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Properties of entire solutions of non-uniformly elliptic equations arising in geometry and in phase transitions

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1 Introduction

The aim of this paper is to study bounded critical points of the following general functional from the calculus of variations

$$\mathcal{E}(u) = \int_{\mathbb{R}^n} \Phi(u, Du)\, dx, \tag{1.1}$$

whose Euler-Lagrange equation is

$$div\, \Phi_\sigma(u, Du) = \Phi_\xi(u, Du). \tag{1.2}$$

Using compactness methods based on the translation invariance of the equation (1.2), and a priori estimates in C^1 norm, we prove various properties of bounded entire solutions of (1.2), such as a sharp inequality for the gradient, energy monotonicity and optimal growth, Liouville type results, and one-dimensional symmetry. An important role in this program is played by the function

$$P = P(x; u) = < \Phi_\sigma(u(x), Du(x)), Du(x) > - \Phi(u(x), Du(x)), \tag{1.3}$$

which incorporates basic analytic and geometric information on u itself. To explain this point let us notice that when the level sets of u are hyper-planes, then

$$u(x) = g(< a, x >), \tag{1.4}$$

for some $g : \mathbb{R} \to \mathbb{R}$, and some vector $a \in \mathbb{R}^n$, with $|a| = 1$. If, in addition, Φ is spherically symmetric in σ, i.e., if we can write

$$\Phi(\xi, \sigma) = \frac{1}{2}\, G(\xi, |\sigma|^2), \tag{1.5}$$

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for some function $G = G(\xi, s)$, then it is easy to recognize that the function P introduced in (1.3) is constant, see Proposition 4.2. Vice-versa, the constancy of $P(\cdot; u)$ hides geometric information on the level sets $\{x \in \mathbb{R}^n \mid u(x) = t\}$ of the solution u, such as the property of being surfaces of zero mean curvature. A basic feature of the function P is that it satisfies a maximum principle, which becomes optimal (in the sense that P becomes constant) for a distinguished geometric configuration of u, namely (1.4). Such result has important connections with the beautiful theory of isoparametric surfaces developed by E. Cartan [C], see also [Tho].

On a bounded entire solution of the model equation $\Delta u = F'(u)$, one has $P = (1/2)|Du(x)|^2 - F(u(x))$. In this case L. Modica in [M2] first established the following important property of P

$$|Du(x)|^2 \le 2F(u(x)), \qquad x \in \mathbb{R}^n, \tag{1.6}$$

under the hypothesis $F \ge 0$. Gradient estimates of entire solutions of uniformly elliptic equations have a long history, which for obvious reasons we will not attempt to describe. The first contributions more closely connected to (1.6), but with different assumptions, are contained in two pioneering papers by Serrin [Se1], [Se2], and in one by Peletier and Serrin [PS].

Using ideas different from those in [M2], the non-positivity of P was generalized in [CGS] to the case

$$\Phi(\xi, \sigma) = \frac{1}{2} G(|\sigma|^2) + F(\xi), \tag{1.7}$$

where $G = G(s)$ is a non-linearity which includes models as diverse as the p-Laplacian and the minimal surface operator, see (2.12) and (2.13) below. In this situation, the function introduced in (1.3) becomes

$$P = G_s(|Du|^2)\,|Du|^2 - \frac{1}{2} G(|Du|^2) - F(u), \tag{1.8}$$

and one of the main results in [CGS] stated that $P \le 0$ on a bounded entire solution of (1.2). It was also shown in [CGS] that, if such P attains its upper bound at one point, then in fact $P \equiv const$, and moreover u is one-dimensional, i.e., of the type (1.4). This latter result provided evidence in favor of the following by now famous conjecture of E. De Giorgi [DG, Open question 3, p. 175]: *Let $u \in C^2(\mathbb{R}^n)$ be an entire solution of*

$$\Delta u = u^3 - u, \tag{1.9}$$

such that $|u| \le 1$. If

$$\frac{\partial u}{\partial x_n} > 0 \qquad \textit{in} \quad \mathbb{R}^n \tag{1.10}$$

holds, then the level sets of u are hyper-planes, i.e., u must be of the type (1.4)*, at least if $n \le 8$.*

The limitation in the dimension is suggested by the deep connection with the Bernstein problem in the theory of minimal surfaces, see [DG], [BDG], [M1],

[AAC], and Sect. 4. It is worth noting that the above mentioned "evidence in favor" in [CGS], is however in discrepancy with the conjecture, in that it establishes the one-dimensional symmetry irregardless of the dimension of the ambient space. The following family of explicit solutions of (1.9) has long been known

$$(1.11) \qquad u(x) = u_{a,\lambda}(x) = tanh\left(\frac{< a, x >}{\sqrt{2}} + \lambda\right), \qquad x \in \mathbb{R}^n,$$

where $a \in \mathbb{R}^n$ is such that $|a| = 1$, and $\lambda \in \mathbb{R}$. Therefore, the conjecture of De Giorgi implicitly states that, at least if $n \leq 8$, under the monotonicity hypothesis (1.10), a bounded entire solution of (1.9) must be of the form $u_{a,\lambda}$, for some $|a| = 1$, and $\lambda \in \mathbb{R}$.

De Giorgi's conjecture, and some variants of it, have received considerable attention over the past few years. The first result goes back to a pioneering paper of L. Modica and S. Mortola [MM], in which the authors proved that in dimension $n = 2$ the conjecture is true, under the additional hypothesis that the level sets of u constitute an equi-Lipschitzian family of curves. A complete solution in the two-dimensional case was only given in 1998, in a beautiful paper by Ghoussoub and Guy [GG]. In fact, these authors proved the conjecture true not just for the Ginzburg-Landau model, but for the equation $\Delta u = F'(u)$, with $F \in C^2(\mathbb{R})$.

A modified version of the conjecture, known as *Gibbons' conjecture* [Ca], contains the additional assumption that $u(x', x_n)$ tend to its extremum values as $x_n \to \mp\infty$, but *uniformly in* $x' \in \mathbb{R}^{n-1}$. Such conjecture has been independently answered in the affirmative in every dimension, and with very different approaches, in the recent papers by Berestycki, Hamel and Monneau [BHM], and by Barlow, Bass and Guy [BBG]. Again, there is a discrepancy between De Giorgi's and Gibbons' conjectures, since the latter has been established irregardless of the dimension. Under a similar assumption of uniform limit at infinity, but for equations in a cylinder, a positive answer for the degenerate model (2.12) has been given by Farina [F1] using rearrangement techniques, see also the paper by Brock [Bro], for a prior related result of one-dimensional symmetry. For a stronger version of the Gibbons' conjecture, one should also consult the recent paper by Farina [F2].

A new major development in the problem proposed by De Giorgi has recently come with the work of L. Ambrosio and X. Cabré. In their beautiful paper [AC] the authors have proved the conjecture true in $\mathbb{R}^3$. In fact, [AC] contains a positive answer to a stronger form of the conjecture, see Theorem 10.1. The double-well potential for the Ginzburg-Landau model with two equal wells $F(u) = \frac{1}{4}(1 - u^2)^2$, for which $F'(u) = u^3 - u$, satisfies the requirements in Theorem 10.1, with $m = -1$, $M = 1$, thus the conjecture follows for $\mathbb{R}^3$. Subsequently, in the joint work with Alberti [AAC], the authors have succeeded in removing the additional assumptions on the non-linearity $F(u)$ in Theorem 10.1, thus establishing the validity of the conjecture in $\mathbb{R}^3$ for the equation $\Delta u = F'(u)$, where $F(u)$ is an arbitrary function in $C^2(\mathbb{R})$, see Theorem 10.3.

The aim of this paper is to generalize various results in [CGS], [BCN], [GG], [AC], and [AAC], to equations of the general type (1.2). A distinctive aspect of our results is that they do not distinguish between Laplace equation, and the two important, yet very different, models given by the the minimal surface operator, and by the p-Laplacian. In addition to this, the general setting in which we work clarifies for the first time the role of invariance under the action of the orthogonal group $O(n)$. As we will see, such invariance pays no role in low dimension ($n = 2$), whereas, when $n \geq 3$, it becomes important, or remains irrelevant, depending on the situation at hand. The paper is composed of ten sections. A short description can be obtained by glancing at the titles of the sections.

After this paper was accepted for publication we received the preprint [F3] by A. Farina, in which the author obtains a one-dimensional symmetry result in $\mathbb{R}^2$, for equations having energy as in (1.7). This result implies the validity of De Giorgi's conjecture in the plane, and within its more restricted range, it provides a very interesting independent proof of our Theorem 7.1.

2 Structural assumptions

In this section we list the general structural hypothesis for this paper. Since we are not interested in the weakest regularity requirements on Φ we assume that

$$\Phi \in C^3(\mathbb{R} \times (\mathbb{R}^n \setminus \{0\})) \cap C^1(\mathbb{R} \times \mathbb{R}^n),$$

(although, in most cases, the weaker requirement $\Phi \in C^2(\mathbb{R}\times(\mathbb{R}^n\setminus\{0\}))\cap C^1(\mathbb{R}\times\mathbb{R}^n)$ would suffice). The function Φ will be supposed normalized as follows

$$\Phi_{\sigma_i}(u, 0) = 0, \qquad i = 1, ..., n. \tag{2.1}$$

Since we want to include the very diverse models (2.12) and (2.13) below, we will list two separate sets of structural hypothesis, (H 1) and (H 2).

(H 1) There exist $p > 1$, $\epsilon \geq 0$, and for every $C > 0$ there exist constants $c_1, c_2 > 0$ such that for any $\xi \in \mathbb{R}$, with $|\xi| \leq C$, and every $\sigma, \zeta \in \mathbb{R}^n \setminus \{0\}$, one has

$$c_1\, |\sigma|^2\, (\epsilon + |\sigma|)^{p-2} \leq \Phi(\xi,\sigma) - \Phi(\xi,0) \leq c_2\, (\epsilon + |\sigma|)^p. \tag{2.2}$$

$$|\Phi_\sigma(\xi,\sigma)| \leq c_2\, (\epsilon + |\sigma|)^{p-1}. \tag{2.3}$$

$$|\Phi_\xi(\xi,\sigma)| \leq c_2\, (1 + |\sigma|)^p. \tag{2.4}$$

$$c_1\, (\epsilon + |\sigma|)^{p-2}\, |\zeta|^2 \leq < \Phi_{\sigma\sigma}(\xi,\sigma)\zeta, \zeta > \leq c_2\, (\epsilon + |\sigma|)^{p-2}\, |\zeta|^2, \tag{2.5}$$

where $\Phi_{\sigma\sigma}$ denotes the Hessian matrix of Φ.

(H 2) For every $C > 0$ there exist constants $c_1, c_2 > 0$ such that for every $\xi \in \mathbb{R}$, with $|\xi| \leq C$, for any $\sigma \in \mathbb{R}^n$, and every $\zeta' = (\zeta, \zeta_{n+1}) \in \mathbb{R}^{n+1}$ which is orthogonal to the vector $(-\sigma, 1) \in \mathbb{R}^{n+1}$, one has

$$\Phi(\xi,\sigma) - \Phi(\xi,0) \geq c_1 \sqrt{1 + |\sigma|^2}. \tag{2.6}$$

$$|\Phi_\sigma(\xi,\sigma)| \leq c_2. \tag{2.7}$$

$$|\Phi_\xi(\xi,\sigma)| \ \leq \ c_2 \tag{2.8}$$

$$c_1 \, \frac{|\zeta'|^2}{\sqrt{1 \, + \, |\sigma|^2}} \ \leq \, < \Phi_{\sigma\sigma}(\xi,\sigma)\zeta,\zeta > \, \leq \ c_2 \, \frac{|\zeta'|^2}{\sqrt{1 \, + \, |\sigma|^2}} \tag{2.9}$$

Remark 2.1. We emphasize that when Φ has the special structure (1.7), then the following weaker regularity hypothesis suffices: In the case (H 1) we assume $G \in C^3(\mathbb{R} \setminus \{0\}) \cap C^1(\mathbb{R})$, $F \in C^2(\mathbb{R})$, whereas in case (H 2) we suppose $G \in C^3(\mathbb{R})$, $F \in C^2(\mathbb{R})$. It is then clear that the results in this paper do include the special case of the non-linear Poisson equation $\Delta u = F'(u)$, with $F \in C^2(\mathbb{R})$.

It is important to note that either when $\epsilon > 0$ in *(H 1)* (and even when *(H 1)* holds with $\epsilon = 0$, but $p \geq 2$), or when *(H 2)* is in force, we can actually assume, and will do so, that $\Phi \in \ C^3(\mathbb{R} \times \mathbb{R}^n)$, since the gradient of Φ with respect to σ has in such cases no singularity at $\sigma = 0$. Such hypothesis will become effective after Remark 9.3, for the remaining part of Sect. 8, and also for sections nine and eleven.

Remark 2.2. Assume *(H 2)*. For every $\sigma \in \mathbb{R}^n$ the choice $\zeta' = (\sigma, |\sigma|^2)$ in (2.9) gives

$$c_1 \, |\sigma|^2 \, \sqrt{1 \, + \, |\sigma|^2} \ \leq \, < \Phi_{\sigma\sigma}(\xi,\sigma)\sigma,\sigma > \, \leq \ c_2 \, |\sigma|^2 \, \sqrt{1 \, + \, |\sigma|^2}. \tag{2.10}$$

This inequality, when used in the proof of Lemma 6.1, guarantees the conclusion

$$< \sigma, \Phi_\sigma(\xi,\sigma) > \, \geq \ \Phi(\xi,\sigma) \ - \ \Phi(\xi,0).$$

From the latter and from (2.6) we obtain

$$< \sigma, \Phi_\sigma(\xi,\sigma) > \, \geq \ c_1 \, \sqrt{1 \, + \, |\sigma|^2}, \tag{2.11}$$

which gives the structural assumption (2.3) in [LU2] with $\mu_1 = c_1$ and $\mu_2 = 0$.

The basic models for (H 1) and (H 2) are, respectively,

$$\Phi(\xi,\sigma) \ = \ \frac{1}{p} \, \left(\epsilon^2 \, + \, |\sigma|^2\right)^{p/2} \, + \ F(\xi), \qquad 1 < p < \infty, \quad \epsilon \geq 0, \tag{2.12}$$

and

$$\Phi(\xi,\sigma) \ = \ \sqrt{1 \, + \, |\sigma|^2} \ + \ F(\xi), \tag{2.13}$$

with corresponding Euler-Lagrange equations

$$div \ \left((\epsilon^2 \, + \, |Du|^2)^{(p-2)/2} \, Du \right) \ = \ F'(u), \qquad 1 < p < \infty,$$

and

$$div \left(\frac{Du}{\sqrt{1 \, + \, |Du|^2}} \right) \ = \ F'(u).$$

When (H 1) holds, by an *entire (weak) solution* to (1.2) we mean a function $u \in W^{1,p}_{loc}(\mathbb{R}^n)$ such that for every $\phi \in W^{1,p}_o(\mathbb{R}^n)$ with compact support

$$\int_{\mathbb{R}^n} < \Phi_\sigma(u, Du), D\phi > \, dx \; + \int_{\mathbb{R}^n} \Phi_\xi(u, Du)\, \phi \, dx \; = \; 0. \tag{2.14}$$

If, instead, (H 2) is in force, then an entire solution to (1.2) will be a function $u \in C^2(\mathbb{R}^n)$ which satisfies the equation in the classical sense.

3 The analysis of the ode

In this section we analyze the ordinary differential equation associated with (1.2), namely

$$(\Phi_\sigma(u, u_x))_x \; = \; \Phi_\xi(u, u_x). \tag{3.1}$$

We assume that

$$\Phi_{\sigma\sigma}(\xi, \sigma) \; > \; 0 \qquad\qquad \text{for every} \quad (\xi, \sigma) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\}),$$

see (2.5). It will be useful in the sequel to also have the expression of (3.1) in non-variational form, at those points $x \in \mathbb{R}$ where $u_x(x) \neq 0$

$$\Phi_{\xi\sigma}(u, u_x)\, u_x \; + \; \Phi_{\sigma\sigma}(u, u_x)\, u_{xx} \; = \; \Phi_\xi(u, u_x). \tag{3.2}$$

We introduce the function

$$P \; = \; P(x; u) \; = \; \Phi(u, u_x) - \; u_x\, \Phi_\sigma(u, u_x). \tag{3.3}$$

Lemma 3.1. *There exists a number P_o such that if u is a solution to* (3.1), *and if moreover $u_x(x) \neq 0$ for every $x \in \mathbb{R}$ when (H 1) holds with $\epsilon = 0$, then*

$$P(x; u) \; \equiv \; P_o.$$

Proof. If either (H 1) holds with $\epsilon > 0$, or (H 2) is valid, then the regularity theory of ode's guarantee that $u \in C^2(\mathbb{R})$. The same conclusion is true when $\epsilon = 0$ in assumption (H 1), but $u_x(x) \neq 0$ for every $x \in \mathbb{R}$. Differentiating (3.3) with respect to x we find

$$\begin{aligned} P_x \; &= \Phi_\xi(u, u_x)\, u_x \; + \; \Phi_\sigma(u, u_x)\, u_{xx} \; - \; \Phi_{\xi\sigma}(u, u_x)\, u_x^2 \\ &- \Phi_{\sigma\sigma}(u, u_x)\, u_{xx}\, u_x \; - \; \Phi_\sigma(u, u_x)\, u_{xx} \\ &= [\Phi_\xi(u, u_x) \; - \; \Phi_{\xi\sigma}(u, u_x)\, u_x \; - \; \Phi_{\sigma\sigma}(u, u_x)\, u_{xx}] \; u_x \; = \; 0, \end{aligned}$$

where in the last equality we have used (3.2). □

Remark 3.2. It is worth observing that the assumption $u_x \neq 0$ in the statement of Lemma 3.1 has only been made to give a sense to the quantity $\Phi_{\sigma\sigma}(u, u_x)$. Such assumption is clearly not needed when the equation is non-degenerate at $u_x = 0$, as it is the case for (H 1) with $\epsilon > 0$, or for (H 2).

Lemma 3.3. *Let u be a bounded solution to* (3.1) *satisfying $u_x > 0$ in $\mathbb{R}$ and set*

$$A = \inf_{\mathbb{R}} u, \qquad B = \sup_{\mathbb{R}} u.$$

One has

$$\Phi(A,0) = \Phi(B,0), \qquad \Phi_\xi(A,0) = \Phi_\xi(B,0) = 0,$$

and

$$\Phi(\xi,0) > \Phi(A,0) = \Phi(B,0) \qquad \xi \in (A,B).$$

Furthermore,

$$\int_{\mathbb{R}} \{\Phi(u,u_x) - \Phi(B,0)\}\, dx < \infty.$$

Proof. Lemma 3.1 implies

(3.4) $$\Phi(u,u_x) - u_x\,\Phi_\sigma(u,u_x) \equiv P_o.$$

Since $A = \lim_{x\to-\infty} u(x), B = \lim_{x\to\infty} u(x)$, and moreover the boundedness of u forces

(3.5) $$\underline{lim}_{x\to\pm\infty}\, u_x(x) = 0,$$

we conclude from (3.4), (3.5)

(3.6) $$\Phi(A,0) = P_o = \Phi(B,0).$$

Observe next that

(3.7) $$\sigma\,\Phi_\sigma(\xi,\sigma) - [\Phi(\xi,\sigma) - \Phi(\xi,0)] > 0, \qquad (\xi,\sigma) \in \mathbb{R}\times(0,\infty).$$

The proof of (3.7) follows noting that the function

(3.8) $$\Psi(\xi,\sigma) = \sigma\,\Phi_\sigma(\xi,\sigma) - [\Phi(\xi,\sigma) - \Phi(\xi,0)]$$

satisfies $\Psi(\xi,0) = 0$ and that furthermore

$$\Psi_\sigma(\xi,\sigma) = \sigma\,\Phi_{\sigma\sigma}(\xi,\sigma) > 0, \qquad (\xi,\sigma) \in \mathbb{R}\times(0,\infty).$$

Once this is known we obtain from (3.4), (3.6)

$$\Phi(u,0) - \Phi(A,0) = \Phi(u,0) + u_x\,\Phi_\sigma(u,u_x) - \Phi(u,u_x) > 0,$$

where in the last inequality we have used (3.7). An analogous inequality holds if we replace $\Phi(A,0)$ with $\Phi(B,0)$. This proves $\Phi(\xi,0) > \Phi(A,0) = \Phi(B,0)$ for $\xi \in (A,B)$. For every fixed $\xi \in \mathbb{R}$ let us denote by

$$H(\xi,\cdot) = \Psi(\xi,\cdot)^{-1}$$

the inverse function of $\Psi(\xi, \cdot)$. Re-writing (3.4) as

$$\Psi(u, u_x) \;=\; \Phi(u, 0) \;-\; \Phi(B, 0),$$

we obtain

$$u_x \;=\; H(u, \Phi(u, 0) - \Phi(B, 0)). \tag{3.9}$$

If we consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(\xi, \sigma) = (\xi, \Psi(\xi, \sigma))$, then f is one-to-one and continuous, and one easily sees that its inverse $f^{-1}(\eta, z) = (\eta, H(\eta, z))$ is also continuous. In particular, the function $(\eta, z) \to H(\eta, z)$ is continuous. From this observation and from (3.9) we infer that in addition to (3.5) one has in fact

$$\lim_{x \to \pm\infty} u_x(x) \;=\; 0. \tag{3.10}$$

Furthermore, (3.9) implies the existence of a constant $M = M(||u||_{L^\infty(\mathbb{R})}) > 0$ such that

$$|u_x(x)| \;\leq\; M, \qquad\qquad x \in \mathbb{R}.$$

Using (3.10), the equation (3.1), and the mean-value theorem, one easily obtains

$$\Phi_\xi(A, 0) \;=\; \Phi_\xi(B, 0) \;=\; 0.$$

Finally, (3.4) and (3.6) give

$$\begin{aligned} &\int_{\mathbb{R}} \{\Phi(u, u_x) \;-\; \Phi(B, 0)\} \; dx \\ &= \int_{\mathbb{R}} u_x \, \Phi_\sigma(u, u_x) \, dx \end{aligned}$$

therefore to estimate the energy it suffices to control the latter integral. For every $\zeta > 0$ one has

$$\begin{aligned} &\int_{-\zeta}^{\zeta} u_x(x) \, \Phi_\sigma(u(x), u_x(x)) \, dx \;=\; \int_{u(-\zeta)}^{u(\zeta)} \Phi_\sigma(t, u_x(u^{-1}(t)) \, dt \\ &\leq \int_A^B \Phi_\sigma(t, u_x(u^{-1}(t))) \, dt \;\leq\; (B \;-\; A) \left(\max_{(\xi,\sigma) \in [A,B] \times [-M,M]} \Phi_\sigma(\xi, \sigma) \right) \end{aligned}$$

Letting $\zeta \to \infty$ we reach the conclusion

$$\int_{\mathbb{R}} u_x \, \Phi_\sigma(u, u_x) \, dx \;\leq\; (B \;-\; A) \left(\max_{(\xi,\sigma) \in [A,B] \times [-M,M]} \Phi_\sigma(\xi, \sigma) \right) \;<\; \infty.$$

□

4 Higher-dimensional analysis of the P-function

Before proving the main results in this section we develop some preparatory considerations. We have the following basic result, see [U], and also [To], [DB] and [Le].

Theorem 4.1. *Assume* (H 1)*, and let* u *be a bounded entire solution to* (1.2)*. There exist positive numbers* M, α *and* γ *depending only on* n*, the parameters* p, ϵ *in the structural assumptions* (H 1)*, and on* $||u||_\infty = ||u||_{L^\infty(\mathbb{R}^n)}$*, such that*

$$||Du||_\infty \ \leq \ M, \tag{4.1}$$

and

$$|Du(x) \ - \ Du(y)| \ \leq M \ R^{-\gamma} \ \left(\frac{|x-y|}{R}\right)^\alpha \tag{4.2}$$

for every $x_o \in \mathbb{R}^n, R > 0$*, and any* $x, y \in B_R(x_o) = \{\xi \in \mathbb{R}^n \mid |\xi - x_o| < R\}$.

For bounded entire solutions of (1.2), with the structural assumptions (H 2), interior a priori bounds for the gradient have been obtained under additional requirements on the energy function Φ. For the special model (2.13) with $F \equiv 0$, the following celebrated result of Bombieri, De Giorgi and Miranda [BDM] holds: *Let* u *be a* C^2 *solution of the minimal surface equation in a ball* $B(x,R) \subset \mathbb{R}^n, n \geq 2$*, then*

$$|Du(x)| \ \leq \ C_1 \ \exp \ \left[C_2 \ \frac{\sup_{y\in B(x,R)} \ (u(y) - u(x))}{R}\right],$$

for appropriate positive numbers C_1, C_2 *depending only on* n. See also [K] for a simpler proof based of the maximum principle. It follows that bounded entire solutions of the minimal surface equation have bounded gradient. For a detailed description of conditions under which it is possible to obtain similar a priori bounds of the gradient for (1.2) with (H 2), we refer the reader to [LU2], p.691-94, where even the more general setting (7.11) is treated, and also to the subsequent work [Si1]. For our purposes it will be important to know that there exist situations in which bounded entire solutions have bounded gradient and we will always work within this framework. This means that when (H 2) is in force we will always *a priori* assume the existence of a constant $M > 0$, depending on n, and on $||u||_\infty$, such that (4.1) hold. Under these circumstances the equation (1.2) becomes uniformly elliptic. We can thus appeal to the classical Schauder estimates, see [LU1], [GT], to conclude that $u \in C^{2,\gamma}_{loc}(\mathbb{R}^n)$ and that (4.2) is valid also.

For the structural hypothesis (H 1), with $\epsilon = 0$, it is well known that the optimal regularity of weak solution is expressed by Theorem 4.1. If, however, in an open set $\Omega \subset \mathbb{R}^n$ we have

$$\inf_{\Omega} \ |Du| \ > \ 0, \tag{4.3}$$

then appealing to the regularity theory for non-degenerate quasi-linear equations [LU1] one infers that actually $u \in C^{2,\beta}_{loc}(\Omega)$, for some $\beta \in (0,1)$ depending on

$||u||_\infty$, on the structural constants, and on the quantity in the left-hand side in (4.3). If either (H 1) and (4.3) hold, or we are in the situation (H 2), we are thus allowed to take second derivatives of the solution u. Observe that if $u_k = D_k u$, then in a classical fashion one recognizes that in weak form the linear equation satisfied by u_k in Ω is

(4.4)
$$\int_\Omega < \Phi_{\sigma\sigma}(u, Du)\ D(u_k), D\phi > dx \ = \ - \int_\Omega < \Phi_{\xi\sigma}(u, Du), D\phi > u_k\ dx$$
$$- \int_\Omega < \Phi_{\xi\sigma}(u, Du), D(u_k) > \phi\ dx \ - \ \int_\Omega \Phi_{\xi\xi}(u, Du)\ u_k\ \phi\ dx,$$

where ϕ is a test function in Ω. Hereafter, we adopt the summation convention over repeated indices. The latter equation can be re-written as follows

$$(a_{ij}\ (u_k)_i)_j \ = \ [\Phi_{\xi\xi} \ - \ div\ \Phi_{\xi\sigma}]\ u_k, \tag{4.5}$$

with a_{ij} given by

$$a_{ij} \ = \ a_{ij}(\xi, \sigma) \ = \ \Phi_{\sigma_i\sigma_j}(\xi, \sigma). \tag{4.6}$$

In the sequel it will be useful to have (1.2) also in the non-variational form

$$a_{ij}\ u_{ij} \ = \ \Phi_\xi - \ < \Phi_{\xi\sigma}, Du >, \tag{4.7}$$

which makes clearly sense when either (H 1) and (4.3) hold, or (H 2) is in force.

We now let

$$\Lambda \ = \ \Lambda(\xi, \sigma) \ = \ \frac{a_{hk}\ \sigma_h\ \sigma_k}{|\sigma|^2}, \qquad (\xi, \sigma) \in \mathbb{R} \times (\mathbb{R}^n \setminus \{0\}) \tag{4.8}$$

and set

$$d_{ij} \ = \ d_{ij}(\xi, \sigma) \ = \ \frac{a_{ij}(\xi, \sigma)}{\Lambda(\xi, \sigma)}. \tag{4.9}$$

We note explicitly that

$$d_{ij}(u, Du)\ u_i\ u_j \ = \ |Du|^2. \tag{4.10}$$

Guided by the analysis of the ode in Sect. 2 we introduce the function $\Psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$\Psi(\xi, \sigma) \ = \ 2 < \sigma, \Phi_\sigma(\xi, \sigma) > \ - 2\left[\Phi(\xi, \sigma) \ - \ \Phi(\xi, 0)\right], \tag{4.11}$$

and consider the quantity

$$\begin{aligned} P \ = \ P(x; u) \ &\overset{def}{=} \ 2 < Du, \Phi_\sigma(u, Du) > \ - 2\, \Phi(u, Du) \\ &= \ \Psi(u, Du) \ - \ 2\, \Phi(u, 0). \end{aligned} \tag{4.12}$$

In the remainder of this section we suppose that $\Phi(\xi,\sigma)$ has the structure (1.5). We let $s = |\sigma|^2$ so that (1.5) gives

$$\Phi_\sigma(\xi,\sigma) \;=\; G_s(\xi,|\sigma|^2)\,\sigma. \tag{4.13}$$

The equation (4.7) presently takes the form

$$a_{ij}\,u_{ij} \;=\; \frac{1}{2}\,G_\xi \;-\; G_{\xi s}\,|Du|^2, \tag{4.14}$$

with

$$a_{ij}(\xi,\sigma) \;=\; 2\,G_{ss}(\xi,|\sigma|^2)\,\sigma_i\,\sigma_j \;+\; G_s(\xi,|\sigma|^2)\,\delta_{ij}. \tag{4.15}$$

For the function in (4.8) we have

$$\Lambda \;=\; \Lambda(\xi,s) \;=\; 2\,s\,G_{ss}(\xi,s) + G_s(\xi,s) \;>0 \qquad (\xi,s)\in\mathbb{R}\times(0,\infty). \tag{4.16}$$

The last inequality is nothing but a reformulation of the ellipticity of the matrix $a_{ij} = \Phi_{\sigma_i\sigma_j}$ which is guaranteed by (2.5), (2.9). We obtain from (4.11)

$$\Psi \;=\; \Psi(\xi,s) \;=\; 2\,s\,G_s(\xi,s) \;-\; G(\xi,s) \;+\; G(\xi,0). \tag{4.17}$$

Since

$$\Psi(\xi,0) \;=\; 0, \tag{4.18}$$

and

$$\Psi_s \;=\; 2\,s\,G_{ss} \;+\; G_s \;=\; \Lambda, \tag{4.19}$$

we conclude from (4.16) that must be

$$\Psi(\xi,s) \;>\; 0 \qquad\qquad (\xi,s)\in\mathbb{R}\times\mathbb{R}^+. \tag{4.20}$$

If we let $F(\xi) = G(\xi,0)$, then we can write the non-linear quantity P in (4.12) as follows

$$P \;=\; 2\,G_s(u,|Du|^2)\,|Du|^2 \;-\; G(u,|Du|^2) \;=\; \Psi(u,|Du|^2) \;-\; F(u). \tag{4.21}$$

It is obvious that if $u\equiv const$, then the same is true for P. The next proposition motivates the introduction of the function P and also the subsequent development in this section.

Proposition 4.2. *Let u be a non-constant entire solution to* (1.2), *with Φ satisfying* (1.5). *If*

$$u(x) \;=\; g(<a,x>),$$

for some $g\in C^2(\mathbb{R})$ and $a\in\mathbb{R}^n$ with $|a|=1$, and if when (H 1) holds with $\epsilon=0$ one has $g'(t)\neq 0$ for every $t\in\mathbb{R}$, then the P-function relative to such a u is identically constant.

Proof. We observe that

$$u_i \ = \ g'(< a, x >)\, a_i, \qquad\qquad u_{ij} \ = \ g''(< a, x >)\, a_i\, a_j. \tag{4.22}$$

By the assumption on g, we know that $Du(x) \neq 0$ for every $x \in \mathbb{R}^n$ when (H 1) holds with $\epsilon = 0$. Therefore, using (4.22) and (4.15), the equation (4.14) now becomes

$$\begin{aligned}&\left\{2\, G_{ss}(g, (g')^2)\, (g')^2 \ + \ G_s(g, (g')^2)\right\}\ g'' \ + \ G_{\xi s}(g, (g')^2)\, (g')^2 \\ &= \ \frac{1}{2}\, G_\xi(g, (g')^2),\end{aligned} \tag{4.23}$$

where we have omitted the argument $< a, x >$ of g, g', g''. Letting $t =< a, x >$, and $\sigma \ = \ s$ in (4.23), we conclude that g is a solution to (3.1) with $\Phi(\xi, \sigma) \ = \ (1/2)G(\xi, \sigma^2)$. By Lemma 3.1 we infer that $P(x; u) \equiv const.$ □

Theorem 4.3. *Assume* (1.5)*, and let* u *be a bounded entire solution to* (1.2) *such that*

$$\inf_{\Omega}\ |Du| \ > \ 0$$

in a bounded open set $\Omega \subset \mathbb{R}^n$*. The following differential inequality holds in* Ω *for the function* P *in* (4.21)

$$\sum_{ij=1}^{n}\ D_i\, (d_{ij}(u, Du)\ D_j P) \ + \ \sum_{i=1}^{n}\ B_i\ D_i P \ \geq \ \frac{|DP|^2}{2\ \Lambda\ |Du|^2}.$$

Here,

$$B_i \ = \ \frac{G_s\ G_{\xi s} \ - \ G_\xi\ (|Du|^2\ G_{ss} \ + \ G_s)}{G_s\ \Lambda}\ D_i u,$$

where all the functions entering in the right-hand side of the latter equation are evaluated in $(u, |Du|^2)$ *.*

Remark 4.4. We stress that although we assumed that Φ is of class C^3, in the expression of B_i only second partial derivatives of G appear. Third derivatives do appear in the calculations needed in the proof on Theorem 4.3, but they eventually cancel.

Proof. Differentiating (4.21) with respect to x_i and using (4.19) gives

$$\begin{aligned}P_i \ &= \ \Psi_\xi\ u_i \ + \ 2\, \Psi_s\ u_{ki}\ u_k \ - \ F'\ u_i. \\ &= \ 2\ \Lambda\ u_{ki}\ u_k \ + \ [\Psi_\xi \ - \ F']\ u_i\end{aligned} \tag{4.24}$$

The following expression will be useful

$$< Du, DP >= \ 2\ \Lambda\ u_{ij}\ u_i\ u_j \ + \ [\Psi_\xi \ - \ F']\ |Du|^2. \tag{4.25}$$

In the sequel, a_{ij} will be a short notation for $a_{ij}(u, Du)$. Similarly, we will write d_{ij}, instead of $d_{ij}(u, Du)$. One has from (4.24)

(4.26)
$$\left(d_{ij}\, P_i\right)_j \;=\; 2\;\left(a_{ij}\;(u_k)_i\right)_j\; u_k \;+\; 2\, a_{ij}\, u_{ki}\, u_{kj} \\ +\, d_{ij}\, u_{ij}\,(\Psi_\xi \;-\; F') \;+\; d_{ij,j}\, u_i\,(\Psi_\xi \;-\; F') \;+\; d_{ij}\, u_i\,(\Psi_\xi \;-\; F')_j\;.$$

Using (4.14), and differentiating (4.17) with respect to ξ, we obtain

$$d_{ij}\, u_{ij}\,(\Psi_\xi \;-\; F') \;=\; \frac{1}{\Lambda}\;\left(\frac{1}{2}\, G_\xi \;-\; G_{\xi s}\,|Du|^2\right) \\ (\Psi_\xi \;-\; F') \;=\; -\,\frac{1}{2\Lambda}\,(\Psi_\xi \;-\; F')^2\;. \tag{4.27}$$

Inserting (4.5) and (4.27) in (4.26), we find

$$\left(d_{ij}\, P_i\right)_j \;=\; 2\, a_{ij}\, u_{ki}\, u_{kj} \;-\; \frac{1}{2\Lambda}\,(\Psi_\xi \;-\; F')^2 \\ +\; 2\,(\Phi_{\xi\xi} \;-\; div\, \Phi_{\xi\sigma})\,|Du|^2 \\ +\; d_{ij,j}\, u_i\,(\Psi_\xi \;-\; F') \;+\; d_{ij}\, u_i\,(\Psi_\xi \;-\; F')_j. \tag{4.28}$$

We next estimate from below the term $2a_{ij}u_{ki}u_{kj}$. The equation (4.15) gives

$$2\, a_{ij}\, u_{ki}\, u_{kj} \;=\; 4\, G_{ss}\, u_{ki}\, u_{kj}\, u_i\, u_j \;+\; 2\, G_s\, u_{ki}\, u_{ki}. \tag{4.29}$$

Schwarz inequality implies

$$u_{ki}\, u_{ki} \;\geq\; \frac{u_{ki}\, u_{kj}\, u_i\, u_j}{|Du|^2} \tag{4.30}$$

Substituting (4.30) in (4.29), one finds

$$2\, a_{ij}\, u_{ki}\, u_{kj} \;\geq\; \frac{2\,\Lambda}{|Du|^2}\, u_{ki}\, u_{kj}\, u_i\, u_j.$$

We now employ (4.24) in the latter inequality, obtaining

(4.31)
$$2\, a_{ij}\, u_{ki}\, u_{kj} \;\geq\; \frac{2\,\Lambda}{|Du|^2}\;\frac{P_k \;-\; (\Psi_\xi \;-\; F')\, u_k}{2\,\Lambda}\;\frac{P_k \;-\; (\Psi_\xi \;-\; F')\, u_k}{2\,\Lambda} \\ =\; \frac{|DP|^2}{2\,\Lambda\,|Du|^2} \;+\; \frac{(\Psi_\xi \;-\; F')^2}{2\,\Lambda} \;-\; \frac{(\Psi_\xi \;-\; F')}{\Lambda\,|Du|^2}\; < Du, DP > .$$

Substitution of (4.31) in (4.28) gives

$$\left(d_{ij}\, P_i\right)_j \;+\; \frac{(\Psi_\xi \;-\; F')}{\Lambda\,|Du|^2}\; < Du, DP > \\ \geq\; \frac{|DP|^2}{2\,\Lambda\,|Du|^2} \;+\; 2\,(\Phi_{\xi\xi} \;-\; div\, \Phi_{\xi\sigma})\,|Du|^2 \;+ d_{ij,j}\, u_i\,(\Psi_\xi \;-\; F') \\ +\; d_{ij}\, u_i\,(\Psi_\xi \;-\; F')_j. \tag{4.32}$$

The proof of the theorem will be completed if we show that for some vector field $\vec{C}$

$$R \overset{def}{=} 2\,(\Phi_{\xi\xi} \,-\, div\,\Phi_{\xi\sigma})\,|Du|^2 \,+\, d_{ij,j}\,u_i\,(\Psi_\xi \,-\, F') \\ +\, d_{ij}\,u_i\,(\Psi_\xi \,-\, F')_j \,=<\vec{C}, DP> . \tag{4.33}$$

First, we have

$$d_{ij}\,u_i\,(\Psi_\xi \,-\, F')_j = d_{ij}\,u_i\,u_j\,(\Psi_{\xi\xi} \,-\, F'') \,+\, d_{ij}\,u_i\,\Psi_{\xi\sigma_k}\,u_{kj} \\ = \big(2\,|Du|^2\,G_{\xi\xi s} \,-\, G_{\xi\xi}\big)\,|Du|^2 \,+\, d_{ij}\,u_i\,\Psi_{\xi\sigma_k}\,u_{kj}. \tag{4.34}$$

From

$$\Psi_{\xi\sigma_k} \,=\, \big(4\,|Du|^2\,G_{\xi ss} \,+\, 2\,G_{\xi s}\big)\,u_k\,,$$

and (4.15), we find

$$d_{ij}\,u_i\,u_{kj}\,\Psi_{\xi\sigma_k} \,=\, \big(4\,|Du|^2\,G_{\xi ss} \,+\, 2\,G_{\xi s}\big)\,u_{ij}\,u_i\,u_j.$$

Substituting the latter expression in (4.34), noting that

$$2\,(\Phi_{\xi\xi} \,-\, div\,\Phi_{\xi\sigma})\,|Du|^2 \,=\, G_{\xi\xi}\,|Du|^2 \,-\, 2\,G_{\xi s}\,\Delta u \\ -\,2\,G_{\xi\xi s}\,|Du|^2 \,-\, 4\,G_{\xi ss}\,u_{ij}\,u_i\,u_j$$

and that (4.17) gives

$$\Psi_\xi \,-\, F' \,=\, 2\,s\,G_{\xi s} \,-\, G_\xi,$$

we conclude

$$R \,=\, 2\,G_{\xi s}\,\big[|Du|^2\,\Delta u \,-\, u_{ij}\,u_i\,u_j\big] \,+\, d_{ij,j}\,u_i\,(2\,|Du|^2\,G_{\xi s} \,-\, G_\xi). \tag{4.35}$$

The second main step in the proof of (4.33) is the computation of the term $d_{ij,j}\,u_i$. Since the latter is very long, and the details are rather tedious and uninformative, we only give the final outcome

$$d_{ij,j}\,u_i \,=\, \frac{2\,G_{ss}}{\Lambda}\,\big[|Du|^2\,\Delta u \,-\, u_{ij}\,u_i\,u_j\big]\,. \tag{4.36}$$

Once the latter equation is substituted in (4.35) one has

$$R \,=\, -\,\frac{2}{\Lambda}\,(G_\xi\,G_{ss} \,+\, G_s\,G_{\xi s})\,\big(|Du|^2\,\Delta u \,-\, u_{ij}\,u_i\,u_j\big)\,. \tag{4.37}$$

At this point we use the equation (4.14) to obtain

$$|Du|^2\,\Delta u \,-\, u_{ij}\,u_i\,u_j \,=\, \frac{\frac{1}{2}G_\xi - G_{\xi s}|Du|^2}{G_s}\,|Du|^2 \,-\, \frac{\Lambda u_{ij}\,u_i\,u_j}{G_s}.$$

Finally, (4.25) gives

$$\Lambda u_{ij}\, u_i\, u_j \;=\; \frac{1}{2} < Du, DP > \;-\; \left(G_{\xi s}|Du|^2 \;-\; \frac{1}{2}\, G_\xi \right)\; |Du|^2,$$

and therefore we find

$$|Du|^2\; \Delta u \;-\; u_{ij}\, u_i\, u_j \;=\; -\,\frac{1}{2G_s}\; < Du, DP > . \tag{4.38}$$

Using the latter equation in (4.37) we conclude

$$R \;=\; \frac{G_\xi\, G_{ss} \;+\; G_s\, G_{\xi s}}{G_s \Lambda}\; < Du, DP >, \tag{4.39}$$

which establishes (4.33) and completes the proof . Finally, the specific form of the vector field $\overrightarrow{B} = (B_1, ..., B_n)$ in the statement of the theorem follows from (4.39) and from (4.32). □

Remark 4.5. The reader should notice the appearance of the geometric quantity

$$|Du|^2\; \Delta u \;-\; u_{ij}\, u_i\, u_j \tag{4.40}$$

in the expressions (4.35), (4.36), (4.37), and in the directional derivative (4.38) of P with respect to Du. We will return to this observation in the proof of Proposition 4.11.

Remark 4.6. In [PP] Payne and Philippin considered quasi-linear equations

$$div\; A(u, |Du|^2) \;=\; B(u, |Du|^2),$$

which are not necessarily the Euler-Lagrange equation of an elliptic integrand, and derived maximum principles for some appropriate P-functions. Due to the greater generality, however, the relevant P and the conditions under which the latter satisfies an elliptic differential inequality are rather implicitly given. Our presentation (which is inspired to an idea introduced in [GL], see also [CGS] and [GS]) is somewhat different from that in [PP].

Theorem 4.7. *Assuming* (1.5), *let* u *be a bounded entire solution to* (1.2) *such that*

$$\inf_{\overline{\Omega}}\; |Du| \;>\; 0,$$

in a certain connected, bounded open set $\Omega \subset \mathbb{R}^n$. *If there exists* $x_o \in \Omega$ *such that*

$$P(x_o; u) \;=\; \sup_{x \in \Omega}\; P(x; u),$$

then $P \equiv P(x_o; u)$ *in* $\overline{\Omega}$.

Proof. It is a direct consequence of Theorem 4.3 and of the maximum principle for quasi-linear uniformly elliptic equations, see [GT]. □

The next thorem provides an a priori pointwise estimate of the gradient of a weak solution to (1.2). It generalizes the results of L. Modica [M2], and of Caffarelli, Segala and one of us [CGS], mentioned in the introduction. Since its proof is similar to that of Theorem 1. in [CGS], we omit it, referring the reader to that source.

Theorem 4.8. *Let u be a bounded entire solution to* (1.2)*, with Φ given by* (1.5)*. With Ψ as in* (4.17)*, under the hypothesis that $G(\xi,0) \geq 0$ for every $\xi \in \mathbb{R}$ one has*

$$\Psi(u(x), |Du(x)|^2) \ \leq \ G(u(x), 0), \qquad\qquad x \in \mathbb{R}^n.$$

The following result is an immediate consequence of Theorem 4.8.

Corollary 4.9. *Let u be a bounded entire solution to* (1.2)*, with Φ as in* (1.5)*. If*

$$G^u \ = \ min \left\{ G(\xi, 0) \mid \inf_{\mathbb{R}^n} \ u \ \leq \ \xi \ \leq \ \sup_{\mathbb{R}^n} \ u \right\},$$

then

$$2\,|Du|^2\, G_s(u, |Du|^2) \ \leq \ G(u, |Du|^2) \ - \ G^u. \tag{4.41}$$

Proof. It is enough to observe that if we let $\Theta(\xi,\sigma) = (1/2)[G(\xi,|\sigma|^2) - G^u]$, then $\Theta_\sigma = \Phi_\sigma$, and $\Theta_\xi = \Phi_\xi$, therefore u is also a solution to

$$div\ \Theta_\sigma(u, Du) \ = \ \Theta_\xi(u, Du).$$

Moreover, $\Theta(\xi,\sigma)$ satisfies the same structural assumptions, (H 1) or (H 2), of the function $\Phi(\xi,\sigma)$. Since $\Theta(\xi,0) = (1/2)[G(\xi,0) - G^u] \geq 0$, the conclusion follows from Theorem 4.8. □

The next theorem of Liouville type can be easily derived from Theorem 4.8. For its proof we refer the reader to that of Theorem 1.8 in [CGS], see also the preceding paper by Modica [M2]. In connection with Theorem 4.10, we cite the remarkable recent paper [SZ], in which the authors establish results of Liouville type, different from Theorem 4.10, for non-linear equations of the form (2.12).

Theorem 4.10. *Suppose that Φ is as in* (1.5)*, and when (H 1) holds and $p \geq 2$ assume that if $G(\xi_o, 0) = G^u$, then*

$$G(\xi, 0) \ - \ G^u \ = \ O(|\xi \ - \ \xi_o|^p) \qquad\qquad \text{as} \quad \xi \to \xi_o.$$

Let u be a bounded entire solution to (1.2)*. If there exists $x_o \in \mathbb{R}^n$ such that $G(u(x_o), 0) = G^u$, then $u \equiv const.$ in $\mathbb{R}^n$.*

The next result is dual to Proposition 4.2.

Proposition 4.11. *Let $u \not\equiv const.$ Under the hypothesis of Theorem 4.10, assume that $P(u;x) \equiv 0$, i.e.,*

$$2\,|Du|^2\, G_s(u, |Du|^2) \ \equiv \ G(u, |Du|^2) \ - \ G^u, \qquad\qquad \text{in} \quad \mathbb{R}^n, \tag{4.42}$$

then the level sets of u

$$\mathcal{L}_u(t) \ = \ \{x \in \mathbb{R}^n \mid u(x) = t\}$$

are embedded $(n-1)$-dimensional manifolds of zero mean curvature.

Proof. We claim that it must be $Du(x) \neq 0$ for every $x \in \mathbb{R}^n$. If in fact there existed $x_1 \in \mathbb{R}^n$ such that $Du(x_1) = 0$, then (4.42) would give $G(u(x_1), 0) = G^u$. But then, Theorem 4.10 would imply $u \equiv u(x_1)$, against the assumption $u \not\equiv const$.

Let now $t \in [A, B]$, where $A = \inf_{\mathbb{R}^n} u$, $B = \sup_{\mathbb{R}^n} u$, be such that $\mathcal{L}_u(t) \neq \emptyset$. The non-vanishing of Du implies that $\mathcal{L}_u(t)$ is an embedded $(n-1)-$dimensional orientable manifold. The mean curvature $H = H(x)$ at a point $x \in \mathcal{L}_u(t)$ is given by the formula

$$\text{(4.43)} \quad \pm (n-1)\, H \;=\; div\left(\frac{Du}{|Du|}\right) \;=\; \frac{1}{|Du|^3}\,\left[|Du|^2\,\Delta u \;-\; u_{ij}\,u_i\,u_j\right].$$

According to (4.38), the vanishing of P implies that of the right-hand side of (4.43). This concludes the proof. □

Proposition 4.11 displays the close connection between the analytic properties of the P-function and the geometric properties of the levels sets of u. Typically, the constancy of P implies that the non-critical level sets of u are isoparametric surfaces. This aspect has already been exploited in the past in several contexts, see [Ka] for a survey. For instance, in the exterior p-capacitary problem, a fine analysis of the asymptotic properties of the relevant P-function in [GS], led to establish the spherical symmetry of the capacitary potential and of the *free boundary*. One of the important ingredients there was A.D. Alexandrov's characterization of the spheres as the only smooth, compact embedded surfaces in $\mathbb{R}^n$ having constant mean curvature. In the conjecture of De Giorgi the role of Alexandrov's theorem is played by the following Liouville type theorem for the minimal surface equation established by Bernstein ($N = 2$), Fleming (different proof, still $N = 2$), De Giorgi ($N = 3$), Almgren ($N = 4$), Simons ($N \leq 7$): *Every entire solution of the minimal surface equation in* $\mathbb{R}^N$ *is an affine function provided that* $N \leq 7$, see, e.g., [G], [B] [Si2]. In the celebrated work [BDG] it was proved that the Bernstein property fails if $N \geq 8$. In fact, the authors showed that: *If* $N \geq 8$ *there exist complete minimal graphs in* $\mathbb{R}^{N+1}$ *which are not hyper-planes*.

The role of the dimension in the Bernstein problem suggests that a possible attack to the conjecture of De Giorgi should ultimately rely on the theory of minimal surfaces. Here is the heuristic argument. Let u be a bounded entire solution to (1.9) satisfying (1.10). If we consider a non-critical level set $\mathcal{L}_u(t)$ of u, then by the implicit function theorem there exists $\phi_t : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $x = (x', x_n) \in \mathcal{L}_u(t)$, if and only if $x_n = \phi_t(x')$. If one could prove that ϕ_t is an entire solution of the minimal surface equation in $\mathbb{R}^N$, with $N = n-1$, then the Bernstein property would imply

$$\phi_t(x') \;=\; c_1 x_1 \;+\; ... \;+\; c_{n-1}x_{n-1} \;+\; \beta$$

if $N = n-1 \leq 7$, i.e., $n \leq 8$. Since $u(x', \phi_t(x')) = t$, this would lead to the conclusion $D_k u = -c_k D_n u$ for $k = 1, ..., n-1$, and therefore u would have to be of the type (1.4).

Despite its obvious appeal, such heuristic argument hides some serious obstacles. Proposition 4.11 suggests that one should look at the relevant P-function, and try to establish its constancy. However, our next result Theorem 4.12 evidentiates a discrepancy between the conjecture of De Giorgi and the corresponding properties of the $P-$function. Irregardless of the dimension, if the latter becomes zero at one single point, then it must be identically zero, and, furthermore, u must be one-dimensional. The next result extends Theorem 5.1 in [CGS] to the more general setting of this paper. Due to the fact that the energy function Φ also depends on u, its proof does not follow straightforwardly from the former.

Theorem 4.12. *Assuming that Φ satisfy the hypothesis of Theorem 4.10, consider a bounded entire solution u of* (1.2). *If for one $x_o \in \mathbb{R}^n$ equality holds in* (4.41), *then we must have $P(\cdot;u) \equiv 0$, and, furthermore, u must be of the type* (1.4).

Proof. We begin by considering the set

$$A \;=\; \{x \in \mathbb{R}^n \mid \; P(x;u) = 0\},$$

which, thanks to the continuity of P is closed, and non-empty, since $x_o \in A$. We claim that A is also open, and therefore $A = \mathbb{R}^n$. To see this let $x_1 \in A$. If $Du(x_1) = 0$, then we must have $G(u(x_1), 0) = G^u$, and Theorem 4.10 implies $u \equiv u(x_1)$. In particular, $Du \equiv 0$ and therefore $P(x;u) \equiv 0$ in $\mathbb{R}^n$. If, instead, $Du(x_1) \neq 0$, then by continuity $\inf_{B(x_1,R)} |Du| > 0$ for some $R > 0$. On the other hand, Theorem 4.8 guarantees that $P \leq 0$, whereas by the definition of A we have $P(x_1;u) = 0$. Theorem 4.7 then shows that $P(x;u) \equiv 0$ in $B(x_1, R)$. In conclusion, we have proved that A is open, and thus $A = \mathbb{R}^n$. This gives

$$2\,|Du|^2\, G_s(u, |Du|^2) \;\equiv\; G(u, |Du|^2) \;-\; G^u, \qquad \text{in} \quad \mathbb{R}^n.$$

Using (4.17), we re-write the latter identity as follows

$$\Psi(u, |Du|^2) \;\equiv\; G(u,0) \;-\; G^u \;=\; F(u) \;-\; G^u \qquad \text{in} \quad \mathbb{R}^n, \tag{4.44}$$

where, as in the proof of Theorem 4.3, we have let $G(u,0) = F(u)$. If we assume $u \not\equiv const$ (when $u \equiv const$ there is nothing to prove), the proof of Proposition 4.11 implies that we must have $Du(x) \neq 0$ for every $x \in \mathbb{R}^n$, and therefore by the regularity theory $u \in C^{2,\alpha}_{loc}(\mathbb{R}^n)$. Denoting by $H(\xi, \cdot) = \Psi(\xi, \cdot)^{-1}$ the inverse of $\Psi(\xi, \cdot)$ (see the discussion following (4.17)), we obtain from (4.44)

$$|Du|^2 \;\equiv\; H(u, F(u) \;-\; G^u) \;\overset{def}{=}\; h(u) \qquad \text{in} \quad \mathbb{R}^n. \tag{4.45}$$

We now consider $f : \mathbb{R} \times (0,\infty) \to \mathbb{R} \times (0,\infty)$ defined by $f(\xi, s) = (\xi, \Psi(\xi, s))$. Thanks to the properties of Ψ, the function f is invertible, with $f^{-1}(\eta, t) = (\eta, H(\eta, t))$. The regularity hypothesis on Φ imply that $\Psi \in C^2(\mathbb{R} \times (0,\infty))$ (we stress that the non-vanishing of Du allows to restrict the attention to the "good" region $s = |Du|^2 > 0$). Since

$$\det\; Jac_f(\xi, s) \;=\; \det \begin{pmatrix} 1 & 0 \\ \Psi_\xi(\xi, s) & \Psi_s(\xi, s) \end{pmatrix} \;=\; \Psi_s(\xi, s) \;>\; 0,$$

we conclude that f and f^{-1} are C^2 diffeomorphisms. This implies, in particular, that $h(\xi) = H(\xi, F(\xi) - G^u)$ is in $C^2(\mathbb{R})$. The inverse function theorem gives

$$Jac_{f^{-1}}(\xi, t) \;=\; \begin{pmatrix} 1 & 0 \\ H_\xi(\xi, t) & H_t(\xi, t) \end{pmatrix},$$

with

$$H_\xi(\xi, \Psi(\xi, s)) \;=\; -\,\frac{\Psi_\xi(\xi, s)}{\Psi(\xi, s)}, \qquad H_t(\xi, \Psi(\xi, s)) \;=\; -\,\frac{1}{\Psi(\xi, s)}. \tag{4.46}$$

Using the above considerations, and (4.44), (4.46), we conclude

$$\begin{aligned} h'(u) \;&=\; H_\xi(u, F(u) - G^u) \;+\; H_t(u, F(u) - G^u) \\ &=\; H_\xi(u, \Psi(u, |Du|^2)) \;+\; H_t(u, \Psi(u, |Du|^2)) \\ &=\; \frac{F'(u) \;-\; \Psi_\xi(u, |Du|^2)}{\Psi_s(u, |Du|^2)}. \end{aligned} \tag{4.47}$$

We now set $v = \mathcal{Y}(u)$, where $\mathcal{Y}$ is to be determined. One has

$$|Dv|^2 \;=\; \mathcal{Y}'(u)^2\, |Du|^2, \qquad \Delta v \;=\; \mathcal{Y}''(u)\, |Du|^2 \;+\; \mathcal{Y}'(u)\, \Delta u. \tag{4.48}$$

The first equation in (4.48), along with (4.45), suggests that we choose $\mathcal{Y}$ in such a way that

$$|Dv|^2 \;=\; \mathcal{Y}'(u)^2\, h(u) \;\equiv\; 1. \tag{4.49}$$

This is clearly possible if we take $\mathcal{Y} \in C^2(\mathbb{R})$ as follows

$$\mathcal{Y}(\xi) \;=\; \int_{u_o}^{\xi} \frac{1}{\sqrt{h(\tau)}}\, d\tau \;=\; \int_{u_o}^{\xi} \frac{1}{\sqrt{H(\tau, F(\tau) - G^u)}}\, d\tau,$$

where u_o is a number arbitrarily fixed in the range of u. We note explicitly that, in view of (4.45), the function h is strictly positive. Differentiating the second equality in (4.49), we also find

$$\mathcal{Y}''(u)\, h(u) \;+\; \frac{1}{2}\, \mathcal{Y}'(u)\, h'(u) \;\equiv\; 0. \tag{4.50}$$

At this point we notice that the fact $P(\cdot; u) \equiv 0$, and (4.38), imply

$$\Delta u \;\equiv\; \frac{u_{ij} u_i u_j}{|Du|^2}.$$

This identity, and (4.25), give

$$\Delta u \;\equiv\; \frac{1}{2}\, \frac{F'(u) \;-\; \Psi_\xi(u, |Du|^2)}{\Psi_s(u, |Du|^2)}.$$

Thanks to the latter equation, to (4.45), and to (4.47), we finally obtain for the second equation in (4.48)

$$\Delta v \;=\; \mathcal{Y}''(u)\, h(u) \;+\; \frac{1}{2}\, \mathcal{Y}'(u)\, h'(u) \;\equiv\; 0 \qquad \text{in} \quad \mathbb{R}^n,$$

where in the last equality we have used (4.50). In view of Liouville theorem, the harmonicity and v, and (4.45), allow to conclude that

$$v(x) \ = < a, x > \ + \beta,$$

for some $a \in \mathbb{R}^n$, with $|a| = 1$, and $\beta \in \mathbb{R}$. The invertibility of $\mathcal{Y}$ implies that

$$u(x) \ = \ \mathcal{Y}^{-1}(v(x)) \ = \ \mathcal{Y}^{-1}(< a, x > \ + \beta),$$

thus u is of the type (1.4), with $g(s) = \mathcal{Y}^{-1}(s + \beta)$. This completes the proof. □

5 Energy monotonicity

In this section we establish an important monotonicity property of the energy of a bounded entire solution to (1.2). It should be emphasized that the derivation of such property relies on a deep a priori quantitative information, namely the non-negativity of the relative P-function expressed by Theorem 4.8 and Corollary 4.9. We denote by Φ^u the number

$$\Phi^u \ = \ \min \left\{ \Phi(\xi, 0) \mid \inf_{\mathbb{R}^n} \ u \ \le \ \xi \ \le \ \sup_{\mathbb{R}^n} \ u \right\}. \tag{5.1}$$

For every $r > 0$ we consider the energy of u in the ball $B_r = \{x \in \mathbb{R}^n \mid |x| < r\}$

$$E(r) = \int_{B_r} \left[\Phi(u, Du) \ - \ \Phi^u \right] \, dx. \tag{5.2}$$

Theorem 5.1. *Let u be a bounded entire solution to* (1.2) *in $\mathbb{R}^n$, $n \ge 2$, with Φ having the form* (1.5)*. The function $I(r) = r^{1-n} E(r)$ is increasing on $(0, \infty)$. In particular, one has*

$$\int_{B_r} \left[\Phi(u, Du) \ - \ \Phi^u \right] \, dx \ \ge \ E(1) \, r^{n-1} \qquad \textit{for every} \quad r \ge 1.$$

Proof. Keeping in mind (1.5), we see that up to an irrelevant multiplicative factor of 2

$$\begin{aligned} I'(r) \ &= \ - \frac{n-1}{r^n} \int_{B_r} \left[G(u, |Du|^2) \ - \ G^u \right] \, dx \\ &+ \ \frac{1}{r^{n-1}} \int_{\partial B_r} \left[G(u, |Du|^2) \ - \ G^u \right] \, d\sigma, \end{aligned} \tag{5.3}$$

where G^u is the number introduced in Corollary 4.9. The computation of the boundary integral in the right-hand side of (5.3) is obtained by an appropriate version of Rellich identity. In the case in which (H 1) holds with $\epsilon = 0$, the latter should be supplemented by an approximation argument based on the elliptic regularization of (1.2) and on the boundary $C^{1,\alpha}$ regularity in [L]. We leave it to the reader to provide the by now classical details. We only give the final product

(5.4)
$$\frac{1}{r^{n-1}}\int_{\partial B_r}\left[G(u,|Du|^2)\,-\,G^u\right]\,d\sigma \,=\, \frac{n}{r^n}\int_{B_r}\left[G(u,|Du|^2)\,-\,G^u\right]\,dx$$
$$-\,\frac{2}{r^n}\int_{B_r} G_s(u,|Du^2|)\,dx\,-\,\frac{2}{r^{n-1}}\int_{\partial B_r}\left(\frac{\partial u}{\partial\eta}\right)^2 G_s(u,|Du^2|)\,d\sigma.$$

Inserting (5.4) into (5.3), we conclude

$$I'(r)\,=\,\int_{\partial B_r}\left(\frac{\partial u}{\partial\eta}\right)^2 G_s(u,|Du^2|)\,d\sigma$$
$$+\,\frac{1}{r^n}\int_{B_r}\left[G(u,|Du|^2)\,-\,G^u\,-\,2\,|Du|^2\,G_s(u,|Du^2|)\right]\,dx$$

The boundary integral in the right-hand side of the above equality is non-negative. Invoking Corollary 4.9 we infer that also the second integral is non-negative, thus reaching the conclusion $I'(r)\geq 0$. This completes the proof of the theorem. □

Remark 5.2. For the non-linear Poisson equation $\Delta u = F'(u)$, L. Modica obtained the monotonicity of the energy in [M3] as a consequence of (1.6). Such result was subsequently extended in [CGS] to quasi-linear equations having the special structure (1.7).

We have seen in Lemma 3.3 that bounded entire solution of the ordinary differential equation (3.1) always have finite energy. This is not the case when $n\geq 2$. For instance, the two-parameter family of entire solutions (1.11) for the Ginzburg-Landau model (1.9) clearly have infinite energy in $\mathbb{R}^n$ with $n\geq 2$. Indeed, Theorem 5.1 implies that the only situation in which the energy is finite is the trivial one.

Theorem 5.3. *Assume* (1.5)*, and let u be a bounded entire solution to* (1.2) *in $\mathbb{R}^n$, with $n\geq 2$. If*

$$\mathcal{E}(u)\,\stackrel{def}{=}\,\int_{\mathbb{R}^n}\left[\Phi(u,Du)\,-\,\Phi^u\right]\,dx\,<\,\infty,$$

then $u\equiv const.$

Proof. Consider the normalized energy $I(r)$ introduced above. Since $\lim_{r\to 0^+} I(r) = 0$, Theorem 5.1 guarantees that $I(r)\geq 0$ for $r\geq 0$. Suppose that $\mathcal{E}(u)<\infty$, then

$$0\,\leq\,\frac{1}{r^{n-1}}\int_{B_r}\left[\Phi(u,Du)\,-\,\Phi^u\right]\,dx\,<\,\frac{\mathcal{E}(u)}{r^{n-1}}\,\to\,0,$$

as $r\to\infty$. The monotonicity of $I(r)$ forces the conclusion

$$\int_{B_r}\left[\Phi(u,Du)\,-\,\Phi^u\right]\,dx\,\equiv\,0.$$

We now observe that

$$\begin{aligned} \Phi(u, Du) \,-\, \Phi^u \;&=\; [\Phi(u, Du) \,-\, \Phi(u, 0)] \,+\, [\Phi(u, 0) \,-\, \Phi^u] \\ &\geq\; \Phi(u, Du) \,-\, \Phi(u, 0) \end{aligned}$$

and the latter difference is ≥ 0, thanks to (2.2), or to (2.6). Hence, $\Phi(u, Du) - \Phi^u \equiv 0$ in $\mathbb{R}^n$. In view of Corollary 4.9 we reach the conclusion $Du \equiv 0$, which gives $u \equiv const.$ □

6 Optimal energy growth

In this section we show that under certain conditions the inequality in the conclusion of Theorem 5.1 can be reversed. The result includes equations of the general type (1.2), and there is no need to assume the more restricted structure (1.5). Its proof is based on an adaptation of a simple, yet ingenious idea due to Ambrosio and Cabré in the case of Laplace equation [AC]. We begin with an elementary lemma which plays an important role in the sequel.

Lemma 6.1. *For every $\xi \in \mathbb{R}$ and $\sigma \in \mathbb{R}^n$ consider the function*

$$\Psi(\xi, \sigma) \;=<\sigma, \Phi_\sigma(\xi, \sigma)> \,-\, [\Phi(\xi, \sigma) \,-\, \Phi(\xi, 0)]\,,$$

which, up to the multiplicative factor $1/2$, coincides with that introduced in (4.11). *One has*

$$\Psi(\xi, \sigma) \;\geq\; 0,$$

with equality holding only in $\sigma = 0$.

Proof. One has $\Psi(\xi, 0) = 0$ for every $\xi \in \mathbb{R}$. To prove the lemma it is enough to show that the origin is the only critical point of $\Psi(\xi, \cdot)$ and that furthermore this function is strictly increasing in every direction. This follows at once if we show that

$$<\sigma, \Psi_\sigma(\xi, \sigma)> >\; 0, \qquad\qquad (\xi, \sigma) \in \mathbb{R} \times (\mathbb{R}^n \setminus \{0\}).$$

The latter inequality is a consequence of the convexity of the function Φ with respect to the variable σ. We have in fact if (H 1) holds

$$(6.1)\quad <\sigma, \Psi_\sigma(\xi, \sigma)> = \sum_{i,j}^{n} \Phi_{\sigma_i \sigma_j}(\xi, \sigma)\, \sigma_i\, \sigma_j \;\geq\; c_1\, (\epsilon \,+\, |\sigma|)^{p-2}\, |\sigma|^2 \;>\; 0,$$

where, in the second to the last inequality, (2.5) has been used. On the other hand, when (H 2) is in force, we obtain from (2.10)

$$(6.2)\qquad <\sigma, \Psi_\sigma(\xi, \sigma)> = <\Phi_{\sigma\sigma}(\xi, \sigma)\sigma, \sigma> \geq\; c_1\, |\sigma|^2\, \sqrt{1 \,+\, |\sigma|^2} \;>\; 0.$$

□

Theorem 6.2. *Let u be a bounded entire solution to* (1.2) *satisfying* (1.10). *Suppose in addition that*

$$\lim_{x_n \to \infty} u(x', x_n) \;=\; \sup_{\mathbb{R}^n} u \;=\; B. \tag{6.3}$$

There exists a constant $C > 0$, depending on n, on $||u||_\infty$, and on the structural parameters in either (H 1) or (H 2), such that

$$E(r) \;=\; \int_{B_r} [\Phi(u, Du) \;-\; \Phi(B, 0)] \; dx \;\leq\; C\, r^{n-1}, \qquad \textit{for every} \quad r > 0.$$

Proof. As in [AC], we define for every $x = (x', x_n) \in \mathbb{R}^n$, and $\lambda \in \mathbb{R}$,

$$u^\lambda(x) \;=\; u(x', x_n + \lambda). \tag{6.4}$$

Similarly to the proof of Theorem 4.8, we exploit the translation invariance of (1.2) to infer that for every $\lambda \in \mathbb{R}$ the function u^λ is also a bounded entire solution of (1.2) (satisfying $u^\lambda \leq B$), i.e.,

$$div\, \Phi_\sigma(u^\lambda, Du^\lambda) \;=\; \Phi_\xi(u^\lambda, Du^\lambda). \tag{6.5}$$

As in (4.1) we have

$$||Du^\lambda||_\infty \;\leq\; M \qquad \text{for every} \quad \lambda \in \mathbb{R}. \tag{6.6}$$

Thanks to (1.10), (6.3), we have presently

$$\lim_{\lambda \to +\infty} u^\lambda(x) \;=\; B, \qquad \frac{\partial u^\lambda}{\partial \lambda}(x) \;>\; 0, \qquad x \in \mathbb{R}^n. \tag{6.7}$$

Consider now for a fixed ball B_r the energy of u^λ in B_r

$$E(r; u^\lambda) \;=\; \int_{B_r} \left[\Phi(u^\lambda, Du^\lambda) \;-\; \Phi(B, 0)\right] \; dx. \tag{6.8}$$

If we are under the hypothesis (H 1), then using the fact that u^λ satisfies (6.5) one finds

$$\begin{aligned}
\frac{d}{d\lambda}\, E(r; u^\lambda) \;&=\; \int_{B_r} \Phi_\xi(u^\lambda, Du^\lambda)\, \frac{\partial u^\lambda}{\partial \lambda}\, dx \\
&\quad + \int_{B_r} \;< \Phi_\sigma(u^\lambda, Du^\lambda), D\left(\frac{\partial u^\lambda}{\partial \lambda}\right) > \; dx \\
&= \int_{\partial B_r} \;< \Phi_\sigma(u^\lambda, Du^\lambda), \eta > \frac{\partial u^\lambda}{\partial \lambda} d\sigma \\
&\geq \; - \; c_2\, (\epsilon + M)^{p-1} \int_{\partial B_r} \frac{\partial u^\lambda}{\partial \lambda} d\sigma,
\end{aligned}$$

where in the last inequality we have used the second equation in (6.7) and the structural assumption (2.3). We conclude for every $r, \lambda > 0$

$$\begin{aligned} & E(r;u^\lambda) \;-\; E(r;u) \\ & = \int_0^\lambda \frac{d}{d\mu}\, E(r;u^\mu)\, d\mu \;\geq\; -\, c_2\, (\epsilon + M)^{p-1} \int_{\partial B_r} \int_0^\lambda \frac{\partial u^\mu}{\partial \mu}\, d\mu\, d\sigma \\ & = C \int_{\partial B_r} [u \;-\; u^\lambda]\, d\sigma\ , \end{aligned}$$

and therefore

(6.9) $E(r;u) \leq 2\sigma_{n-1}||u||_\infty C r^{n-1} + \ E(r;u^\lambda) \;=\; C'\ r^{n-1} \;+\; E(r;u^\lambda).$

If instead (H 2) holds, then we use (2.7) to obtain

$$\frac{d}{d\lambda}\, E(r;u^\lambda) \;\geq -\, c_2 \int_{\partial B_r} \frac{\partial u^\lambda}{\partial \lambda} d\sigma,$$

which again gives the estimate (6.9), but with a different constant. It is at this point that the assumption (6.3), or equivalently the first equation in (6.7), is used to prove that

$$\lim_{\lambda \to \infty} E(r;u^\lambda) \;=\; 0. \tag{6.10}$$

To see this we multiply (6.5) (with u replaced by u^λ) by $(u^\lambda - B)$ and integrate by parts on B_r to obtain

$$\begin{aligned} & \int_{B_r} < \Phi_\sigma(u^\lambda, Du^\lambda), Du^\lambda > \ dx \\ & = \int_{\partial B_r} (u^\lambda - B) < \Phi_\sigma(u^\lambda, Du^\lambda), \eta > d\sigma \;-\; \int_{B_r} \Phi_\xi(u^\lambda, Du^\lambda)(u^\lambda - B)\, dx. \end{aligned}$$

Passing to the limit as $\lambda \to +\infty$, using the uniform boundedness of u^λ and of Du^λ, as well as the continuity of Φ_σ and Φ_ξ, we obtain by dominated convergence

$$\lim_{\lambda \to +\infty} \int_{B_r} < \Phi_\sigma(u^\lambda, Du^\lambda), Du^\lambda > \ dx \;=\; 0. \tag{6.11}$$

We now invoke Lemma 6.1, and the left-hand side of (2.2) in case (H 1), or (2.6) when (H 2) holds, to conclude from (6.11)

$$\lim_{\lambda \to +\infty} \int_{B_r} \left[\Phi(u^\lambda, Du^\lambda) \;-\; \Phi(u^\lambda, 0)\right]\ dx \;=\; 0.$$

Since by dominated convergence

$$\lim_{\lambda \to +\infty} \int_{B_r} \left[\Phi(u^\lambda, 0) \;-\; \Phi(B, 0)\right]\ dx \;=\; 0,$$

we obtain (6.10). With this result in hands we finally have from (6.9)

$$E(r) \;=\; E(r;u) \;\leq\; C\, r^{n-1}. \tag{6.12}$$

□

7 A generalized version of a conjecture of De Giorgi in $\mathbb{R}^2$

In this section we prove that in the plane the conjecture of De Giorgi admits an affirmative answer for the general class of variational equations (1.2), without any restriction on the integrand $\Phi(u, Du)$.

Theorem 7.1. *Let u be a bounded entire solution to* (1.2) *in $\mathbb{R}^2$, and suppose that*

$$\frac{\partial u}{\partial x_2}(x_1, x_2) \; > \; 0. \tag{7.1}$$

There exists a function $g \in C^2(\mathbb{R})$ such that $u(x) = g(a_1 x_1 \; + \; a_2 x_2)$ for some $a = (a_1, a_2)$ with $a_1^2 + a_2^2 = 1$.

Proof. Let us assume for the moment that the dimension n is arbitrary and consider a bounded entire solution to (1.2) satisfying (1.10). Since $Du(x) \neq 0$ for every $x \in \mathbb{R}^n$, by the regularity theory we know that $u \in C^{2,\alpha}_{loc}(\mathbb{R}^n)$. We consider for a fixed $k = 1, ..., n-1$, the function

$$\zeta \; = \; \frac{D_k u}{D_n u}$$

and notice that letting $\sqrt{\omega} = D_n u$ one has

$$\omega \, D\zeta = \; D_n u \, D(D_k u) \; - \; D_k u \, D(D_n u). \tag{7.2}$$

We observe that, thanks to (4.1), we have

$$\omega \, \zeta^2 \; = \; (D_k u)^2 \; \leq \; M. \tag{7.3}$$

To simplify the notation we let henceforth

$$B(x) \; \overset{def}{=} \; \Phi_{\sigma\sigma}(u(x), Du(x)), \tag{7.4}$$

and note that this matrix is symmetric and, thanks to (2.5) or (2.9), positive definite. We re-write equation (4.5) as follows

$$div \; (B(x) \, D(u_k)) \; = \; [\Phi_{\xi\xi} \; - \; div \, \Phi_{\xi\sigma}] \; u_k, \qquad k = 1, ..., n. \tag{7.5}$$

It is then easy to recognize from (7.2) and (7.5) that

$$div \; (\omega \, B(x) \, D\zeta) \; = \; 0. \tag{7.6}$$

Having observed (7.6), the proof follows by a variation on the theme of the classical Caccioppoli inequality, noted in [BCN]. Let $\alpha \in C^\infty_o(\mathbb{R}^n)$, such that $0 \leq \alpha \leq 1$, $supp \, \alpha \subset \{|x| \leq 2\}$, and $\alpha \equiv 1$ on $|x| \leq 1$. Letting $\alpha_R(x) = \alpha(x/R)$, we choose the test function $\phi = \alpha_R^2 \zeta$ in the weak form of (7.6) obtaining in a standard fashion

$$\int_{\mathbb{R}^n} \alpha_R^2 \, \omega < B(x) \, D\zeta, D\zeta > dx \; \le \; 2 \left(\int_{\mathbb{R}^n} \alpha_R^2 \omega < B(x) \, D\zeta, D\zeta > dx \right)^{1/2} \times \left(\int_{\mathbb{R}^n} \omega \zeta^2 < B(x) \, D\alpha_R, D\alpha_R > dx \right)^{1/2} . \tag{7.7}$$

Suppose now that there exist $C > 0$, independent of $R > 0$, such that

$$\int_{\mathbb{R}^n} \omega \, \zeta^2 < B(x) \, D\alpha_R, D\alpha_R > \; dx \; \le \; C. \tag{7.8}$$

This would imply for every $R > 0$

$$\int_{\mathbb{R}^n} \alpha_R^2 \, \omega < B(x) \, D\zeta, D\zeta > dx \; \le \; 4 \, C,$$

hence, by monotone convergence,

$$\int_{\mathbb{R}^n} \omega < B(x) \, D\zeta, D\zeta > dx \; < \; \infty.$$

Using this information and noting that the first integral in the right-hand side of (7.7) is actually performed on the set $\{R \le |x| \le 2R\}$, we would finally obtain letting $R \to \infty$ in (7.7)

$$\int_{\mathbb{R}^n} \omega < B(x) \, D\zeta, D\zeta > dx \; = \; 0.$$

The strict positivity of ω and the local ellipticity of the matrix $B(x)$ (remember (4.3)) finally give $D\zeta \equiv 0$, which is like saying that $D_k u = c_k D_n u$, for some constant c_k. Repeating the same argument for every $k = 1, ..., n-1$ we would conclude that

$$u(x) \; = \; g(c_1 x_1 + c_2 x_2 + ... + c_{n-1} x_{n-1} + x_n)$$

for some function $g \in C^2(\mathbb{R})$. To complete the proof of the theorem we are thus left with establishing (7.8). When $n = 2$ the latter inequality is a consequence of the structural assumptions, of the boundedness of Du, and of the crucial fact that $|B_R| = cR^2$. If (H 1) holds one has in fact from (2.5) and (7.3)

$$\begin{aligned} \int_{\mathbb{R}^n} \omega \zeta^2 \; < B(x) \, D\alpha_R, D\alpha_R > \; dx \\ \le \; c_2' \int_{B_{2R}} |Du|^2 (\epsilon + |Du|)^{p-2} |D\alpha_R|^2 dx \\ \le \; \frac{C}{R^2} \int_{B_{2R}} (\epsilon + |Du|)^p dx \; \le \; C(\epsilon + M)^p. \end{aligned} \tag{7.9}$$

In the case (H 2) we proceed slightly differently. Observing that the vector

$$(D\alpha_R, < Du, D\alpha_R >) \in \mathbb{R}^{n+1}$$

is orthogonal to the vector $(-Du, 1)$, and that

$$|(D\alpha_R, < Du, D\alpha_R >)|^2 \leq \left(1 + |Du|^2\right) |D\alpha_R|^2,$$

we obtain from (2.9)

$$\begin{aligned} &\int_{\mathbb{R}^n} \omega\zeta^2 < B(x)\, D\alpha_R, D\alpha_R > dx \\ &\leq c_2 \int_{B_{2R}} |Du|^2 \sqrt{1+|Du|^2}\, |D\alpha_R|^2\, dx \\ &\leq \frac{C}{R^2} \int_{B_{2R}} dx \leq C. \end{aligned} \tag{7.10}$$

□

Remark 7.2. The idea of studying the function (7.2) in connection with the conjecture of De Giorgi was first introduced in [MM] (see also [BCN], [GG] and [AC]), except that in [MM] the approach was different from the one outlined above based on an idea of Caffarelli, Berestycki and Nirenberg [BCN]. It is clear that the above simple proof of the conjecture is possible thanks to the special role played by the volume of the ball in $\mathbb{R}^2$, namely $|B_R| \leq cR^2$. In dimension higher than two the stronger growth of the volume of the balls at infinity poses a serious obstruction.

Remark 7.3. It would be of interest to extend Theorem 7.1 to generalized variational equations. By this we mean equations of the type

$$div\ A(u, Du) = B(u, Du) \tag{7.11}$$

with regularity and structural assumptions on A and B similar to those made above for the equation (1.2), but no other hypothesis otherwise, i.e., without assuming that $A(\xi,\sigma) = \Phi_\sigma(\xi,\sigma)$ and $B(\xi,\sigma) = \Phi_\xi(\xi,\sigma)$, for some function $\Phi(\xi,\sigma)$. However, if one allows dependence on Du in the right-hand side of (7.11), then a difficulty arises in the above arguments.

We close this section by noting an interesting corollary of Theorem 7.1 and of the results in Sect. 2.

Theorem 7.4. *Let u be a bounded entire solution to* (1.2) *in $\mathbb{R}^2$ satisfying* (7.1), *and let Φ^u be as in* (5.1). *There exists a constant $C > 0$, depending on $||u||_\infty$ and on the structural parameters in either (H 1) or (H 2), such that for every $r > 1$*

$$\int_{B_r} [\Phi(u, Du) - \Phi^u]\, dx \leq C\, r.$$

We do not give the details of the proof of Theorem 7.4 since it follows directly from Theorem 7.1 and from the finiteness of the energy established in Lemma 3.3.

8 A weaker form of the generalized conjecture of De Giorgi in $\mathbb{R}^3$

The aim of this section is to provide, similarly to Theorem 7.1, a general positive answer in $\mathbb{R}^3$ to the problem proposed by De Giorgi, but when an additional assumption is introduced. Namely, that the entire solution u tends to its extremum values along its direction of monotonicity. It is worth noting that, interestingly, similarly to the case of two variables, the invariance of the energy (1.5) under the action of $O(3)$ is not needed.

Theorem 8.1. *Let u be a bounded entire solution to* (1.2) *in $\mathbb{R}^3$ satisfying* (1.10). *Suppose that*

(8.1)
$$\lim_{x_3\to-\infty} u(x',x_3) = \inf_{\mathbb{R}^3} u \overset{def}{=} A, \qquad \lim_{x_3\to\infty} u(x',x_3) = \sup_{\mathbb{R}^3} u \overset{def}{=} B.$$

If one has

$$\Phi(\xi,0) \geq min\{\Phi(A,0),\Phi(B,0)\}, \qquad \textit{for every} \quad \xi\in(A,B), \tag{8.2}$$

then u is of the type (1.4).

Proof. We assume without loss of generality that $min\ \{\Phi(A,0),\Phi(B,0)\} = \Phi(B,0)$. Theorem 6.2 gives (now $n=3$)

$$E(r) = \int_{B_r} [\Phi(u,Du) - \Phi(B,0)]\ dx \leq C\, r^2, \qquad \text{for every} \quad r>0.$$

The latter inequality, together with the assumption (8.2), implies

$$\int_{B_r} [\Phi(u,Du) - \Phi(u,0)]\ dx \leq C\, r^2, \qquad \text{for every} \quad r>0. \tag{8.3}$$

This is precisely what is needed to implement the argument in the proof of Theorem 7.1. In fact, one only needs to prove the existence of $C>0$ independent of R such that

$$\int_{\mathbb{R}^n} \omega\, \zeta^2\ < B(x)D\alpha_R, D\alpha_R >\ dx \leq C, \tag{8.4}$$

where $B(x)$ is the matrix-valued function defined in (7.4). Returning to (7.9) we now find, when (H 1) holds,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \omega\, \zeta^2\ < B(x)D\alpha_R, D\alpha_R >\ dx \\
&\leq c_2 \int_{B_{2R}} |Du|^2(\epsilon+|Du|)^{p-2}|D\alpha_R|^2 dx \\
&\leq \frac{C}{R^2}\int_{B_{2R}} |Du|^2\ (\epsilon+|Du|)^{p-2} dx \\
&\leq \frac{C}{R^2}\int_{B_{2R}} [\Phi(u,Du) - \Phi(u,0)]\, dx \leq C,
\end{aligned}$$

where in the second to the last inequality we have used (2.2), and in the last the crucial estimate (8.3) has been employed. This establishes (8.4) and completes the proof of the theorem in this case. If, instead, (H 2) is in force, then proceeding as in (7.10), and using (2.6) and (8.3) (recall that we are assuming that bounded entire solutions have bounded gradient) we obtain

$$\begin{aligned}(8.5)\qquad &\int_{\mathbb{R}^n} \omega\zeta^2 \, < B(x)\, D\alpha_R, D\alpha_R > \, dx \\ &\leq \, c_2 \int_{B_{2R}} |Du|^2 \sqrt{1+|Du|^2}\, |D\alpha_R|^2 \, dx \\ &\leq \, \frac{C}{R^2} \int_{B_{2R}} [\Phi(u, Du) \, - \, \Phi(u,0)] \, dx \, \leq \, C.\end{aligned}$$

This finishes the proof. □

Remark 8.2. Theorem 8.1 generalizes an analogous result in [AC] concerning the equation $\Delta u = F'(u)$.

9 Lowering the dimension

In the sequel we consider an energy function $\Phi = \Phi(\xi, \sigma)$s satisfying the structural hypothesis (H 1) or (H 2). Given such a Φ we introduce the function $\overline{\Phi} : \mathbb{R}\times\mathbb{R}^{n-1} \to \mathbb{R}$ defined by

$$\overline{\Phi}(\xi, \sigma') \, = \, \overline{\Phi}(\xi, \sigma_1, ..., \sigma_{n-1}) \, = \, \Phi(\xi, \sigma_1, ..., \sigma_{n-1}, 0).$$

It is not difficult to check that the function $\overline{\Phi}$ verifies the same assumptions of Φ, (H 1) or (H 2), but in $\mathbb{R} \times \mathbb{R}^{n-1}$. We have the following basic lemma.

Lemma 9.1. *Let u be a bounded entire solution to* (1.2) *satisfying* (1.10)*. The function*

$$(9.1)\qquad \overline{u}(x') \overset{def}{=} \lim_{x_n \to +\infty} u(x', x_n),$$

is a bounded entire solution in $\mathbb{R}^{n-1}$ of the equation

$$(9.2)\qquad div_{x'} \overline{\Phi}_{\sigma'}(\overline{u}, D_{x'}\overline{u}) = \overline{\Phi}_{\xi}(\overline{u}, D_{x'}\overline{u}),$$

i.e., one has for every $\eta \in C^\infty_o(\mathbb{R}^{n-1})$

$$(9.3)\qquad \int_{\mathbb{R}^{n-1}} < \overline{\Phi}_{\sigma'}(\overline{u}, D_{x'}\overline{u}), D_{x'}\eta > dx' \, + \, \int_{\mathbb{R}^{n-1}} \overline{\Phi}_{\xi}(\overline{u}, D_{x'}\overline{u})\, \eta \, dx' \, = \, 0.$$

A similar statement holds for the function $\underline{u}(x') \overset{def}{=} \lim_{x_n \to -\infty} u(x', x_n)$.

Proof. We only give the proof for $\overline{u}$. Consider the one-parameter family of functions u^λ defined in (6.4). Thanks to (1.10) we have

$$u^\lambda(x) \;<\; u^\mu(x) \qquad \text{if} \quad \lambda < \mu, \quad \text{for every} \quad x \in \mathbb{R}^n. \tag{9.4}$$

For every compact set $K \subset \mathbb{R}^n$ the Theorem of Dini and (9.4) guarantee that

$$u^\lambda(x) \;\nearrow\; \overline{u}(x') \qquad \text{as} \quad \lambda \to \infty \tag{9.5}$$

uniformly in $x \in K$ (we think of $\overline{u}$ as a function of n variables, independent of x_n). The Hölder estimates of the gradient, see (4.2) and the discussion following Theorem 4.1, imply the existence of $C, \alpha > 0$, depending on n, $||u||_\infty$, the structural constants in (H 1) or (H 2), and on K, such that

$$|Du^\lambda(x) \;-\; Du^\lambda(y)| \;\le\; C\,|x-y|^\alpha, \qquad \text{for every} \quad x, y \in K, \lambda \in \mathbb{R}.$$

We infer the existence of a sub-sequence $\{u^{\lambda_j}\}_{j\in\mathbb{N}}$ which converges uniformly on K in C^1 norm to $\overline{u}$. Considering the sequence of compact sets $K_m = \{x \in \mathbb{R}^n \mid |x| \le m\} \nearrow \mathbb{R}^n$, by a diagonal process it is possible to extract a sub-sequence $\{u_m\}_{m\in\mathbb{N}}$ of $\{u^\lambda\}_{\lambda\in\mathbb{R}}$, which converges in C^1 norm on compact subsets of $\mathbb{R}^n$. In the sequel, abusing the notation for the sake of brevity, when we write

$$u^\lambda \;\to\; \overline{u}, \qquad Du^\lambda \;\to\; D_{x'}\overline{u}, \quad \text{as} \quad \lambda \to \infty, \tag{9.6}$$

we really mean that the convergence is for the sub-sequence $\{u_m\}_{m\in\mathbb{N}}$ of $\{u^\lambda\}_{\lambda\in\mathbb{R}}$ constructed as above. This being said, one can easily see that (9.4) and (9.6) imply the following

$$u(x', x_n) \;\to\; \overline{u}(x'), \qquad Du(x', x_n) \;\to\; D_{x'}\overline{u}(x') \quad \text{as} \quad x_n \to \infty, \tag{9.7}$$

uniformly on compact subsets of $\mathbb{R}^{n-1}$ (again, (9.7) must be interpreted as taking place on an appropriate sub-sequence). Using this information we can show that $\overline{u}$ satisfies (9.3). Given in fact a function $\eta \in C_o^\infty(\mathbb{R}^{n-1})$ one takes $\phi(x) = \alpha_\lambda^{-1}\eta(x')\zeta_\lambda(x_n)$ in (2.14), where $\zeta_\lambda \in C_o^\infty(\mathbb{R})$, $0 \le \zeta_\lambda \le 1$, $supp\,\zeta_\lambda \subset [\lambda, 2\lambda+2]$, $\zeta_\lambda \equiv 1$ on $[\lambda+1, 2\lambda+1]$, $|\zeta_\lambda'| \le 2$, and $\alpha_\lambda = \int_{\mathbb{R}} \zeta_\lambda dx_n$. The resulting equation is

$$\begin{aligned}
0 \;=\; & \int_{\mathbb{R}} \frac{\zeta_\lambda(x_n)}{\alpha_\lambda} \int_{\mathbb{R}^{n-1}} < \Phi_{\sigma'}(u(x', x_n), Du(x', x_n)), D_{x'}\eta(x') > dx'\, dx_n \\
+ & \;\frac{1}{\alpha_\lambda} \int_{\mathbb{R}} \zeta_\lambda'(x_n) \int_{\mathbb{R}^{n-1}} \Phi_{\sigma_n}(u(x', x_n), Du(x', x_n))\, \eta(x')\, dx'\, dx_n \\
+ & \int_{\mathbb{R}} \frac{\zeta_\lambda(x_n)}{\alpha_\lambda} \int_{\mathbb{R}^{n-1}} \Phi_\xi(u(x', x_n), Du(x', x_n))\, \eta(x')\, dx'\, dx_n \\
= & \; I(\lambda) \;+\; II(\lambda) \;+\; III(\lambda).
\end{aligned}$$

To estimate the first term we proceed as follows

$$
\begin{aligned}
I(\lambda) \; = \; & \\
& \int_{\mathbb{R}} \frac{\zeta_\lambda(x_n)}{\alpha_\lambda} \int_{\mathbb{R}^{n-1}} < \Phi_{\sigma'}(u(x',x_n), Du(x',x_n)) \\
& - \overline{\Phi}_{\sigma'}(\overline{u}(x'), D_{x'}\overline{u}(x')), D_{x'}\eta(x') > dx'\, dx_n \\
& + \int_{\mathbb{R}} \frac{\zeta_\lambda(x_n)}{\alpha_\lambda} \int_{\mathbb{R}^{n-1}} < \overline{\Phi}_{\sigma'}(\overline{u}(x'), D_{x'}\overline{u}(x')), D_{x'}\eta(x') > dx'\, dx_n \\
& = \; I'(\lambda) \; + \; I''(\lambda).
\end{aligned}
$$

Clearly,

$$
I''(\lambda) \; \equiv \; \int_{\mathbb{R}^{n-1}} < \overline{\Phi}_{\sigma'}(\overline{u}(x'), D_{x'}\overline{u}(x')), D_{x'}\eta(x') > dx'.
$$

If we denote $K = supp\, \eta \subset \mathbb{R}^{n-1}$, then

$$
\begin{aligned}
|I'(\lambda)| \; \le \; & \sup_{\lambda \le x_n \le 2\lambda+2} \int_K |\Phi_{\sigma'}(u(x',x_n), Du(x',x_n)) \\
& - \; \overline{\Phi}_{\sigma'}(\overline{u}(x'), D_{x'}\overline{u}(x'))| \; |D_{x'}\eta(x')| \; dx'
\end{aligned}
$$

and the right-hand side tends to zero as $\lambda \to \infty$ in view of (9.7).

To evaluate $II(\lambda)$ we proceed as for $I(\lambda)$, but use the fact that, due to the support properties of ζ'_λ, the integral in x_n is actually performed on the set $[\lambda, \lambda + 1] \cup [2\lambda+1, 2\lambda+2]$, and $\alpha_\lambda^{-1} \le \lambda^{-1} \to 0$ as $\lambda \to \infty$. Letting $\lambda \to \infty$ in the resulting equation one has $II(\lambda) \to 0$. Finally, proceeding similarly to $I(\lambda)$, one obtains

$$
III(\lambda) \; \to \; \int_{\mathbb{R}^{n-1}} \overline{\Phi}_\xi(\overline{u}(x'), D_{x'}\overline{u}(x'))\; \eta(x')\; dx'.
$$

This completes the proof of (9.3). □

Remark 9.2. The idea of dimensional reduction via the stability properties of the functions $\underline{u}$, $\overline{u}$ was introduced in [BCN].

Remark 9.3. To proceed in the analysis we will need to know that the Hessian matrix $\Phi_{\sigma\sigma}$ has continuous entries. Henceforth in this section we thus assume that $\Phi \in C^3(\mathbb{R} \times \mathbb{R}^n)$. As already mentioned in Remark 2.1 such hypothesis is natural when $\epsilon > 0$ in (H 1), or for (H 2). It is also consistent with some important situations in which there is degeneracy in the gradient, such as (2.12) with $p > 2$. The model (2.12) with $1 < p < 2$ is however excluded.

In the sequel we continue to denote by u a bounded entire solution to (1.2) satisfying (1.10). Let $\zeta \in C_o^\infty(\mathbb{R}^n)$ and set $K = supp\, \zeta$. If $\Omega \subset \mathbb{R}^n$ is a bounded open set such that $K \subset \Omega$, then (4.3) holds in Ω. Therefore, there exists $\beta \in (0,1)$ which depends on $n,\ ||u||_{L^\infty(\mathbb{R}^n)},\ \Omega$, the bound in (4.3), and on the structural constants in (H 1) or (H 2), such that $u \in C^{2,\beta}(\overline{\Omega})$. The function $v = D_n u$ is a positive solution to (4.4). We will indicate with $B = B(x)$ the matrix-valued function defined in (7.4). For what follows it will be convenient to introduce the quantities

$$b(x) \overset{def}{=} \Phi_{\xi\sigma}(u(x), Du(x)),$$
$$V(x) \overset{def}{=} \Phi_{\xi\xi}(u(x), Du(x)).$$

Letting

$$\phi \;=\; \frac{\zeta^2}{v}$$

in (4.4), one obtains

$$\int_{\mathbb{R}^n} \frac{< B(x)\, Dv, Dv >}{v^2}\, \zeta^2\, dx \;=$$
$$2 \int_{\mathbb{R}^n} \frac{< B(x)\, Dv, D\zeta >}{v}\, \zeta\, dx \;+\; 2 \int_{\mathbb{R}^n} \zeta < b, D\zeta > \; dx \;+\; \int_{\mathbb{R}^n} V\, \zeta^2\, dx.$$

Schwarz inequality gives

$$0 \le \int_{\mathbb{R}^n} < B(x)\, D\zeta, D\zeta > dx + 2 \int_{\mathbb{R}^n} \zeta < b, D\zeta > \; dx + \int_{\mathbb{R}^n} V\, \zeta^2\, dx. \tag{9.8}$$

This crucial inequality constitutes the starting point for the following dimension-reduction arguments. We introduce the new quantities

$$\overline{B}(x') \;=\; \left(\overline{\Phi}_{\sigma_i' \sigma_j'}(\overline{u}(x'), D_{x'}\overline{u}(x'))\right)_{i,j=1,\dots,n-1},$$

$$\overline{b}(x') \;=\; \overline{\Phi}_{\xi\sigma'}(\overline{u}(x'), D_{x'}\overline{u}(x')),$$

$$\overline{V}(x') \;=\; \overline{\Phi}_{\xi\xi}(\overline{u}(x'), D_{x'}\overline{u}(x')).$$

Lemma 9.4. *For any* $\eta \in C_o^\infty(\mathbb{R}^{n-1})$ *one has*

$$\begin{aligned} 0 \;\le\; & \int_{\mathbb{R}^{n-1}} < \overline{B}(x')\, D\eta, D\eta > dx' \;+\; 2 \int_{\mathbb{R}^{n-1}} \eta < \overline{b}, D\eta > \; dx' \\ & + \int_{\mathbb{R}^{n-1}} \overline{V}\, \eta^2\, dx'. \end{aligned} \tag{9.9}$$

Proof. Let $\eta \in C_o^\infty(\mathbb{R}^{n-1})$. With $\zeta_\lambda \in C_o^\infty(\mathbb{R})$ as in the proof of Lemma 9.1 we let $\beta_\lambda = \int_{\mathbb{R}} \zeta_\lambda^2(x_n) dx_n$ and consider the test function

$$\zeta(x) \;=\; \frac{\eta(x')\zeta_\lambda(x_n)}{\sqrt{\beta_\lambda}}$$

in (9.8). Our aim is to show that, passing to the limit as $\lambda \to \infty$ in (9.8), produces (9.9). We write

$$\int_{\mathbb{R}^n} < B(x)\, D\zeta, D\zeta > dx \;=\; \sum_{i,j=1}^{n-1} \int_{\mathbb{R}^n} \Phi_{\sigma_i \sigma_j}(u, Du)\, D_i\zeta\, D_j\zeta\, dx$$

$$+\; 2 \sum_{j=1}^{n-1} \int_{\mathbb{R}^n} \Phi_{\sigma_n \sigma_j}(u, Du)\, D_n\zeta\, D_j\zeta\, dx \;+\; \int_{\mathbb{R}^n} \Phi_{\sigma_n \sigma_n}(u, Du)\, (D_n\zeta)^2\, dx$$

$$=\; I(\lambda) \;+\; II(\lambda) \;+\; III(\lambda).$$

One has

$$I(\lambda) \;=$$

$$\int_{\mathbb{R}} \frac{\zeta(x_n)^2}{\beta_\lambda} \int_{\mathbb{R}^{n-1}} \Big[\Phi_{\sigma_i \sigma_j}(u(x', x_n), Du(x', x_n))$$

$$-\; \overline{\Phi}_{\sigma'_i \sigma'_j}(\overline{u}(x'), D_{x'}\overline{u}(x'))\Big]\, D_i\eta(x') D_j\eta(x') dx' dx_n$$

$$+\; \int_{\mathbb{R}} \frac{\zeta(x_n)^2}{\beta_\lambda} \int_{\mathbb{R}^{n-1}} \overline{\Phi}_{\sigma_{i'} \sigma_{j'}}(\overline{u}(x'), D_{x'}\overline{u}(x'))\, D_i\eta(x')\, D_j\eta(x')\, dx'\, dx_n$$

$$=\; I'(\lambda) \;+\; I''(\lambda).$$

It is clear that

$$I''(\lambda) \;\equiv\; \int_{\mathbb{R}^{n-1}} \overline{\Phi}_{\sigma_{i'} \sigma_{j'}}(\overline{u}(x'), D_{x'}\overline{u}(x'))\, D_i\eta(x')\, D_j\eta(x')\, dx'.$$

In estimating $I'(\lambda)$ we use the uniform convergence (9.7) on compact subsets of $\mathbb{R}^{n-1}$, the support property of ζ_λ, and the continuity of $\Phi_{\sigma\sigma}$, to obtain $I'(\lambda) \to 0$ as $\lambda \to \infty$.

To estimate $II(\lambda)$ and $III(\lambda)$ we proceed similarly to the proof of Lemma 9.1. Using the support property of ζ'_λ and the observation that $\beta_\lambda^{-1} \le \lambda^{-1}$, we conclude that $II(\lambda), III(\lambda) \to 0$, as $\lambda \to \infty$. Summarizing, we have proved

$$\int_{\mathbb{R}^n} < B(x)\, D\zeta, D\zeta > dx \;\to\; \int_{\mathbb{R}^{n-1}} < \overline{B}(x')\, D\eta, D\eta > dx', \qquad \lambda \to \infty.$$

By analogous arguments one treats the remaining two integrals in the right-hand side of (9.8) concluding that

$$2 \int_{\mathbb{R}^n} \zeta < b, D\zeta > \; dx \;+\; \int_{\mathbb{R}^n} V\, \zeta^2\, dx$$

$$\to\; 2 \int_{\mathbb{R}^{n-1}} \eta < \overline{b}, D\eta > \; dx' \;+\; \int_{\mathbb{R}^{n-1}} \overline{V}\, \eta^2\, dx',$$

as $\lambda \to \infty$. This completes the proof of the lemma. □

Lemma 9.4 implies the following important result.

Theorem 9.5. *There exists* $\psi \in C^1(\mathbb{R}^{n-1})$, $\psi > 0$, *such that*

$$\begin{aligned} & div_{x'}\left(\overline{B}(x')\, D_{x'}\psi \;+\; \psi\, \overline{b}(x')\right) \\ & -\; < \overline{b}(x'), \psi > \;-\, \overline{V}(x')\, \psi \;\leq\; 0 \qquad \textit{in } \mathbb{R}^{n-1}. \end{aligned} \tag{9.10}$$

Proof. Consider the linear equation in $\mathbb{R}^{n-1}$

$$div_{x'}\left(\overline{B}(x')\, D_{x'}w \;+\; w\, \overline{b}(x')\right) \;-\; < \overline{b}(x'), w > \;-\, \overline{V}(x')\, w \;=\; 0. \tag{9.11}$$

On any bounded open set $\Omega \subset \mathbb{R}^{n-1}$ the Rayleigh quotient associated to (9.11) is

$$\mathcal{R}(\eta) \;=\; \frac{1}{||\eta||_{L^2(\Omega)}} \int_\Omega \left[< \overline{B}(x')\, D\eta, D\eta > \;+ 2\, \eta < \overline{b}, D\eta > \;+ \overline{V}\, \eta^2\right]\, dx'.$$

The first Dirichlet eigenvalue is defined by

$$\lambda_\Omega \;=\; \inf_{\eta \in W^{1,2}_o(\Omega), \eta \not\equiv 0} \mathcal{R}(\eta).$$

Lemma 9.4 asserts that $\lambda_\Omega \geq 0$. Furthermore, by Theorem 8.38 in [GT] the first Dirichlet eigenfunction ψ_Ω is strictly positive in Ω. We follow the argument in the proof of Theorem 1.7 in [BCN]. Let λ_R and ψ_R respectively denote the first eigenvalue and eigenfunction for the ball $B'_R = \{x' \in \mathbb{R}^{n-1} \mid |x'| < R\}$, then one has trivially $0 \leq \lambda_{R^*} \leq \lambda_R \leq \lambda_1$ for every $R^* > R > 1$. Normalize ψ_R so that $\psi_R(0) = 1$ for every $R \geq 1$. By the Harnack inequality Theorem 8.20 in [GT] we infer the existence of constants $C_R, \epsilon_R > 0$ such that for every $x' \in B'_{R/2}$

$$\epsilon_R \;\leq\; \psi_{R^*}(x') \;\leq\; C_R \qquad\qquad R^* \geq R.$$

From elliptic theory we can thus find a sequence $R_k \to \infty$, and a function $\psi > 0$ in $\mathbb{R}^{n-1}$, such that $\psi_{R_k} \to \psi$ in $C^{1,\delta}$ on every compact set. Furthermore, since for each $R > 0$ the corresponding eigenvalue λ_R is ≥ 0, we conclude that ψ solves the differential inequality (9.10). □

Having obtained Theorem 9.5 we now prove the following.

Theorem 9.6. *If* $n = 3$, *then either*

(i) $\overline{u} \;\equiv\; B$, *a constant which satisfies*

$$\Phi_{\xi\xi}(B, 0) \;\geq\; 0,$$

or the function $\overline{u}$ *is one-dimensional, i.e.,*

(ii) $\overline{u}(x') \;=\; g(< c, x' >)$ *for some* $g \in C^2(\mathbb{R})$ *with* $g' > 0$ *and some* $c \in \mathbb{R}^2$ *such that* $|c| = 1$.

Proof. We proceed as in the proof of Theorem 7.1, letting this time for $k = 1, 2$

$$\omega \;=\; \psi^2, \qquad\qquad \zeta \;=\; \frac{D_k\overline{u}}{\psi}.$$

In what follows we write for simplicity $v = D_k\overline{u}$, for fixed $k = 1$ or 2, then

$$\omega\, D\zeta \;=\; \psi\, Dv \;-\; v\, D\psi. \tag{9.12}$$

Re-writing (9.10) in weak form one has for any $\eta \in C_o^\infty(\mathbb{R}^{n-1}), \eta \geq 0$,

$$\begin{aligned} -\int_{\mathbb{R}^{n-1}} <\overline{B}D\psi, D\eta> dx' &\leq \int_{\mathbb{R}^{n-1}} \psi <\overline{b}, D\eta> dx' \\ &+ \int_{\mathbb{R}^{n-1}} \eta <\overline{b}, D\psi> dx' \;+\; \int_{\mathbb{R}^{n-1}} \overline{V}\psi\eta\, dx'. \end{aligned} \tag{9.13}$$

On the other hand, since $\overline{u}$ is a bounded solution of (9.2), its derivatives $v = D_k\overline{u}$ satisfy the linearized equation in $\mathbb{R}^{n-1}$, see (4.4),

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} <\overline{B}Dv, D\eta> dx' &= -\int_{\mathbb{R}^{n-1}} v <\overline{b}, D\eta> dx' \\ &- \int_{\mathbb{R}^{n-1}} \eta <\overline{b}, Dv> dx' \;-\; \int_{\mathbb{R}^{n-1}} \overline{V}v\eta\, dx' \end{aligned} \tag{9.14}$$

where $\eta \in C_o^\infty(\mathbb{R}^{n-1})$ is arbitrary.

We now claim that (9.12), (9.13) and (9.14) imply the following crucial differential inequality

$$\int_{\mathbb{R}^{n-1}} \omega <\overline{B}D\zeta, D(\eta\zeta)> dx' \;\leq\; 0, \tag{9.15}$$

for $\eta \in C_o^\infty(\mathbb{R}^{n-1})$, with $\eta \;\geq\; 0$. To prove this claim we proceed as follows

$$\begin{aligned} &\int_{\mathbb{R}^{n-1}} \omega <\overline{B}D\zeta, D(\eta\zeta)> dx' \\ &= \int_{\mathbb{R}^{n-1}} \psi <\overline{B}Dv, D(\eta\zeta)> dx' \;-\; \int_{\mathbb{R}^{n-1}} v <\overline{B}D\psi, D(\eta\zeta)> dx' \\ &= \int_{\mathbb{R}^{n-1}} <\overline{B}Dv, D(\eta\psi\zeta)> dx' \;-\; \int_{\mathbb{R}^{n-1}} <\overline{B}D\psi, D(\eta v\zeta)> dx' \\ &\leq -\int_{\mathbb{R}^{n-1}} v <\overline{b}, D(\eta\psi\zeta)> dx' \;-\; \int_{\mathbb{R}^{n-1}} <\overline{b}, Dv> \eta\psi\zeta\, dx' \\ &+ \int_{\mathbb{R}^{n-1}} \psi <\overline{b}, D(\eta v\zeta)> dx' \;+\; \int_{\mathbb{R}^{n-1}} <\overline{b}, D\psi> \eta v\zeta\, dx' \;=\; 0. \end{aligned}$$

Once (9.15) is established we follow the argument in the proof of Theorem 7.1 (here, the fact that $n = 2$ is used!) to conclude that

$$D_k\overline{u} \;=\; c_k\, \psi, \qquad\qquad k \;=\; 1, 2.$$

If $c_1 = c_2 = 0$, then $\overline{u} \equiv B$ (a constant) and we obtain from Lemma 9.4

$$\int_{\mathbb{R}^2} < \overline{\Phi}_{\sigma'\sigma'}(B,0)\ D_{x'}\eta, D_{x'}\eta >\ dx' \ +\ \overline{\Phi}_{\xi\xi}(B,0) \int_{\mathbb{R}^2} \eta^2\ dx' \ \geq\ 0,$$

for every $\eta \in C_o^\infty(\mathbb{R}^2)$. Since $\overline{\Phi}_{\xi\xi}(B,0) = \Phi_{\xi\xi}(B,0)$, and the matrix $\overline{\Phi}_{\sigma'\sigma'}(B,0)$ is positive definite, the latter inequality implies $\Phi_{\xi\xi}(B,0) \geq 0$.

If instead at least one c_k is not zero, then one clearly has $\overline{u}(x_1,x_2) = g(b_1x_1 + b_2x_2)$ with $b_k = (c_1^2 + c_2^2)^{-1/2}c_k, k = 1,2$, and the positivity of ψ implies $g' > 0$. The proof is complete. □

10 A generalization of the theorem of Ambrosio and Cabré in $\mathbb{R}^3$

In [AC] the authors have given a positive answer to the conjecture of De Giorgi for $n = 3$. In fact, they have proved the stronger result.

Theorem 10.1 **(**Ambrosio and Cabré). *Let u be a bounded solution in $\mathbb{R}^3$ of the equation*

$$\Delta u \ =\ F'(u),$$

where $F \in C^2(\mathbb{R})$ and

$$F \ \geq\ min\{F(m), F(M)\} \qquad \text{in} \qquad (m, M)$$

for each pair of real numbers $m < M$ satisfying $F'(m) = F'(M) = 0, F''(m) \geq 0, F''(M) \geq\ 0$. If (1.10) *holds, then the level sets of u are planes, i.e., u is of the type* (1.4).

The aim of this section is to establish the following generalization of Theorem 10.1.

Theorem 10.2. *Let u be a bounded entire solution to* (1.2) *in $\mathbb{R}^3$ with $\Phi \in C^3(\mathbb{R} \times \mathbb{R}^3)$ of the type* (1.5)*. Suppose that*

$$\Phi(\xi,0) \ \geq\ min\{\Phi(A,0), \Phi(B,0)\} \qquad \xi \in (A,B) \tag{10.1}$$

for each pair of real numbers $A < B$ satisfying

$$\Phi_\xi(A,0) \ =\ \Phi_\xi(B,0) \ = 0,$$

and

$$\Phi_{\xi\xi}(A,0) \ \geq\ 0, \qquad \Phi_{\xi\xi}(B,0) \ \geq\ 0.$$

If

$$\frac{\partial u}{\partial x_3} > 0 \qquad \text{in} \qquad \mathbb{R}^3,$$

then the level sets of u are planes, i.e., u is of the type (1.4).

Proof. Let

$$A = \inf_{\mathbb{R}^3} u, \qquad B = \sup_{\mathbb{R}^3} u,$$

and set $\underline{u}(x') = \lim_{x_3\to-\infty} u(x', x_3)$, $\overline{u}(x') = \lim_{x_3\to+\infty} u(x', x_3)$. Clearly, $\underline{u} < \overline{u}$ in $\mathbb{R}^2$ and $A = \inf_{\mathbb{R}^2} \underline{u}$, $B = \sup_{\mathbb{R}^2} \overline{u}$. We apply Theorem 9.6. If case (i) occurs, then $\overline{u} \equiv B$ and one has $\Phi_{\xi\xi}(B, 0) \geq 0$, whereas the equation gives $\Phi_{\xi}(B, 0) = 0$. If instead case (ii) is verified, then due to the fact that $\Phi(u, \sigma) = (1/2)G(u, |\sigma|^2)$ (the spherical symmetry of Φ in σ plays a crucial role at this point) we infer that the function g satisfies the ode

(10.2)

$$G_{\xi s}(g, g'^2)\,(g')^2 \;+\; \Big(G_s(g, g'^2) \;+\; 2\,g'^2\, G_{ss}(g, g'^2)\Big)\; g'' \;=\; \frac{1}{2}\, G_{\xi}(g, g'^2),$$

and moreover $g' > 0$ in $\mathbb{R}$. Applying Lemma 3.3 with $\inf_{\mathbb{R}^2}\, \overline{u} = A_1$ and $\sup_{\mathbb{R}^2}\, \overline{u} = B$, we conclude that

$$\Phi(A_1, 0) \;=\; \Phi(B, 0), \qquad \Phi_{\xi}(A_1, 0) \;=\; \Phi_{\xi}(B, 0) \;=\; 0,$$

and that

$$\Phi(\xi, 0) \;>\; \Phi(A_1, 0) \;=\; \Phi(B, 0).$$

These properties, and the C^2 smoothness of $\xi \to \Phi(\xi, 0)$, also imply

$$\Phi_{\xi\xi}(B, 0) \;\geq\; 0.$$

A similar analysis of $\underline{u}$ proves that

$$\Phi_{\xi}(A, 0) \;=\; 0, \qquad \text{and} \qquad \Phi_{\xi\xi}(A, 0) \;\geq\; 0.$$

According to (10.1) we conclude $\Phi(\xi, 0) \;\geq\; min\{\Phi(A, 0), \Phi(B, 0)\}$. Without loss of generality we now assume that $min\{\Phi(A, 0), \Phi(B, 0)\} = \Phi(0, B)$.

As in the proof of Theorem 8.1, the final goal is to show that

$$E(r; u) \;=\; \int_{B_r} [\Phi(u, Du) \;-\; \Phi(B, 0)]\; dx \;\leq\; C\, r^2, \qquad \text{for every } r > 1.$$

If one considers the functions u^{λ} introduced in (6.4), then using the hypothesis (1.10) one obtains, as in the proof of Theorem 6.2,

$$E(r; u) \;\leq\; C\, r^{n-1} \;+\; E(r; u^{\lambda}). \tag{10.3}$$

The proof will be completed if we can show

$$\overline{\lim}_{\lambda\to\infty}\, E(R; u^{\lambda}) \;\leq\; C\, R^2. \tag{10.4}$$

It is clear that if $\overline{u} \equiv const = B$, then $\lim_{\lambda\to\infty}\; E(R; u^{\lambda}) = 0$. To prove (10.4), in the case $\overline{u} \not\equiv const$, we use the uniform convergence in C^1 norm on compact subsets of $\mathbb{R}^n$ of u^λ to $\overline{u}$, see (9.6). From the latter, and from (ii) of Theorem 9.6, we obtain

$$\begin{aligned}\lim_{\lambda\to\infty} E(R;u^\lambda) &= \int_{B_r} \left[\overline{\Phi}(\overline{u}(x'), D_{x'}\overline{u}(x')) - \Phi(B,0)\right] dx \\ &\leq C\,R \int_{\{x'\in\mathbb{R}^2 \mid |x'|\leq r\}} \left[\overline{\Phi}(\overline{u}(x'), D_{x'}\overline{u}(x')) - \Phi(B,0)\right] dx' \\ &\leq C'\,R^2 \int_{\mathbb{R}} \left[\overline{\Phi}(g(t), g'(t)) - \Phi(B,0)\right] dt \leq C''\,R^2,\end{aligned}$$

where in the last inequality we have used the finiteness of the energy for the solution $g = g(t)$ of (10.2) deriving from Lemma 3.3. This completes the proof of the theorem. □

After this paper was completed we received from L. Ambrosio the preprint [AAC] in which the authors use ideas from the calculus of variations to improve on Theorem 10.1 by removing the extra assumptions on the non-linearity F. They establish the following.

Theorem 10.3. *Assume that $F \in C^2(\mathbb{R})$. Let u be a bounded solution to $\Delta u = F'(u)$ in $\mathbb{R}^3$ satisfying* (1.10)*, then u must be of the type* (1.4)*.*

The proof of Theorem 10.3 is based on the observation that if the solution u were a local minimum, in a suitable sense, of the relative energy, then a simple comparison argument would provide the improved energy growth

$$\int_{B_r} \left[|Du|^2 + F(u)\right] dx \leq C\,r^{n-1}, \qquad r > 1.$$

This observation was made in Lemma 1 in [CC]. The main new idea in [AAC] consists in showing that the monotonicity assumption (1.10) does in fact imply the local minimality of u. Such implication is by no means trivial and it is based on the construction of a so-called *calibration* associated to the energy functional. Such notion is intimately connected to the theory of null Lagrangians, see [GH], chap.1, sec.4, and chap.4, sec.2.6. Interestingly, although the authors work with the special case $\Phi(\xi,\sigma) = (1/2)|Du|^2 + F(\xi)$, they carry the construction of the appropriate calibration for general integrands of the calculus of variations, see Theorem 4.4 in [AAC]. Such construction relies explicitly on the P-function which we have introduced in (1.3), and thanks to its generality covers the setting of the present paper. Here is the main consequence.

Theorem 10.4. *Let u be a bounded entire solution to* (1.2) *satisfying the assumption* (1.10)*, and $\overline{u}, \underline{u}$ be as in* (9.1)*. In a bounded C^1 domain $\Omega \subset \mathbb{R}^n$ consider the energy functional associated with* (1.1)

(10.5) $$\mathcal{E}(v;\Omega) = \int_\Omega \Phi(v, Dv)\,dx,$$

and the class of functions

$$\mathcal{C}^{\sharp}_{\Omega} \;=\; \{v \in C^1(\overline{\Omega}) \mid \underline{u}(x') \le v(x) \le \underline{u}(x') \\ \textit{for every} \quad x = (x', x_n) \in \Omega,\ v \equiv u \quad \textit{on} \quad \partial\Omega\}.$$

The function u minimizes the energy over the collection $\mathcal{C}^{\sharp}_{\Omega}$, i.e.,

$$\mathcal{E}(u;\Omega) \;\le\; \mathcal{E}(v;\Omega), \qquad\qquad \textit{for every} \quad v \in \mathcal{C}^{\sharp}_{\Omega}.$$

Using Theorem 10.4, and the results in sections 3, 9 and 10, we can remove the additional assumptions on Φ in Theorem 10.2, thus obtaining a generalization of Theorem 10.3.

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