcal{S}$ is expressed by the functional

$$P_H(\mathcal{S}) \;=\; \mathcal{P}(\phi) \;=\; \int_\Omega \sqrt{1+\mathcal{B}_\phi(\phi)^2}\, dudv\ ,$$

where $\mathcal{B}_\phi(\phi)$ denotes the nonlinear inviscid Burger operator acting on ϕ, see (4.20). If we set $\xi = \phi_u$, $\eta = \phi_v$, then the integrand of the above functional is given by

$$F(\phi,\xi,\eta) \;=\; \sqrt{1+(\xi+\phi\eta)^2}\ .$$

The Hessian of F is given by

$$\nabla^2 F \;=\; F^{-3}\begin{pmatrix} \eta^2 & \eta & \phi\eta + (\xi+\phi\eta)F^2 \\ \eta & 1 & \phi \\ \phi\eta + (\xi+\phi\eta)F^2 & \phi & \phi^2 \end{pmatrix}\ ,$$

and therefore $det\ \nabla^2 F = -(\xi+\phi\eta)^2 F^{-5}$, which shows that F is not convex.

These considerations suggest that the above conjecture could be repaired as follows: *In* $\mathbb{H}^1$ *the vertical planes are the only stable entire* H*-minimal graphs.* As a corollary of our results, we will answer affirmatively this amended conjecture in Theorem 1.8. We note that in Theorem 3.5 below we show that the vertical planes (1.2) are stable. An H-minimal surface with empty characteristic locus is called *stable* if the second variation of the H-perimeter is nonnegative for every compactly supported deformation, see Definition 3.3 below. The role of stability in the sub-Riemannian Bernstein problem has been recently highlighted in [DGN2], where the first three named authors have proved the instability of the H-minimal entire graphs $x = y(\alpha t + \beta)$, with $\alpha > 0$, $\beta \in \mathbb{R}$. This result also clarified an incorrect belief by several experts in the field, namely that such surfaces should constitute a counterexample to the above formulated amended form of the Bernstein conjecture.

We are ready to give a summary of our results. In $\mathbb{H}^1$ we single out a large class of H-minimal surfaces, which we call graphical strips, see Definition 1.3 below, and which, after possibly a left-translation and rotation about the t-axis, can be represented in one of the two forms (1.4), or (1.5). If for the function G in these definitions we have $G' > 0$ on some sub-interval, we call the relative surface a strict graphical strip. In Theorem 1.5 we show that graphical strips are H-minimal, and have empty characteristic locus. Our first main result shows that every strict graphical strip is unstable, in the sense that there exist local deformations of the surface which strictly increase the horizontal perimeter, see Theorem 1.6. Our second main result, Theorem 1.7, shows that, modulo left-translations and rotations about the group center, every H-minimal entire graph in $\mathbb{H}^1$, with empty characteristic locus, and which is not itself a vertical plane, contains a strict graphical strip. Combining this result with Theorem 1.6, we prove that the only stable H-minimal entire graphs in $\mathbb{H}^1$, with empty characteristic locus, are the vertical planes, see Theorem 1.8.

To state our main theorems we begin with a definition which plays a central role in our work.

Definition 1.3. *We say that a C^2 surface $\mathcal{S} \subset \mathbb{H}^1$ is a* graphical strip *if there exist an interval $I \subset \mathbb{R}$, and $G \in C^2(I)$, with $G' \geq 0$ on I, such that, after possibly a left-translation and a rotation about the t-axis, then either*

$$\mathcal{S} \ = \ \{(x,y,t) \in \mathbb{H}^1 \mid (y,t) \in \mathbb{R} \times I, x = yG(t)\} \ , \tag{1.4}$$

or

$$\mathcal{S} \ = \ \{(x,y,t) \in \mathbb{H}^1 \mid (x,t) \in \mathbb{R} \times I, y = -xG(t)\} \ . \tag{1.5}$$

If there exists $J \subset I$ such that $G' > 0$ on J, then we call $\mathcal{S}$ a strict graphical strip.

Remark 1.4. *We stress the importance of the assumed strict positivity of G' in the definition of strict graphical strip, as opposed to the weaker requirement $G' \geq 0$. This positivity will play a crucial role in the proof of Theorem 1.6. We also mention explicitly that while a vertical plane such as* (1.2) *is a graphical strip, it is not a strict graphical strip. If for instance $a \neq 0$, then we can re-write* (1.2) *as $x = \alpha y + \beta$, with $\alpha = -b/a$, and $\beta = \gamma/a$. Assuming $\beta = 0$, which can be always achieved by a left-translation (see* (1.6) *below), we would have $G(t) \equiv \alpha$, and therefore $G' \equiv 0$, against the assumption in Definition 1.3.*

We now state a first theorem which, besides having an interest in its own right, also serves to motivate our main results.

Theorem 1.5. *Let $\mathcal{S} \subset \mathbb{H}^1$ be a graphical strip, then $\mathcal{S}$ is H-minimal and it has empty characteristic locus. When $I = \mathbb{R}$, then every surface such as* (1.4) *or* (1.5) *is a global intrinsic H-minimal graph.*

We stress that in the Heisenberg group the left-translations (2.2) are affine transformations, thereby they preserve planes and lines. For instance, if P denotes the plane $ax + by + ct = \gamma$, then denoting by $P' = g_0 \circ P$, where $g_0 = (x_0, y_0, t_0)$, one easily sees that P' is given by

$$\left(a + \frac{cy_0}{2}\right) x + \left(b - \frac{cx_0}{2}\right) y + ct \ = \ \gamma + ax_0 + by_0 + ct_0 \ . \tag{1.6}$$

Notice that a vertical plane ($c = 0$) is mapped into a vertical plane by a left-translation. More in general, the left-translations preserve the property of a surface of having empty characteristic locus. Furthermore, they preserve the H-mean curvature, and therefore the H-minimality, the

H-perimeter, and the property of a surface of being stable. We also notice that rotations about the t-axis (the group center), also have the same properties.

The notion of intrinsic graph in the second part of Theorem 1.5 is that introduced in [FSS3], but see also [FSS2]. The proof of Theorem 1.5 shows that if $\mathcal{S}$ is of type (1.4) with $I=\mathbb{R}$, then it is a global X_1-graph, whereas if it is of type (1.5), then $\mathcal{S}$ is a global X_2-graph. We recall that $\mathcal{S}$ is said an intrinsic X_1-graph if there exist an open set $\Omega\subset\mathbb{R}^2_{u,v}$, and a C^2 function $\phi:\Omega\to\mathbb{R}$, such that on Ω we can parameterize $\mathcal{S}$ as follows $(x,y,t)=(0,u,v)\circ\phi(u,v)e_1=(0,u,v)\circ(\phi(u,v),0,0)$. This means that

$$\mathcal{S} \;=\; \Big\{(x,y,t)\in\mathbb{H}^1 \mid (u,v)\in\Omega,(x,y,t)=\Big(\phi(u,v),u,v-\frac{u}{2}\phi(u,v)\Big)\Big\}\ . \tag{1.7}$$

When we can take $\Omega=\mathbb{R}^2_{(u,v)}$, then $\mathcal{S}$ is called a global X_1-graph. Similarly, if the points of $\mathcal{S}$ can be described by $(x,y,t)=(u,0,v)\circ\phi(u,v)e_2=(u,0,v)\circ(0,\phi(u,v),0)$, i.e., if

$$\mathcal{S} \;=\; \Big\{(x,y,t)\in\mathbb{H}^1 \mid (u,v)\in\Omega,(x,y,t)=\Big(u,\phi(u,v),v+\frac{u}{2}\phi(u,v)\Big)\Big\}\ , \tag{1.8}$$

then $\mathcal{S}$ is said an intrinsic X_2-graph (global, if $\Omega=\mathbb{R}^2_{(u,v)}$). Clearly, the vertical planes (1.2) are global intrinsic graphs. If, for instance, $a\neq 0$, then $\tilde{P}_0$ can be parameterized as in (1.7), with

$$\phi(u,v) \;=\; -\,\frac{b}{a}u+\frac{\gamma}{a}\ .$$

Before proceeding, we pause to give some examples which illustrate the situation of Theorem 1.5.

Example 1: The choice $G(t)=\tan\tanh(t)$ makes (1.3) a special case of Theorem 1.5. We stress that in this example $G'(t)>0$ for every $t\in\mathbb{R}$, and therefore the surface $\mathcal{S}$ is a strict graphical strip with $I=\mathbb{R}$. According to Theorem 1.5, we conclude that $\mathcal{S}$ is also a C^ω global intrinsic X_1-graph.

Example 2: The class of strict graphical strips is not contained in that of global intrinsic graphs. Consider, for instance, the function $G(t)=\cot(-t+\frac{\pi}{2})$, with $-\frac{\pi}{2}<t<\frac{\pi}{2}$. We note that $G'(t)>0$, and therefore the corresponding surface in (1.4),

$$x \;=\; y\ \cot(-t+\frac{\pi}{2})\ ,$$

is a strict graphical strip, with $I=(-\frac{\pi}{2},\frac{\pi}{2})$. This surface is the helicoid

$$x \;=\; r\ \cos\theta\ ,\ y \;=\; r\ \sin\theta\ ,\ t \;=-\theta+\frac{\pi}{2}\ ,\ 0\le r<\infty\ ,\ 0<\theta<\pi\ .$$

If we set $D=\mathbb{R}\times I$, then the map $\Phi(y,t)=(y,t+\frac{y^2}{2}G(t))$ is not a diffeomorphism of D onto the whole (u,v)-plane. However, the arguments in the proof of Theorem 1.5 show that it is a diffeomorphism onto the connected open subset $\Omega\subset\mathbb{R}^2_{u,v}$ defined by $\Omega=\mathbb{R}^2\setminus(L^+\cup L^-)$, where $L^+=\{(0,v)\in\mathbb{R}^2\mid v\ge\frac{\pi}{2}\}$, $L^-=\{(0,v)\in\mathbb{R}^2\mid v\le-\frac{\pi}{2}\}$. As a consequence, the helicoid is *not* a global intrinsic graph, although it is a C^ω intrinsic X_1-graph on the domain Ω. If we denote by $\Psi(u,v)=(\Psi_1(u,v),\Psi_2(u,v))=\Phi^{-1}(u,v)$, then the function $\phi(u,v)$ in (1.7) is given by

$$\phi(u,v) \;=\; u\ G(\Psi_2(u,v))\ , \qquad (u,v)\in\Omega\ .$$

Example 3: The choice $G(t)=\alpha t+\beta$, with $\alpha>0$, $\beta\in\mathbb{R}$, gives the strict graphical strips $x=y(\alpha t+\beta)$ studied in [DGN2], where it was proved that these surfaces are unstable. We note that such surfaces are C^ω global intrinsic H-minimal graphs . We also observe that in this example it is possible to compute explicitly the function $\phi(u,v)$ in (1.7) which describes $\mathcal{S}$ as a global X_1-graph. One finds

$$\phi(u,v) \;=\; \frac{2u(\alpha v+\beta)}{2+\alpha u^2}\ . \tag{1.9}$$

Example 4: The surface described by

$$\mathcal{S} \ = \ \{(x,y,t) \in \mathbb{H}^1 \mid (y,t) \in \mathbb{R} \times (0,\infty), x = yt^2\} \ ,$$

is a strict graphical strip with $I = (0,\infty)$, but like the surface in example 2, it is not a global intrinsic graph.

We are now ready to state the first main result of this paper. In what follows we indicate with $\boldsymbol{\nu}^H = \boldsymbol{N}^H/|\boldsymbol{N}^H|$ the horizontal Gauss map of $\mathcal{S}$, see section 2, and by $\mathcal{V}^H_{II}(\mathcal{S};\mathcal{X})$ the second variation of the H-perimeter with respect to a deformation of $\mathcal{S}$ in the direction of the vector field $\mathcal{X}$, see Definitions 3.1 and 3.3 below. An H-minimal surface $\mathcal{S}$ is called *stable* if $\mathcal{V}^H_{II}(\mathcal{S};\mathcal{X}) \geq 0$ for every compactly supported $\mathcal{X} = aX_1 + bX_2 + kT$. Otherwise, it is called unstable. We note that, since thanks to Theorem 1.5 every graphical strip has empty characteristic locus, the horizontal Gauss map $\boldsymbol{\nu}^H$ of such a surface is globally defined.

Theorem 1.6. *Let $\mathcal{S}$ be a strict graphical strip, then $\mathcal{S}$ is unstable. In fact, there exists a continuum of $h \in C^2_0(\mathcal{S})$, for which $\mathcal{V}^H_{II}(\mathcal{S};h\boldsymbol{\nu}^H) < 0$.*

As a consequence of Theorem 1.6, the H-minimal surfaces corresponding to Examples 1, 2, 3 and 4, are all unstable, i.e., they are not local minimizers of the H-perimeter. We emphasize that the class of strict graphical strips is very wide. For instance, as we show in Theorem 1.7 below, every H-minimal entire graph in $\mathbb{H}^1$, with empty characteristic locus, and which is not itself a vertical plane, contains a strict graphical strip. The main ingredients in the proof of Theorem 1.6 are the second variation formulas for the H-perimeter, see Theorem 3.4 below, and the explicit construction of deformations of the surface along which the H-perimeter decreases strictly. This part is delicate and it has been influenced by the recent paper [DGN2]. In connection with Theorem 1.6 we mention that, after the present work was completed, we have received the interesting paper [BSV] in which the authors, using the construction in [DGN2], in combination with other tools in part also independently developed in [GS], establish the instability of global intrinsic graphs, thus answering affirmatively the above formulated Bernstein type conjecture in this setting. As we have seen, Theorem 1.6 includes surfaces, such as for instance the helicoid in example 2, or that in example 4, which are not global intrinsic graphs.

Here is our second main result.

Theorem 1.7. *Let $\mathcal{S} \subset \mathbb{H}^1$ be an H-minimal entire graph, with empty characteristic locus, and that is not itself a vertical plane such as* (1.2)*, then there exists a strict graphical strip $\mathcal{S}_0 \subset \mathcal{S}$.*

We mention explicitly that the above statement should be interpreted in the sense that, after possibly composing with a suitable rotation about the t-axis and a left-translation, the transformed surface contains a portion of the type (1.4), or (1.5). The proof of Theorem 1.7 is based on some of the main results in [GP]. For clarity of exposition we present in Section 5 a detailed account of the main reduction steps. We mention that, alternatively, one could use the results independently obtained in [CHMY] and [CH].

Finally, by combining Theorems 1.6 and 1.7, we answer affirmatively the above formulated Bernstein conjecture.

Theorem 1.8 (**of Bernstein type**). *In $\mathbb{H}^1$ the only stable H-minimal entire graphs, with empty characteristic locus, are the vertical planes* (1.2)*.*

Concerning the higher-dimensional case of Theorem 1.8 we mention that, using the construction in [BDG], we obtain the following negative result.

Theorem 1.9. *In $\mathbb{H}^n$, with $n \geq 5$, there exist C^ω stable H-minimal graphs, with empty characteristic locus, which are not vertical hyperplanes.*

What happens when $n = 2, 3, 4$ is, presently, terra incognita. We will now briefly describe the organization of the paper. In section 2 we collect some basic facts about the Heisenberg group, and introduce the main geometric set-up. In section 3 we collect some results from sub-Riemannian geometric measure theory, specifically, the first and second variation formulas for the horizontal perimeter. Sections 4 and 5 are the central parts of the paper. After proving Theorem 1.5, the remainder of section 4 is devoted to proving Theorem 1.6. In section 5 we prove Theorems 1.7 and 1.8. Finally, in section 6 we prove Theorem 1.9.

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2. **Preliminaries**

In this section we collect some definitions and known results which will be needed in the paper. We recall that the Heisenberg group $\mathbb{H}^n$ is the graded, nilpotent Lie group of step $r = 2$ whose underlying manifold is $\mathbb{C}^n \times \mathbb{R} \cong \mathbb{R}^{2n+1}$, with non-Abelian left-translation

$$L_{(z,t)}(z', t') \ = \ (z, t) \circ (z', t') \ = \ \left(z + z', t + t' - \frac{1}{2} Im(z \cdot \overline{z'}) \right) \ ,$$

and non-isotropic dilations

$$\delta_\lambda(z, t) \ = \ (\lambda z, \lambda^2 t) \ , \qquad \lambda > 0 \ . \tag{2.1}$$

These dilations provide a natural scaling associated with the grading of the Heisenberg algebra $\mathfrak{h}_n = V_1 \oplus V_2$, where $V_1 = \mathbb{C}^n \times \{0\}$, $V_2 = \{0\} \times \mathbb{R}$. The homogeneous dimension associated with (2.1) is $Q = 2n + 2$. We recall that identifying $\mathfrak{h}_n$ with $\mathbb{R}^{2n+1}$, by identifying $z = x + iy \in \mathbb{C}^n$ with $(x, y) \in \mathbb{R}^{2n}$, we have for the bracket of $\xi = (x, y, t)$, $\xi' = (x', y', t') \in \mathfrak{h}_n$

$$[\xi, \xi'] \ = \ (0, 0, x \cdot y' - x' \cdot y) \ .$$

Here, and throughout the paper, we will use $v \cdot w$ to denote the standard Euclidean inner product of two vectors v and w in $\mathbb{R}^n$. It is then clear that $[V_1, V_1] = V_2$, and that V_2 is the group center. Via the Caley map, $\mathbb{H}^n$ can be identified with the boundary of the Siegel upper half-space $\mathcal{U}^n = \{z \in \mathbb{C}^{n+1} \mid Im\ z_{n+1} > 2 \sum_{j=1}^n |z_j|^2\}$, see Ch.12 in [S]. In the real coordinates $g = (x, y, t) \in \mathbb{R}^{2n+1}$ the non-Abelian group law of $\mathbb{H}^n$ is given by

$$g \circ g' \ = \ (x, y, t) \circ (x', y', t') \ = \ (x + x', y + y', t + t' + \frac{1}{2}(x \cdot y' - x' \cdot y)) \ . \tag{2.2}$$

Let $(L_g)_*$ be the differential of the left-translation $L_g(g') = g \circ g'$. A simple computation shows that

$$\begin{aligned} (L_g)_* \left(\frac{\partial}{\partial x_i} \right) &\overset{def}{=} X_i \ = \ \frac{\partial}{\partial x_i} \ - \ \frac{y_i}{2} \frac{\partial}{\partial t} \ , \qquad i = 1, ..., n \ , \\ (L_g)_* \left(\frac{\partial}{\partial y_i} \right) &\overset{def}{=} X_{n+i} \ = \ \frac{\partial}{\partial y_i} \ + \ \frac{x_i}{2} \frac{\partial}{\partial t} \ , \qquad i = 1, ..., n \ , \\ (L_g)_* \left(\frac{\partial}{\partial t} \right) &\overset{def}{=} T \ = \ \frac{\partial}{\partial t} \end{aligned} \tag{2.3}$$

We note that the only non-trivial commutator is

$$[X_i, X_{n+j}] \ = \ \delta_{ij}\ T\ , \qquad\qquad i,j = 1,...,n\ ,$$

therefore the vector fields $\{X_1,...,X_{2n}\}$ generate the Lie algebra $\mathfrak{h}_n$.

Henceforth, $\mathbb{H}^n$ will be endowed with a left-invariant inner product $< \cdot,\cdot >$, with respect to which $\{X_1,...,X_{2n},T\}$ constitute an orthonormal basis. With the exception of the Euclidean inner product, which, as we have said above, we denote $v\cdot w$, no other inner product will be used in this paper, so when we write $< \cdot,\cdot >$ there will be no danger of confusion. The horizontal bundle is $H\mathbb{H}^n = \cup_{g\in\mathbb{H}^n} H_g\mathbb{H}^n$, where

$$H_g\mathbb{H}^n \ = \ span\{X_1(g),...,X_{2n}(g)\}\ .$$

We denote by $\nabla_X Y$ the Levi-Civita connection with respect to $< \cdot,\cdot >$. Projecting such connection onto the horizontal subbundle $H\mathbb{H}^n \subset T\mathbb{H}^n$, we obtain a connection $\nabla^H_X Y$ on $H\mathbb{H}^n$, which we call the *horizontal Levi-Civita connection*. This idea goes back to that of E. Cartan's *non-holonomic connection*, see [C]. For any $X\in\Gamma(T\mathbb{H}^n)$, $Y\in\Gamma(H\mathbb{H}^n)$ we let

$$\nabla^H_X Y \ = \ \sum_{i=1}^{2n} < \nabla_X Y, X_i > X_i\ , \tag{2.4}$$

and one can easily verify that $\nabla^H_X Y$ is metric preserving and torsion free, in the sense that if we define the horizontal torsion of $\mathcal{S}$ as

$$T^H(X,Y) \ = \ \nabla^H_X Y \ - \ \nabla^H_Y X \ - \ [X,Y]^H\ ,$$

where $[X,Y]^H = \sum_{i=1}^{2n} < [X,Y], X_i > X_i$, then $T^H(X,Y) = 0$. If $f\in C^1(\mathbb{H}^n)$, we let

$$\nabla^H f \ = \ \sum_{i=1}^{2n} < \nabla f, X_i > X_i\ ,$$

where we have denoted by ∇f the Riemannian gradient of f.

Given an oriented hypersurface $\mathcal{S}\subset\mathbb{H}^n$, we denote by $\boldsymbol{N}$ the Riemannian non-unit normal to $\mathcal{S}$. Throughout the paper, we will indicate with

$$\Sigma(\mathcal{S}) \ = \ \{g\in\mathcal{S} \mid T_g\mathcal{S} = H_g\mathbb{H}^n\}\ ,$$

characteristic locus of $\mathcal{S}$. Notice that $g\in\Sigma(\mathcal{S})$ is equivalent to having $< \boldsymbol{N}, X_i >= 0$, $i = 1,...,2n$, at g. We recall that it was proved in [B] that $H^{Q-1}(\Sigma(\mathcal{S})) = 0$, where H^s indicates the s-dimensional Hausdorff measure constructed with the Carnot-Carathéodory distance associated with the subbundle $H\mathbb{H}^n$. If $\mathcal{S}$ is of class C^2, then we define the non-unit horizontal normal of $\mathcal{S}$ by

$$\boldsymbol{N}^H \ = \ \sum_{i=1}^{2n} < \boldsymbol{N}, X_i > X_i\ .$$

It is clear that at a given $g\in\mathcal{S}$ one has $\boldsymbol{N}^H \neq 0$ if and only if $g\notin\Sigma(\mathcal{S})$. If $g\notin\Sigma(\mathcal{S})$, then the horizontal tangent space to $\mathcal{S}$ in g is defined by

$$HT_g\mathcal{S} \ = \ \{\boldsymbol{v}\in H_g\mathbb{H}^n \mid < \boldsymbol{v}, \boldsymbol{N}^H >= 0\}\ . \tag{2.5}$$

It is easy to recognize that $HT_g\mathcal{S} = T_g\mathcal{S}\cap H_g\mathbb{H}^n$. The horizontal tangent bundle of $\mathcal{S}$ is $HT\mathcal{S} = \underset{g\in\mathcal{S}}{\cup} T^H_g\mathcal{S}$. At every non-characteristic point of $\mathcal{S}$ we define the horizontal Gauss map by letting

$$\boldsymbol{\nu}^H \ = \ \frac{\boldsymbol{N}^H}{|\boldsymbol{N}^H|}\ .$$

Since

$$H_g\mathbb{H}^n \ = \ HT_g\mathcal{S} \ \oplus \ span\{\boldsymbol{\nu}_g^H\} \ ,$$

we have $dim(HT_g\mathcal{S}) = 2n-1$. For instance, in $\mathbb{H}^1$ we simply have $HT\mathcal{S} = span\{\boldsymbol{e}_1\}$, where

$$\boldsymbol{e}_1 \ = \ (\boldsymbol{\nu}^H)^\perp \ = <\boldsymbol{\nu}^H, X_2> X_1 - \ <\boldsymbol{\nu}^H, X_1> X_2 \ .$$

Definition 2.1. *Let $\mathcal{S} \subset \mathbb{H}^n$ be a C^k hypersurface, $k \geq 2$, with $\Sigma(\mathcal{S}) = \varnothing$, then we define the* horizontal connection *on $\mathcal{S}$ as follows. Let ∇^H denote the horizontal Levi-Civita connection introduced above. For every $X, Y \in C^1(\mathcal{S}; HT\mathcal{S})$ we let*

$$\nabla^{H,\mathcal{S}}_X Y \ = \ \nabla^H_{\overline{X}}\overline{Y} - \ <\nabla^H_{\overline{X}}\overline{Y}, \boldsymbol{\nu}^H> \boldsymbol{\nu}^H \ ,$$

where $\overline{X}, \overline{Y} \in C^1(\mathbb{H}^n; H\mathbb{H}^n)$ are such that $\overline{X} = X$, $\overline{Y} = Y$ on $\mathcal{S}$.

One can check that Definition 2.1 is well-posed, i.e., it is independent of the extensions $\overline{X}, \overline{Y}$ of the vector fields X, Y. For every $X, Y \in C^1(\mathcal{S}; HT\mathcal{S})$ one has

$$\nabla^{H,\mathcal{S}}_X Y \ - \ \nabla^{H,\mathcal{S}}_Y X \ = \ [X,Y]^H - \ <[X,Y]^H, \boldsymbol{\nu}^H> \boldsymbol{\nu}^H \ , \tag{2.6}$$

in other words $\nabla^{H,\mathcal{S}}_X Y - \nabla^{H,\mathcal{S}}_Y X$ equals the projection of $[X,Y]^H$ onto $HT\mathcal{S}$.

It is clear from (2.6) that the horizontal connection $\nabla^{H,\mathcal{S}}$ on $\mathcal{S}$ is not necessarily torsion free. This depends on the fact that it is not true in general that, if $X, Y \in C^1(S; HT\mathcal{S})$, then $[X,Y]^H \in C^1(\mathcal{S}; HT\mathcal{S})$. However, since in the first Heisenberg group $\mathbb{H}^1$ this is trivially true, in this setting $\nabla^{H,\mathcal{S}}$ is torsion free, and therefore it has the properties of a Levi-Civita connection.

Given a function $f \in C^1(\mathcal{S})$ we will denote

$$\nabla^{H,\mathcal{S}} f \ = \ \nabla^H \overline{f} - \ <\nabla^H \overline{f}, \boldsymbol{\nu}^H> \boldsymbol{\nu}^H \ , \tag{2.7}$$

where $\overline{f} \in C^1(\mathbb{H}^n)$ denotes any extension of f. Henceforth, we will let

$$\nabla^{H,\mathcal{S}}_i f \ = <\nabla^{H,\mathcal{S}} f, X_i> = \ X_i \overline{f} - \ <\nabla^H \overline{f}, \boldsymbol{\nu}^H><\boldsymbol{\nu}^H, X_i> \ .$$

Definition 2.2. *Let $\mathcal{S} \subset \mathbb{H}^n$ be a C^k hypersurface, $k \geq 2$, with $\Sigma(\mathcal{S}) = \varnothing$, then for every $X, Y \in C^1(\mathcal{S}; HT\mathcal{S})$ we define a tensor field of type $(0,2)$ on $\mathcal{S}$, as follows*

$$II^{H,\mathcal{S}}(X,Y) \ = <\nabla^H_X Y, \boldsymbol{\nu}^H> \boldsymbol{\nu}^H \ . \tag{2.8}$$

We call $II^{H,\mathcal{S}}(\cdot,\cdot)$ the horizontal second fundamental form *of $\mathcal{S}$. We also define $\mathcal{A}^{H,\mathcal{S}} : HT\mathcal{S} \to HT\mathcal{S}$ by letting for every $g \in \mathcal{S}$ and $\boldsymbol{u}, \boldsymbol{v} \in HT_g\mathcal{S}$*

$$<\mathcal{A}^{H,\mathcal{S}}\boldsymbol{u}, \boldsymbol{v}> = \ - \ <II^{H,\mathcal{S}}(\boldsymbol{u},\boldsymbol{v}), \boldsymbol{\nu}^H> = \ - \ <\nabla^H_X Y, \boldsymbol{\nu}^H> \ , \tag{2.9}$$

where $X, Y \in C^1(\mathcal{S}, HT\mathcal{S})$ are such that $X_g = \boldsymbol{u}$, $Y_g = \boldsymbol{v}$. We call the linear map $\mathcal{A}^{H,\mathcal{S}} : HT_g\mathcal{S} \to HT_g\mathcal{S}$ the horizontal shape operator *If $\boldsymbol{e}_1, ..., \boldsymbol{e}_{2n-1}$ denotes a local orthonormal frame for $HT\mathcal{S}$, then the matrix of the horizontal shape operator with respect to the basis $\boldsymbol{e}_1, ..., \boldsymbol{e}_{2n-1}$ is given by the $(2n-1) \times (2n-1)$ matrix $-\big[<\nabla^H_{\boldsymbol{e}_i}\boldsymbol{e}_j, \boldsymbol{\nu}^H> \big]_{i,j=1,...,2n-1}$.*

Definitions 2.1 and 2.2 are taken from [DGN1]. A different notion of the second fundamental form has been explored by the last named author and R. Hladky [HP] using a generalization of the Webster-Tanaka connection on a wide class of sub-Riemannian manifolds. This second fundamental form and the (un-symmetrized) operator $\mathcal{A}^{H,\mathcal{S}}$ are used to analyze the minimal and constant mean curvature surfaces in this setting. We emphasize that, when restricted to the case of the Carnot groups, these two formulations are equivalent.

We call *horizontal principal curvatures* the real eigenvalues $\kappa_1, ..., \kappa_{2n-1}$ of the symmetrized operator $\mathcal{A}^{H,\mathcal{S}}_{sym} = \frac{1}{2}(\mathcal{A}^{H,\mathcal{S}} + (\mathcal{A}^{H,\mathcal{S}})^t)$. The *horizontal mean curvature* of $\mathcal{S}$ is defined as follows

$$\mathcal{H} \ = \ \kappa_1 \ + \ ... \ + \ \kappa_{2n-1} \ . \tag{2.10}$$

When the hypersurface $\mathcal{S}$ has non empty characteristic locus, then at every $g_0 \in \Sigma(\mathcal{S})$ we define

$$\mathcal{H}(g_0) \ = \ \lim_{g \to g_0, g \in \mathcal{S}\setminus\Sigma} \mathcal{H}(g) \ ,$$

provided that such limit exists, finite or infinite. We do not define the H-mean curvature at those points $g_0 \in \Sigma$ at which the above limit does not exist.

Definition 2.3. *A C^2 surface $\mathcal{S} \subset \mathbb{H}^1$ is called H-minimal if $\mathcal{H} \equiv 0$ as a continuous function on $\mathcal{S}$.*

We will need the following result, which will prove useful for computing the H-mean curvature, see Proposition 9.8 in [DGN1].

Proposition 2.4. *The H-mean curvature in definition* (2.10) *coincides with the function defined by the equation*

$$\mathcal{H} \ = \ \sum_{i=1}^{2n} \nabla_i^{H,\mathcal{S}} \ < \boldsymbol{\nu}^H, X_i > \ .$$

Given a vector field $X \in C^1(\mathbb{H}^n; H\mathbb{H}^n)$, we denote by

$$div^H X \ = \ \sum_{i=1}^{2n} X_i < X, X_i > \ ,$$

the horizontal divergence of X.

3. First and second variation of the H-perimeter

In this section we introduce the relevant notions of stationary and stable surface which enter in the statement of Theorem 1.8, and we recall the first and second variation formulas from [DGN1] and [DGN2] which will be used in its proof.

Given an oriented C^2 surface $S \subset \mathbb{H}^1$, with Riemannian (non-unit) normal $\boldsymbol{N}$, we introduce the quantities

$$p \ = < \boldsymbol{N}, X_1 > \ , \quad q \ = < \boldsymbol{N}, X_2 > \ , \quad \omega \ = < \boldsymbol{N}, T > \ , \quad W \ = \ \sqrt{p^2 + q^2} \ , \tag{3.1}$$

and, at every point where $W \neq 0$, we set

$$\overline{p} \ = \ \frac{p}{W} \ , \quad \overline{q} \ = \ \frac{q}{W} \ , \quad \overline{\omega} \ = \ \frac{\omega}{W} \ . \tag{3.2}$$

Notice that

$$\boldsymbol{N}^H \ = \ pX_1 + qX_2 \ , \quad \boldsymbol{\nu}^H \ = \ \overline{p}X_1 + \overline{q}X_2 \ , \quad < \boldsymbol{N}^H, \boldsymbol{N} > = \ W^2 \ . \tag{3.3}$$

¿From (3.3) we easily recognize that

$$\cos(\boldsymbol{N}^H \angle \boldsymbol{N}) \ = \ \frac{W}{|\boldsymbol{N}|} \ . \tag{3.4}$$

In the classical theory of minimal surfaces, the concept of area or perimeter occupies a central position, see [DG1], [DG2], [G], [MM], [Si], [CM]. In sub-Riemannian geometry there exists an appropriate variational notion of perimeter. Given an open set $\mathcal{U} \subset \mathbb{H}^1$ we denote $\mathcal{F}(\mathcal{U}) = \{\zeta \in C_0^1(\mathcal{U}; H\mathbb{H}^1) \mid ||\zeta||_{L^\infty(\mathcal{U})} \leq 1\}$. A function $f \in L^1(\mathcal{U})$ is said to belong to $BV_H(\mathcal{U})$ (the space of functions with finite horizontal bounded variation), if

$$Var_H(f;\mathcal{U}) \ = \ \sup_{\zeta \in \mathcal{F}(\mathcal{U})} \int_{\mathcal{U}} f \ div^H \zeta \ dg \ < \ \infty \ .$$

This space becomes a Banach space with the norm $||f||_{BV_H(\Omega)} = ||f||_{L^1(\Omega)} + Var_H(u;\Omega)$. Given a measurable set $\mathcal{E} \subset \mathbb{H}^1$, the *H-perimeter* of $\mathcal{E}$ with respect to the open set $\mathcal{U} \subset \mathbb{H}^1$ is defined as follows, see for instance [CDG], and [GN],

$$P_H(\mathcal{E};\mathcal{U}) \ = \ Var_H(\chi_{\mathcal{E}};\mathcal{U}) \ .$$

When $\mathcal{E}$ is a C^1 domain, one can recognize that

$$P_H(\mathcal{E};\mathcal{U}) \ = \ \int_{\partial\mathcal{E}\cap\mathcal{U}} \cos(\boldsymbol{N}^H \angle \boldsymbol{N}) \ d\sigma \ , \tag{3.5}$$

where $d\sigma$ indicates the standard surface measure on $\partial\mathcal{E}$. We will denote by $d\sigma_H$ the H-perimeter measure concentrated on $\mathcal{S} = \partial\mathcal{E}$. According to (3.4), (3.5), we have for any Borel subset $E \subset \mathcal{S}$ such that $P_H(E) < \infty$,

$$\sigma_H(E) \ = \ \int_E \frac{W}{|\boldsymbol{N}|} \ d\sigma \ . \tag{3.6}$$

Two important properties of the H-perimeter are its invariance with respect to the dilations (2.1) and the left-translations (2.2). The former, is expressed by the equation

$$\sigma_H(\delta_\lambda(\mathcal{S})) \ = \ \lambda^{Q-1} \ \sigma_H(\mathcal{S}) \ .$$

In keeping up with the notation of [DGN2] it will be convenient to indicate with $Y\zeta$ and $Z\zeta$ the respective actions of the vector fields $\boldsymbol{\nu}^H$ and $(\boldsymbol{\nu}^H)^\perp$ on a function $\zeta \in C^1(\mathcal{S})$, thus

$$Y\zeta \ \overset{def}{=} \ \overline{p} \ X_1\zeta \ + \ \overline{q} \ X_2\zeta \ , \qquad Z\zeta \ \overset{def}{=} \ \overline{q} \ X_1\zeta \ - \ \overline{p} \ X_2\zeta \ . \tag{3.7}$$

The frame $\{Z, Y, T\}$ is orthonormal. It is worth observing that, since the metric tensor $\{g_{ij}\}$ with respect to the inner product $< \cdot, \cdot >$ has the property $det\{g_{ij}\} = 1$, then the (Riemannian) divergence in $\mathbb{H}^1$ of these vector fields is given by

$$div \ Y \ = \ X_1\overline{p} \ + \ X_2\overline{q} \ = \ \mathcal{H} \ , \qquad div \ Z \ = \ X_1\overline{q} \ - \ X_2\overline{p} \ , \tag{3.8}$$

where the first equality is justified by Proposition 2.4 and by the fact that $|\boldsymbol{\nu}^H| = 1$. Using Cramer's rule one easily obtains from (3.7)

$$X_1\zeta \ = \ \overline{q} \ Z\zeta \ + \ \overline{p} \ Y\zeta \ , \qquad X_2\zeta \ = \ \overline{q} \ Y\zeta \ - \ \overline{p} \ Z\zeta \ . \tag{3.9}$$

One also has

$$\nabla_1^{H,\mathcal{S}}\zeta \ = \ \overline{q} \ Z\zeta \ , \qquad \nabla_2^{H,\mathcal{S}}\zeta \ = \ - \ \overline{p} \ Z\zeta \ , \tag{3.10}$$

so that

$$|\nabla^{H,\mathcal{S}}\zeta|^2 \ = \ (Z\zeta)^2 \ . \tag{3.11}$$

We notice that

$$\overline{q}Z\overline{p} \ - \ \overline{p}Z\overline{q} \ = \ \mathcal{H} \ . \tag{3.12}$$

This can be easily recognized using the first equation in (3.8), and (3.9), as follows

$$\mathcal{H} \ = \ X_1\overline{p} \ + \ X_2\overline{q} \ = \ \overline{q}Z\overline{p} - \overline{p}Z\overline{q} + \overline{p}Y\overline{p} + \overline{q}Y\overline{q} \ = \ \overline{q}Z\overline{p} - \overline{p}Z\overline{q} \ ,$$

where we have used the fact that $0 = \frac{1}{2}Y(\overline{p}^2 + \overline{q}^2) = \overline{p}Y\overline{p} + \overline{q}Y\overline{q}$.

Definition 3.1. *Let $\mathcal{S} \subset \mathbb{H}^1$ be an oriented C^2 surface, with $\Sigma(\mathcal{S}) = \varnothing$. Consider the family of vector fields $\mathcal{X} = aX_1 + bX_2 + kT$, with $a, b, k \in C_0^2(\mathcal{S})$, and the family of surfaces $\mathcal{S}^\lambda$, where for small $\lambda \in \mathbb{R}$ we have let*

$$\mathcal{S}^\lambda = J_\lambda(\mathcal{S}) \ = \ \mathcal{S} \ + \ \lambda\mathcal{X} \ . \tag{3.13}$$

We define the first variation *of the H-perimeter with respect to the deformation* (3.13) *as*

$$\mathcal{V}_I^H(\mathcal{S};\mathcal{X}) \ = \ \frac{d}{d\lambda} \ P_H(\mathcal{S}^\lambda)\Big|_{\lambda=0} \ .$$

We say that $\mathcal{S}$ is stationary *if $\mathcal{V}_I^H(\mathcal{S};\mathcal{X}) = 0$, for every $\mathcal{X}$.*

Classical minimal surfaces are stationary points of the perimeter (the area functional for graphs). It is natural to ask what is the connection between the notion of H-minimal surface and that of H-perimeter. To answer this question we recall the following results from [DGN1].

Theorem 3.2. *Let $\mathcal{S} \subset \mathbb{H}^1$ be an oriented C^2 surface, with $\Sigma(\mathcal{S}) = \varnothing$, then*

$$\mathcal{V}_I^H(\mathcal{S};\mathcal{X}) \ = \ \int_{\mathcal{S}} \mathcal{H} \ \frac{\cos(\mathcal{X}\angle \boldsymbol{N})}{\cos(\boldsymbol{\nu}^H\angle \boldsymbol{N})} \ |\mathcal{X}| \ d\sigma_H \ . \tag{3.14}$$

In particular, $\mathcal{S}$ is stationary if and only if it is H-minimal.

Versions of Theorem 3.2 have also been obtained independently by other people. An approach based on motion by H-mean curvature can be found in [BC]. When $\mathcal{X} = a\boldsymbol{\nu}^H + kT$, then a proof based on CR-geometry can be found in [CHMY], and [RR2]. Recently, a general first variation formula for a wide class of sub-Riemannian spaces has been found in [HP].

Definition 3.3. *Given an oriented C^2 surface $\mathcal{S} \subset \mathbb{H}^1$, with $\Sigma(\mathcal{S}) = \varnothing$, we define the* second variation *of the H-perimeter with respect to the deformation* (3.13) *as*

$$\mathcal{V}_{II}^H(\mathcal{S};\mathcal{X}) \ = \ \frac{d^2}{d\lambda^2} \ P_H(\mathcal{S}^\lambda)\Big|_{\lambda=0} \ .$$

We say that $\mathcal{S}$ is stable*, if it is stationary (i.e., H-minimal), and if*

$$\mathcal{V}_{II}^H(\mathcal{S};\mathcal{X}) \ \geq \ 0 \ , \qquad \text{for every} \quad \mathcal{X}.$$

If there exists $\mathcal{X} \neq 0$ such that $\mathcal{V}_{II}^H(\mathcal{S};\mathcal{X}) < 0$, then we say that $\mathcal{S}$ is unstable.

The following second variation formula from [DGN1], see also Theorem 3.3 in [DGN2], will play a crucial role in the proof of Theorem 1.6.

Theorem 3.4. *Let $\mathcal{S} \subset \mathbb{H}^1$ be a C^2 oriented surface, with empty characteristic locus, then the second variation of the H-perimeter with respect to the deformation* (3.13)*, with $\mathcal{X} = h\boldsymbol{\nu}^H$, $h \in C_0^2(\mathcal{S})$, is given by*

$$\mathcal{V}_{II}^H(\mathcal{S};h\boldsymbol{\nu}^H) \ = \ \int_{\mathcal{S}} \Big\{ (Zh)^2 + h^2\big[2(\overline{p}T\overline{q} - \overline{q}T\overline{p}) + 2\overline{\omega}(\overline{q}Y\overline{p} - \overline{p}Y\overline{q}) + \overline{\omega}^2\big] \Big\} \ d\sigma_H \ . \tag{3.15}$$

If instead we choose $\mathcal{X} = aX_1$, $a \in C_0^2(\mathcal{S})$, in (3.13)*, then the corresponding second variation is given by*

$$\begin{aligned} \mathcal{V}_{II}^H(\mathcal{S};aX_1) \ = \ \int_{\mathcal{S}} \Big\{ & \overline{p}^2(Za)^2 + \overline{p}^2 \ \overline{\omega}^2 \ a^2 \\ & + \ \overline{\omega}Z(a^2) - \overline{p} \ \overline{q}\left(T(a^2) - \overline{\omega}Y(a^2)\right) \Big\} \ d\sigma_H \ . \end{aligned} \tag{3.16}$$

Theorem 3.4 ia special case of a general second variation formula found in [DGN1], see also Theorem 3.3 in [DGN2]. Using such general result, in combination with some integration by parts formulas from [DGN1], we can prove the following theorem.

Theorem 3.5. *Every vertical plane such as* (1.2) *is stable.*

Proof. Since the notion of stability is invariant under left-translations, and rotations about the t-axis, we can assume without restriction the $\tilde{P}_0 = \mathcal{S}$ is the plane $x = 0$. Since for this surface we have $\overline{p} \equiv 1, \overline{q} = \overline{\omega} \equiv 0$, the formula (3.3) in Theorem 3.3 in [DGN2] gives for $\mathcal{X} = aX_1 + bX_2 + kT$,

$$\mathcal{V}_{II}^H(\mathcal{S};\mathcal{X}) \ = \ \int_{\mathcal{S}} \left\{ (Za)^2 + 2(TbZk - TkZb) + T(ab) \right\} d\sigma_H \ .$$

Since a, b, c are compactly supported in $\mathcal{S}$, Lemma 3.7 in [DGN2] gives

$$\int_{\mathcal{S}} T(ab) d\sigma_H \ = \ 0 \ .$$

We also have

$$\begin{aligned} \int_{\mathcal{S}} TbZk \ d\sigma_H \ &= \ \int_{\mathcal{S}} T(bZk) \ d\sigma_H \ - \ \int_{\mathcal{S}} bT(Zk) \ d\sigma_H \\ &= \ - \int_{\mathcal{S}} bZ(Tk) \ d\sigma_H \ + \ \int_{\mathcal{S}} b[Z,T]k \ d\sigma_H \end{aligned}$$

We now using the following commutator formula from [DGN1]

$$[Z,T] \ = \ (\overline{q}T\overline{p} - \overline{p}T\overline{q}) \ Y \ ,$$

valid on any non-characteristic surface. In the present case, such formula gives $[Z,T] = 0$, and we thus conclude

$$\int_{\mathcal{S}} TbZk \ d\sigma_H \ = \ - \int_{\mathcal{S}} bZ(Tk) \ d\sigma_H \ = \ - \int_{\mathcal{S}} Z(bTk) \ d\sigma_H \ + \ \int_{\mathcal{S}} TkZb \ d\sigma_H \ .$$

Since Lemma 3.6 in [DGN2] gives for every $\zeta \in C_0^1(\mathcal{S})$,

$$\int_{\mathcal{S}} Z\zeta \ d\sigma_H \ = \ - \int_{\mathcal{S}} \zeta \, \overline{\omega} \ d\sigma_H \ = \ 0 \ ,$$

we finally obtain

$$\mathcal{V}_{II}^H(\mathcal{S};\mathcal{X}) \ = \ \int_{\mathcal{S}} (Za)^2 d\sigma_H \ \geq \ 0 \ .$$

This proves the stability of $\mathcal{S}$.

□

4. **Instability of strict graphical strips**

After these preparations we turn to the core of the proof of Theorem 1.6. To put the subsequent discussion on a solid ground, we begin with proving Theorem 1.5.

Proof of Theorem 1.5. We provide the proof only for the class of surfaces in (1.4), leaving it to the interested reader to develop the completely analogous details for the second class. It is obvious from the definition (1.4) that $\mathcal{S}$ is a C^2 graph over the open subset $\mathbb{R} \times I$ of the

(y,t)-plane. We next observe that $\mathcal{S}$ has empty characteristic locus. We can use the global defining function

$$\phi(x,y,t) \;=\; x - yG(t) \;, \tag{4.1}$$

and assume that $\mathcal{S}$ is oriented in such a way that $\boldsymbol{N} = \nabla\phi = (X_1\phi)X_1 + (X_2\phi)X_2 + (T\phi)T$. Recalling (3.1), we find

$$p = X_1\phi = 1 + \frac{y^2}{2}G'(t) \;, \qquad q = X_2\phi = -G(t) - \frac{xy}{2}G'(t) \;, \qquad \omega \;= T\phi = -yG'(t) \;. \tag{4.2}$$

Since $p \geq 1 > 0$, we see from (4.2) that $\Sigma(\mathcal{S}) = \varnothing$. In order to prove the H-minimality of $\mathcal{S}$, we use (3.8), which gives

$$\mathcal{H} \;=\; X_1\overline{p} \;+\; X_2\overline{q} \;.$$

¿From now on, to simplify the notation, we will omit the variable t in all expressions involving $G(t), G'(t), G''(t)$. The second equation in (4.2) becomes on $\mathcal{S}$

$$q \;=\; -\,G\,\left(1 \;+\; \frac{y^2}{2}G'\right) \;. \tag{4.3}$$

We thus find on $\mathcal{S}$

$$W^2 \;=\; p^2 + q^2 \;=\; \left(1 + G^2\right)\left(1 \;+\; \frac{y^2}{2}G'\right)^2 \;. \tag{4.4}$$

Since they will be useful in the proof of Lemma 4.1, in what follows we compute several quantities, even if they are not strictly necessary for the calculation of $\mathcal{H}$. From (4.2) we find

$$X_1p \;=\; -\frac{y^3}{4}G'' \;, \qquad X_2p \;=\; yG' + \frac{xy^2}{4}G'' \;, \tag{4.5}$$

$$X_1q \;=\; \frac{xy^2}{4}G'' \;, \qquad X_2q \;=\; -xG' - \frac{x^2y}{4}G'' \;. \tag{4.6}$$

¿From (4.2), (4.3), (4.5), (4.6) we find on $\mathcal{S}$

$$X_2q \;=\; -\,yG\left(G' + \frac{y^2}{4}GG''\right) \;, \tag{4.7}$$

and

$$X_1W \;=\; \frac{pX_1p + qX_1q}{W} \;=\; -\frac{y^3}{4}G''\left(1+G^2\right)^{\frac{1}{2}} \;, \tag{4.8}$$

$$X_2W \;=\; \frac{pX_2p + qX_2q}{W} \;=\; y\left(1+G^2\right)^{\frac{1}{2}}\left\{G' + \frac{y^2}{4}GG''\right\} \;. \tag{4.9}$$

We now compute

$$\mathcal{H} \;=\; \frac{X_1p}{W} - p\frac{X_1W}{W^2} + \frac{X_2q}{W} - q\frac{X_2W}{W^2} \;. \tag{4.10}$$

Using (4.2), (4.3), (4.5), (4.7), (4.8) and (4.9), we find

$$\frac{X_1p}{W} - p\frac{X_1W}{W^2} \;=\; 0 \;, \qquad \frac{X_2q}{W} - q\frac{X_2W}{W^2} \;=\; 0 \;.$$

Inserting the latter two equations in (4.10) we conclude that $\mathcal{S}$ is H-minimal. To complete the proof of the theorem we are left with showing that, when $I = \mathbb{R}$, the surface $\mathcal{S}$ in (1.4) is a global intrinsic X_1-graph according to [FSS3]. To prove this, we want to show that there exist curvilinear coordinates $(u,v) \in \mathbb{R}^2$, and $\phi \in C^2(\mathbb{R}^2_{u,v})$, such that $\mathcal{S}$ can be globally parameterized by

$$\theta(u,v) \;=\; \left(\phi(u,v), u, v - \frac{u}{2}\phi(u,v)\right) \ . \tag{4.11}$$

We thus see that we must have

$$\phi(u,v) \;=\; x \ , \quad u \;= y \ , \qquad v - \frac{u}{2}\phi(u,v) \;=\; t \ .$$

These equations give

$$u = y \ , \ v = t + \frac{u}{2}\phi(u,v) \;=\; t + \frac{y}{2}x \;=\; t + \frac{y^2}{2}G(t) \ .$$

We want to show next that the map $\Phi : \mathbb{R}^2_{y,t} \to \mathbb{R}^2_{u,v}$ given by

$$\Phi(y,t) \;=\; \left(y, t + \frac{y^2}{2}G(t)\right) \ ,$$

defines a global diffeomorphism onto. Now, its Jacobian is given by

$$det \begin{pmatrix} 1 & 0 \\ yG(t) & 1 + \frac{y^2}{2}G'(t) \end{pmatrix} \;=\; 1 + \frac{y^2}{2}G'(t) \;\neq\; 0 \ , \ \text{for every } (y,t) \in \mathbb{R}^2 \ . \tag{4.12}$$

Furthermore, Φ is globally one-to-one. Assume in fact that $\Phi(y_1,t_1) = \Phi(y_2,t_2)$, then we have

$$y_1 = y_2 \ , \quad t_1 + \frac{y_1^2}{2}G(t_1) = t_2 + \frac{y_2^2}{2}G(t_2) \ . \tag{4.13}$$

Now, let $\alpha = y_1 = y_2$, then either $\alpha = 0$, in which case (4.13) gives $t_1 = t_2$, or $\alpha \neq 0$. In this second case, we look at the function $f(t) = t + (\alpha^2/2)G(t)$, and we see that f is strictly increasing over $\mathbb{R}$. Therefore, (4.13) forces again the conclusion $t_1 = t_2$. It is also easy to see that Φ is onto. Thanks to the assumption $G' > 0$ one has in fact for every $y \in \mathbb{R}$

$$\lim_{t \to \pm\infty} t + \frac{y^2}{2}G(t) \;=\; \pm\,\infty \ .$$

Therefore, given $(u,v) \in \mathbb{R}^2$, if we choose $y = u$, then we can always find $t \in R$ such that $t + \frac{y^2}{2}G(t) = v$. In conclusion, Φ is globally invertible on $\mathbb{R}^2$. Denote by $\Psi(u,v) = (\Psi_1(u,v), \Psi_2(u,v))$ the inverse of Φ. Clearly, $y(u,v) = u$. But then the function

$$\phi(u,v) \;=\; uG(\Psi_2(u,v)) \ , \tag{4.14}$$

defines $\mathcal{S}$ as a global intrinsic X_1-graph. We note that, using (4.12) and the inverse function theorem, we obtain for the Jacobian matrix of Ψ

$$J_\Psi(u,v) \;=\; \begin{pmatrix} 1 & 0 \\ -\frac{uG}{1+\frac{u^2}{2}G'} & \frac{1}{1+\frac{u^2}{2}G'} \end{pmatrix} \ , \tag{4.15}$$

where for brevity we have written G instead of $G(\Psi_2(u,v))$, and similarly for G'.

In closing, for the benefit of the reader, we provide a second derivation of the H-minimality of the surface (1.4) based on the fact that it is locally an intrinsic X_1-graph as in (4.11), with $\phi(u,v)$ given by (4.14). We stress that the following computations only use the fact that $\mathcal{S}$ be locally defined as in (4.11) in the neighborhood of any fixed point, for some C^2 function $\phi(u,v)$. We will use the following formula, found in [GS], [BSV], for the H-mean curvature of an intrinsic graph in $\mathbb{H}^1$

$$\mathcal{H} \;=\; -\,\mathcal{B}_\phi\left(\frac{\mathcal{B}_\phi(\phi)}{\sqrt{1+\mathcal{B}_\phi(\phi)^2}}\right) \ , \tag{4.16}$$

where for a function $f \in C^1(\mathbb{R})$

$$B_\phi(f) \;=\; f_u + \phi f_v \tag{4.17}$$

denotes the linear transport operator. One can easily verify that

$$\mathcal{H} \;=\; -\,\frac{\mathcal{B}_\phi(\mathcal{B}_\phi(\phi))}{(1+\mathcal{B}_\phi(\phi)^2)^{\frac{3}{2}}}\ , \tag{4.18}$$

and therefore the condition that $\mathcal{S}$ be H-minimal becomes

$$\mathcal{B}_\phi(\mathcal{B}_\phi(\phi)) \;=\; 0\ , \qquad \phi \in C^2(\mathbb{R}^2)\ , \tag{4.19}$$

where now

$$\mathcal{B}_\phi(\phi) \;=\; \phi_u + \phi\phi_v\ , \tag{4.20}$$

denotes the nonlinear inviscid Burger operator. Using (4.14), (4.15), we compute

$$\phi_u \;=\; G + uG'\frac{\partial \Psi_2}{\partial u} \;=\; G + uG'\left(-\frac{uG}{1+\frac{u^2}{2}G'}\right) \;=\; \frac{G\left(1-\frac{u^2}{2}G'\right)}{1+\frac{u^2}{2}G'}\ ,$$

$$\phi_v \;=\; uG'\frac{\partial \Psi_2}{\partial v} \;==\; \frac{uG'}{1+\frac{u^2}{2}G'}\ .$$

¿From the last two formulas we find

$$\mathcal{B}_\phi(\phi) \;=\; G\ .$$

This gives

$$\mathcal{B}_\phi(\mathcal{B}_\phi(\phi)) \;=\; \mathcal{B}_\phi(G) \;=\; G_u + \phi G_v \;=\; G'\frac{\partial \Psi_2}{\partial u} + uGG'\frac{\partial \Psi_2}{\partial v} \;=\; -\frac{uGG'}{1+\frac{u^2}{2}G'} + \frac{uGG'}{1+\frac{u^2}{2}G'} \;=\; 0\ ,$$

which, according to (4.19), proves the H-minimality of $\mathcal{S}$.

□

We now turn to the proof of Theorem 1.6. Since we will want to compute the second variation of a graphical strip such as (1.4) with respect to deformations along the horizontal normal $\boldsymbol{\nu}^H$, we will need to use formula (3.15) in Theorem 3.4. As a first step, we will compute the quantities which appear as the coefficient of h^2 in the integral in the right-hand side of (3.15). This is the content of the next lemma.

Lemma 4.1. *Let $\mathcal{S}$ be the H-minimal surface given by* (1.4), *then one has*

$$2(\overline{p}T\overline{q} - \overline{q}T\overline{p}) + 2\overline{\omega}(\overline{q}Y\overline{p} - \overline{p}Y\overline{q}) + \overline{\omega}^2 \;=\; -\,\frac{2G'(t)}{W^2}\ . \tag{4.21}$$

Proof. As in the proof of Theorem 1.5, we use the global defining function (4.1), and we obtain (4.2). Using (3.7) and (4.5), (4.6), we obtain on $\mathcal{S}$

$$Yp \;=\; \frac{1}{W}\{pX_1p + qX_2p\} \;=\; -\,\frac{y}{(1+G^2)^{\frac{1}{2}}}\left\{\frac{y^2}{4}G''(1+G^2) + GG'\right\}\ , \tag{4.22}$$

$$Yq \;=\; \frac{1}{W}\{pX_1q + qX_2q\} \;=\; \frac{yG}{(1+G^2)^{\frac{1}{2}}}\left\{\frac{y^2}{4}G''(1+G^2) + GG'\right\}\ . \tag{4.23}$$

Combining (4.22), (4.23) we find

$$YW \;=\; \frac{1}{W}\{pYp + qYq\} \;=\; -\,y\left\{\frac{y^2}{4}G''(1+G^2) + GG'\right\}\ . \tag{4.24}$$

Combining (4.22), (4.23) and (4.24), two small miracles happen, namely

$$Y\overline{p} \ = \ \frac{WYp - pYW}{W^2} \ = \ 0 \ , \qquad Y\overline{q} \ = \ \frac{WYq - qYW}{W^2} \ = \ 0 \ . \tag{4.25}$$

We now turn to the computation of the derivatives along the characteristic direction T. Differentiating (4.2) we obtain on $\mathcal{S}$

$$Tp \ = \ \frac{y^2}{2}G'' \ , \qquad Tq \ = \ - \ \left(G' + \frac{y^2}{2}GG''\right) \ . \tag{4.26}$$

¿From (4.26) we find

$$TW \ = \ \frac{1}{W}\left\{pTp + qTq\right\} \ = \ \frac{1}{(1+G^2)^{\frac{1}{2}}}\left\{GG' + \frac{y^2}{2}G''(1+G^2)\right\} \ . \tag{4.27}$$

Using (4.26) and (4.27) we obtain

$$T\overline{p} \ = \ - \ \frac{GG'}{W(1+G^2)} \ , \qquad T\overline{q} \ = \ - \ \frac{G'}{W(1+G^2)} \ . \tag{4.28}$$

¿From (4.2) and (4.28) we conclude that

$$\overline{p} \ T\overline{q} \ - \ \overline{q} \ T\overline{p} \ = \ - \ \frac{G'}{W(1+G^2)^{\frac{1}{2}}} \ . \tag{4.29}$$

Finally, combining (4.29) with (4.26) and (4.2), we obtain

$$2(\overline{p}T\overline{q} - \overline{q}T\overline{p}) + 2\overline{\omega}(\overline{q}Y\overline{p} - \overline{p}Y\overline{q}) + \overline{\omega}^2 \ = \ - \ \frac{2G'(t)}{W^2} \ ,$$

which proves (4.21).

□

¿From Theorem 3.4 and Lemma 4.1, we obtain the following corollary.

Corollary 4.2. *Let $\mathcal{S}$ be an H-minimal surface given as in* (1.4)*. For any $h \in C^2_0(\mathcal{S})$, one has*

$$\mathcal{V}^H_{II}(\mathcal{S}; h\boldsymbol{\nu}^H) \ = \ \int_{\mathcal{S}} |\nabla^{H,\mathcal{S}}h|^2 d\sigma_H \ - \ 2 \int_{\mathcal{S}} \frac{h^2 \ G'(t)}{W^2} \ d\sigma_H \ . \tag{4.30}$$

For any $a \in C^2_0(\mathcal{S})$, one has

$$\mathcal{V}^H_{II}(\mathcal{S}; aX_1) \ = \ \int_{\mathcal{S}} \frac{\left(1 + \frac{y^2}{2}G'(t)\right)^2}{W^2} |\nabla^{H,\mathcal{S}}a|^2 d\sigma_H \ - \ 2 \int_{\mathcal{S}} \frac{a^2}{W^2(1+G(t)^2)} d\sigma_H \ . \tag{4.31}$$

To proceed further we next project onto the (y,t) plane the formulas (4.30), (4.31) by means of the C^2 parametrization $\theta : \mathbb{R} \times I \to \mathbb{R}^3$ of the surface $\mathcal{S}$ given by $\theta(y,t) = (yG(t), y, t)$.

Lemma 4.3. *Let $\mathcal{S}$ be the H-minimal surface given by* (1.4)*. For any $h \in C^2_0(\mathcal{S})$, one has*

$$\begin{aligned} \mathcal{V}^H_{II}(\mathcal{S}; h\boldsymbol{\nu}^H) \ = \ & \int_{\mathbb{R}\times I} \frac{\left(1 + \frac{y^2}{2}G'(t)\right) \ u_y^2}{(1+G(t)^2)^{1/2}} \ dydt \\ & - \ 2\int_{\mathbb{R}\times I} \frac{u^2 \ G'(t)}{\left(1 + \frac{y^2}{2}G'(t)\right)(1+G(t)^2)^{1/2}} \ dydt \ , \end{aligned} \tag{4.32}$$

where we have set $u = h\circ\theta \in C^2_0(\mathbb{R}\times I)$. *For any* $a\in C^2_0(\mathcal{S})$, *one has*

$$\mathcal{V}^H_{II}(\mathcal{S};aX_1) \ = \ \int_{\mathbb{R}\times I} \frac{\left(1+\frac{y^2}{2}G'(t)\right)\ u_y^2}{(1+G(t)^2)^{3/2}}\ dydt \tag{4.33}$$
$$- \ 2\int_{\mathbb{R}\times I} \frac{u^2\ G'(t)}{\left(1+\frac{y^2}{2}G'(t)\right)(1+G(t)^2)^{3/2}}\ dydt\ ,$$

where this time we have let $u = a\circ\theta \in C^2_0(\mathbb{R}\times I)$.

Proof. In order to prove (4.32) we make some reductions. Keeping in mind (3.6), from (4.4) we obtain

$$\int_{\mathcal{S}} \frac{h^2\ G'(t)}{W^2} d\sigma_H \ = \ \int_{\mathbb{R}\times I} \frac{u^2\ G'(t)}{\left(1+\frac{y^2}{2}G'(t)\right)(1+G(t)^2)^{1/2}}\ dydt\ . \tag{4.34}$$

In order to express the first integral in the right-hand side of (4.30) as an integral on $\mathbb{R}\times I$, we compute $|\nabla^{H,\mathcal{S}}h|^2$. We have from (3.7), (4.2) and (4.3)

$$|\nabla^{H,\mathcal{S}}h|^2 \ = \ (Zh)^2 \ = \ (\overline{q}X_1h - \overline{p}X_2h)^2 \tag{4.35}$$
$$= \ \frac{(G(t)X_1h + X_2h)^2}{1+G(t)^2}\ .$$

Now, the chain rule gives $u_y = G(t)h_x + h_y$, and therefore we see that we have on $\mathcal{S}$

$$G(t)X_1h + X_2h \ = \ G(t)\ h_x + h_y \ = \ u_y\ .$$

¿From (4.35) we thus conclude that

$$\int_{\mathcal{S}} \left|\nabla^{H,\mathcal{S}}h\right|^2 d\sigma_H \ = \ \int_{\mathbb{R}\times I} \frac{\left(1\ +\ \frac{y^2}{2}G'(t)\right)\ u_y^2}{(1+G(t)^2)^{1/2}}\ dydt\ . \tag{4.36}$$

Combining (4.34) and (4.36) we obtain (4.32). The proof of (4.33) proceeds analogously, and we omit the details.

□

The next lemma is the keystone to the proof of Theorem 1.6. In order to state it, given an interval $I_\delta = (-4\delta, 4\delta)$, with $\delta > 0$, we fix a function $\chi\in C^\infty_0(\mathbb{R})$, such that $0\le\chi(s)\le 1$, $\chi\equiv 1$ on $|s|\le\delta$, $\chi\equiv 0$ for $|s|\ge 2\delta$, $|\chi'|\le C = C(\delta)$, and $\int_{\mathbb{R}}\chi(s)ds = 1$. For every $k\in\mathbb{N}$ we define $\chi_k(s) = \chi(s/k)$, so that $\chi_k(s)\equiv 1$ for $|s|\le\delta k$, $\chi_k(s)\equiv 0$ for $|s|\ge 2\delta k$, and $|\chi_k'(s)|\le C/k$, with C independent of k. We also let $\tilde\chi_k(s) = k\chi(ks)$, and notice that the $\int_{\mathbb{R}}\tilde\chi_k(s)ds = 1$ for every $k\in\mathbb{N}$, and that $supp(\tilde\chi_k)\subset[-2\delta/k, 2\delta/k]$.

Lemma 4.4. *Let* $G\in C^2(I_\delta)$ *be such that* $G' > 0$ *on* I_δ. *Define for* $k\in\mathbb{N}$

$$f_k(y,t) \ = \ \frac{\chi_k(y)\chi(t)}{\sqrt{1+\frac{y^2}{2}\,G_k'(t)}}$$

where we have let $G_k(t) = G\star\tilde\chi_k(t)$. *We have* $f_k\in C^\infty_0(\mathbb{R}\times I_\delta)$, *and there exists* $k_0\in\mathbb{N}$ *such that for all* $k > k_0$

$$\int_{\mathbb{R}\times I_\delta} \frac{1+\frac{y^2}{2}\,G'(t)}{\sqrt{1+G(t)^2}}\ \left(\frac{\partial f_k}{\partial y}(y,t)\right)^2\ dydt \ < \ 2\int_{\mathbb{R}\times I_\delta} \frac{G'(t)}{(1+\frac{y^2}{2}\,G'(t))\sqrt{1+G(t)^2}}\ f_k(y,t)^2\, dydt\ . \tag{4.37}$$

Proof. We begin by observing that since $G' \in C(I_\delta)$, such function is uniformly continuous on $[-2\delta, 2\delta]$. We recall from basic properties of approximations to the identity that $G'_k = G' \star \tilde{\chi}_k \to G'$ uniformly on $[-2\delta, 2\delta]$. As a consequence of this, and of the fact that there exists $\epsilon > 0$ such that $G' \geq \epsilon$ on $[-2\delta, 2\delta]$, we can find $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, and for all $t \in [-2\delta, 2\delta]$, one has

$$\begin{cases} \frac{1}{2}\,G'(t) \ \leq \ G'_k(t) \ \leq \ 2\,G'(t) \ , \\ \frac{1}{2}\,(1 + \frac{y^2}{2}\,G'(t)) \ \leq \ 1 + \frac{y^2}{2}\,G'_k(t) \ \leq \ 2\,(1 + \frac{y^2}{2}\,G'(t)) \ . \end{cases} \tag{4.38}$$

We begin with the right-hand side of (4.37).

$$\begin{aligned} (RHS) \ &= 4 \int_{\mathbb{R}} f_k(y,t)^2 \left(\frac{G'(t)}{(2 + y^2\,G'(t))\sqrt{1 + G(t)^2}} \right) dydt \\ &= \ 8 \int_{-2\delta}^{2\delta} \chi(t)^2\, \frac{G'(t)}{\sqrt{1 + G(t)^2}} \left(\int_{\mathbb{R}} \frac{\chi_k(y)^2}{(2 + y^2\,G'(t))\,(2 + y^2\,G'_k(t))}\, dy \right) dt \\ &\longrightarrow \ 8 \int_{-2\delta}^{2\delta} \chi(t)^2\, \frac{G'(t)}{\sqrt{1 + G(t)^2}} \left(\int_{\mathbb{R}} \frac{1}{(2 + y^2\,G'(t))^2}\, dy \right) dt \\ &= \ 8 \int_{-2\delta}^{2\delta} \chi(t)^2\, \frac{G'(t)}{\sqrt{1 + G(t)^2}} \left(\frac{\sqrt{2}\,\pi}{8\,\sqrt{G'(t)}} \right) dt \\ &= \ \sqrt{2}\,\pi \int_{-2\delta}^{2\delta} \chi(t)^2 \sqrt{\frac{G'(t)}{1 + G(t)^2}}\, dt \qquad \text{as } k \to \infty \ , \end{aligned} \tag{4.39}$$

where we have used (4.38) and Lebesgue dominated convergence theorem.

On the other hand, we obtain for the integral in the left-hand side of (4.37)

$$\begin{aligned} (LHS) \ &= \int_{\mathbb{R}\times I_\delta} \frac{W}{1 + G(t)^2} \left(\frac{\partial}{\partial y} f_k(y,t) \right)^2 dydt \\ &= \int_{\mathbb{R}\times I_\delta} \chi(t)^2 \frac{2 + y^2\,G'(t)}{\sqrt{1 + G(t)^2}} \left\{ \frac{\chi'_k(y)^2}{2 + y^2\,G'_k(t)} - \frac{2\,y\,\chi_k(y)\chi'_k(y)\,G'_k(t)}{(2 + y^2\,G'_k(t))^2} + \frac{y^2\,\chi_k(y)^2\,G'_k(t)^2}{(2 + y^2\,G'_k(t))^3} \right\} dydt \\ &= \int_{\mathbb{R}\times I_\delta} \frac{\chi(t)^2}{\sqrt{1 + G(t)^2}}\ \chi'_k(y)^2\ \frac{2 + y^2\,G'(t)}{2 + y^2\,G'_k(t)}\, dydt \\ &\quad - \int_{\mathbb{R}\times I_\delta} \chi(t)^2\, \frac{G'_k(t)}{\sqrt{1 + G(t)^2}} \frac{y\,(2 + y^2\,G'(t))}{(2 + y^2\,G'_k(t))^2} \big(\chi_k(y)^2\big)'\, dydt \\ &\quad + \int_{\mathbb{R}\times I_\delta} \chi(t)^2\, \frac{G'_k(t)^2}{\sqrt{1 + G(t)^2}}\ \chi_k(y)^2\ \frac{y^2\,(2 + y^2\,G'(t))}{(2 + y^2\,G'_k(t))^3}\, dydt \ . \end{aligned} \tag{4.40}$$

Using the fact that

$$\frac{\partial}{\partial y} \left(\frac{y}{2 + y^2\,G'_k(t)} \right) \ = \ \frac{2 - y^2\,G'_k(t)}{(2 + y^2\,G'_k(t))^2} \ , \qquad \frac{\partial}{\partial y} \left(\frac{2 + y^2\,G'(t)}{2 + y^2\,G'_k(t)} \right) \ = \ \frac{4\,y\,(G'(t) - G'_k(t))}{(2 + y^2\,G'_k(t))^2} \ ,$$

we integrate by parts the integral containing the term $\left(\chi_k(y)^2\right)'$, obtaining

$$
\begin{aligned}
(4.41)\qquad & -\int_{\mathbb{R}\times I_\delta} \chi(t)^2\,\frac{G_k'(t)}{\sqrt{1+G(t)^2}}\,\frac{y\,(2+y^2\,G'(t))}{(2+y^2\,G_k'(t))^2}\left(\chi_k(y)^2\right)'\,dydt \\
&= 2\int_{I_\delta}\chi(t)^2\frac{G_k'(t)}{\sqrt{1+G(t)^2}}\left(\int_{\mathbb{R}}\frac{\chi_k(y)^2}{(2+y^2\,G_k(t)^2)^2}\,\frac{2+y^2\,G'(t)}{2+y^2\,G_k'(t)}\,dy\right)dt \\
&\quad - \int_{\mathbb{R}\times I_\delta}\chi(t)^2\,\frac{G_k'(t)^2}{\sqrt{1+G(t)^2}}\,\chi_k(y)^2\,\frac{y^2\,(2+y^2\,G'(t))}{(2+y^2\,G_k'(t))^3}\,dydt \\
&\quad + \int_{I_\delta}\chi(t)^2\,\frac{G_k'(t)(G'(t)-G_k'(t))}{\sqrt{1+G(t)^2}}\left(\int_{\mathbb{R}}\chi_k(y)^2\,\frac{4\,y^2}{(2+y^2\,G_k'(t))^2}\,dy\right)dt\ .
\end{aligned}
$$

Using (4.41) in (4.40), and after canceling one term, we obtain for the left-hand side of (4.37)

$$
\begin{aligned}
(4.42)\qquad (LHS) &= \int_{\mathbb{R}\times I_\delta}\frac{W}{1+G(t)^2}\left(\frac{\partial}{\partial y}f_k(y,t)\right)^2 dydt \\
&= \int_{\mathbb{R}\times I_\delta}\frac{\chi(t)^2}{\sqrt{1+G(t)^2}}\,\chi_k'(y)^2\,\frac{2+y^2\,G'(t)}{2+y^2\,G_k'(t)}\,dydt \\
&\quad + 2\int_{-2\delta}^{2\delta}\chi(t)^2\frac{G_k'(t)}{\sqrt{1+G(t)^2}}\left(\int_{\mathbb{R}}\frac{\chi_k(y)^2}{(2+y^2\,G_k'(t))^2}\,\frac{2+y^2\,G'(t)}{2+y^2\,G_k'(t)}\,dy\right)dt \\
&\quad + \int_{I_\delta}\chi(t)^2\,\frac{G_k'(t)(G'(t)-G_k'(t))}{\sqrt{1+G(t)^2}}\left(\int_{\mathbb{R}}\chi_k(y)^2\,\frac{4\,y^2}{(2+y^2\,G_k'(t))^2}\,dy\right)dt \\
&= I_k\ +\ II_k\ +\ III_k\ .
\end{aligned}
$$

We now analyze each of the three integrals in (4.42). Using the fact that $G'(t) > 0$ on I_δ, we also have $G_k'(t) = G' \star \tilde{\chi}_k(t) > 0$ on I_δ. When $k \to \infty$ the first integral satisfies

$$
\begin{aligned}
(4.43)\qquad I_k &= \int_{\mathbb{R}\times I}\frac{\chi(t)^2}{\sqrt{1+G(t)^2}}\,\chi_k'(y)^2\,\frac{2+y^2\,G'(t)}{2+y^2\,G_k'(t)}\,dydt \\
&\le \frac{C}{k^2}\int_{-2\delta}^{2\delta}\frac{1}{\sqrt{1+G(t)^2}}\left(\int_{-2k\delta}^{2k\delta}\frac{2+y^2\,G'(t)}{2+y^2\,G_k'(t)}\,dy\right)dt \\
\text{(by (4.38))}\quad &\le \frac{C}{k^2}\int_{-2\delta}^{2\delta}\frac{1}{\sqrt{1+G'(t)^2}}\left(\int_{-2k\delta}^{2k\delta}2\,dy\right)dt\ =\ \frac{8C\delta}{k}\int_{-2\delta}^{2\delta}\frac{1}{\sqrt{1+G'(t)^2}}\,dt \longrightarrow\ 0\ .
\end{aligned}
$$

Similarly, letting $k \to \infty$ we have for the third integral

$$
\begin{aligned}
(4.44)\qquad III_k &= \int_{I_\delta}\chi(t)^2\,\frac{G_k'(t)(G'(t)-G_k'(t))}{\sqrt{1+G(t)^2}}\left(\int_{\mathbb{R}}\chi_k(y)^2\,\frac{4\,y^2}{(2+y^2\,G_k'(t))^2}\,dy\right)dt \\
\text{(by (4.38))}\quad &\le 16\sup_{t\in[-2\delta,2\delta]}\left|G_k'(t)-G'(t)\right|\int_{-2\delta}^{2\delta}\frac{G'(t)}{\sqrt{1+G(t)^2}}\left(\int_{\mathbb{R}}\frac{y^2}{(2+y^2\,G'(t))^2}\,dy\right)dt \longrightarrow\ 0\ ,
\end{aligned}
$$

from the uniform convergence of G_k' to G' on $[-2\delta, 2\delta]$. Finally, we have

(4.45)

$$\begin{aligned} II_k &= 2\int_{-2\delta}^{2\delta} \chi(t)^2 \frac{G_k'(t)}{\sqrt{1+G(t)^2}} \left(\int_{\mathbb{R}} \frac{\chi_k(y)^2}{(2+y^2\,G_k'(t))^2}\,\frac{2+y^2\,G'(t)}{2+y^2\,G_k'(t)}\,dy\right) dt \\ &\longrightarrow\ 2\int_{-2\delta}^{2\delta} \chi(t)^2\, \frac{G'(t)}{\sqrt{1+G(t)^2}} \left(\int_{\mathbb{R}} \frac{1}{(2+y^2\,G'(t)^2)^2}\,dy\right) dt \\ &=\ 2\int_{-2\delta}^{2\delta} \chi(t)^2\, \frac{G'(t)}{\sqrt{1+G(t)^2}} \left(\frac{\sqrt{2}\,\pi}{8\,\sqrt{G'(t)}}\right) dt \ =\ \frac{\sqrt{2}\,\pi}{4}\int_{-2\delta}^{2\delta} \chi(t)^2 \sqrt{\frac{G'(t)}{1+G(t)^2}}\,dt\ . \end{aligned}$$

In the above, we have used (4.38) to deduce that for large enough k

$$\frac{G_k'(t)}{\sqrt{1+G(t)^2}} \left(\frac{\chi_k(y)^2}{(2+y^2\,G_k'(t))^2}\,\frac{2+y^2\,G'(t)}{2+y^2\,G_k'(t)}\right) \le 16\,\frac{G'(t)}{\sqrt{1+G(t)^2}}\,\frac{1}{(2+y^2\,G'(t))^2} \in L^1(\mathbb{R}\times[-2\delta,2\delta])$$

and therefore, Lebesgue dominated convergence theorem applies. To summarize, we have as $k \to \infty$

$$(LHS)\ =\ \int_{\mathbb{R}\times I_\delta} \frac{W}{1+G(t)^2}\left(\frac{\partial}{\partial y} f_k(y,t)\right)^2 dydt \longrightarrow \frac{\sqrt{2}\,\pi}{4}\int_{-2\delta}^{2\delta} \chi(t)^2 \sqrt{\frac{G'(t)}{1+G(t)^2}}\,dt\ .$$

Combining this with(4.39), we reach the sought for conclusion.

□

We are now ready to prove Theorem 1.6.

Proof of Theorem 1.6. Let $\tilde{\mathcal{S}}$ be a graphical strip, then there exist $I \subset \mathbb{R}$, $\tilde{G} \in C^2(\mathbb{R})$, with $G \ge 0$, such that, after possibly a left-translation and a rotation about the t-axis, $\tilde{\mathcal{S}}$ can be represented in the form $x = y\tilde{G}(t)$ for $(y,t) \in \mathbb{R}\times I$. If we assume further that $\tilde{\mathcal{S}}$ is a strict graphical strip, then we can find an interval $J = (a,b) \subset I$, such that $\tilde{G}' > 0$ on J. Since the stability, or the instability, are invariant under left-translations and rotations, it will suffice to prove that $\tilde{\mathcal{S}}$ is unstable. Assume without restriction that $-\infty < a < b < \infty$, and set $t_0 = (a+b)/2$, $g_0 = (0,0,-t_0)$. Consider the left-translated surface $\mathcal{S} = g_0 \circ \tilde{\mathcal{S}}$, see (2.2), then $\mathcal{S}$ is described by

$$x\ =\ y\,G(t)\ , \qquad (y,t) \in \mathbb{R}\times I_\delta\ ,$$

with $4\delta = (b-a)/2$, and $G(t) = \tilde{G}(t_0+t)$. Since it is clear that $G' > 0$ on I_δ, we can apply Lemma 4.4, and conclude that there exists $k_0 \in \mathbb{N}$ such that for $k \ge k_0$ the sequence f_k satisfies (4.37). This being said, we now define $h_k : \mathbb{H}^1 \to \mathbb{R}$ as follows

$$h_k(x,y,t)\ =\ \frac{1}{\sqrt{1+\frac{y^2}{2}G_k'(t)}}\,\chi_k(y)\chi(t)\chi_k(x-y\,G(t))\ .$$

We observe that $h_k(\theta(y,t)) = f_k(y,t)\chi_k(0) = f_k(y,t)$, and therefore $h_k \in C_0^2(\mathcal{S})$. At this point, appealing to (4.32) in Lemma 4.3, and to Lemma 4.4, we conclude that for every fixed $k \ge k_0$, we have

$$\mathcal{V}_{II}^H(\mathcal{S}; h_k\boldsymbol{\nu}^H)\ <\ 0\ .$$

This proves that $\mathcal{S}$ is unstable, and therefore such is also the surface $\tilde{\mathcal{S}}$.

□

Remark 4.5. *In particular, since every global minimizer is also a local one, we have also shown that $\mathcal{S}$ cannot be a global minimizer of the H-perimeter.*

5. Instability of H-minimal entire graphs and proof of the Bernstein conjecture

In this section we prove Theorems 1.7 and 1.8. Our strategy will be to first establish Theorem 1.7, and then combine this result with Theorem 1.6 to obtain Theorem 1.8. As we have mentioned in the introduction, our proof of Theorem 1.7 is based on the main results in [GP], but we reiterate that an alternative proof could be obtained combining the results in the two papers [CHMY] and [CH].

We begin by recalling the basic notion of seed curve from [GP]. In what follows, $\Omega \subset \mathbb{R}^2$ denotes a given connected, open set of the (x,y)-plane, and $f \in C^k(\Omega)$, with $k \geq 2$. We consider the graph of f over Ω

$$\mathcal{S} \ = \ \{(x,y,t) \in \mathbb{H}^1 \mid (x,y) \in \Omega, t = f(x,y)\} \ . \tag{5.1}$$

We assume that $\mathcal{S}$ be oriented in such a way that $\boldsymbol{N} = \nabla\phi = (X_1\phi)X_1 + (X_2\phi)X_2 + (T\phi)T$, were $\phi(x,y,t) = t - f(x,y)$. From (3.1) we obtain

$$p \ = \ X_1\phi \ = \ -f_x - \frac{y}{2} \ , \quad q = X_2\phi \ = \ -f_y + \frac{x}{2} \ , \qquad W = \sqrt{\left(f_x + \frac{y}{2}\right)^2 + \left(f_y - \frac{x}{2}\right)^2} \ ,$$

and thus we have from (3.3),

$$\boldsymbol{\nu}^H \ = \ - \frac{f_x + \frac{y}{2}}{\sqrt{\left(f_x + \frac{y}{2}\right)^2 + \left(f_y - \frac{x}{2}\right)^2}} \ X_1 \ - \ \frac{f_y - \frac{x}{2}}{\sqrt{\left(f_x + \frac{y}{2}\right)^2 + \left(f_y - \frac{x}{2}\right)^2}} \ X_2 \ ,$$

away from $\Sigma(\mathcal{S})$. We stress that, since $\mathcal{S}$ is a graph over the (x,y)-plane, $\boldsymbol{\nu}^H$ is independent of the variable t. Such crucial property would not be true for a graphical portion over the coordinate planes (y,t) or (x,t). We can thus identify in a natural fashion $\boldsymbol{\nu}^H$ with a unit C^{k-1} vector field $\tilde{\boldsymbol{\nu}}^H$ onto the (x,y)-plane as follows (recall that $k \geq 2$). Given a point $g = (z,t) \in \mathcal{S}$, with $z = (x,y)$, we let

$$\tilde{\boldsymbol{\nu}}^H(z) \ = \ \left(- \frac{f_x + \frac{y}{2}}{\sqrt{\left(f_x + \frac{y}{2}\right)^2 + \left(f_y - \frac{x}{2}\right)^2}} \ , \ - \frac{f_y - \frac{x}{2}}{\sqrt{\left(f_x + \frac{y}{2}\right)^2 + \left(f_y - \frac{x}{2}\right)^2}} \right) \ .$$

The notation

$$(\tilde{\boldsymbol{\nu}}^H)^\perp \ = \ \left(- \frac{f_y - \frac{x}{2}}{\sqrt{\left(f_x + \frac{y}{2}\right)^2 + \left(f_y - \frac{x}{2}\right)^2}} \ , \ \frac{f_x + \frac{y}{2}}{\sqrt{\left(f_x + \frac{y}{2}\right)^2 + \left(f_y - \frac{x}{2}\right)^2}} \right) \ ,$$

will indicate the unit vector field in Ω perpendicular to $\tilde{\boldsymbol{\nu}}^H$ (with respect to the Euclidean inner product $u \cdot v$ in $\mathbb{R}^2$).

Definition 5.1. *Let $\mathcal{S}$ be a C^2 graph as in (5.1), with $\Sigma(\mathcal{S}) = \varnothing$, and suppose that $\mathcal{S}$ be H-minimal. Given a point $z \in \Omega \subset \mathbb{R}^2$, a* seed curve *of $\mathcal{S}$ based at z is defined to be the integral curve of the vector field $\tilde{\boldsymbol{\nu}}^H$ with initial point z. Denoting such a seed curve by $\gamma_z(s)$, we then have*

$$\gamma_z'(s) \ = \ \tilde{\boldsymbol{\nu}}^H(\gamma_z(s)) \ , \qquad \gamma_z(0) \ = \ z \ . \tag{5.2}$$

We will indicate by $\tilde{\mathcal{L}}_z(r)$ the integral curve of $(\tilde{\boldsymbol{\nu}}^H)^\perp$ starting at the point z, i.e.,

$$\tilde{\mathcal{L}}_z'(r) \ = \ (\tilde{\boldsymbol{\nu}}^H)^\perp(\tilde{\mathcal{L}}_z(r)) \ , \qquad \tilde{\mathcal{L}}_z(0) \ = \ z \ . \tag{5.3}$$

If the base point z is understood or irrelevant, we simply denote the seed curve by $\gamma(s)$, and $\tilde{\mathcal{L}}_z(r)$ by $\tilde{\mathcal{L}}(r)$.

We emphasize that the assumption of H-minimality on $\mathcal{S}$, implies the crucial property that the vector field $\tilde{\boldsymbol{\nu}}^H$ be divergence free in Ω, with respect to the standard divergence operator in $\mathbb{R}^2$. As a consequence of this fact, it is proved in [GP] that the integral curves $\tilde{\mathcal{L}}_z$ of $(\tilde{\boldsymbol{\nu}}^H)^\perp$ are straight-line segments. Furthermore, since $|\tilde{\boldsymbol{\nu}}^H| \equiv 1$ in Ω, every seed curve is parameterized by arc-length, and it is a C^1 embedded curve in $\mathbb{R}^2$ over its interval of definition. The curves $\{\tilde{\mathcal{L}}_z, \gamma_z\}$ are used in [GP] to define a local C^1 diffeomorphism of the (x,y)-plane given by

$$(s,r) \ \to \ (x(s,r), y(s,r)) \ \stackrel{def}{=} \ F(s,r) \ = \ \gamma(s) \ + \ r \ (\tilde{\boldsymbol{\nu}}^H)^\perp(\gamma(s)) \ . \tag{5.4}$$

We note explicitly that F maps the straight line $r = 0$ into the seed curve $\gamma(s)$, i.e.,

$$F(s,0) \ = \ \gamma(s) \ .$$

On the other hand, the straight line $s = 0$ is mapped into the straight line passing through the base point z of the seed curve and having direction vector $(\tilde{\boldsymbol{\nu}}^H)^\perp(z)$, i.e.,

$$F(0,r) \ = \ z \ + \ r \ (\tilde{\boldsymbol{\nu}}^H)^\perp(z) \ .$$

One recognizes that $F(0,r) = \tilde{\mathcal{L}}_z(r)$. In particular, when $z = \gamma(s)$ we obtain from this identity

$$\tilde{\mathcal{L}}_{\gamma(s)}(r) \ = \ \gamma(s) \ + \ r \ (\tilde{\boldsymbol{\nu}}^H)^\perp(\gamma(s)) \ .$$

Along the seed curve we have $(\tilde{\boldsymbol{\nu}}^H)^\perp(\gamma(s)) = \gamma'(s)^\perp = (\gamma_2'(s), -\gamma_1'(s))$. One thus has the following explicit expression for $F(s,r)$

$$F(s,r) \ = \ \gamma(s) + r\gamma'(s)^\perp \ = \ \Big(\gamma_1(s) + r\gamma_2'(s) \ , \ \gamma_2(s) - r\gamma_1'(s)\Big) \ . \tag{5.5}$$

Henceforth, given $f \in C^k(\Omega)$ as in (5.1), we will use the notation

$$h(s,r) \ = \ f(F(s,r)) \ = \ f(\gamma_1(s) + r\gamma_2'(s) \ , \ \gamma_2(s) - r\gamma_1'(s)) \ ,$$

for all values (s,r) for which the right-hand side is defined, and also let $h_0(s) = h(s,0)$. Notice that

$$h_0(0) \ = \ h(0,0) \ = \ f(\gamma(0)) \ = \ f(z) \ .$$

We next define

$$(s,r) \ \longrightarrow \ \mathcal{F}(s,r) \ = \ (\gamma_1(s) + r\gamma_2'(s), \gamma_2(s) - r\gamma_1'(s), h(s,r)) \ , \tag{5.6}$$

and observe explicitly that $\mathcal{F}(0,0) = (\gamma(0), h(0,0)) = (z, f(z))$. We will need the following result, which is Theorem A in [GP]. This result shows, in particular, that, thanks to the assumption that $\mathcal{S}$ be H-minimal, the function $h(s,r)$ must take up a special structure, see (5.7) below.

Theorem 5.2. *Let $\mathcal{S} \subset \mathbb{H}^1$ be a C^k graph of the type* (5.1)*, with $\Sigma(\mathcal{S}) = \varnothing$. If $\mathcal{S}$ is H-minimal, then for every $g = (z,t) \in \mathcal{S}$, there exist intervals I, J, and an open neighborhood of g on $\mathcal{S}$ that can be parameterized by* (5.6)*, where $h(s,r)$ is given by*

$$h(s,r) \ = \ h_0(s) \ - \ \frac{r}{2} \, \gamma(s) \cdot \gamma'(s) \ , \qquad (s,r) \in I \times J \ , \tag{5.7}$$

with

$$\gamma \in C^{k+1}(I), \ h_0 \in C^k(I) \ .$$

Corollary 5.3. *Let $\mathcal{S} \subset \mathbb{H}^1$ be given as in Theorem 5.2. At every $g = \mathcal{F}(s,r) \in \mathcal{S}$, one has*

$$W \ = \ \left| h_0' - r + \frac{r^2}{2}\kappa + \frac{1}{2}\gamma' \cdot \gamma^\perp \right| \ , \tag{5.8}$$

where W is the angle function defined in (3.1)*. Moreover, the horizontal Gauss map is given by*

$$\boldsymbol{\nu}^H \;=\; sgn\left(h_0' - r + \frac{r^2}{2}\kappa + \frac{1}{2}\gamma'\cdot\gamma^{\perp}\right)\left\{\gamma_1'(s)\ X_1 + \gamma_2'(s)\ X_2\right\}\,. \tag{5.9}$$

Proof. Suppose that $\theta : \Omega \subset \mathbb{R}^2 \to \mathbb{H}^1$, with $\theta(u,v) = x(u,v)X_1 + y(u,v)X_2 + t(u,v)T$ be a local parameterization os $\mathcal{S}$, then a direct calculation shows that the coefficients of $\boldsymbol{N} = \theta_u \wedge \theta_v$, and $\boldsymbol{N}^H$ as in (3.3), with respect to the orthonormal basis $\{X_1, X_2, T\}$, are given by the equations

$$\begin{cases} p \;=\; y_u t_v - y_v t_u \;-\; \frac{y}{2}(x_u y_v - x_v y_u)\ , \\ q \;=\; x_v t_u - x_u t_v \;+\; \frac{x}{2}(x_u y_v - x_v y_u)\ , \\ \omega \;=\; x_u y_v - x_v y_u\ . \end{cases} \tag{5.10}$$

Applying the formulas (5.10) to the parameterization $\theta(u,v) = \mathcal{F}(s,r)$ given by (5.6), (5.7), we find

$$\begin{cases} p \;=\; \gamma_1'\left(h_0' - r + \frac{r^2}{2}\kappa + \frac{1}{2}\gamma'\cdot\gamma^{\perp}\right)\ , \\ q \;=\; \gamma_2'\left(h_0' - r + \frac{r^2}{2}\kappa + \frac{1}{2}\gamma'\cdot\gamma^{\perp}\right)\ , \\ \omega \;=\; -\ (1 - r\kappa)\ , \end{cases} \tag{5.11}$$

where, following [GP], we have denoted by

$$\kappa(s) \;=\; \gamma''(s)\cdot\gamma'(s)^{\perp}$$

the signed curvature of the seed curve $\gamma(s)$. From (5.11) we obtain

$$W \;=\; \sqrt{p^2+q^2} \;=\; \left|h_0' - r + \frac{r^2}{2}\kappa + \frac{1}{2}\gamma'\cdot\gamma^{\perp}\right|\ .$$

The assumption that the characteristic locus of $\mathcal{S}$ be empty implies that for every (s,r) in the domain of $W(s,r)$, one has

$$h_0' - r + \frac{r^2}{2}\kappa + \frac{1}{2}\gamma'\cdot\gamma^{\perp} \;\neq\; 0\ .$$

¿From the expression of W, and from (5.11), we thus conclude that the horizontal Gauss map of $\mathcal{S}$ is given by (5.9).

□

Theorem 5.2 admits the following converse, which is Theorem B in [GP].

Theorem 5.4. *Given an interval $I \subset \mathbb{R}$, $k \geq 2$, a properly embedded plane curve $\gamma \in C^k(I)$, parameterized by arc-length, and a function $h_0 \in C^k(I)$, let $\mathcal{S} \subset \mathbb{H}^1$ be the surface parameterized by $\mathcal{F} : I \times \mathbb{R}$ as in* (5.6)*, with $h(s,r)$ given by* (5.7)*. Then, $\mathcal{S}$ is a C^{k-1} H-minimal surface.*

Combining Theorems 5.2 and 5.4, we conclude that to specify a patch of a smooth H-minimal surface, one must specify a single curve in $\mathbb{H}^1$ determined by a seed curve γ, parameterized by arc-length, and an initial height function h_0. We will also need the following result, which is either Theorem A in [CHMY], or Theorem E in [GP].

Theorem 5.5. *Suppose that $\mathcal{S} \subset \mathbb{H}^1$ be a connected H-minimal entire graph over the (x,y)-plane, then:*

(1) *Either $\mathcal{S}$ is a plane of the form $ax + by + ct = \gamma$ for some real numbers a, b, c, γ, with $c \neq 0$.*

(2) *Or, there exist* $g_0 = (x_0, y_0, t_0) \in \mathbb{H}^1$, $a, b \in \mathbb{R}$ *such that* $a^2 + b^2 = 1$, *and* $h_0 \in C^2(\mathbb{R})$, *such that* $\mathcal{S}$ *is globally parameterized by*

$$\left(x + x_0, y + y_0, t_0 - \frac{1}{2}ab(x^2 - y^2) - \frac{1}{2}(b^2 - a^2)xy + h_0(ax + by) + \frac{1}{2}x_0 y - \frac{1}{2}xy_0\right) .$$

We emphasize that both types of surfaces arising in this theorem have non-empty characteristic loci. For instance, in the case (1) we have $\Sigma(\mathcal{S}) = \{(-2b/c, 2a/c, \gamma/c))\}$. Finally, we recall a basic result in H-minimal surface theory. We mention that the next result is one half of Theorem C in [GP], and of Corollary 4.2 in [CHMY].

Theorem 5.6. *Let S be a C^2, H-minimal surface without boundary, which is complete, connected and embedded, then S is a ruled surface, all of whose rules are horizontal straight lines, which are the integral curves of $(\boldsymbol{\nu}^H)^\perp$.*

After these preliminaries, we turn to the proof of Theorem 1.7. To prepare for it, we first show that an H-minimal surface satisfying the hypothesis of Theorem 1.7, can be reduced to an H-minimal graph over the (y, t)-plane, having similar properties. We will need some simple lemmas which clarify the effect of left-translations on a graph. As we have noted in the introduction, the left-translations (2.2) are affine transformations, thereby they preserve planes and lines, see (1.6). Moreover, the left-translations preserve the property of a surface of having empty characteristic locus, they preserve the H-mean curvature, and therefore the H-minimality, the H-perimeter, and the property of a surface of being stable or unstable. We note that rotations about the t-axis (the group center), also have the same properties.

Lemma 5.7. *Suppose $P \subset \mathbb{H}^1$ be a plane with Euclidean normal of the form $\boldsymbol{N}^e_P = (a, 0, c)$, with $a^2 + c^2 \neq 0$, and let $\mathcal{S}$ be a graph over P. For a fixed $g_0 = (x_0, y_0, t_0) \in \mathbb{H}^1$, $g_0 \circ \mathcal{S}$ is a graph over a plane $\tilde{P}$ with Euclidean normal vector given by $\boldsymbol{N}^e_{\tilde{P}} = \left(a, 0, c - \frac{ay_0}{2}\right)$.*

Proof. For $g = (x, y, t) \in P$, consider the straight-line through g and parallel (with respect to the Euclidean inner product in $\mathbb{R}^3$) to $\boldsymbol{N}^e_P$, $L(g) = \{\ell_g(s) = g + s(a, 0, c) \in \mathbb{H}^1 \mid s \in \mathbb{R}\}$. The assumption that $\mathcal{S}$ be a graph over P implies the existence of a unique $s_0 \in \mathbb{R}$ such that $L(g) \cap \mathcal{S} = \{\ell_g(s_0)\}$. Consider the left-translated line $g_0 \circ L(g)$, which is given by

$$g_0 \circ l_g(s) \;=\; g_0 \circ g + s\left(a, 0, c - \frac{ay_0}{2}\right) , \qquad s \in \mathbb{R} ,$$

and note that $g_0 \circ l_g(s_0) \in g_0 \circ \mathcal{S}$. We see that $g_0 \circ \mathcal{S}$ is a graph over a plane $\tilde{P}$ with Euclidean normal $\left(a, 0, c - \frac{ay_0}{2}\right)$, unless the Euclidean normal to the plane $g_0 \circ P$ is perpendicular (with respect to the standard inner product in $\mathbb{R}^3$) to $\left(a, 0, c - \frac{ay_0}{2}\right)$. But this cannot happen. To verify this, observe that from (1.6) the Euclidean normal to $g_0 \circ P$ is given by $\left(a + \frac{cy_0}{2}, -\frac{cx_0}{2}, c\right)$. Since the Euclidean inner product of this vector with $\left(a, 0, c - \frac{y_0 a}{2}\right)$ is $a^2 + c^2 \neq 0$, we reach the desired conclusion.

□

We are now ready to accomplish our first reduction.

Lemma 5.8. *Let $\mathcal{S} \subset \mathbb{H}^1$ be an H-minimal entire graph, with $\Sigma(\mathcal{S}) = \varnothing$, and assume that $\mathcal{S}$ is not itself a vertical plane such as (1.2). After composing with a suitable rotation about the t-axis and with a left-translation, we may assume that there exist $\psi \in C^2(\mathbb{R}^2)$ for which*

$$\mathcal{S} \;=\; \{(x, y, t) \in \mathbb{H}^1 \mid (y, t) \in \mathbb{R}^2, x = \psi(y, t)\} .$$

Proof. Suppose that $\mathcal{S}$ be a graph over the plane P given by $ax+by+ct=\gamma$ for $a,b,c,\gamma\in\mathbb{R}$, with $a^2+b^2+c^2\neq 0$. Suppose first that $b=0$, and consider the two cases, $a=0$, and $a\neq 0$. In the former case, keeping in mind that $c\neq 0$, we see that $\mathcal{S}$ is a graph over the plane $t=\gamma/c$. The left-translation by $g_0=(0,0,-\gamma/c)$, sends this to the plane $t=0$, and the surface $g_0\circ\mathcal{S}$ becomes a global H-minimal graph, with empty characteristic locus, over the (x,y)-plane. This however contradicts Theorem 5.5, since this result forces such a graph to have non-empty characteristic locus. We must thus have $a\neq 0$. In this case, left-translating by $g_0=(0,-2c/a,0)$, Lemma 5.7 shows that $g_0\circ\mathcal{S}$ is an entire graph over the (y,t)-plane. Thus, we can find $\psi\in C^2(\mathbb{R}^2)$ which defines $g_0\circ\mathcal{S}$, thus yielding the desired conclusion.

We are thus left with the situation $b\neq 0$. In this case, however, performing a rotation of an angle $\theta=cotan^{-1}(-a/b)\in(0,\pi)$ about the t-axis, which preserves H-minimality, we obtain a new surface which is an entire graph over a plane having Euclidean normal in the form $(a,0,c)$, and we can thus argue as in the first part to reach the sought for conclusion.

□

As it will be useful later, we next prove that the notions of seed curve and height function are preserved under left-translation.

Lemma 5.9. *Suppose there exist intervals $I,J\subset\mathbb{R}$ so that a portion, $\mathcal{S}_0$, of an H-minimal surface, $\mathcal{S}$, is parameterized by a seed curve, γ, and height function, h_0, as in* (5.6) *and* (5.7)*, with $s\in I, r\in J$. If $g_0=(x_0,y_0,t_0)\in\mathbb{H}^1$, then the surface $g_0\circ\mathcal{S}_0$ is also parameterized as in* (5.6) *and* (5.7)*, using a seed curve, $\hat\gamma$, and height function, $\hat h_0$, given by $\hat\gamma(s)=(x_0+\gamma_1(s),y_0+\gamma_2(s))$, and $\hat h_0(s)=h_0(s)+\frac{x_0}{2}\gamma_2(s)-\frac{y_0}{2}\gamma_1(s)$, for $s\in I, r\in J$.*

Proof. With $F(s,r)$ as in (5.5), consider $g_0\circ\mathcal{S}_0$, which is parameterized by $g_0\circ\mathcal{F}(s,r)=g_0\circ(F(s,r),h(s,r))$. Using (2.2), we see that $g_0\circ\mathcal{F}(s,r)$ is given by

$$\hat{\mathcal{F}}(s,r) \;=\; (\hat F(s,r),\hat h(s,r))\ ,$$

with

$$\hat F(s,r)=\hat\gamma(s)+r(\hat\gamma'(s))^\perp=\gamma(s)+r(\gamma'(s))^\perp+(x_0,y_0)\ ,$$

and

$$\hat h(s,r)=\hat h_0(s)-\frac{r}{2}\hat\gamma(s)\cdot\hat\gamma'(s)$$

where $\hat h_0(s)=h_0(s)+\frac{x_0}{2}\gamma_2(s)-\frac{y_0}{2}\gamma_1(s)$. Applying Corollary 5.3 to the parameterization $\hat{\mathcal{F}}(s,r)$, we find $\hat W(s,r)=W(s,r)$, and therefore from (5.9) we obtain for horizontal Gauss map of $g_0\circ\mathcal{S}_0$ at the point $g_0\circ(F(s,r),h(s,r))$,

$$sgn\left(h_0'-r+\frac{r^2}{2}\kappa+\frac{1}{2}\gamma'\cdot\gamma^\perp\right)\left\{\hat\gamma_1'(s)\ X_1+\hat\gamma_2'(s)\ X_2\right\}\ .$$

Since $\hat\gamma_i'(s)=\gamma_i'(s)$, we have that the components of the horizontal Gauss map are the same as those of the unit horizontal Gauss map for $\mathcal{S}_0$ at the point $(F(s,r),h(s,r))$, see Corollary 5.3. Upon projection to the plane $t=0$, we have that $\hat\gamma'$, as a vector field on $\mathbb{R}^2$, is just a translation of γ' by the vector (x_0,y_0). Thus, $\hat\gamma$ is a seed curve for $\mathcal{S}_0$.

□

We apply these results to a useful special case.

Corollary 5.10. *Suppose that $\mathcal{S}$ be a H-minimal entire graph over the (y,t)-plane with empty characteristic locus, and that $\mathcal{S}$ is not itself a vertical plane. There exist a point $g_0\in\mathcal{S}$, an interval $I\subset\mathbb{R}$, $\gamma\in C^3(I)$, $h_0\in C^2(I)$, so that a neighborhood $\mathcal{S}_0$ of g_0 can be parameterized by* (5.6),(5.7) *for $s\in I$ and $r\in\mathbb{R}$.*

Proof. Since we assume $\mathcal{S}$ is a graph over the (y,t)-plane, there exists $\psi \in C^2(\mathbb{R}^2)$ such that $\mathcal{S}$ is described by $x = \psi(y,t)$. Consider the defining function $\Psi(x,y,t) = x - \psi(y,t)$. If $\Psi_t = \psi_t \equiv 0$, then we would have $\psi(y,t) = f(y)$, and by the H-minimality of $\mathcal{S}$, we would conclude that $f(y) = \alpha y + \beta$, against the hypothesis that $\mathcal{S}$ is not a vertical plane. Therefore, there exists $g_0 \in \mathcal{S}$ such that $\psi_t(g_0) \neq 0$. The Implicit Function Theorem implies the existence of a neighborhood of g_0 on $\mathcal{S}$ which may be written as a graph over the plane $t = 0$ (with empty characteristic locus). Applying Theorem 5.2, we obtain intervals $I, \subset \mathbb{R}$, $\gamma \in C^3(I), h_0 \in C^2(I)$, so that a neighborhood of g_0 is parameterized by (5.6), (5.7) for $s \in I$, $r \in J$. To finish the proof, we need to show that for every $s \in I$, we may extend the domain of r to the whole of $\mathbb{R}$. To see this, we note that for each $s_0 \in I$, the curve

$$r \;\to\; \mathcal{L}_{(\gamma(s_0),h_0(s_0))}(r) = (\gamma(s_0) + r\gamma'(s_0)^{\perp}, h(s_0,r)) \;,$$

with $h(s,r)$ given by (5.7), and $r \in J$, is a horizontal straight line segment in $\mathbb{H}^1$. Using (5.7), we see that the tangent vector to this line is simply

$$\left(\gamma'(s_0)^{\perp}, -\frac{\gamma(s_0)\cdot\gamma'(s_0)}{2}\right) \;=\; \gamma_2'(s_0)\; X_1(\mathcal{F}(s_0,r)) - \gamma_1'(s_0)\; X_2(\mathcal{F}(s_0,r)) \;, \tag{5.12}$$

which, by Corollary 5.3, is precisely $(\boldsymbol{\nu}^H)^{\perp}$. Thus, by the standard uniqueness theory for solutions to ordinary differential equations, these line segments must coincide with subsets of the horizontal line foliation of $\mathcal{S}$ guaranteed by Theorem 5.6. As the entirety of these horizontal lines are contained in $\mathcal{S}$, we conclude that the parameterization given by (5.6), (5.7) extends to $(s,r) \in I \times \mathbb{R}$.

□

In order to extract some crucial additional information from our assumption that $\mathcal{S}$ be a an entire graph with $\Sigma(\mathcal{S}) = \varnothing$, the following elementary lemma will be useful.

Lemma 5.11. *Let $g_1, g_2 \in \mathbb{H}^1$, $v = (v_1,v_2,v_3), w = (w_1,w_2,w_3) \in \mathbb{R}^3$, and consider the straight-lines $L_i = \{g_i + rv \mid r \in \mathbb{R}\}$, $i = 1,2$. If $\pi : \mathbb{H}^1 \to \mathbb{R}^2$ denotes the projection to the (y,t)-plane given by $\pi(z_1,z_2,t) = (0,z_2,t)$, then $\pi(L_1) \cap \pi(L_2) = \varnothing$ if and only if $\pi(g_1) \neq \pi(g_2)$, and $v \times w$ is (Euclidean) perpendicular to $(1,0,0)$.*

Proof. Suppose $\pi(L_1) \cap \pi(L_2) = \varnothing$, then it is obvious that it must be $\pi(g_1) \neq \pi(g_2)$. Furthermore, we must also have $\pi(v) \times \pi(w) = (v_2w_3 - w_2v_3, 0, 0) = 0$. Thus, we conclude that (v_2,v_3) and (w_2,w_3) are constant multiples of one another. But then $v \times w$ takes the form $(v_2w_3 - w_2v_3, \star, \star) = (0, \star, \star)$, and we conclude that $v \times w$ is (Euclidean) perpendicular to $(1,0,0)$, one direction of the lemma. The opposite direction follows by simply reversing the previous argument.

□

Next, we apply Lemma 5.11 in the case of the parameterization given in Corollary 5.10. The following lemma plays a crucial role in the proof of Theorem 1.7.

Lemma 5.12. *Let $I \subset \mathbb{R}$, $\gamma \in C^3(I), h_0 \in C^2(I)$, and consider a portion $\mathcal{S}_0$ of an H-minimal entire graph $\mathcal{S}$ over the (y,t)-plane having empty characteristic locus. Suppose that $\mathcal{S}_0$ be parameterized as in (5.6),(5.7) for $(s,r) \in I \times \mathbb{R}$. There exists a subinterval, $J \subset I$, such that $\gamma(J)$ is either a straight line segment, or a circular arc.*

Proof. By re-parameterizing γ, we may assume $0 \in I$. As in the proof of Corollary 5.10, for every fixed $s \in I$, we consider the horizontal straight lines contained in $\mathcal{S}$, and defined by

$$\mathcal{L}_{(\gamma(s),h_0(s))}(r) = \left(\gamma(s) + r\gamma'(s)^{\perp}, h_0(s) - \frac{r}{2}\gamma(s)\cdot\gamma'(s)\right) \;, \qquad r \in \mathbb{R} \;. \tag{5.13}$$

For every $s \in I$ we consider the two lines $L_1 = \mathcal{L}_{(\gamma(0),h_0(0))}$, $L_2 = \mathcal{L}_{(\gamma(s),h_0(s))}$. The assumption that $\mathcal{S}$ be an entire graph over the (y,t)-plane implies, in particular, that for every $s \in I$ we must have $\pi(L_1) \cap \pi(L_2) = \varnothing$. We can thus use Lemma 5.11 to infer that the directional vectors of L_1 and L_2, $v = \big(\gamma'(0)^\perp, -\frac{1}{2}\gamma(0)\cdot\gamma'(0)\big)$ and $w = \big(\gamma'(s)^\perp, -\frac{1}{2}\gamma(s)\cdot\gamma'(s)\big)$, satisfy the condition that $v \times w$ be perpendicular to $(1,0,0)$. A simple computation now gives

$$v \times w \cdot (1,0,0) = \frac{1}{2}\Big(\gamma_1'(0)(\gamma(s)\cdot\gamma'(s)) - \gamma_1'(s)(\gamma(0)\cdot\gamma'(0))\Big) = 0 \ . \tag{5.14}$$

We first discuss three special cases. If $\gamma_1'(0) = 0$ and $\gamma(0)\cdot\gamma'(0) \neq 0$, then for (5.14) to be satisfied we must have $\gamma_1'(s) = 0$ for all s in a neighborhood J of $s = 0$. Hence, $\gamma(J)$ is a line segment and we have reached one of our conclusions. Similarly, if $\gamma(0)\cdot\gamma'(0) = 0$, but $\gamma_1'(0) \neq 0$, then (5.14) is only satisfied if $\gamma(s)\cdot\gamma'(s) = 0$ for all s in a neighborhood J of $s = 0$. In this case, we claim that $\gamma(J)$ is a circular arc. To see this, we note that $d/ds(|\gamma(s)|^2) = 2\gamma(s)\cdot\gamma'(s) \equiv 0$ on J. Thus, $|\gamma(s)| \equiv C > 0$ on J, hence $\gamma(J)$ is a circular arc, reaching the second of our conclusions. Last, if both $\gamma(0)\cdot\gamma'(0) = 0$ and $\gamma_1'(0) = 0$, then by the fact that $|\gamma'(0)| = 1$, we obtain that $|\gamma_2'(0)| = 1$. and, using (5.13), we conclude that $\mathcal{S}_0$ contains the straight line

$$\mathcal{L}_{(\gamma(0),h_0(0))}(r) = (\gamma_1(0) \pm r, \gamma_2(0), h_0(0)) \ , \qquad r \in \mathbb{R} \ .$$

This clearly violates our assumption that $\mathcal{S}_0$ is a graph over the (y,t)-plane, as the whole line projects to the same point, $(0, \gamma_2(0), h_0(0))$, in the (y,t)-plane, thus this case cannot occur.

If we are not in any of these cases, then we must have that both $\gamma_1'(0) \neq 0$ and $\gamma\cdot\gamma'(0) \neq 0$. By the continuity of γ and γ', there exists a neighborhood of $s = 0$, J, such that $\gamma_1'(s) \neq 0$, and $\gamma(s)\cdot\gamma'(s) \neq 0$ for any $s \in J$. In this case we conclude that (5.14) is equivalent to the existence, for every $s \in J$, of a $C(s) \neq 0$ such that

$$\begin{pmatrix} \gamma_1'(0) \\ \gamma_1'(s) \end{pmatrix} \ = \ C(s) \ \begin{pmatrix} \gamma(0)\cdot\gamma'(0) \\ \gamma(s)\cdot\gamma'(s) \end{pmatrix} \ .$$

Equating the first entries we find that $C(s) \equiv C \neq 0$ for $s \in J$, with

$$C \ = \frac{\gamma_1'(0)}{\gamma(0)\cdot\gamma'(0)} \ .$$

We conclude that

$$\gamma_1'(s) \ = \ C \ \gamma(s)\cdot\gamma'(s) \ , \qquad s \in J \ . \tag{5.15}$$

Recalling that $\frac{d}{ds}|\gamma(s)|^2 = 2\gamma(s)\cdot\gamma'(s)$, (5.15) implies

$$|\gamma(s)|^2 = \frac{2}{C}\gamma_1(s) + C_0 \tag{5.16}$$

where C_0 is an integration constant. We claim that this implies that $\gamma(J)$ is a circular arc. To see this, we left-translate $\mathcal{S}_0$ by $(-1/C, 0, 0)$. Lemma 5.9 shows that the translated surface will have a seed curve given by $\hat\gamma(s) = (\gamma_1(s) - 1/C, \gamma_2(s))$. Computing $|\hat\gamma(s)|^2$ and using (5.16), we have

$$|\hat\gamma(s)|^2 = |\gamma(s)|^2 - \frac{2}{C}\gamma_1(s) + \frac{1}{C^2} = \frac{2}{C}\gamma_1(s) - \frac{2}{C}\gamma_1(s) + \frac{1}{C^2} + C_0 = \frac{1}{C^2} + C_0 \ .$$

Thus, $\hat\gamma(J)$ is a circular arc and hence, such is $\gamma(J)$ as well. This completes the proof.

□

With these preliminary computations in place, we are finally ready to establish our main reduction result.

Proof of Theorem 1.7. Let $\mathcal{S}$ be an H-minimal entire graph over a plane P with empty characteristic locus and which is not itself a vertical plane. In view of Lemma 5.8, after possibly a left-translation and a rotation about the t-axis, we may assume that P is the plane $x = 0$, and that $\mathcal{S}$ is given by $x = \psi(y,t)$ for some $\psi \in C^2(\mathbb{R}^2)$. Corollary 5.10 guarantees that there exists $g_0 \in \mathcal{S}$, an interval $I \subset \mathbb{R}$, a unit-speed $\gamma \in C^3(I)$, $h_0 \in C^2(I)$, so that a neighborhood $\mathcal{S}_0$ of g_0 can be parameterized by

$$\mathcal{F}(s,r) \; = \; \Big(\gamma(s) + r\gamma'(s)^{\perp}, h_0(s) - \frac{r}{2}\gamma(s)\cdot\gamma'(s)\Big) \; , \qquad (s,r) \in I \times \mathbb{R} \; . \tag{5.17}$$

Since $\mathcal{S}$ is a graph over the plane $x = 0$, Lemma 5.12 yields that $\gamma(J)$ is either a straight-line segment, or a circular arc. We next show that assumption of empty characteristic locus on $\mathcal{S}$ rules out the possibility that $\gamma(J)$ be a straight line segment. If, in fact, γ were linear, then it would have the form

$$\gamma(s) = (x_0 + a_1 s, y_0 + a_2 s) \; ,$$

with $a_1^2 + a_2^2 = 1$. In this case, the signed curvature $\kappa(s) \equiv 0$, and the angle function W, given by (5.8), becomes

$$W(r,s) \; = \; \left| h_0'(s) - r + \frac{1}{2}(a_1 y_0 - a_2 x_0) \right| \; , \qquad (s,r) \in J \times \mathbb{R} \; .$$

Since it is clear that, for each fixed $s \in J$, there exists $r \in \mathbb{R}$, $r = h_0'(s) + \frac{1}{2}(a_1 y_0 - a_2 x_0)$, where $W(s,r) = 0$, we would conclude that $\mathcal{S}$ has a characteristic point at $g = \mathcal{F}(s,r)$, against our hypothesis.

Therefore, $\gamma(J)$ must be a unit-speed circular arc, i.e.,

$$\gamma(s) = (x_0 + R\cos(s/R), y_0 + R\sin(s/R)) \; , \tag{5.18}$$

for some $(x_0, y_0) \in \mathbb{R}^2$, and $R > 0$, and (5.17) becomes

(5.19)

$$\mathcal{F}(s,r) \; = \; \Big(x_0 + (R+r)\cos(s/R), y_0 + (R+r)\sin(s/R), h_0(s) + \frac{r}{2}(x_0\sin(s/R) - y_0\cos(s/R))\Big) \; .$$

Consider the left-translated surface $\tilde{\mathcal{S}}_0 = (-x_0, -y_0, 0) \circ \mathcal{S}_0$ parameterized by

$$\tilde{\mathcal{F}}(s,r) = (-x_0,-y_0,0)\circ\mathcal{F}(s,r) \; = \; \Big((R+r)\cos\frac{s}{R}, (R+r)\sin\frac{s}{R}, \overline{h}_0(s)\Big) \; , \tag{5.20}$$

where

$$\tilde{h}_0(s) = h_0(s) + \frac{R}{2}\left(y_0\cos\frac{s}{R} - x_0\sin\frac{s}{R}\right) \; .$$

By Lemma 5.7, we know that the $\tilde{\mathcal{S}} = (-x_0, -y_0, 0)\circ\mathcal{S}$ is a non-characteristic entire graph over a plane $\tilde{P}$, having Euclidean normal $\tilde{\boldsymbol{N}}^e = (1, 0, -y_0/2)$. Applying (5.10) to the parametrization $\tilde{\mathcal{F}}(s,r)$ we obtain

$$\begin{cases} \tilde{p} \; = \; - \; \sin\frac{s}{R}\; \left(\frac{(R+r)^2}{2R} - \tilde{h}_0'(s)\right) \; , \\ \tilde{q} \; = \; \cos\frac{s}{R}\; \left(\frac{(R+r)^2}{2R} - \tilde{h}_0'(s)\right) \; , \\ \tilde{\omega} \; = \; - \; \frac{R+r}{R} \; . \end{cases} \tag{5.21}$$

Using (5.21) we immediately recognize that the angle function $\overline{W}$ for $\overline{\mathcal{S}}_0$ is given by

$$\overline{W}(s,r) \; = \; \left| \overline{h}_0'(s) - \frac{(R+r)^2}{2R} \right| \; .$$

Since $\overline{W}(s,r) \neq 0$ for any $(s,r) \in J \times \mathbb{R}$, we conclude that we must either have

$$\frac{(R+r)^2}{2R} \; > \; \tilde{h}_0'(s) \; , \quad \text{or} \quad \frac{(R+r)^2}{2R} \; < \; \tilde{h}_0'(s)$$

for every $(s,r) \in J \times \mathbb{R}$. It is clear that for no fixed $s \in J$ can the second inequality hold for every $r \in \mathbb{R}$, and therefore we must have for every fixed $s \in J$

$$\tilde{h}_0'(s) \ < \ \frac{(R+r)^2}{2R} \ , \qquad \text{for every } r \in \mathbb{R} \ .$$

This imposes that we must have the crucial property

$$\tilde{h}_0'(s) \ < \ 0 \ , \qquad \text{for every} \quad s \in J \ . \tag{5.22}$$

Having achieved this conclusion, consider the portion $\tilde{\mathcal{S}}_0$ of $\tilde{\mathcal{S}}$ given by (5.20). The non-unit Euclidean normal to $\tilde{\mathcal{S}}_0$ is given by

$$\tilde{\boldsymbol{N}}^e \ = \ \tilde{\mathcal{F}}_s \times \ \tilde{\mathcal{F}}_r \ = \left(- \ \tilde{h}_0'(s) \ \sin\frac{s}{R} \ , \ \tilde{h}_0'(s) \ \cos\frac{s}{R} \ , \ - \ \frac{R+r}{R} \right) \ ,$$

where the cross product is taken with respect to the Euclidean inner product in $\mathbb{R}^3$. The Euclidean Gauss map of $\tilde{\mathcal{S}}_0$ is thus given by

$$\tilde{\boldsymbol{\nu}}^e \ = \ \frac{1}{\sqrt{\left(\frac{R+r}{R}\right)^2 + \tilde{h}_0'(s)^2}} \ \left(- \ \tilde{h}_0'(s) \ \sin\frac{s}{R} \ , \ \tilde{h}_0'(s) \ \cos\frac{s}{R} \ , \ - \ \frac{R+r}{R} \right)$$

For any fixed $s \in J$, letting r range over $\mathbb{R}$, we see that the image of $\tilde{\boldsymbol{\nu}}^e$ in $S^2 \subset \mathbb{R}^3$ describes a curve, denoted Γ, which is an open arc of a great circle. We observe that Γ passes through the point $\frac{-\tilde{h}_0'(s)}{|\tilde{h}_0'(s)|}(\sin(s), -\cos(s), 0) = (\sin(s), -\cos(s), 0)$, and that the closure of Γ contains the points $(0,0,\pm 1)$. Since $\tilde{\mathcal{S}}_0$ is a graph over the plane $\tilde{P}$, we must have that $\tilde{\boldsymbol{\nu}}^e(\tilde{\mathcal{S}}_0) \subset S^2$ lies entirely to one side of the plane determined by the vector $\frac{1}{\sqrt{1+y_0^2/4}}(1,0,-y_0/2) \in S^2$. Now, if $y_0 \neq 0$, then the points $(0,0,\pm 1) \in \Gamma$ would lie on opposite sides of this plane, thus reaching a contradiction. We conclude that we must have $y_0 = 0$, and therefore $\tilde{\mathcal{S}}_0$ is a graph over a portion of the (y,t)-plane.

Since from (5.20) we see that $x/y = \cot(s/R)$, when $s \neq 0$, we fix an open sub-interval of J, $\tilde{J} = (a,b)$, with either $0 < a < b < \pi$, or $-\pi < a < b < 0$, and we consider the open interval $I = \tilde{h}_0(\tilde{J})$. We stress that, in view of (5.22) we know that $\tilde{h}_0^{-1} : I \to \tilde{J}$ exists, and, depending on the choice that we have made of $\tilde{J}$, we have that either $0 < \tilde{h}_0^{-1}(t) < \pi$, or $-\pi < \tilde{h}_0^{-1}(t) < 0$, for every $t \in I$. We may thus re-write $\tilde{\mathcal{S}}_0$ as

$$x \ = \ y \ \cot(\tilde{h}_0^{-1}(t)) \ = \ y \ G(t) \ . \tag{5.23}$$

We note that $G \in C^2(I)$ since $\tilde{h}_0^{-1} \in C^2(I)$, and that moreover, thanks to (5.22) we have

$$G'(t) \ = \ - \ (1 + \cot^2(\tilde{h}_0^{-1}(t)))\frac{1}{\tilde{h}_0'(\tilde{h}_0^{-1}(t))} \ > \ 0 \ .$$

Furthermore, since $y = (R+r)\sin(s/R)$ for $s \in \tilde{J}$ and $r \in \mathbb{R}$, we conclude that y attains every real number. In addition, we claim the map $(r,s) \in \mathbb{R} \times I \to ((R+r)\sin(s/R), \tilde{h}_0(s)) \in \mathbb{R} \times J$ is one to one. To see this, we consider $(r_1,s_1),(r_2,s_2)$ so that $((R+r_1)\sin(s_1/R), \tilde{h}_0(s_1)) = ((R+r_2)\sin(s_2/R), \tilde{h}_0(s_2))$. The injectivity of $\tilde{h}_0$ implies that $s_1 = s_2$, and so we must have $r_1 = r_2$ as well. We thus see that the parametrization (5.23) of $\tilde{\mathcal{S}}$ is valid for $(y,t) \in \mathbb{R} \times I$. This completes the proof.

□

With Theorem 1.7 in hands, we can now establish our main result of Bernstein type.

Proof of Theorem 1.8. Let $\mathcal{S} \subset \mathbb{H}^1$ be a stable H-minimal entire graph, with $\Sigma(\mathcal{S}) = \varnothing$. Assume by contradiction that $\mathcal{S}$ is not a vertical plane. By Theorem 1.7, after possibly a left-translation and a rotation about the t-axis, the resulting surface $\mathcal{S}$ contains a strict graphical strip $\mathcal{S}_0$. By Theorem 1.6 we know that $\mathcal{S}_0$ is unstable, and therefore also $\mathcal{S}$ must be unstable, thus reaching a contradiction. We conclude that $\mathcal{S}$ must be a vertical plane.

□

6. **Obstruction to the higher-dimensional Bernstein problem**

In this section we prove Theorem 1.9. We begin with a simple proposition, which is fact valid in any Carnot group.

Proposition 6.1. *Suppose that the hypersurface $\mathcal{S} \subset \mathbb{H}^n$ be a* vertical cylinder*, i.e., it can be represented in the form*

$$\mathcal{S} \;=\; \{g = (x,y,t) \in \mathbb{H}^n \mid \mathfrak{h}(x,y) = 0\}\ , \tag{6.1}$$

where $\mathfrak{h} \in C^2(\mathbb{R}^{2n})$, and there exist an open set $\omega \subset \mathbb{R}^{2n}$ and $\alpha > 0$ such that $|\nabla \mathfrak{h}| \geq \alpha$ in ω. Under these assumptions, the characteristic locus of $\mathcal{S}$ is empty, and the H-mean curvature of $\mathcal{S}$ is therefore globally defined and it is given by

$$\mathcal{H}(x,y,t) \;=\; (2n-1)\ H(x,y)\ , \tag{6.2}$$

where $H(x,y)$ represents the Riemannian mean curvature of the projection $\pi(\mathcal{S})$ of $\mathcal{S}$ onto $\mathbb{R}^{2n} \times \{0\}$. In particular, $\mathcal{S}$ is H-minimal if and only if $\pi(\mathcal{S})$ is a classical minimal surface in $\mathbb{R}^{2n}$. Furthermore, the H-perimeter of $\mathcal{S}$, $\sigma_H(\mathcal{S})$, is given by

$$\sigma_H(\mathcal{S}) \;=\; H^{2n}(\mathcal{S})\ , \tag{6.3}$$

where H^{2n} is the standard $2n$-dimensional Hausdorff measure in $\mathbb{R}^{2n+1}$, i.e., the surface measure.

Proof. Consider the defining function for $\mathcal{S}$ given by $\phi(x,y,t) = \mathfrak{h}(x,y)$. We first observe that for $i = 1, ..., n$,

$$X_i\phi(x,y,t) \;=\; \frac{\partial \mathfrak{h}}{\partial x_i}\ , \qquad\qquad X_{n+i}\phi(x,y,t) \;=\; \frac{\partial \mathfrak{h}}{\partial y_i}\ ,$$

hence, since $T\phi \equiv 0$, we conclude that

$$\nabla^H \phi \;=\; \nabla \phi \;=\; \nabla \mathfrak{h}\ , \tag{6.4}$$

which, thanks to the assumption $|\nabla \mathfrak{h}| \geq \alpha > 0$, proves in particular that $\Sigma(\mathcal{S}) = \varnothing$, and that $\boldsymbol{\nu}^H = \frac{\nabla \mathfrak{h}}{|\nabla \mathfrak{h}|} = \boldsymbol{\nu}$, where $\boldsymbol{\nu}$ denotes the Riemannian unit normal of $\mathcal{S}$. Thanks to Proposition 2.4, the H-mean curvature of $\mathcal{S}$ is given by

$$\mathcal{H} \;=\; \sum_{i=1}^{2n} \nabla_i^{H,\mathcal{S}} < \boldsymbol{\nu}^H, X_i > = \sum_{i=1}^{2n} X_i \boldsymbol{\nu}_i \;=\; div\ \boldsymbol{\nu} \;=\; (2n-1)\ H\ ,$$

where in the third and second to the last equality we have used (6.4). This proves (6.2). Finally, (6.3) derives from (6.4) and from (3.5), or equivalently (3.6).

□

Proof of Theorem 1.9. Consider the Heisenberg group $\mathbb{H}^n$, and denote by $N+1=2n$ the dimension of the horizontal layer $\mathbb{R}^{2n}\times\{0\}$. For $(x,y)\in\mathbb{R}^{2n}$, we write $y=(y',y_n)$, with $y'\in\mathbb{R}^{n-1}$, and denote by $\mathbb{R}^N=\mathbb{R}^n_x\times\mathbb{R}^{n-1}_{y'}$. By the fundamental results in [BDG], given any $N\geq 8$ there exists a non-affine $f\in C^\omega(\mathbb{R}^N)$ such that $\mathcal{S}_0=\{(x,y)\in\mathbb{R}^{2n}\mid y_n=f(x,y')\}$ is an entire minimal graph. Clearly, if we consider the defining function $\mathfrak{h}(x,y)=y_n-f(x,y')$ for $\mathcal{S}_0$, then

$$|\nabla\mathfrak{h}(x,y')| \ = \ \sqrt{1+|\nabla_{x,y'}f(x,y')|^2} \ \geq \ 1 \ , \qquad \text{for every} \quad (x,y')\in\mathbb{R}^N \ .$$

Consider the vertical cylinder $\mathcal{S}\subset\mathbb{H}^n$ such that $\pi(\mathcal{S})=\mathcal{S}_0$. Thanks to Proposition 6.1, $\mathcal{S}$ is an H-minimal entire graph, over the hyperplane $\{(x,y,t)\in\mathbb{H}^n\mid y_n=0\}$, with empty characteristic locus, and which is not a vertical hyperplane. Using the fact that the unit vector field on $\mathcal{S}$

$$(x,y,t)\ \to\ \boldsymbol{\nu}^H \ = \ \frac{1}{\sqrt{1+|\nabla_{x,y'}f|^2}}\ \left\{\sum_{i=1}^n(-f_{x_i})X_i \ + \ \sum_{i=1}^{n-1}(-f_{y_i})X_{n+i} \ + \ X_{2n}\right\} \ ,$$

is independent of the t-variable, and moreover $div^H\boldsymbol{\nu}^H=div\boldsymbol{\nu}=0$, we can easily prove the stability of $\mathcal{S}$ similarly to the classical case, see [CM], pagg.1-4, and also [BSV] for a general discussion of sub-Riemannian calibrations in $\mathbb{H}^n$. Finally, we observe that the condition $N\geq 8$, translates into $n\geq 9/2$, hence a counterexample to the Bernstein problem can be found for any $n\geq 5$.

□

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