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## A minimum problem with free boundary for a degenerate quasilinear operator

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**Abstract.** In this paper we prove  $C^{1,\alpha}$  regularity (near flat points) of the free boundary  $\partial\{u > 0\} \cap \Omega$  in the Alt-Caffarelli type minimum problem for the  $p$ -Laplace operator:

$$J(u) = \int_{\Omega} (|\nabla u|^p + \lambda^p \chi_{\{u>0\}}) dx \rightarrow \min \quad (1 < p < \infty).$$

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### 1 Introduction

For a given domain  $\Omega$  in  $\mathbf{R}^n$  consider the problem of minimizing the energy functional

$$(1.1) \quad J(u) = \int_{\Omega} (|\nabla u|^p + \lambda^p \chi_{\{u>0\}}) dx \quad (1 < p < \infty)$$

among all functions  $u \in W^{1,p}(\Omega)$  with  $u - u_0 \in W_0^{1,p}(\Omega)$  for a prescribed  $u_0 \geq 0$ . We assume that  $\lambda$  is a positive constant. Then the minimizer  $u$  satisfies (in a certain weak sense, see Theorem 2.1) the following overdetermined system

$$(1.2) \quad \Delta_p u = 0 \quad \text{in } \{u > 0\}, \quad u = 0, \quad |\nabla u| = c \quad \text{on } \partial\{u > 0\} \cap \Omega$$

with  $c = \lambda/(p-1)^{1/p}$ . Here  $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$  is the  $p$ -Laplace operator. We are interested in the regularity properties of the so-called *free boundary*

$$\Gamma = \partial\{u > 0\} \cap \Omega.$$

Problems of this kind, known as *Bernoulli-type problems*, appear for instance in two dimensional flows [5], heat flows [1], electrochemical machining [13], etc.

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Also, the same free boundary condition appears under simplifying assumptions in the limit of high activation energy in combustion theory, see e.g. [4].

For  $p = 2$ , the problem has been studied in the by now classical paper of Alt and Caffarelli [2]. Our objective in this paper is to prove the regularity of the free boundary for any  $1 < p < \infty$ . The difficulty of the problem and its main difference from [2] is that the governing operator, the  $p$ -Laplacian, is not uniformly elliptic (degenerate for  $p > 2$  and singular for  $1 < p < 2$ ). The case of uniformly elliptic quasilinear equations has been treated by Alt, Caffarelli and Friedman [3]. The main result of [3], similar to the one of [2], states that near *flat* free boundary points the free boundary is  $C^{1,\alpha}$  regular. On the other hand, the regularity of the free boundary in (1.2) will imply nondegeneracy of  $|\nabla u|$  near the free boundary, which will make the  $p$ -Laplacian uniformly elliptic. However, it appears not to be easy to establish the nondegeneracy up to the free boundary without the regularity. We overcome this difficulty by proving both properties, nondegeneracy of the gradient and the regularity of the free boundary, simultaneously.

Finally, we have to mention that under suitable convexity assumptions, one can establish the existence and uniqueness of classical solutions to the free boundary problem (1.2) by using a Perron-type method, see [10], [11]. For general configurations, certain weak solutions can be obtained in the limit of a singular perturbation problem related to combustion theory

$$\Delta_p u^\varepsilon = \frac{1}{\varepsilon} \beta \left( \frac{u^\varepsilon}{\varepsilon} \right)$$

as  $\varepsilon \rightarrow 0$ , where  $\beta \geq 0$  and  $\text{supp } \beta = [0, 1]$ , see [6].

The structure of the paper is as follows. In Sects. 3 and 4 we establish the uniform Lipschitz continuity and a certain nondegeneracy of the minimizers at any free boundary point. As a consequence, we obtain in Sect. 5 that the free boundary has locally finite perimeter. Section 6 contains the key flatness-nondegeneracy theorems (Theorems 6.3 and 6.4), which together with Sects. 7– 8 imply our main result, Theorem 9.1, that the free boundary is  $C^{1,\alpha}$  regular near flat free boundary points.

## 2 Preliminaries

The existence of minimizers of (1.1) as well as their relation with the free boundary problem (1.2) can be established precisely as in [2].

**Theorem 2.1** *If  $J(u_0) < \infty$ , then there exists an absolute minimizer  $u$  of  $J$  in the class  $\mathcal{K} = \{v \in W^{1,p}(\Omega) : v - u_0 \in W_0^{1,p}(\Omega)\}$ , i.e. a function  $u \in \mathcal{K}$  such that*

$$J(u) \leq J(v) \quad \text{for any } v \in \mathcal{K}.$$

*Any absolute minimizer  $u$  is nonnegative,  $\Delta_p u = 0$  in  $\{u > 0\}$  and, moreover, it satisfies the free boundary condition (1.2) in the following very weak sense:*

$$\lim_{\varepsilon \searrow 0} \int_{\partial\{u>\varepsilon\} \cap \Omega} ((p-1)|\nabla u|^p - \lambda^p)\eta \cdot \nu = 0$$

*for any  $\eta \in W_0^{1,p}(\Omega)^n$  and where  $\nu$  is the outward normal.  $\square$*

Note that every absolute minimizer  $u$  is  $p$ -subharmonic in  $\Omega$ , since  $J(u) \leq J(u - \varepsilon\eta)$  and  $\chi_{\{u-\varepsilon\eta>0\}} \leq \chi_{\{u>0\}}$  for any nonnegative  $\eta \in C_0^\infty(\Omega)$  and  $\varepsilon > 0$ . In particular,  $\Delta_p u$  is a nonnegative Radon measure with support in  $\Omega \cap \partial\{u > 0\}$ . Also, observe that if  $u_0 \geq 0$  is uniformly bounded in  $\Omega$ , then

$$0 \leq \inf_{\Omega} u \leq \sup_{\Omega} u \leq \sup_{\Omega} u_0$$

(here inf and sup are understood in the essential sense).

Most of the results in this paper are proved also for so-called *local minimizers*  $u \in \mathcal{K}$  such that  $J(u) \leq J(v)$  for any  $v \in \mathcal{K}$  with

$$\|\nabla u - \nabla v\|_{L^p(\Omega)} + \|\chi_{\{u>0\}} - \chi_{\{v>0\}}\|_{L^1(\Omega)} < \varepsilon$$

for some  $\varepsilon > 0$ .

Next, we remark that if  $u$  is a (local) minimizer of  $J$  in  $\Omega$ , then the rescaling of  $u$  around  $x_0$  by a factor of  $r$

$$u_r(x) := \frac{1}{r} u(rx + x_0)$$

is a (local) minimizer in  $\Omega_r := \{(x - x_0)/r : x \in \Omega\}$ . Rescalings are especially useful since, as we prove in Sect. 3, the (local) minimizers are uniformly Lipschitz continuous and therefore we can extract a subsequence  $u_r$  converging as  $r \rightarrow 0$  to a function  $u_0$  in  $\mathbf{R}^n$ . The latter process is referred to as *blow-up*.

Finally, without loss of generality, we assume throughout the paper that

$$\lambda = \lambda_p := (p - 1)^{1/p}$$

in the functional (1.1), so that we have  $c = 1$  in (1.2).

### 3 Lipschitz continuity

In this section we establish the Lipschitz continuity of minimizers.

Let  $u$  be an absolute (local) minimizer of  $J$  in  $\Omega$ . Then  $u$  is  $p$ -subharmonic and we can assume that  $u$  is upper semicontinuous. For any (small)  $B = B_r(y) \subset\subset \Omega$ , denote by  $v = v_B$  the solution of the Dirichlet problem

$$\Delta_p v = 0 \text{ in } B, \quad v - u \in W_0^{1,p}(B).$$

From the minimality of  $u$  in  $B$ , we have

$$\int_B (|\nabla u|^p + \lambda^p \chi_{\{u>0\}}) \leq \int_B (|\nabla v|^p + \lambda^p)$$

or

$$\int_B (|\nabla u|^p - |\nabla v|^p) \leq \lambda^p \int_B \chi_{\{u=0\}}.$$

Set now

$$u^s(x) = s u(x) + (1 - s)v(x), \quad 0 \leq s \leq 1.$$

Clearly,  $u^0 = v$  and  $u^1 = u$ . We thus obtain

$$\begin{aligned} \int_B (|\nabla u|^p - |\nabla v|^p) &= p \int_0^1 ds \int_B |\nabla u^s|^{p-2} \nabla u^s \cdot \nabla(u - v) \\ &= p \int_0^1 ds \int_B (|\nabla u^s|^{p-2} \nabla u^s - |\nabla v|^{p-2} \nabla v) \cdot \nabla(u - v) \\ &= p \int_0^1 \frac{ds}{s} \int_B (|\nabla u^s|^{p-2} \nabla u^s - |\nabla v|^{p-2} \nabla v) \cdot \nabla(u^s - v), \end{aligned}$$

where in the second step we have used that  $\int_B |\nabla v|^{p-2} \nabla v \cdot \nabla(u - v) = 0$ .

Next, we apply the well-known inequality

$$(|\xi|^{p-2} \xi - |\eta|^{p-2} \eta) \cdot (\xi - \eta) \geq \gamma \begin{cases} |\xi - \eta|^2 (|\xi| + |\eta|)^{p-2}, & 1 < p \leq 2 \\ |\xi - \eta|^p & p \geq 2 \end{cases}$$

for any nonzero  $\xi, \eta \in \mathbf{R}^n$  and a constant  $\gamma = \gamma(n, p) > 0$ .

For  $p \geq 2$  we obtain

$$\int_B (|\nabla u|^p - |\nabla v|^p) \geq \gamma p \int_0^1 \frac{ds}{s} \int_B |\nabla(u^s - v)|^p = \gamma p \int_0^1 s^{p-1} ds \int_B |\nabla(u - v)|^p$$

and consequently

$$(3.1) \quad \int_B |\nabla(u - v)|^p \leq C \int_B \chi_{\{u=0\}}.$$

In the case  $1 < p \leq 2$  we have

$$\begin{aligned} \int_B (|\nabla u|^p - |\nabla v|^p) &\geq \gamma p \int_0^1 \frac{ds}{s} \int_B |\nabla(u^s - v)|^2 (|\nabla u^s| + |\nabla v|)^{p-2} \\ &\geq c \int_0^1 s ds \int_B |\nabla(u - v)|^2 (|\nabla u| + |\nabla v|)^{p-2} \\ &\geq c \int_B |\nabla(u - v)|^2 (|\nabla u| + |\nabla v|)^{p-2}. \end{aligned}$$

On the other hand, using the Hölder inequality, we have

$$\begin{aligned} \int_B |\nabla(u - v)|^p &\leq \left( \int_B |\nabla(u - v)|^2 (|\nabla u| + |\nabla v|)^{p-2} \right)^{p/2} \\ &\quad \times \left( \int_B (|\nabla u| + |\nabla v|)^p \right)^{1-p/2}, \end{aligned}$$

hence

$$(3.2) \quad \int_B |\nabla(u - v)|^p \leq C \left( \int_B \chi_{\{u=0\}} \right)^{p/2} \left( \int_B |\nabla u|^p \right)^{1-p/2},$$

where  $C = C(n, p) > 0$  and we have used that  $\int_B |\nabla v|^p \leq \int_B |\nabla u|^p$ .

We are now ready to prove the first result on the regularity of minimizers.

**Lemma 3.1** *Let  $u$  be a bounded absolute minimizer in  $B_1$ . Then  $u$  is  $C^\alpha$  regular in  $B_{7/8}$  for some  $\alpha = \alpha(n, p) \in (0, 1)$  and*

$$\|u\|_{C^\alpha(B_{7/8})} \leq C(n, p, \|u\|_{L^\infty(B_1)}).$$

*Proof.* Let  $M = \|u\|_{L^\infty(B_1)}$  and  $B = B_r(y)$  with  $y \in B_{7/8}$  and  $0 < r \leq 1/16$ . Since  $u$  is a  $p$ -subsolution, a Caccioppoli type inequality (see [9], Lemma 3.27) implies that

$$\int_B |\nabla u|^p \leq \frac{C}{r^p} \int_{2B} u^p \leq C M^p r^{n-p},$$

where  $\alpha B = B_{\alpha r}(y)$ . On the other hand, if  $v = v_B$  is the  $p$ -harmonic function with  $u - v \in W_0^{1,p}(B)$ , we have

$$\sup_{\frac{1}{2}B} |\nabla v| \leq \left( \frac{C}{r^n} \int_B |\nabla v|^p \right)^{1/p} \leq \frac{C M}{r}$$

Now, let us take a small  $\varepsilon > 0$ , to be specified below, and  $0 < r \leq r_0(\varepsilon)$  such that  $r^\varepsilon \leq 1/2$ . Then

$$\begin{aligned} \|\nabla u\|_{L^p(B_{r^{1+\varepsilon}}(y))} &\leq \|\nabla(u - v)\|_{L^p(B_{r^{1+\varepsilon}}(y))} + \|\nabla v\|_{L^p(B_{r^{1+\varepsilon}}(y))} \\ &\leq \|\nabla(u - v)\|_{L^p(B_r(y))} + C r^{(1+\varepsilon)(n/p)} \|\nabla v\|_{L^\infty(B_{r/2}(y))} \\ &\leq C(M, n, p) \begin{cases} r^{n/p - (1-p/2) + r^{(1+\varepsilon)(n/p) - 1}}, & \text{for } 1 < p < 2 \\ r^{n/p} + r^{(1+\varepsilon)(n/p) - 1}, & \text{for } p \geq 2 \end{cases} \end{aligned}$$

Thus, for  $\rho = r^{1+\varepsilon}$ , we have

$$(3.3) \quad \|\nabla u\|_{L^p(B_\rho(y))} \leq C(M, n, p) \rho^{n/p - (1-\alpha)},$$

where  $\alpha = \alpha(n, p) > 0$ , if we take  $\varepsilon = \varepsilon(n, p) > 0$  sufficiently small. Applying Morrey's theorem, see e.g. [14], Theorem 1.53, we conclude the proof of the lemma.  $\square$

The next lemma is the main step in establishing the Lipschitz regularity of minimizers.

**Lemma 3.2** *Let  $u$  be a bounded absolute minimizer in  $B_1$  and  $u(0) = 0$ . Then there exists a constant  $C = C(n, p) > 0$  such that*

$$\|u\|_{L^\infty(B_{1/4})} \leq C.$$

*Proof.* Indeed, assume the contrary. Then there exists a sequence of bounded absolute minimizers  $u_k$  in  $B_1$ ,  $k = 1, 2, \dots$ , such that

$$\max_{\overline{B_{1/4}}} u_k(x) > k.$$

Set

$$d_k(x) = \text{dist}(x, \{u_k = 0\}) \quad \text{in } B_1$$

and define

$$\mathcal{O}_k = \{x \in B_1 : d_k(x) \leq (1 - |x|)/3\}.$$

Observe that  $\overline{B}_{1/4} \subset \mathcal{O}_k$ . In particular,

$$m_k := \sup_{\mathcal{O}_k} (1 - |x|) u_k(x) \geq \frac{3}{4} \max_{\overline{B}_{1/4}} u_k(x) > \frac{3}{4} k.$$

Since  $u_k(x)$  is bounded (for fixed  $k$ ), we will have  $(1 - |x|) u_k(x) \rightarrow 0$  as  $|x| \rightarrow 1$ , and therefore  $m_k$  will be attained at some point  $x_k \in \mathcal{O}_k$ :

$$(3.4) \quad (1 - |x_k|) u_k(x_k) = \max_{\mathcal{O}_k} (1 - |x|) u_k(x).$$

Clearly,

$$u_k(x_k) = \frac{m_k}{1 - |x_k|} \geq m_k > \frac{3}{4} k.$$

Since  $x_k \in \mathcal{O}_k$ , by the definition we will have

$$(3.5) \quad \delta_k := d_k(x_k) \leq (1 - |x_k|)/3.$$

Let now  $y_k \in \partial\{u_k > 0\} \cap B_1$  be such that

$$(3.6) \quad |y_k - x_k| = \delta_k.$$

Then (3.5)–(3.6) will imply the inclusions

$$B_{2\delta_k}(y_k) \subset B_1 \quad \text{and} \quad B_{\delta_k/2}(y_k) \subset \mathcal{O}_k.$$

In particular, for  $z \in B_{\delta_k/2}(y_k)$

$$(1 - |z|) \geq (1 - |x_k|) - |x_k - z| \geq (1 - |x_k|) - (3/2)\delta_k \geq (1 - |x_k|)/2.$$

This, in conjunction with (3.4), will imply that

$$\max_{\overline{B}_{\delta_k/2}(y_k)} u_k \leq 2u_k(x_k).$$

Next, since  $B_{\delta_k}(x_k) \subset \{u_k > 0\}$ ,  $u_k$  satisfies  $\Delta_p u_k = 0$  in  $B_{\delta_k}(x_k)$ . By the Harnack inequality for  $p$ -harmonic functions, there is a constant  $c = c(n, p) > 0$  such that

$$\min_{\overline{B}_{3\delta_k/4}(x_k)} u_k \geq c u_k(x_k).$$

In particular,

$$\max_{\overline{B}_{\delta_k/4}(y_k)} u_k \geq c u_k(x_k).$$

Consider now

$$w_k(x) = \frac{u_k(y_k + (\delta_k/2)x)}{u_k(x_k)}.$$

From the properties of  $u_k$  above, we obtain

$$(3.7) \quad \max_{\overline{B_1}} w_k \leq 2, \quad \max_{\overline{B_{1/2}}} w_k \geq c > 0, \quad w_k(0) = 0.$$

We will also have that  $w_k$  is an absolute minimizer of

$$J_k(w) = \int |\nabla w|^p + \lambda_k^p \chi_{\{w>0\}}, \quad \lambda_k^p = \frac{\delta_k \lambda^p}{2u_k(x_k)} \rightarrow 0,$$

in  $B_1$ . Let now  $v_k$  be such that  $v_k - w_k \in W_0^{1,p}(B_{3/4})$  and  $\Delta_p v_k = 0$  in  $B_{3/4}$ . From the minimality of  $w_k$ ,

$$(3.8) \quad \int_{B_{3/4}} (|\nabla w_k|^p + \lambda_k^p \chi_{\{w_k>0\}}) \leq \int_{B_{3/4}} (|\nabla v_k|^p + \lambda_k^p).$$

Arguing as in the proof of (3.1)–(3.2) and Lemma 3.1, we obtain that

$$(3.9) \quad \int_{B_{3/4}} |\nabla(w_k - v_k)|^p \leq C(\lambda_k^p) \rightarrow 0$$

and that  $w_k$  and  $v_k$  are uniformly  $C^\alpha$  in  $B_{5/8}$ . Thus, we can extract subsequences (still denoted by  $w_k$  and  $v_k$ ) such that  $w_k \rightarrow w_0$  and  $v_k \rightarrow v_0$  uniformly on  $B_{5/8}$ . Observe that  $\Delta_p v_0 = 0$  in  $B_{5/8}$  and that (3.9) implies that  $w_0 \equiv v_0 + c$ . Hence,  $\Delta_p w_0 = 0$  in  $B_{5/8}$ . By the strong minimum principle,  $w_0 = 0$  in  $B_{5/8}$ , since  $w_0 \geq 0$  and  $w_0(0) = 0$ . On the other hand, (3.7) implies

$$\max_{\overline{B_{1/2}}} w_0 \geq c > 0,$$

a contradiction.

The lemma is proved. □

**Theorem 3.3** *If  $u$  is a local minimizer in  $\Omega$ , then  $u \in \text{Lip}(\Omega)$ . Moreover, for every  $K \subset\subset \Omega$  such that  $K \cap \partial\{u > 0\} \neq \emptyset$  there exist a constant  $C = C(n, p, \text{dist}(K, \partial\Omega)) > 0$  such that*

$$\|\nabla u\|_{L^\infty(K)} \leq C.$$

*Proof.* The statement follows easily from Lemma 3.2. We refer to the proof of Theorem 2.3 in [3] for more details. See also the proof of Theorem 2.1 in [6]. □

### 4 Nondegeneracy

As a simple corollary of Theorem 3.3, we obtain the following statement.

**Lemma 4.1** *For every  $K \subset\subset \Omega$  there exists a constant  $C = C(n, p, \text{dist}(K, \partial\Omega)) > 0$  such that for any absolute (local) minimizer  $u$*

$$\frac{1}{r} \int_{\partial B_r} u > C \quad \text{implies} \quad u > 0 \quad \text{in } B_r$$

for any (small) ball  $B_r \subset K$ .

*Proof.* Indeed, otherwise we will have that  $B_r \subset K$  contains a free boundary point and therefore, by Theorem 3.3,  $u \leq Cr$  on  $\partial B_r$ , a contradiction.  $\square$

Next, we prove a key nondegeneracy lemma.

**Lemma 4.2** *For any  $\gamma > p - 1$  and  $0 < \kappa < 1$  there exists a constant  $c = c(\kappa, \gamma, n, p) > 0$  such that for every absolute (local) minimizer  $u$  and for any (small) ball  $B_r \subset \Omega$*

$$\frac{1}{r} \left( \int_{B_r} u^\gamma \right)^{1/\gamma} < c \quad \text{implies} \quad u = 0 \text{ in } B_{\kappa r}.$$

*Proof.* Without loss of generality, we may consider the case  $r = 1$ . Set

$$\varepsilon = \frac{1}{\sqrt{\kappa}} \sup_{B_{\sqrt{\kappa}}} u.$$

By the interior estimate for  $\sup u$  for  $p$ -subharmonic functions (see Theorem 1.3 in [15])

$$\varepsilon \leq C \left( \int_{B_1} u^\gamma \right)^{1/\gamma}.$$

Let  $\phi(x) = \phi_\kappa(|x|)$  be the solution of

$$\Delta_p \phi = 0 \text{ in } B_{\sqrt{\kappa}} \setminus B_\kappa, \quad \phi = 0 \text{ on } \partial B_\kappa, \quad \phi = 1 \text{ on } \partial B_{\sqrt{\kappa}}$$

and put  $\phi = 0$  on  $B_\kappa$ . Set  $v = \varepsilon \sqrt{\kappa} \phi$  in  $B_{\sqrt{\kappa}}$ . Then  $v \geq u$  on  $\partial B_{\sqrt{\kappa}}$  and  $w = \min(u, v)$  is an admissible function. Therefore,  $J(u) \leq J(w)$  or, equivalently,

$$\int_{B_{\sqrt{\kappa}}} (|\nabla u|^p + \lambda^p \chi_{\{u>0\}}) \leq \int_{B_{\sqrt{\kappa}} \setminus B_\kappa} (|\nabla w|^p + \lambda^p \chi_{\{w>0\}}).$$

Hence,

$$\begin{aligned} \int_{B_\kappa} (|\nabla u|^p + \lambda^p \chi_{\{u>0\}}) &\leq \int_{B_{\sqrt{\kappa}} \setminus B_\kappa} (|\nabla w|^p - |\nabla u|^p) \\ &\leq p \int_{B_{\sqrt{\kappa}} \setminus B_\kappa} |\nabla w|^{p-2} \nabla w \cdot \nabla(w - u) \\ &= -p \int_{\partial B_\kappa} |\nabla w|^{p-2} (w - u) \nabla w \cdot \nu \\ &= p \int_{\partial B_\kappa} u |\nabla v|^{p-2} \nabla v \cdot \nu. \end{aligned}$$

Since also  $|\nabla v| \leq C\varepsilon$  on  $\partial B_\kappa$ , we find that

$$\int_{B_\kappa} (|\nabla u|^p + \lambda^p \chi_{\{u>0\}}) \leq C\varepsilon^{p-1} \int_{\partial B_\kappa} u.$$

On the other hand,

$$\begin{aligned} \int_{\partial B_\kappa} u &\leq C(n, \kappa) \left( \int_{B_\kappa} u + \int_{B_\kappa} |\nabla u| \right) \\ &\leq C(n, \kappa, p) \left( \int_{B_\kappa} \varepsilon \lambda^p \chi_{\{u>0\}} + \int_{B_\kappa} (|\nabla u|^p + \lambda^p \chi_{\{u>0\}}) \right) \\ &\leq C(n, \kappa, p)(1 + \varepsilon) \int_{B_\kappa} (|\nabla u|^p + \lambda^p \chi_{\{u>0\}}). \end{aligned}$$

Therefore, if  $\varepsilon$  is small enough, we obtain  $u = 0$  in  $B_\kappa$ . □

**Corollary 4.3** *For any  $K \subset\subset \Omega$  there exist constants  $c, C > 0$  such that if  $B_r(x) \subset K \cap \{u > 0\}$  touches  $\partial\{u > 0\}$  then*

$$cr \leq u(x) \leq Cr. \quad \square$$

**Theorem 4.4** *For any  $K \subset\subset \Omega$  there exist a constant  $c = c(n, p, K, \Omega)$ ,  $0 < c < 1$  such that for any absolute (local) minimizer  $u$  and for any (small) ball  $B_r = B_r(x) \subset K$  with  $x \in \partial\{u > 0\}$ ,*

$$c < \frac{\mathcal{L}^n(B_r \cap \{u > 0\})}{\mathcal{L}^n(B_r)} < 1 - c.$$

*Proof.* By Lemma 4.2, there exists  $y \in B_{r/2}$  such that  $u(y) \geq cr > 0$ . By Lipschitz continuity,  $u > 0$  in  $B_{\kappa r}(y)$ , for a small  $\kappa > 0$ . Thus, the estimate from below follows.

To prove the estimate from above, it is enough to consider the case  $r = 1$ . Assume the contrary. Then there exists a sequence of absolute minimizers  $u_k$  in  $B_1(0)$ , such that  $0 \in \partial\{u_k > 0\}$  and

$$\mathcal{L}^n(\{u_k = 0\}) =: \varepsilon_k \rightarrow 0.$$

Let  $v_k$  be such that  $v_k - u_k \in W_0^{1,p}(B_{1/2})$  and  $\Delta_p v_k = 0$  in  $B_{1/2}$ . Arguing as in the proof (3.1)–(3.2), we obtain that

$$\int_{B_{1/2}} |\nabla v_k - \nabla u_k|^p \leq C(\varepsilon_k) \rightarrow 0.$$

Since  $u_k$  and  $v_k$  are uniformly Lipschitz continuous in  $\overline{B}_{1/4}$ , we may assume that  $u_k \rightarrow u_0$  and  $v_k \rightarrow v_0$  uniformly in  $B_{1/4}$ . Observe that  $\Delta_p v_0 = 0$  and that the estimate above implies that  $u_0 = v_0 + c$ . Hence,  $\Delta_p u_0 = 0$  in  $B_{1/4}$  and from the strong minimum principle it follows that  $u_0 \equiv 0$  in  $B_{1/4}$ , since  $u_0 \geq 0$  and  $u_0(0) = 0$ . On the other hand, we know

$$\left( \int_{B_{1/4}} u_k^\gamma \right)^{1/\gamma} \geq c > 0, \quad \text{for } \gamma > p - 1,$$

which implies a similar inequality for  $u_0$ , a contradiction.

The theorem is proved. □

*Remark 4.5* Theorem 4.4 implies that the free boundary  $\partial\{u > 0\}$  has Lebesgue measure zero for every local minimizer. Moreover, it implies that for every  $K \subset\subset \Omega$ , the intersection  $\partial\{u > 0\} \cap K$  has Hausdorff dimension less than  $n$ . In fact, to prove these statements, it is enough to use only the left-hand side estimate in Theorem 4.4.

## 5 The measure $\Lambda = \Delta_p u$

The main objective of this section is to show that for any absolute (local) minimizer  $u$  the free boundary is locally of finite perimeter. For that purpose, set

$$\Lambda = \Delta_p u.$$

Then  $\Lambda$  is a nonnegative Radon measure.

**Theorem 5.1** *For any  $K \subset\subset \Omega$  there exist constants  $c, C > 0$  such that for any (local) minimizer  $u$*

$$cr^{n-1} \leq \int_{B_r} d\Lambda \leq Cr^{n-1}$$

for any (small) ball  $B_r = B_r(x) \subset K$  with  $x \in \partial\{u > 0\}$ .

*Proof.* Let  $\zeta \in C^\infty(\Omega)$ ,  $\zeta \geq 0$ , be a test function. Then

$$\int_{B_r} \zeta d\Lambda = - \int_{B_r} |\nabla u|^{p-2} \nabla u \cdot \nabla \zeta.$$

Approximating  $\chi_{B_r}$  by suitable test functions  $\zeta$ , we obtain that (for almost all  $r > 0$ )

$$\int_{B_r} d\Lambda = \int_{\partial B_r} |\nabla u|^{p-2} \nabla u \cdot \nu \leq Cr^{n-1},$$

where in the last step we used that  $u$  is Lipschitz continuous.

To prove the estimate from below, it is enough to consider the case  $r = 1$ . Assume the contrary. Then there exists a sequence of absolute minimizers  $u_k$  in the unit ball  $B_1(0)$ , such that  $0 \in \partial\{u_k > 0\}$  and for the measures  $\Lambda_k = \Delta_p u_k$

$$\int_{B_1} d\Lambda_k =: \varepsilon_k \rightarrow 0.$$

Since the functions  $u_k$  are uniformly Lipschitz continuous, we may assume that  $u_k \rightarrow u_0$  uniformly on  $B_{1/2}$ , where  $u_0$  is Lipschitz continuous as well. Consider then the uniformly bounded sequence  $g_k = |\nabla u_k|^{p-2} \nabla u_k$ . We may extract a subsequence (still denoted by  $g_k$ ) such that  $g_k \rightarrow g_0$  weakly- $*$  in  $L^\infty(B_{1/2})$ . We claim that  $g_0 = |\nabla u_0|^{p-2} \nabla u_0$ . Indeed, if  $B_\rho = B_\rho(y) \subset \{u_0 > 0\}$  then one can extract a subsequence of  $u_k$  locally converging to  $u_0$  in  $C^{1,\alpha}(B_\rho)$ . Hence,  $g_0 = |\nabla u_0|^{p-2} \nabla u_0$  in  $B_\rho$ . Next, if  $B_\rho \subset \{u_0 = 0\}$ ,  $u_k = 0$  in  $B_{\rho(1-\delta)}$  for sufficiently large  $k \geq k(\delta)$ . Otherwise, we will have  $\int_{B_\rho} u_0^\gamma \geq c(\delta)$  for  $\gamma > p-1$ . Thus,  $g_0 = 0 = |\nabla u_0|^{p-2} \nabla u_0$  in  $B_\rho$  also in this case. Finally, we show that  $\partial\{u_0 > 0\} \cap B_{1/2}$  has a vanishing Lebesgue measure. Indeed, every point  $x_0 \in$

$\partial\{u_0 > 0\} \cap B_{1/2}$  is a limit of a sequence  $x_k \in \partial\{u_k > 0\} \cap B_{1/2}$ . Using this, we can prove that  $u_0$  satisfies the nondegeneracy condition

$$\left( \int_{B_r(x_0)} u_0^\gamma \right)^{1/\gamma} \geq cr^n$$

for any ball  $B_r(x_0) \subset B_{1/2}$ . Along with the Lipschitz continuity, this is enough to prove that  $\mathcal{L}^n(B_r(x_0) \cap \{u_0 > 0\}) \geq c\mathcal{L}^n(B_r(x_0))$  for  $c > 0$ . This implies that  $\mathcal{L}^n(\partial\{u_0 > 0\}) = 0$ , see Remark 4.5.

Thus,  $|\nabla u_k|^{p-2}\nabla u_k$  converges weakly-\* to  $|\nabla u_0|^{p-2}\nabla u_0$  in  $L^\infty(B_{1/2})$ . Hence, for every  $\zeta \in C_0^\infty(B_{1/2})$ ,  $\zeta \geq 0$ , one has

$$\int_{B_{1/2}} |\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla \zeta = \lim_{k \rightarrow \infty} \int_{B_{1/2}} |\nabla u_k|^{p-2}\nabla u_k \cdot \nabla \zeta$$

or

$$\int_{B_{1/2}} \zeta d\Lambda_0 = \lim_{k \rightarrow \infty} \int_{B_{1/2}} \zeta d\Lambda_k \leq \|\zeta\|_{L^\infty(B_{1/2})} \lim_{k \rightarrow \infty} \varepsilon_k = 0.$$

Thus,  $\Lambda_0 = 0$  in  $B_{1/2}$ , which means that  $u_0$  is  $p$ -harmonic. From the facts  $u_0 \geq 0$ ,  $u_0(0) = 0$ , and the strong minimum principle, we infer  $u_0 = 0$  in  $B_{1/2}$ . On the other hand, since  $0 \in \partial\{u_k > 0\}$ , by nondegeneracy we have  $\left(\int_{B_{1/4}} u_k^\gamma\right)^{1/\gamma} \geq c > 0$  and  $\gamma > p - 1$ , and therefore a similar inequality holds for  $u_0$ . Hence, we have reached a contradiction.

The theorem is proved. □

**Theorem 5.2** *Let  $u$  be a local minimizer in  $\Omega$ . Then*

- (i)  $\mathcal{H}^{n-1}(K \cap \partial\{u > 0\}) < \infty$  for every  $K \subset\subset \Omega$ .
- (ii) *There is a Borel function  $q_u$  such that*

$$\Delta_p u = q_u \mathcal{H}^{n-1} \llcorner \partial\{u > 0\},$$

*that is for every  $\zeta \in C_0^\infty(\Omega)$*

$$-\int_{\Omega} |\nabla u|^{p-2}\nabla u \cdot \nabla \zeta = \int_{\Omega \cap \partial\{u > 0\}} \zeta q_u d\mathcal{H}^{n-1}.$$

- (iii) *For any  $K \subset\subset \Omega$  there exist constants  $c, C > 0$  such that*

$$c \leq q_u(x) \leq C, \quad cr^{n-1} \leq \mathcal{H}^{n-1}(B_r(x) \cap \partial\{u > 0\}) \leq Cr^{n-1}$$

*for every ball  $B_r(x) \subset K$  with  $x \in \partial\{u > 0\}$ .*

*Proof.* This follows easily from Theorem 5.1. For more details, see the proof of Theorem 4.5 in [2]. □

From (i) in Theorem 5.2 it follows that the set  $A = \Omega \cap \{u > 0\}$  has finite perimeter locally in  $\Omega$ , see [7], Chapter 4, in the sense that

$$\mu_u = -\nabla\chi_A$$

is a Borel measure and the total variation  $|\mu_u|$  is a Radon measure. We define the reduced boundary of  $A$  by

$$\partial_{\text{red}}A = \{x \in \Omega : |\nu_u(x)| = 1\},$$

where  $\nu_u(x)$  is the unique unit vector with

$$\int_{B_r(x)} |\chi_A - \chi_{\{y: (y-x) \cdot \nu_u(x) < 0\}}| = o(r^n),$$

if such vector exists, and  $\nu_u(x) = 0$  otherwise. In what follows, we will use some results about sets of finite perimeter, mainly from [7], Chapter 4, such as

$$\mu_u = \nu_u \mathcal{H}^{n-1} \llcorner \partial_{\text{red}}\{u > 0\}.$$

To proceed, we will need some properties of the so-called blow-up limits. Let  $u$  be a local minimizer in  $\Omega$ ,  $K \subset\subset \Omega$  and  $B_{\rho_k}(x_k) \subset K$  be a sequence of balls with  $\rho_k \rightarrow 0$ ,  $x_k \rightarrow x_0 \in \Omega$  and  $u(x_k) = 0$ . Consider then the blow-up sequence

$$(5.1) \quad u_k(x) = \frac{1}{\rho_k} u(x_k + \rho_k x).$$

Since  $u_k$  are uniformly Lipschitz continuous, for a subsequence,

$$(5.2) \quad u_k \rightarrow u_0 \quad \text{in } C_{\text{loc}}^\alpha(\mathbf{R}^n) \text{ for every } 0 < \alpha < 1,$$

$$(5.3) \quad \nabla u_k \rightarrow \nabla u_0 \quad \text{weakly-* in } L_{\text{loc}}^\infty(\mathbf{R}^n),$$

$$(5.4) \quad \partial\{u_k > 0\} \rightarrow \partial\{u_0 > 0\} \quad \text{locally in the Hausdorff distance,}$$

$$(5.5) \quad \chi_{\{u_k > 0\}} \rightarrow \chi_{\{u_0 > 0\}} \quad \text{in } L_{\text{loc}}^1(\mathbf{R}^n).$$

Moreover, if  $x_k \in \partial\{u_k > 0\}$  then  $x_0 \in \partial\{u_0 > 0\}$ . For the proof we refer to Sect. 4.7 in [2] and pp. 19–20 in [3]; see also our proof of Theorem 5.1.

The following lemma is an analogue of Lemma 3.3 in [3] with the same proof.

**Lemma 5.3** *The limit  $u_0$  of a blow-up sequence of  $u$  with respect to balls  $B_{\rho_k}(x_k)$ ,  $u(x_k) = 0$ , is an absolute minimizer of  $J$  in any ball.  $\square$*

**Lemma 5.4** *Let  $u$  be a local minimizer and  $x_0 \in \partial\{u > 0\}$ . Then*

$$\limsup_{x \rightarrow x_0, u(x) > 0} |\nabla u(x)| = 1.$$

*Proof.* Denote the lim sup by  $\ell$  and let the sequence  $y_k \rightarrow x_0, u(y_k) > 0$ , be such that  $|\nabla u(y_k)| \rightarrow \ell$ . Set  $\rho_k = \text{dist}(y_k, \partial\{u > 0\})$  and let  $x_k \in \partial B_{\rho_k}(y_k) \cap \partial\{u > 0\}$ . Consider then the blow-up sequence (5.1) and assume that (5.2)–(5.5) hold. Also, assume that  $e_k := (x_k - y_k)/|x_k - y_k|$  converges to the unit vector  $e_n$ . Now, observe that  $0 \in \partial\{u_0 > 0\}, B_1(-e_n) \subset \{u_0 > 0\}$ , and

$$|\nabla u_0| \leq \ell \text{ in } \{u_0 > 0\}, \quad |\nabla u_0(-e_n)| = \ell.$$

This implies  $\ell > 0$ . Since  $u_0$  is also  $p$ -harmonic in  $\{u_0 > 0\}$ , it is locally  $C^{1,\alpha}$  there. In particular, there exists a small  $\delta > 0$  such that  $|\nabla u_0| > \ell/2$  in  $B_\delta(-e_n)$ . If  $e$  denotes a unit vector such that  $\nabla u_0(-e_n) = |\nabla u_0(-e_n)|e$ , the directional derivative  $v = D_e u_0$  will satisfy in  $B_\delta(-e_n)$  a uniformly elliptic equation

$$D_i(a_{ij}D_jv) = 0, \quad a_{ij} = |\nabla u_0|^{p-2} \left( \delta_{ij} + (p-2) \frac{D_i u_0 D_j u_0}{|\nabla u_0|^2} \right).$$

By the strong maximum principle, we must have  $D_e u_0 = \ell$  in  $B_\delta(-e_n)$ , implying that  $\nabla u_0 = \ell e$ . By continuation, we can prove that this is true in the whole  $B_1(-e_n)$ . Therefore

$$u_0(x) = \ell(x \cdot e) + C \quad \text{in } B_1(-e_n).$$

Since  $u_0(0) = 0$  and  $u_0 > 0$  in  $B_1(-e_n)$ , we obtain that  $C = 0$  and  $e = -e_n$ . Thus,

$$u_0(x) = -\ell x_n \quad \text{in } B_1(-e_n).$$

Using the continuation method one more time, we see that

$$(5.6) \quad u_0(x) = -\ell x_n \quad \text{in } \{x_n < 0\}.$$

Next, we claim that

$$(5.7) \quad u_0 = 0 \quad \text{in } \{0 < x_n < \varepsilon_0\}$$

for some  $\varepsilon_0 > 0$ . Indeed, let

$$s = \limsup_{\substack{x_n \rightarrow 0+, x' \in \mathbf{R}^n \\ u(x', x_n) > 0}} D_n u_0(x', x_n).$$

Then  $s < \infty$ , since  $u_0$  is uniformly Lipschitz. Assume that  $s > 0$ . Consider a sequence  $(z_k, h_k), h_k \rightarrow 0+$ , such that  $D_n u_0(z_k, h_k) \rightarrow s$ . Arguing as above, one can show that the blow-up limit  $u_{00}$  of  $u_0$  with respect to the balls  $B_{h_k}(z_k, 0)$  satisfies

$$u_{00}(x) = s x_n \quad \text{in } \{x_n > 0\}.$$

On the other hand, we have

$$u_{00}(x) = -\ell x_n \quad \text{in } \{x_n < 0\}.$$

We have reached a contradiction, since by Lemma 5.3  $u_{00}$  is a minimizer and therefore by Theorem 4.4 the set  $\{u_{00} = 0\}$  must have positive density. Thus,  $s = 0$  and consequently  $u_0(x', x_n) = o(x_n)$ . Hence, for every  $\varepsilon > 0$

$$\frac{1}{r} \left( \int_{B_r(x_0)} u_0^\gamma \right)^{1/\gamma} < \varepsilon$$

for any  $x_0 = (z_0, h_0)$ ,  $r = h_0$ , if  $h_0$  is small enough. But then the nondegeneracy lemma (Lemma 4.2) implies that  $u_0 = 0$  in some strip  $\{0 < x_n < \varepsilon_0\}$ .

Having proved (5.6) and (5.7), we deduce from Theorem 2.1 that  $\ell = 1$ .  $\square$

**Theorem 5.5** *Let  $x_0 \in \partial_{\text{red}}\{u > 0\}$  and suppose that the upper  $\mathcal{H}^{n-1}$ -density satisfies*

$$\Theta^{*n-1} \left( \mathcal{H}^{n-1} \llcorner \partial\{u > 0\}, x_0 \right) \leq 1.$$

*Then the topological tangent plane  $\text{Tan}(\partial\{u > 0\}, x_0)$  of  $\partial\{u > 0\}$  at  $x_0$  is given by  $\{x : x \cdot \nu_u(x_0) = 0\}$ . If, in addition,*

$$\int_{B_r(x_0) \cap \partial\{u > 0\}} |q_u - q_u(x_0)| d\mathcal{H}^{n-1} = o(r^{n-1}), \quad \text{as } r \rightarrow 0,$$

*then  $q_u(x_0) = 1$  and*

$$u(x_0 + x) = (-x \cdot \nu_u(x_0))^+ + o(|x|), \quad \text{as } x \rightarrow 0.$$

For definitions of  $\Theta^{*m}(\mu, a)$  and  $\text{Tan}(S, a)$ , see [7], 2.10.19 and 3.1.21.

*Proof.* Without loss of generality, assume  $\nu_u(x_0) = e_n$ . Consider then the blow-up limit  $u_0$  of  $u$  with respect to the sequence of balls  $B_{\rho_k}(x_0)$ ,  $\rho_k \rightarrow 0$ . Using that  $x_0 \in \partial_{\text{red}}\{u > 0\}$  and that the upper  $\mathcal{H}^{n-1}$ -density of  $\partial\{u > 0\}$  at  $x_0$  does not exceed 1, precisely as in the proofs of Theorem 4.8 in [2] and Theorem 3.5 in [3], one can show that  $u_0 > 0$  in  $\{x_n < 0\}$  and  $u_0 = 0$  on  $\{x_n > 0\}$ . As a consequence,  $\{x_n = 0\}$  is the tangent space to  $\partial\{u > 0\}$  at  $x_0$ .

To prove the second part of the theorem, we again repeat the arguments from [3] and obtain that

$$|\nabla u_0|^{p-2} \nabla u_0 \cdot e_n = q_u(x_0) \quad \text{on } \{x_n = 0\}$$

in the sense that

$$- \int_{B_r \cap \{x_n < 0\}} |\nabla u_0|^{p-2} \nabla u_0 \cdot \nabla \zeta = q_u(x_0) \int_{B_r^+} \zeta(x', 0) d\mathcal{H}^{n-1}$$

for every  $\zeta \in C_0^1(B_r)$ . Since  $\Delta_p u_0 = 0$  in  $\{x_n < 0\}$ , from the boundary regularity it follows that the boundary condition above is satisfied in the classical sense. Hence, from the Lemma 5.4 we obtain  $q_u(x_0) = 1$ .

Finally, we need to show that  $u_0 = (-x_n)^+$ . Define the function  $w_0$  by

$$w_0(x) = \begin{cases} u_0(x) & \text{in } \{x_n < 0\} \\ -u_0(x^*) & \text{in } \{x_n > 0\}, \end{cases}$$

where  $x^* = (x', -x_n)$  for  $x = (x', x_n)$ . It is easy to see that  $w_0$  is  $p$ -harmonic in the whole  $\mathbf{R}^n$ . Observe now, since  $|\nabla u_0| \leq 1$  in  $\{x_n < 0\}$  by Lemma 5.4,

$$|\nabla w_0| \leq 1 \quad \text{in } \mathbf{R}^n.$$

On the other hand,

$$D_n w_0 = -1 \quad \text{on } \{x_n = 0\}.$$

In a small ball  $B_\delta(0)$ , we have  $|\nabla w_0| > 1/2$ , hence  $D_n w_0$  satisfies a uniformly elliptic equation in divergence form and from the strong comparison principle we infer that

$$D_n w_0 = -1 \quad \text{in } B_\delta(0).$$

By continuation, we can prove that, in fact,

$$D_n w_0 = -1 \quad \text{in } \mathbf{R}^n.$$

This implies that  $w_0 = -x_n$  in  $\mathbf{R}^n$ , or equivalently,  $u_0 = (-x_n)^+$  in  $\mathbf{R}^n$ . The proof is complete.  $\square$

**Theorem 5.6** For  $\mathcal{H}^{n-1}$  a.e.  $x$  in  $\partial_{\text{red}}\{u > 0\}$

$$q_u(x) = 1.$$

Since also  $\mathcal{H}^{n-1}(\partial\{u > 0\} \setminus \partial_{\text{red}}\{u > 0\}) = 0$  (from the positive density property)

$$\Delta_p u = \mathcal{H}^{n-1} \llcorner \partial_{\text{red}}\{u > 0\}$$

for any local minimizer  $u$  in  $\Omega$ .

*Proof.* Just observe that the condition on  $q_u$  in Theorem 5.5 is satisfied for  $\mathcal{H}^{n-1}$  a.e.  $x_0 \in \partial_{\text{red}}\{u > 0\}$ . This follows from [7], 4.5.6(2) and [7], 2.9.8, 2.9.9 applied to  $\mathcal{H}^{n-1}$  on  $\partial\{u > 0\}$  and the Vitali relation

$$\{(x, B_r(x)) : x \in \partial\{u > 0\} \text{ and } B_r(x) \subset\subset \Omega\}. \quad \square$$

## 6 Flatness and nondegeneracy of the gradient

We define the relevant flatness classes as in [3], Definition 5.1.

**Definition 6.1** Let  $0 \leq \sigma_+, \sigma_- \leq 1$  and  $\tau > 0$ . We say that  $u$  belongs to the class  $F(\sigma_+, \sigma_-; \tau)$  in  $B_\rho = B_\rho(0)$  if  $u$  is a local minimizer in  $B_\rho$  with  $0 \in \partial\{u > 0\}$ , and

$$(6.1) \quad \begin{aligned} u(x) &= 0 & \text{for } x_n \geq \sigma_+ \rho, \\ u(x) &\geq -(x_n + \sigma_- \rho) & \text{for } x_n \leq -\sigma_- \rho, \\ |\nabla u| &\leq 1 + \tau & \text{in } B_\rho. \end{aligned}$$

More generally, changing the direction  $e_n$  by  $\nu$  and the origin by  $x_0$  in the definition above, we obtain the flatness class  $F(\sigma_+, \sigma_-; \tau)$  in  $B_\rho(x_0)$  in direction  $\nu$ .

**Remark 6.2** If  $x_0 \in \partial_{\text{red}}\{u > 0\} \cap \Omega$  then  $u \in F(\sigma_\rho, 1; \infty)$  in  $B_\rho(x_0)$  in direction  $\nu_0 = \nu_u(x_0)$  with  $\sigma_\rho \rightarrow 0$  as  $\rho \rightarrow 0$ . This follows from the fact that any blow-up  $u_0$  at  $x_0$  vanishes on  $\{x : x \cdot \nu_0 \geq 0\}$ .

The following two theorems play an important role in the iteration process of proving the  $C^{1,\alpha}$  regularity of the free boundary.

**Theorem 6.3** *There exists  $\sigma_0 > 0$  and  $C_0 > 0$  such that*

$$u \in F(\sigma, 1; \sigma) \text{ in } B_1 \quad \text{implies} \quad u \in F(2\sigma, C_0\sigma; \sigma) \text{ in } B_{1/2}$$

for  $0 < \sigma < \sigma_0$ .

**Theorem 6.4** *For every  $\delta > 0$  there exists  $\sigma_\delta > 0$  and  $C_\delta > 0$  such that*

$$u \in F(\sigma, 1; \sigma) \text{ in } B_1 \quad \text{implies} \quad |\nabla u| \geq 1 - \delta \text{ in } B_{1/2} \cap \{x_n \leq -C_\delta\sigma\}$$

for  $0 < \sigma < \sigma_\delta$ .

We first prove the following weak forms of the theorems.

**Lemma 6.5** *For every  $\varepsilon > 0$  there exists  $\sigma_\varepsilon > 0$  such that*

$$u \in F(\sigma, 1; \sigma) \text{ in } B_1 \quad \text{implies} \quad u \in F(2\sigma, \varepsilon; \sigma) \text{ in } B_{1/2}$$

for  $0 < \sigma < \sigma_\varepsilon$ .

**Lemma 6.6** *For every  $\varepsilon > 0$  and  $\delta > 0$  there exists  $\sigma_{\varepsilon,\delta} > 0$  such that*

$$u \in F(\sigma, 1; \sigma) \text{ in } B_1 \quad \text{implies} \quad |\nabla u| \geq 1 - \delta \text{ in } B_{1/2} \cap \{x_n \leq -\varepsilon\}$$

for  $0 < \sigma < \sigma_{\varepsilon,\delta}$

*Proof of Lemma 6.5.* We use the following construction from [2,3]. Let

$$\eta(y) = \exp\left(-\frac{9|y|^2}{1-9|y|^2}\right)$$

for  $|y| < 1/3$  and  $\eta(y) = 0$  for  $|y| \geq 1/3$ , and choose  $s \geq 0$  maximal such that

$$B_1 \cap \{u > 0\} \subset D := \{x \in B_1 : x_n < \sigma - s\eta(x')\},$$

where  $x = (x', x_n)$ . Hence, there exists a point

$$z \in B_{1/2} \cap \partial D \cap \partial\{u > 0\}.$$

Observe also that  $s \leq \sigma$  since  $0 \in \partial\{u > 0\}$ .

Now, let  $\xi \in \partial B_{3/4}$  and  $\xi_n \leq -1/2$ . We want to prove an estimate for  $u(\xi)$  from below. Consider the solution  $v = v_{\kappa,\rho}$  of

$$\begin{aligned} \Delta_p v &= 0 && \text{in } D \setminus \overline{B_\rho(\xi)}, \\ v &= 0 && \text{on } \partial D \cap B_1, \\ v &= (1 + \sigma)(\sigma - x_n) && \text{on } \partial D \setminus B_1, \\ v &= -(1 - \kappa\sigma)x_n && \text{on } \partial B_\rho(\xi), \end{aligned}$$

where  $\kappa > 0$  is large and  $\rho > 0$  is small constants to be chosen later. We claim that for large  $\kappa = \kappa(\rho)$

$$(6.2) \quad u(x_\xi) \geq v(\xi) \quad \text{for some } x_\xi \in \partial B_\rho(\xi).$$

Indeed, otherwise  $u \leq v$  on  $\partial(D \setminus B_\rho(\xi))$  and, by the comparison principle, also  $u \leq v$  on  $D \setminus B_\rho(\xi)$ . Then the contradiction follows from the following two statements, applied at the point  $z$ .

*Claim 6.7* If  $B$  is a ball in  $\{u = 0\}$  touching  $\partial\{u > 0\}$  at  $x_0$ , then

$$\limsup_{x \rightarrow x_0} \frac{u(x)}{\text{dist}(x, B)} = 1.$$

*Claim 6.8* The function  $v = v_\kappa$  constructed above satisfies

$$|\nabla v(z)| \leq 1 + C\sigma - c\kappa\sigma \quad \text{for } z \in B_{1/2} \cap (\partial D \setminus \partial B_1)$$

for some positive constants  $C = C(\rho)$  and  $c = c(\rho)$  and if  $0 < \sigma < \sigma(\kappa, \rho)$ .

We postpone the proofs of these claims for a moment.

Now, choose  $\kappa = \kappa(\rho) > C(\rho)/c(\rho)$  and  $\sigma < \sigma(\kappa(\rho), \rho)$ . Then (6.2) holds and

$$\begin{aligned} u(\xi) &\geq u(x_\xi) - \rho(1 + \sigma) \geq v(x_\xi) - \rho(1 + \sigma) \\ &= -(1 - \kappa\sigma)(x_\xi)_n - \rho(1 + \sigma) \geq -(x_\xi)_n - \kappa\sigma - 2\rho \geq -\xi_n - 4\rho \end{aligned}$$

for  $\sigma < \sigma(\rho)$  sufficiently small. That is,

$$(6.3) \quad u(\xi) \geq -\xi_n - 4\rho \quad \text{on } \{\xi \in \partial B_{3/4}, \xi_n < -1/2\}.$$

Integrating along vertical lines and using that  $|\nabla u| \leq 1 + \sigma$ , we obtain

$$u(\xi + \alpha e_n) \geq u(\xi) - \alpha(1 + \sigma) \geq -\xi_n - 4\rho - \alpha - \alpha\sigma \geq -(\xi_n + \alpha) - 5\rho$$

for  $\sigma < \sigma(\rho)$ . Choosing  $\rho = \varepsilon/10$ , we complete the proof of Lemma 6.5.  $\square$

*Proof of Lemma 6.6.* Assume the contrary. Then there exists a sequence  $u_k \in F(1/k, 1; 1/k)$  such that

$$|\nabla u_k(x^k)| \leq 1 - \delta \quad \text{for some } x^k \in B_{1/2} \cap \{x_n \leq -\varepsilon\}.$$

Letting  $k \rightarrow \infty$ , we obtain from Lemma 6.5 that

$$u_k(x) \rightarrow u_0(x) = -x_n \quad \text{uniformly on } \overline{B_{3/4}}.$$

Moreover, on the positivity set  $\{u_0 > 0\} = \{x_n < 0\}$  the convergence is locally  $C^{1,\alpha}$ . This implies that if a subsequence of  $x^k$  converges to  $x^0 \in B_{1/2} \cap \{x_n \leq -\varepsilon\}$ , then  $|\nabla u_0(x^0)| \leq 1 - \delta$ , which contradicts to the fact that  $|\nabla u_0| = 1$ .  $\square$

*Proof of Theorem 6.3.* We revisit the proof of Lemma 6.5. Choose  $\rho = 1/10$  and  $\kappa = \kappa(\rho)$  such that (6.2) holds. We can refine the estimate (6.3) as follows. Set

$$w(x) = (1 + \sigma)(\sigma - x_n) - u(x).$$

Then  $u \in F(\sigma, 1; \sigma)$  implies that  $w(x) \geq 0$  in  $B_{2\rho}(\xi)$  and

$$w(x_\xi) \leq -(x_\xi)_n - v(x_\xi) + C\sigma \leq C\sigma.$$

For  $\sigma$  sufficiently small, we know from Lemma 6.6 that  $|\nabla u| > 1/2$ , hence  $u$  will satisfy a linear nondivergence equation

$$Lu = 0,$$

where the operator  $L$  is defined by

$$(6.4) \quad Lv = a_{ij}D_{ij}v, \quad a_{ij} = a_{ij}(\nabla u) = \delta_{ij} + (p-2) \frac{D_i u D_j u}{|\nabla u|^2}.$$

Observe that the ellipticity constant of  $L$  is  $\lambda = \lambda(p) = \max\{p-1, 1/(p-1)\}$ . As a consequence,  $w$  will also satisfy

$$Lw = 0.$$

Then, applying the Harnack inequality, we obtain that

$$w(\xi) \leq Cw(x_\xi) \leq C\sigma$$

or

$$u(\xi) \geq -\xi_n - C\sigma \quad \text{on } \{\xi \in \partial B_{3/4}, \xi_n \leq -1/2\}.$$

Integrating along vertical lines and using that  $|\nabla u| \leq 1 + \sigma$ , we conclude that

$$u(\xi + \alpha e_n) \geq u(\xi) - (1 + \sigma)\alpha \geq -(\xi_n + \alpha) - C\sigma,$$

which implies that  $u \in F(2\sigma, C\sigma; \sigma)$  in  $B_{1/2}$ .  $\square$

*Proof of Theorem 6.4.* Assume the contrary. Then there exists a sequence  $\sigma_k \rightarrow 0$  and  $u_k \in F(\sigma_k, 1; \sigma_k)$  such that

$$|\nabla u_k(x^k)| \leq 1 - \delta \quad \text{for some } x^k \in B_{1/2} \cap \{x_n \leq -k\sigma_k\}.$$

Let  $d_k = \text{dist}(x^k, \partial\{u_k > 0\})$  and  $y^k \in \partial\{u_k > 0\}$  be such that  $|x^k - y^k| = d_k$ . From Theorem 6.3 it follows that  $d_k \geq (k - C_0)\sigma_k$ . Define now

$$\tilde{u}_k(x) = \frac{u_k(y^k + 2d_k x)}{2d_k}, \quad \tilde{x}^k = \frac{x^k - y^k}{2d_k}.$$

Then one can easily verify that

$$\begin{aligned} \tilde{u}_k &\in F((C_0 + 1)/2(k - C_0), 1; \sigma_k) \quad \text{in } B_1, \\ |\tilde{x}^k| &= \frac{1}{2}, \quad (\tilde{x}^k)_n \leq -\frac{1}{2}(1 - (C_0 + 1)/(k - C_0)), \end{aligned}$$

and

$$|\nabla \tilde{u}_k(\tilde{x}^k)| \leq 1 - \delta.$$

This contradicts to Lemma 6.6. The proof is complete. □

We now prove the claims stated in the proof of Lemma 6.5.

*Proof of Claim 6.7.* Denote

$$\ell = \limsup_{x \rightarrow x^0} u(x)/\text{dist}(x, B)$$

and let the sequence  $x^k \rightarrow x^0$  be such that  $u(x^k) > 0$  and

$$\frac{u(x^k)}{d_k} \rightarrow \ell,$$

where  $d_k = \text{dist}(x^k, B)$ . Moreover, let  $y^k \in \partial B$  be such that  $|x^k - y^k| = d_k$ . By nondegeneracy  $\ell > 0$ . Consider now a blow-up sequence

$$u_k(x) = \frac{u(y_k + d_k x)}{d_k}$$

and assume that for a subsequence

$$(x^k - y^k)/d_k \rightarrow e, \quad u_k \rightarrow u_0.$$

We claim that

$$u_0(x) = \ell(x \cdot e)^+.$$

Indeed, by construction,  $u_0(x) = 0$  if  $x \cdot e \leq 0$ ,  $u_0(x) \leq \ell(x \cdot e)$  if  $x \cdot e > 0$ , and  $u_0(e) = \ell$ . Both  $u_0(x)$  and  $\ell(x \cdot e)^+$  are  $p$ -harmonic in  $\{u_0 > 0\}$  and from the strong maximum principle (applicable here since  $\ell > 0$ ) it follows that they must coincide.

The only constant  $\ell$  for which  $u_0$  can have the form  $\ell(x \cdot e)^+$  is  $\ell = 1$  and the proof is complete. □

*Proof of Claim 6.8.* The idea of the proof is to construct an explicit  $p$ -superharmonic function  $w$  in  $D \setminus B_\rho(\xi)$  to estimate  $v$ . We start with the observation that a  $C^2$  function  $w$  with  $|\nabla w| > 0$  is  $p$ -superharmonic, if it verifies the inequality

$$b_{ij} D_{ij} w \leq 0$$

for every positive matrix  $b_{ij}$  of ellipticity  $\lambda = \lambda(p) = \max\{1/(p-1), p-1\}$ , i.e. such that

$$\lambda^{-1} |\zeta|^2 \leq b_{ij} \zeta_i \zeta_j \leq \lambda |\zeta|^2 \quad \text{for every } \zeta \in \mathbf{R}^n.$$

Indeed, the particular choice of

$$b_{ij} = \delta_{ij} + (p-2) \frac{D_i w D_j w}{|\nabla w|^2}$$

will be equivalent to  $\Delta_p w \leq 0$ .

We construct  $w$  in the form  $v_1 - \kappa\sigma v_2$  with  $v_1, v_2$  defined as follows. Let

$$v_1 = \frac{\gamma_1}{\mu_1}(1 - \exp(-\mu_1 d_1)) \quad \text{in } D,$$

where

$$d_1(x) = -x_n + \sigma - s\eta(x'),$$

and with positive constants  $\mu_1, \gamma_1$  depending on  $\sigma$ . Then

$$1 \leq |\nabla d_1| \leq 1 + C\sigma, \quad |D^2 d_1| \leq C\sigma.$$

Hence, if  $b_{ij}$  is an arbitrary positive matrix of ellipticity  $\lambda(p)$ ,

$$(6.5) \quad \begin{aligned} b_{ij} D_{ij} v_1 &= \gamma_1 \exp(-\mu_1 d_1) b_{ij} (D_{ij} d_1 - \mu_1 D_i d_1 D_j d_1) \\ &\leq \gamma_1 \exp(-\mu_1 d_1) (C\sigma - c\mu_1) < 0, \end{aligned}$$

if we choose

$$\mu_1 = C_1 \sigma$$

with sufficiently large  $C_1$ . Next, if

$$\gamma_1 = 1 + C_2 \sigma$$

with large  $C_2$ , we will obtain that for  $x \in D$

$$|\nabla v_1(x)| = \gamma_1 \exp(-\mu_1 d_1) |\nabla d_1| \geq \gamma_1 (1 - 2\mu_1) \geq 1.$$

Hence,  $v_1$  is  $p$ -superharmonic in  $D$  by (6.5). Moreover, for  $x \in \partial(D \setminus B_\rho(\xi))$

$$v_1(x) \geq \gamma_1 d_1(x) (1 - C\mu_1) \geq (1 + 2\sigma) d_1(x) \geq v(x),$$

if  $\sigma$  is sufficiently small, and therefore the maximum principle yields

$$v_1 \geq v \quad \text{in } D \setminus B_\rho(\xi).$$

We also have that at  $z \in B_{1/2} \cap (\partial D \setminus \partial B_1)$

$$|\nabla v_1(z)| = \gamma_1 |\nabla d_1| \leq 1 + C\sigma.$$

Next, we define  $v_2$  depending on  $B_\rho(\xi)$  by

$$v_2 = \frac{\gamma_2}{\mu_2} (\exp(\mu_2 d_2) - 1) \quad \text{in } \tilde{D} \setminus B_\rho(\xi),$$

with constants  $\gamma_2, \mu_2$  to be specified later. Here  $\tilde{D} \subset D$  is a domain with a smooth boundary containing

$$D \setminus B_{1/10}(\partial B'_1 \times \{0\})$$

and  $d_2$  is a function in  $C^2(D \setminus B_\rho(\xi))$  satisfying

$$\begin{aligned} d_2 &= 0 && \text{on } \partial \tilde{D}, \\ d_2 &= 1 && \text{on } \partial B_\rho(\xi), \\ C &\geq |\nabla d_2| \geq c > 0 && \text{in } \tilde{D} \setminus \overline{B_\rho(\xi)}. \end{aligned}$$

Thus, for any positive matrix  $b_{ij}$  of ellipticity  $\lambda(p)$ ,

$$(6.6) \quad \begin{aligned} b_{ij}D_{ij}v_2 &= \gamma_2 \exp(\mu_2 d_2) b_{ij}(D_{ij}d_2 + \mu_2 D_i d_2 D_j d_2) \\ &\geq \gamma_2 \exp(\mu_2 d_2)(-C + c\mu_2) > 0, \end{aligned}$$

if  $\mu_2$  is large enough. Then choose  $\gamma_2$  such that

$$v_2 = 1 \quad \text{on } \partial B_\rho(\xi)$$

or, explicitly,

$$\gamma_2 = \frac{\mu_2}{\exp(\mu_2) - 1}.$$

In  $\tilde{D} \setminus \overline{B_\rho(\xi)}$  we have

$$|\nabla v_2| = \gamma_2 \exp(\mu_2 d_2) |\nabla d_2| \leq C$$

and at the point  $z$

$$|\nabla v_2(z)| = \gamma_2 |\nabla d_2(z)| \geq c > 0.$$

Thus, the function

$$w = v_1 - \kappa\sigma v_2$$

satisfies  $b_{ij}D_{ij}w \leq 0$  in  $\tilde{D} \setminus \overline{B_\rho(\xi)}$  with

$$w = v_1 \geq v \quad \text{on } \partial\tilde{D},$$

and for  $x \in \partial B_\rho(\xi)$

$$w(x) \geq d_1(x) - \kappa\sigma \geq -(1 - \kappa\sigma)x_n = v(x).$$

Also, the gradient of  $w$  is not degenerate, since

$$|\nabla w| \geq |\nabla v_1| - \kappa\sigma |\nabla v_2| \geq 1 - C\kappa\sigma > 0,$$

if  $\sigma < \sigma(\kappa)$  is small enough. We conclude that  $w$  is  $p$ -superharmonic and the comparison principle yields  $w \geq v$  in  $\tilde{D} \setminus \overline{B_\rho(\xi)}$ . In particular,

$$|\nabla v(z)| \leq |\nabla w(z)| = |\nabla v_1(z)| - \kappa\sigma |\nabla v_2(z)| \leq 1 + C\sigma - \kappa\sigma.$$

The claim is proved. □

## 7 Gradient estimates

**Theorem 7.1** *Let  $u$  be a local minimizer. For any  $D \subset\subset \Omega$  and a ball  $B_r(x) \subset D$  such that  $B_r(x) \cap \partial\{u > 0\}$  is nonempty,*

$$\sup_{B_r(x)} |\nabla u| \leq 1 + Cr^\alpha$$

with  $C > 0$ ,  $0 < \alpha < 1$  (depending only on  $D$ ,  $\Omega$ ,  $n$ ,  $p$ ).

*Proof.* For any  $\varepsilon > 0$  consider the function

$$U_\varepsilon = (|\nabla u|^2 - 1 - \varepsilon)^+$$

Observe that it vanishes in a neighborhood of the free boundary. Since  $U_\varepsilon > 0$  implies  $|\nabla u| > 1 + \varepsilon$ , the closure of  $\{U_\varepsilon > 0\}$  is contained in  $\{|\nabla u| > 1 + \varepsilon/2\}$ . The function  $u$  satisfies the linearized equation

$$Lu = a_{ij}(\nabla u)D_{ij}u = 0,$$

where  $L$  is as in (6.4). We recall that the matrix  $a_{ij}(\nabla u)$  is  $\lambda(p)$ -uniformly elliptic in  $\{|\nabla u| > 1 + \varepsilon/2\}$ . Hence, by [8], Sect. 13.3, the function  $v = |\nabla u|^2$  satisfies

$$Mv = D_i(e^{\gamma v}a_{ij}(\nabla u)D_jv) \geq 0 \quad \text{in } \{|\nabla u| > 1 + \varepsilon/2\},$$

where  $\gamma = \gamma(p, \|\nabla u\|_{L^\infty(\Omega)})$  is some positive constant. Hence,  $U_\varepsilon$  satisfies

$$MU_\varepsilon \geq 0 \quad \text{in } \{|\nabla u| > 1 + \varepsilon/2\}.$$

Extending the operator  $M$  with the uniformly elliptic divergence-form operator

$$\widetilde{M}w = D_i(\widetilde{a}_{ij}(x)D_jw) \quad \text{in } \Omega$$

with measurable coefficients such that

$$\widetilde{a}_{ij}(x) = e^{\gamma v(x)}a_{ij}(\nabla u(x)) \quad \text{in } \{|\nabla u| > 1 + \varepsilon/2\},$$

we obtain

$$\widetilde{M}U_\varepsilon \geq 0 \quad \text{in } \Omega.$$

For any  $r > 0$  set

$$h_\varepsilon(r) = \sup_{B_r} U_\varepsilon,$$

where the origin is taken to be on the free boundary. Then  $h_\varepsilon(r) - U_\varepsilon$  is  $\widetilde{M}$ -supersolution in  $B_r$  and

$$\begin{aligned} h_\varepsilon(r) - U_\varepsilon &\geq 0 && \text{in } B_r \\ &= h_\varepsilon(r) && \text{in } B_r \cap \{u \equiv 0\}. \end{aligned}$$

By [8], Theorem 8.18, with  $1 \leq q < n/(n-2)$

$$\inf_{B_{r/2}} (h_\varepsilon(r) - U_\varepsilon) \geq cr^{-n/q} \|h_\varepsilon(r) - U_\varepsilon\|_{L^q(B_r)} \geq ch_\varepsilon(r),$$

since  $|B_r \cap \{u \equiv 0\}| \geq cr^n$  by the positive density property. Taking  $\varepsilon \rightarrow 0$ , we obtain

$$\inf_{B_{r/2}} (h_0(r) - U_0) \geq ch_0(r) \quad (0 < c < 1)$$

or

$$\sup_{B_{r/2}} U_0 \leq (1 - c)h_0(r).$$

In conclusion,

$$h_0(r/2) \leq (1 - c)h(r)$$

and by a standard argument we deduce that  $h_0(r) \leq Cr^\alpha$  for some  $C > 0$  and  $0 < \alpha < 1$ . This completes the proof of the theorem. □

### 8 Nonhomogeneous blow-up

In the sequel, we will denote points in  $\mathbf{R}^n$  by  $(y, h)$ , with  $y \in \mathbf{R}^{n-1}$  and  $h \in \mathbf{R}$ , and balls in  $\mathbf{R}^{n-1}$  by  $B'_\rho$ .

**Lemma 8.1** *Let  $u_k \in F(\sigma_k, \sigma_k; \tau_k)$  in  $B_{\rho_k}$  with  $\sigma_k \rightarrow 0, \tau_k \sigma_k^{-2} \rightarrow 0$ . For  $y \in B'_1$ , set*

$$\begin{aligned} f_k^+(y) &= \sup\{h : (\rho_k y, \sigma_k \rho_k h) \in \partial\{u_k > 0\}\}, \\ f_k^-(y) &= \inf\{h : (\rho_k y, \sigma_k \rho_k h) \in \partial\{u_k > 0\}\}. \end{aligned}$$

Then, for a subsequence,

$$f(y) := \limsup_{\substack{z \rightarrow y \\ k \rightarrow \infty}} f_k^+(z) = \liminf_{\substack{z \rightarrow y \\ k \rightarrow \infty}} f_k^-(z) \quad \text{for all } y \in B'_1.$$

Further,  $f_k^+ \rightarrow f, f_k^- \rightarrow f$  uniformly,  $f(0) = 0$ , and  $f$  is continuous.

This is the analogue of Lemma 5.3 in [3]. The proof is based on Theorem 6.3 and is identical to the one of Lemma 7.3 in [2].

**Lemma 8.2**  *$f$  is subharmonic.*

The proof is identical to the one of Lemma 5.4 in [3].

**Lemma 8.3** *There exists a positive constant  $C$  such that, for any  $y \in B'_{r/2}$ ,*

$$\int_0^1 \frac{1}{r^2} \left( \int_{\partial B'_r(y)} f - f(y) \right) \leq C.$$

*Proof.* The proof below is a modification of that of Lemmas 5.6–5.7 in [3]. It should be pointed out that Theorem 6.4 is used in an essential way.

Without loss of generality, we assume  $\rho_k = 1$ . Also, it suffices to prove the lemma for  $y = 0$ , since  $u_k \in F(8\sigma_k, 8\sigma_k; \tau_k)$  in  $B_{1/4}(y, \sigma_k f_k^+(y))$ .

Set

$$w_k(y, h) = \frac{u_k(y, h) + h}{\sigma_k}.$$

Since the free boundary of  $u_k$  lies in the strip  $|x_n| \leq \sigma_k$ ,  $|\nabla u_k| \leq 1 + \tau_k$ ,  $\tau_k \leq \sigma_k$ , and we have  $w_k \leq C$  in  $B_1^- = B_1 \cap \{h < 0\}$ . The flatness assumption also implies that  $w_k \geq -C$  in  $B_1^-$ , and thus

$$|w_k| \leq C \quad \text{in } B_1^-.$$

*Claim 8.4* For a subsequence,

$$\lim_{k \rightarrow \infty} w_k =: w \quad \text{exists everywhere in } B_1^-.$$

The convergence is uniform in compact subsets of  $B_1^-$ , and  $w$  satisfies

$$(8.1) \quad a_{ij}(e_n)D_{ij}w = \sum_{i=1}^{n-1} D_{ii}w + (p-1)D_{nn}w = 0 \quad \text{in } B_1^-,$$

$$(8.2) \quad w(0, h) \leq 0,$$

$$(8.3) \quad w(y, 0) = f(y)$$

in the sense that  $\lim_{h \rightarrow 0^-} w(y, h) = f(y)$ ,

$$(8.4) \quad |w| \leq C.$$

Once we prove this claim, the lemma will follow by Lemma 5.5 in [3] after applying an affine transformation.  $\square$

*Proof of Claim 8.4.* By Theorem 6.4 we know that

$$(8.5) \quad |\nabla u_k| \geq 1/2 \quad \text{in } B_1 \cap \{h \leq -C_0\sigma_k\}$$

for  $\sigma_k$  sufficiently small. Then  $u_k$  satisfies

$$a_{ij}(\nabla u_k)D_{ij}u_k = 0, \quad a_{ij}(\nabla u_k) = \delta_{ij} + (p-2) \frac{D_i u_k D_j u_k}{|\nabla u_k|^2}$$

for  $h \leq -C_0\sigma_k$ . Therefore, we have

$$a_{ij}(\nabla u_k)D_{ij}w_k = 0 \quad \text{in } B_1 \cap \{h \leq -C_0\sigma_k\}.$$

From the flatness assumption it is clear that, for a subsequence,

$$u_k(y, h) \rightarrow -h$$

in  $C^2$  in compact subsets of  $B_1^-$ . Also, we may assume

$$w_k \rightarrow w$$

in  $C^2$  in compact subsets of  $B_1^-$  and that  $w$  satisfies (8.1). Clearly, (8.4) is also valid. Also, since

$$(8.6) \quad -D_n w_k = -\frac{1}{\sigma_k}(D_n u_k + 1) \leq \frac{|\nabla u_k| - 1}{\sigma_k} \leq \frac{\tau_k}{\sigma_k}$$

and  $w_k(0, 0) = 0$ , we obtain for  $h \leq 0$

$$w_k(0, h) \leq |h| \frac{\tau_k}{\sigma_k} \rightarrow 0.$$

Thus,  $w(0, h) \leq 0$  and (8.2) follows.

It remains to prove (8.3). First, we show that for any small  $\delta > 0$  and a large constant  $K$

$$(8.7) \quad w_k(y, h\sigma_k) \rightarrow f(y) \quad \text{uniformly for } y \in B'_{1-\delta}, \quad -K \leq h \leq -1.$$

By Lemma 8.1, it suffices to prove that

$$(8.8) \quad w_k(y, h\sigma_k) - f_k^+(y) \rightarrow 0.$$

From (8.6) we obtain

$$\begin{aligned} w_k(y, h\sigma_k) - f_k^+(y) &\leq w_k(y, \sigma_k f_k^+(y)) - f_k^+(y) + (f_k^+(y) - h) \frac{\tau_k}{\sigma_k} \\ &= (f_k^+(y) - h) \frac{\tau_k}{\sigma_k} \leq (1 + K) \frac{\tau_k}{\sigma_k} \rightarrow 0. \end{aligned}$$

To show (8.8) from below, take a sequence  $y_k \in B'_{1-\delta}$ ,  $-K \leq h_k \leq -C_0$  and consider  $u_k$  in  $B_{R\sigma_k}(x_k)$ , where  $x_k$  is the free boundary point

$$x_k = (y_k, \sigma_k f_k^+(y_k))$$

and  $R$  is a large constant. We know that

$$u_k \in F(\tilde{\delta}_k, 1; \tau_k) \quad \text{in } B_{R\sigma_k}(x_k),$$

if

$$\tilde{\delta}_k = \frac{1}{R} \sup_{B'_{R\sigma_k}(y_k)} (f_k^+ - f_k^+(y_k)).$$

Notice that  $\tilde{\delta}_k \rightarrow 0$  by Lemma 8.1. From Theorem 6.3 we have

$$u_k \in F(2\delta_k, C\delta_k; \tau_k) \quad \text{in } B_{(R/2)\sigma_k}(x_k)$$

for

$$\delta_k = \max(\tilde{\delta}_k, \tau_k).$$

Hence, for  $-R/2 \leq h \leq -C(R/2)\delta_k$ ,

$$u_k(x_k + h\sigma_k e_n) \geq -(h\sigma_k + C\delta_k(R/2)\sigma_k).$$

In other words,

$$w_k(x_k + h\sigma_k e_n) - f_k^+(y_k) = \frac{u_k(x_k + h\sigma_k e_n) + h\sigma_k}{\sigma_k} \geq -C(R/2)\delta_k \rightarrow 0.$$

This proves (8.8) and consequently (8.7). To complete the proof of (8.3), we use the following barrier argument. Let  $\Omega_\delta$  be a domain with  $C^\infty$  regular boundary that approximates  $B_{1-\delta}^-$  in such a way that

$$B_{1-2\delta}^- \subset \Omega_\delta \subset B_{1-\delta}^-.$$

For small  $\varepsilon > 0$ , let also  $g_\varepsilon$  be a  $C^\infty$  function on  $\partial\Omega_\delta$ , which satisfies

$$(8.9) \quad \begin{aligned} f - 2\varepsilon &\leq g_\varepsilon \leq f - \varepsilon && \text{on } \partial\Omega_\delta \cap \partial B_{1-3\delta}^- \cap \{h = 0\} \\ g_\varepsilon &\leq f - \varepsilon && \text{on } \partial\Omega_\delta \cap \{h = 0\} \\ g_\varepsilon &\leq w - \varepsilon && \text{on } \partial\Omega_\delta \cap \{h < 0\}. \end{aligned}$$

Let now  $\phi_\varepsilon$  solve the Dirichlet problem

$$\begin{aligned} a_{ij}(e_n)D_{ij}\phi_\varepsilon &= 1 \quad \text{in } \Omega_\delta, \\ \phi_\varepsilon &= g_\varepsilon \quad \text{on } \partial\Omega_\delta. \end{aligned}$$

Then for a fixed large  $K = K(\varepsilon, \delta)$  to be specified below, we obtain from (8.7) and (8.9) that

$$(8.10) \quad w_k > \phi_\varepsilon \quad \text{on } \partial(\Omega_\delta \cap \{h < -K\sigma_k\})$$

for sufficiently large  $k \geq k(\varepsilon, \delta, K)$ . We may assume  $K > 2C_0$ , where  $C_0$  is as in (8.5). The function  $w_k$  is bounded and satisfies a  $\lambda(p)$ -uniformly elliptic equation in  $B_1^- \cap \{h \leq -C_0\sigma_k\}$ . By the interior gradient estimates we deduce that

$$|\nabla w_k| \leq \frac{C}{(K - C_0)\sigma_k} \quad \text{in } \Omega_\delta \cap \{h \leq -K\sigma_k\},$$

where  $C$  is independent of  $k, K$ . In particular,

$$|\nabla u_k - (-e_n)| \leq \frac{C}{K} \quad \text{in } \Omega_\delta \cap \{h \leq -K\sigma_k\}.$$

Hence,

$$\begin{aligned} a_{ij}(\nabla u_k)D_{ij}\phi_\varepsilon &= (a_{ij}(\nabla u_k) - a_{ij}(-e_n))D_{ij}\phi_\varepsilon + 1 \\ &\geq 1 - \frac{C}{K}\|\phi_\varepsilon\|_{C^{1,1}(\Omega_\delta)} > 0 \quad \text{in } \Omega_\delta \cap \{h < -K\sigma_k\}, \end{aligned}$$

for sufficiently large  $K = K(\varepsilon, \delta)$ . Thus,

$$a_{ij}(\nabla u_k)D_{ij}\phi_\varepsilon > a_{ij}(\nabla u_k)D_{ij}w_k \quad \text{in } \Omega_\delta \cap \{h < -K\sigma_k\}$$

and combining with (8.10) we deduce that  $\phi_\varepsilon \leq w_k$  in  $\Omega_\delta \cap \{h < -K\sigma_k\}$ , for sufficiently large  $k$ . Letting  $k \rightarrow \infty$ , we obtain that  $w(y, h) \geq \phi_\varepsilon(y, h)$  in  $\Omega_\delta$  and consequently,

$$\liminf_{h \rightarrow 0^-} w(y, h) \geq f(y) - 2\varepsilon,$$

for  $y \in B'_{1-3\delta}$ . Similarly, by constructing a barrier from above, we obtain

$$\limsup_{h \rightarrow 0^-} w(y, h) \leq f(y) + 2\varepsilon.$$

for  $y \in B'_{1-3\delta}$ . Since  $\varepsilon$  and  $\delta$  were arbitrary, the proof of (8.3) is complete. □

### 9 Regularity of the free boundary

Everything is now ready to prove the  $C^{1,\alpha}$  regularity of the free boundary near flat points.

**Theorem 9.1** *Suppose  $u$  is a local minimizer and  $D \subset\subset \Omega$ . Then there exist positive constants  $\alpha, \beta, \sigma_0, \tau_0, C$  such that if*

$$u \in F(\sigma, 1; \infty) \quad \text{in } B_\rho(x_0) \subset D \text{ in direction } \nu$$

with  $\sigma \leq \sigma_0, \rho \leq \tau_0 \sigma^{2/\beta}$ , then

$$B_{\rho/4}(x_0) \cap \partial\{u > 0\} \quad \text{is a } C^{1,\alpha} \text{ surface.}$$

More precisely,  $B_{\rho/4}(x_0) \cap \partial\{u > 0\}$  is a graph in direction  $\nu$  of a  $C^{1,\alpha}$  function, and for any  $x_1, x_2$  on this surface,

$$|\nu(x_1) - \nu(x_2)| \leq C\sigma \left| \frac{x_1 - x_2}{\rho} \right|^\alpha.$$

*Proof.* The proof follows the same scheme as the proof of Theorem 8.1 in [2]. Assume for a moment that  $B_\rho(x_0) = B_1$  and  $\nu = e_n$ . Then Lemma 8.3 implies  $C(n, p)$ -Lipschitz regularity and then “better than” Lipschitz regularity of  $f$ , precisely as in [2], Lemmas 7.7 and 7.8. Namely, we have that there exists a large constant  $C = C(n, p) < \infty$  and for any  $\theta > 0$  a small constant  $c_\theta = c_\theta(n, p) > 0$  such that we can find a ball  $B'_r$  and a vector  $l \in \mathbf{R}^{n-1}$  with

$$c_\theta \leq r \leq \theta, |l| \leq C \quad \text{and} \quad f(y) \leq l \cdot y + (\theta/2)r, \text{ for } y \in B'_r.$$

This, in conjunction with Theorem 6.3 and the proof of Theorem 7.1, implies the flatness improvement in a smaller ball: if

$$u \in F(\sigma, 1; \tau) \text{ in } B_\rho(x_0) \text{ in direction } \nu$$

with  $\sigma < \sigma_0$  and  $\tau \leq \sigma_0 \sigma^2$  for sufficiently small  $\sigma_0$ , then

$$u \in F(\theta\sigma, \theta\sigma; \theta^2\tau) \text{ in } B_{\bar{\rho}}(x_0) \text{ in direction } \bar{\nu}$$

for some  $\bar{\rho}, \bar{\nu}$  with  $c_\theta \rho \leq \bar{\rho} \leq \rho/4$  and  $|\bar{\nu} - \nu| < C\theta$ , see Lemmas 7.9 and 7.10 in [2]. Finally, using Theorem 7.1 and iteratively applying the flatness improvement, we conclude the proof as in Theorem 8.1 in [2]. □

**Corollary 9.2** *Let  $u$  be a local minimizer. Then  $\partial_{\text{red}}\{u > 0\}$  is an analytic surface locally in  $\Omega$  and the remainder of  $\partial\{u > 0\}$  has  $\mathcal{H}^{n-1}$  measure 0.*

*Proof.* The  $C^{1,\alpha}$  regularity of  $\partial_{\text{red}}\{u > 0\}$  follows from Theorem 9.1 and Remark 6.2. Once we have  $C^{1,\alpha}$  regularity, it follows that  $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$  is uniformly elliptic locally near  $\partial_{\text{red}}\{u > 0\}$  and we obtain the analyticity by [12].  $\square$

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