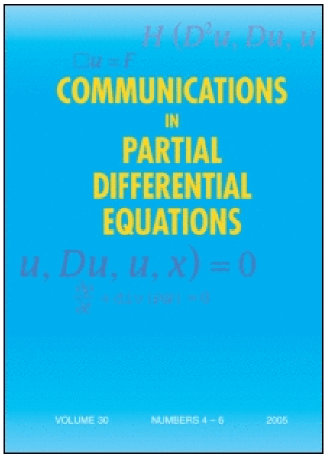


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# AN EMBEDDING THEOREM AND THE HARNACK INEQUALITY FOR NONLINEAR SUBELLIPTIC EQUATIONS

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## 1. Introduction

Let $X_1, \ldots, X_m$ be $C^\infty$ vector fields in $\mathbf{R}^n$ satisfying Hörmander's condition for hypoellipticity **[H]**:

$$\operatorname{rank} \operatorname{Lie}[X_1, \ldots, X_m] = n,$$

at every point $x \in \mathbf{R}^n$. Denote by $X_j^*$ the formal adjoint of $X_j$. The linear operator $\mathcal{L} = \sum_{j=1}^{m} X_j^* X_j$ is the subelliptic Laplacian associated to the vector fields $X_1, \ldots, X_m$. Since the appearance of Hörmander's fundamental work **[H]**, the study of properties of solutions of $\mathcal{L}u = 0$ has received increasing attention, see **[B]**, **[RS]**, **[FP]**, **[S]**, **[J]**, **[JS]**, **[KS1]**, **[KS2]**, **[CGL]**. A large

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part of this development has been related to some fundamental properties of a metric, which is naturally associated to the vector fields $X_1, \ldots, X_m$, see **[NSW]**. Concerning linear equations, there exists nowadays a rather satisfactory picture. On the other hand, the consideration of problems related to the geometry of $CR$ manifolds, see **[JL1]**, **[JL2]**, **[GL]**, suggests that a corresponding nonlinear theory should be developed. An important motivating example is given by the energy functional in the subelliptic Sobolev embedding. Given an open set $U \subset \mathbf{R}^n$, and a function $u \in C^1(U)$, denote by $D_{\mathcal{L}}u = (X_1 u, \ldots, X_m u)$ the subelliptic gradient of $u$. For $1 < p < \infty$ we consider the functional

$$J_p(u) = \int_U |D_{\mathcal{L}}u|^p dx = \int_U \left[\sum_{j=1}^m (X_j u)^2\right]^{p/2} dx\,,$$

and define $\overset{\circ}{S}{}^{1,p}(U)$ to be the completion of $C_0^1(U)$ in the norm generated by $J_p$. The Euler equation of $J_p$ is

$$\sum_{j=1}^m X_j^*(|D_{\mathcal{L}}u|^{p-2} X_j u) = 0. \tag{1.1}$$

We call the operator in (1.1) the *subelliptic p–Laplacian.* Critical points of $J_p$ are (weak) solutions of (1.1), and vice–versa. In the linear case ($p = 2$), the Harnack inequality for nonnegative solutions of (1.1) follows either from the work of Jerison [J] and Lemma 3.2 in this paper, or from the work of Kusuoka and Stroock **[KS1]**.

In this paper, we propose to study a general class of nonlinear subelliptic equations, whose prototype is constituted by (1.1) above. Our objectives are: a) To establish an optimal embedding result of Sobolev type for the subelliptic spaces $\overset{\circ}{S}{}^{1,p}$; b) To prove a Harnack type inequality for nonnegative solutions. From b) the Hölder continuity of solutions with respect to the $(X_1, \ldots, X_m)$–control distance will follow. To generalize equation (1.1) we consider measurable functions $A : \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^m \to \mathbf{R}^m$, $f : \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^m \to \mathbf{R}$, and suppose that $A = (A_1, \ldots, A_m)$. We assume that $A_i$, $i = 1, \ldots, m$, and $f$, satisfy the following structural conditions: *There exist* $p \in (1, \infty)$, $c_1 \geq 0$, *and measurable functions* $f_1, f_2, f_3, g_2, g_3, h_3$ *on* $\mathbf{R}^n$, *such that for a.e.* $x \in \mathbf{R}^n$, $u \in \mathbf{R}$ *and* $\zeta \in \mathbf{R}^m$

$$\text{(S)} \qquad \begin{cases} |A(x,u,\zeta)| \leq c_1|\zeta|^{p-1} + g_2(x)|u|^{p-1} + g_3(x) \\ |f(x,u,\zeta)| \leq f_1(x)|\zeta|^{p-1} + f_2(x)|u|^{p-1} + f_3(x) \\ A(x,u,\zeta) \cdot \zeta \geq |\zeta|^p - f_2(x)|u|^p - h_3(x). \end{cases}$$

Given $C^\infty$ vector fields $X_1, \ldots, X_m$ in $\mathbf{R}^n$, satisfying Hörmander's condition for hypoellipticity, we consider the equation

$$\sum_{j=1}^{m} X_j^* A_j(x, u, X_1 u, \ldots, X_m u) = f(x, u, X_1 u, \ldots, X_m u). \tag{1.2}$$

(1.2) must be interpreted in a suitable weak sense. Given an open set $U \subset \mathbf{R}^n$, denote by $S^{1,p}(U)$ the completion of $C^1(U)$ in the norm

$$\|u\|_{S^{1,p}(U)} = \left[\int_U (|D_{\mathcal{L}} u|^p + |u|^p)\, dx\right]^{\frac{1}{p}}.$$

A function $u \in L^p_{\text{loc}}(U)$ is said to belong to $S^{1,p}_{\text{loc}}(U)$ if $\varphi u \in S^{1,p}(U)$ for every $\varphi \in C_0^1(U)$. Let $u \in S^{1,p}_{\text{loc}}(U)$. We say that $u$ is a (weak) solution to (1.2) if for every $\varphi \in \overset{\circ}{S}{}^{1,p}(U)$

$$\begin{aligned} \sum_{j=1}^{m} \int_U A_j(x, u, X_1 u, \ldots, X_m u) X_j \varphi\, dx = \\ \int_U f(x, u, X_1 u, \ldots, X_m u) \varphi\, dx. \end{aligned} \tag{1.3}$$

It is worth noting that the choice

$$A_j(x,u,\zeta) = A_j(\zeta) = |\zeta|^{p-2}\zeta_j, \quad j = 1, \ldots, m, \quad f \equiv 0,$$

makes (1.1) just a special case of (1.2). Given the structural conditions $(S)$, in order for definition (1.3) to be well posed one needs to specify the relevant integrability assumptions on the functions $f_i$, $g_i$, $h_i$. A distinctive feature of our results is the optimal choice of the Lebesgue spaces, to which the functions $f_i$, $g_i$, $h_i$ in $(S)$ are required to belong. This follows from the sharp exponents in the Sobolev type embedding

$$\overset{\circ}{S}{}^{1,p}(U) \hookrightarrow L^q(U). \tag{1.4}$$

We recall that a piecewise $C^1$ curve $\gamma : [0,T] \to \mathbf{R}^n$ is said to be subunitary if for every $\xi \in \mathbf{R}^n$ and $t \in (0,T)$

$$(\gamma'(t)\cdot\xi)^2 \le \sum_{j=1}^{m}(X_j(\gamma(t))\cdot\xi)^2.$$

Given two points $x, y \in \mathbf{R}^n$, the $(X_1,\ldots,X_m)$–control distance from $x$ to $y$ is defined as follows:

$$\begin{aligned} d(x,y) = \inf\{T > 0 | &\text{There exists a subunitary } \gamma : [0,T] \to \mathbf{R}^n, \\ &\text{with } \gamma(0) = x, \gamma(T) = y\}. \end{aligned} \tag{1.5}$$

For $x \in \mathbf{R}^n$ and $R > 0$, let $B(x,R) = \{y \in \mathbf{R}^n | d(x,y) < R\}$. An important consequence of the work in [**FP**], [**NSW**] is the existence, for any bounded set $U \subset \mathbf{R}^n$, of positive constants, $C$, $R_0$ and $Q$, such that

$$|B(x,tR)| \ge C t^Q |B(x,R)|, \tag{1.6}$$

for every $x \in U$, $R \le R_0$ and $0 < t < 1$. We call the number $Q$ in (1.6) the *homogeneous dimension* relative to $U$. We mention that (1.6) plays a pervasive role in the results of this paper. Also, the following estimates of the fundamental solution of $\mathcal{L} = \sum_{j=1}^{m} X_j^* X_j$, found by A. Sanchez Calle [S], are important to us. Let $\Gamma(x,y)$ denote the (positive) fundamental solution of $\mathcal{L}$. Then, there exists $C > 0$ such that:

$$C\frac{d(x,y)^2}{|B(x,d(x,y))|} \leq \Gamma(x,y) \leq C^{-1}\frac{d(x,y)^2}{|B(x,d(x,y))|}, \tag{1.7}$$

$$|D_{\mathcal{L}}\Gamma(x,y)| \leq C^{-1}\frac{d(x,y)}{|B(x,d(x,y))|}, \quad x \neq y, \tag{1.8}$$

see also [**NSW**].

An outline of the paper is as follows. In section 2 we prove the following embedding theorem of Sobolev type. Suppose $B_R = B(x,R)$ is a ball in the metric (1.5) of sufficiently small radius and consider the subelliptic Sobolev space $\overset{\circ}{S}{}^{1,p}(B_R)$. Let $x \in U \subset\subset \mathbf{R}^n$ and denote by $Q$ the homogeneous dimension relative to $U$. Theorem 2.3 asserts that, if $1 < p < Q$, then $\overset{\circ}{S}{}^{1,p}(B_R) \hookrightarrow L^q(B_R)$ for $1 \leq q \leq \dfrac{Qp}{Q-p}$. Furthermore, we have for $u \in \overset{\circ}{S}{}^{1,p}(B_R)$

$$\left(\frac{1}{|B_R|}\int_{B_R}|u|^q dx\right)^{\frac{1}{q}} \leq CR\left(\frac{1}{|B_R|}\int_{B_R}|D_{\mathcal{L}}u|^p dx\right)^{\frac{1}{p}} \tag{1.9}$$

for some $C = C(U, X_1, \ldots, X_m) > 0$. (1.9) and a standard partition of the unity argument imply (1.4) for any $U \subset\subset \mathbf{R}^n$.

The ideas involved in the proof of the above result have a classical flavor. Our approach relies on some subelliptic representation formulas. Of course, (1.6), (1.7), and (1.8) play an important role throughout. We should mention that in the proof of Theorem 2.3 interpolation is not used. In a classical fashion, we reduce the proof to the study of the mapping properties of a suitable fractional integration operator, see (2.6) and Theorem 2.7. For the latter, we adapt to our context a nice idea, due to Hedberg [**He**], for the classical Hardy–Littlewood–Sobolev theorem. We mention that (1.4) extends previous results of Folland and Stein [**FS**], [**F**], relative to Sobolev spaces on nilpotent groups. If $r$ is the number of commutators necessary to span $\mathbf{R}^n$, then a result of Rothschild and Stein [**RS**] gives $\overset{\circ}{S}{}^{1,p}(U) \hookrightarrow \overset{\circ}{W}{}^{\frac{1}{r},p}(U)$. Here, $\overset{\circ}{W}{}^{\frac{1}{r},p}(U)$ denotes the usual Sobolev space of fractional order $\frac{1}{r}$. By Sobolev's embedding theorem, $\overset{\circ}{W}{}^{\frac{1}{r},p}(U) \hookrightarrow L^q(U)$, with $\frac{1}{q} = \frac{1}{p} - \frac{1}{rn}$. In general, however, the exponent $q$ thus determined is much smaller than the number $Qp/(Q-p)$ in Theorem 2.3. When $p = q$, (1.9) is contained in Jerison's Poincarè inequality [**J**]. We mention that a different proof of (1.9), based on an adaptation of Jerison's work on the Poincarè inequality, has been found by Lu [**L**].

Section 3 of the paper is devoted to the proof of the Harnack inequality for nonnegative solutions of (1.2), see Theorem 3.1. The relevant integrability assumptions on the functions $f_i$, $g_i$, $h_i$ in $(S)$ are stated in (i)–(iii) at the beginning of the section. As mentioned above, the choice of the Lebesgue classes is optimal for Theorem 3.1 to hold. An example is provided by the model case of the Heisenberg group $\mathbf{H}^n$. Consider the function $u(z,t) = \log|\log[(|z|^4 + t^2)^{\frac{1}{4}}]|$, with $(z,t) \in \mathbf{H}^n$. Then $u$ belongs to the Folland–Stein space $\overset{\circ}{S}{}^{1,2}(U)$, with $U = \{(z,t) \in \mathbf{H}^n \mid |z|^4 + t^2 < e^{-4}\}$. Furthermore, it is a nonnegative solution of $\Delta_{\mathbf{H}^n} u = Vu$ in $U$, with $V \in L^{\frac{Q}{2}}(U)$. This example shows that the assumption $f_2 \in L^s_{\text{loc}}$, with $s > Q/p$, in (ii) of section 3, is optimal for the local boundedness of solutions to (1.2), see Theorem 3.4. Similar examples prove the optimality of the other requirements in (i)–(iii). As a consequence of Theorem 3.1, we prove that solutions of (1.2) are Hölder continuous with respect to the $(X_1, \ldots, X_m)$–control distance (1.5), see Theorem 3.35. Our results constitute a generalization of a classical work of Serrin [**Se**] concerned with quasi–linear degenerate elliptic equations. Similarly to the results in [**Se**], our proof relies on a suitable adaptation of Moser's iteration technique [**M**]. This is made possible by the embedding results in section 2, the existence of suitable cut–off functions, see Lemma 3.2, and the Poincarè inequality in [**J**].

We finally mention that the results in sections 2 and 3 have suitable extensions to equations for which the dependence in the $x$ variable is allowed to degenerate with respect to the metric $d(x, y)$. The precise statements of the theorems and their proofs will appear elsewhere.

## 2. Subelliptic Embedding Theorems of Sobolev Type

The main theme of this section will be the use of some representation formulas, which generalize those established in [**CGL**]. By means of these formulas we are able to represent, in a classical fashion, an arbitrary smooth function in terms of a fractional integral involving its subelliptic gradient, see Proposition 2.4. We are thus led to study fractional integration on spaces of homogeneous type. The main result in this context is Theorem 2.7. By the latter, and by Proposition 2.4, we obtain the Sobolev type embedding Theorem 2.3.

We begin with a consequence of the work in [**NSW**], which plays a pervasive role in the results of this section.

PROPOSITION 2.1. *Given a bounded set $U \subset \mathbf{R}^n$ there exist $Q \geq n$, $R_0 > 0$ and $C > 0$, such that for every $x \in U$, $R \leq R_0$ and $0 < t < 1$*

$$|B(x, tR)| \geq Ct^Q |B(x, R)|.$$

*Proof*: By Theorem 1 in [**NSW**] we have for every $x \in U$, and $R \leq R_0$,

$$C_1 \Lambda(x, R) \leq |B(x, R)| \leq C_2 \Lambda(x, R), \tag{2.2}$$

where $\Lambda(x, R)$ is a polynomial function in $R$ with positive coefficients depending on $x$. Recalling that the degree of the polynomial function $\Lambda(x, R)$ is between $n$ and the number $Q(x) = \lim\limits_{r \to 0} \dfrac{\log \Lambda(x, r)}{\log r}$, we obtain for $0 < t < 1$

$$|B(x, tR)| \geq C_1 \Lambda(x, tR) \geq C_1 t^{Q(x)} \Lambda(x, R) \geq C_3 t^{Q(x)} |B(x, R)|.$$

Define now $Q = Q(U) = \sup\limits_{x \in U} Q(x)$. From the latter estimate, Proposition 2.1 follows. ∎

*Remark*: We explicitly observe that (2.2) implies that for any $U \subset\subset \mathbb{R}^n$ there exist $C > 0$ and $R_0 > 0$ such that for any $x \in U$ and $0 < R \leq R_0$

$$|B(x,R)| \geq C|B(x,2R)|.$$

We have the following

THEOREM 2.3 (SOBOLEV EMBEDDING THEOREM). *Let $U \subset \mathbb{R}^n$ be a bounded open set and denote by $Q$ the homogeneous dimension relative to $U$. Let $1 < p < Q$. Then there exist $C > 0$ and $R_0 > 0$ such that for any $x \in U$, $B_R = B(x,R)$, with $R \leq R_0$, we have*

$$\left(\frac{1}{|B_R|}\int_{B_R} |u|^{\kappa p} dx\right)^{\frac{1}{\kappa p}} \leq CR\left(\frac{1}{|B_R|}\int_{B_R} |D_{\mathcal{L}}u|^p dx\right)^{\frac{1}{p}}$$

*for any $u \in \overset{\circ}{S}{}^{1,p}(B_R)$. Here, $1 \leq \kappa \leq \dfrac{Q}{Q-p}$.*

The proof of Theorem 2.3 is based on the following representation result.

PROPOSITION 2.4. *Let $D \subset \mathbb{R}^n$ by a $C^1$ domain and let $u \in C_0^1(D)$. For every $x \in D$ we have*

$$u(x) = \int_D D_{\mathcal{L}}\Gamma(x,\xi) \cdot D_{\mathcal{L}}u(\xi)d\xi.$$

PROOF. It follows from that of Proposition 2.1 in [**CGL**], with minor modifications. ∎

Let now $B_R = B(x,R)$ be as in Theorem 2.3. For $u \in C_0^1(B_R)$ and $x \in B_R$ we obtain from Proposition 2.4

$$|u(x)| \leq \int_{B_R} |D_{\mathcal{L}}\Gamma(x,\xi)|\,|D_{\mathcal{L}}u(\xi)|d\xi.$$

Using (1.8), we have for some constant $C > 0$ and any $x \in B_R$

$$|u(x)| \leq C \int_{B_R} |D_{\mathcal{L}}u(\xi)| \frac{d(x,\xi)}{|B(x,d(x,\xi))|} d\xi. \tag{2.5}$$

We now introduce for $0 < \alpha \leq Q$ the operator of fractional integration

$$I_\alpha(f)(x) = \int_{B_R} |f(\xi)| \frac{d(x,\xi)^\alpha}{|B(x,d(x,\xi))|} d\xi. \tag{2.6}$$

Using (2.6) we can rewrite (2.5) as follows

$$|u(x)| \leq C I_1(|D_{\mathcal{L}}u|)(x) \text{ for } x \in B_R.$$

The proof of Theorem 2.3 will be a direct consequence of this estimate and of the following

THEOREM 2.7. *Let $U \subset \mathbb{R}^n$ and $Q$ be as in Theorem 2.3. Assume $1 \leq p \leq \infty$. Then, $I_\alpha$ maps $L^p(B_R)$ continuously into $L^q(B_R)$, with $0 \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{Q}$. Moreover, there exist $C > 0$ and $R_0 > 0$ such that for any $x_0 \in U$, $B_R = B(x_0, R)$, with $R \leq R_0$, we have*

$$\left( \frac{1}{|B_R|} \int_{B_R} |I_\alpha(f)(x)|^q dx \right)^{\frac{1}{q}} \leq C R^\alpha \left( \frac{1}{|B_R|} \int_{B_R} |f(x)|^p dx \right)^{\frac{1}{p}},$$

*for every $f \in L^p(B_R)$. When $p = 1$ one must have*

$$0 \leq \frac{1}{p} - \frac{1}{q} < \frac{\alpha}{Q}.$$

*Remark*: We note explicitly that the case $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$ forces the restriction $1 < p < \frac{Q}{\alpha}$. When $p = \frac{Q}{\alpha}$, $I_\alpha$ does not map $L^p(B_R)$ into $L^\infty(B_R)$. There is, however, an end–point result analogous to Trudinger's Euclidean inequality, see [**D2**].

PROOF OF THEOREM 2.7. We start with the case $0 \leq \frac{1}{p} - \frac{1}{q} < \frac{\alpha}{Q}$ and suitably adapt the approach in [GT] based on Young's convolution theorem. Let $r \geq 1$ be defined by $1 - \frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. We claim that for a fixed $x \in B_R$, the function $y \mapsto \frac{d(x,y)^\alpha}{|B(x,d(x,y))|} = h(x,y)$ is in $L^r(B_R)$ and, in fact,

$$\sup_{x \in B_R} \|h(x,\cdot)\|^r_{L^r(B_R)} \leq C \frac{R^{\alpha r}}{|B_R|^{r-1}}.$$

To see this we proceed as follows. Let $x \in B_R$. Then, $B(x,2R) \supset B_R$, so that

$$\begin{aligned}
\int_{B_R} h(x,y)^r dy &\leq \int_{B(x,2R)} \frac{d(x,y)^{\alpha r}}{|B(x,d(x,y))|^r} dy \\
&= \sum_{k=0}^{\infty} \int_{2^{-k}R < d(x,y) < 2^{-k+1}R} \frac{d(x,y)^{\alpha r}}{|B(x,d(x,y))|^r} dy \\
&\leq C \sum_{k=0}^{\infty} \frac{(2^{-k}R)^{\alpha r}}{|B(x,2^{-k}R)|^{r-1}} \qquad \text{(by Proposition 2.1)} \\
&\leq C \sum_{k=0}^{\infty} \frac{(2^{-k}R)^{\alpha r}}{2^{-kQ(r-1)}|B(x,R)|^{r-1}} \\
&= C \frac{R^{\alpha r}}{|B(x,R)|^{r-1}} ,
\end{aligned}$$

the series being convergent, since $\alpha r > Q(r-1)$ is equivalent to $\frac{\alpha}{Q} > 1 - \frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. Observing that, by (2.2), $|B(x,R)|^{1-r} \leq C|B(x,2R)|^{1-r} \leq C|B_R|^{1-r}$, we finally obtain the claim. We then write for $f \in L^p(B_R)$

$$|f(y)|h(x,y) = h(x,y)^{\frac{r}{q}} h(x,y)^{\frac{r}{p'}} |f(y)|^{\frac{p}{q}} |f(y)|^{p(1-\frac{1}{r})}.$$

Hölder's inequality gives for $x \in B_R$

$$|I_\alpha(f)(x)| \leq \left( \int_{B_R} |f(y)|^p h(x,y)^r dy \right)^{\frac{1}{q}} \left( \int_{B_R} h(x,y)^r dy \right)^{\frac{1}{p'}} .$$

$$\left(\int_{B_R} |f(y)|^p dy\right)^{\frac{1}{r'}}.$$

Integration of this inequality yields

$$\begin{aligned}\left(\int_{B_R} |I_\alpha(f)(x)|^q dx\right)^{\frac{1}{q}} &\leq \sup_{x\in B_R} \|h(x,\cdot)\|_{L^r(B_R)} \left(\int_{B_R} |f(y)|^p dy\right)^{\frac{1}{p}} \\ &\leq CR^\alpha |B_R|^{\frac{1}{r}-1} \left(\int_{B_R} |f(y)|^p dy\right)^{\frac{1}{p}},\end{aligned}$$

where in the latter inequality we have used the above claim. Recalling that $1 - \dfrac{1}{r} = \dfrac{1}{p} - \dfrac{1}{q}$, we conclude that Theorem 2.7 holds.

Next, we study the case $\dfrac{1}{p} - \dfrac{1}{q} = \dfrac{\alpha}{Q}$. Recall that now must be $1 < p < \dfrac{Q}{\alpha}$. We follow an idea in [**He**]. Set $f \equiv 0$ in $B_R^c$. For any $x \in B_R$ and $0 < \varepsilon < R$ we write

$$I_\alpha(f)(x) = I_\alpha^1(f)(x) + I_\alpha^2(f)(x),$$

where

$$\begin{aligned} I_\alpha^1(f)(x) &= \int_{B(x,\varepsilon)} |f(y)| \frac{d(x,y)^\alpha}{|B(x,d(x,y))|} dy \\ I_\alpha^2(f)(x) &= \int_{B(x,\varepsilon)^c\cap B_R} |f(y)| \frac{d(x,y)^\alpha}{|B(x,d(x,y))|} dy. \end{aligned}$$

We estimate first $I_\alpha^1(f)(x)$.

$$\begin{aligned} I_\alpha^1(f)(x) &= \sum_{k=0}^{\infty} \int_{2^{-(k+1)}\varepsilon < d(x,y) < 2^{-k}\varepsilon} |f(y)| \frac{d(x,y)^\alpha}{|B(x,d(x,y))|} dy \\ &\leq C\varepsilon^\alpha \sum_{k=0}^{\infty} \frac{2^{-k\alpha}}{|B(x,2^{-k}\varepsilon)|} \int_{B(x,2^{-k}\varepsilon)} |f(y)| dy, \end{aligned}$$

where in the latter inequality we have used (2.2). The above gives

$$I_\alpha^1(f)(x) \leq C_1 Mf(x)\varepsilon^\alpha, \tag{2.8}$$

where $Mf(x) = \sup_{x\in B} \frac{1}{|B|}\int_B |f(y)|dy$ denotes the Hardy–Littlewood maximal operator with respect to the metric balls $B$. Next, we estimate $I^2_\alpha(f)(x)$. We observe that for any $x \in B_R$ $B(x,2R) \supset B_R$. Therefore, by Hölder's inequality we obtain

$$
\begin{aligned}
I^2_\alpha(f)(x) &\le \int_{B(x,\varepsilon)^c\cap B(x,2R)} |f(y)|\frac{d(x,y)^\alpha}{|B(x,d(x,y))|}dy \\
&\le \left(\int_{B_R} |f(y)|^p dy\right)^{\frac{1}{p}} \cdot \\
&\quad \left(\int_{B(x,\varepsilon)^c\cap B(x,2R)} \frac{d(x,y)^{\alpha p'}}{|B(x,d(x,y))|^{p'}}dy\right)^{\frac{1}{p'}}.
\end{aligned} \tag{2.9}
$$

We now choose $k_0 \in \mathbf{N}$ such that $2^{k_0}\varepsilon > R \ge 2^{k_0-1}\varepsilon$. Then,

$$
\begin{aligned}
&\left(\int_{B(x,\varepsilon)^c\cap B(x,2R)} \frac{d(x,y)^{\alpha p'}}{|B(x,d(x,y))|^{p'}}dy\right)^{\frac{1}{p'}} = \\
&\left(\sum_{k=0}^{k_0}\int_{2^k\varepsilon<d(x,y)<2^{k+1}\varepsilon} \frac{d(x,y)^{\alpha p'}}{|B(x,d(x,y))|^{p'}}dy\right)^{\frac{1}{p'}} \le \\
&C\left(\sum_{k=0}^{k_0}\frac{(2^k\varepsilon)^{\alpha p'}}{|B(x,2^k\varepsilon)|^{p'-1}}\right)^{\frac{1}{p'}},
\end{aligned}
$$

where in the latter inequality we have used (2.2). By Proposition 2.1 we infer

$$|B(x,2^k\varepsilon)| \ge C2^{-(k_0-k)Q}|B(x,R)|.$$

Substitution in the above yields

$$
\begin{aligned}
&\left(\int_{B(x,\varepsilon)^c\cap B(x,2R)} \frac{d(x,y)^{\alpha p'}}{|B(x,d(x,y))|^{p'}}dy\right)^{\frac{1}{p'}} \le \\
&C\left(\sum_{k=0}^{k_0}\frac{(2^{-(k_0-k)}R)^{\alpha p'}}{(2^{-(k_0-k)Q}|B(x,R)|)^{p'-1}}\right)^{\frac{1}{p'}} \le
\end{aligned} \tag{2.10}
$$

$$\frac{CR^\alpha 2^{-k_0\alpha}}{|B(x,R)|^{1/p}2^{-k_0Q/p}}\left(\sum_{k=0}^{k_0}\frac{2^{k\alpha p'}}{2^{kQ(p'-1)}}\right)^{\frac{1}{p'}}.$$

Recalling that $2^{k_0} \le 2\frac{R}{\varepsilon}$, and noting that $Q(p'-1) > \alpha p'$ since $\alpha < \frac{Q}{p}$, we finally obtain from (2.9) and (2.10)

$$I_\alpha^2(f)(x) \le C\|f\|_{L^p(B_R)}R^{\frac{Q}{p}}|B(x,R)|^{-\frac{1}{p}}\varepsilon^{-\frac{Q}{p}+\alpha}. \tag{2.11}$$

Since by (2.2) we have for any $x \in B_R$,

$$|B(x,R)|^{-\frac{1}{p}} \le C|B(x,2R)|^{-\frac{1}{p}} \le C|B_R|^{-\frac{1}{p}},$$

by (2.8) and (2.11) we finally obtain

$$I_\alpha(f)(x) \le C_1Mf(x)\varepsilon^\alpha + C_2\|f\|_{L^p(B_R)}R^{\frac{Q}{p}}|B_R|^{-\frac{1}{p}}\varepsilon^{-\frac{Q}{p}+\alpha}.$$

Minimizing the right-hand side with respect to $\varepsilon$ yields

$$I_\alpha(f)(x) \le C_3R^\alpha|B_R|^{-\frac{\alpha}{Q}}\|f\|_{L^p(B_R)}^{\frac{p\alpha}{Q}}Mf(x)^{\frac{p}{q}},$$

where, we recall, $q = \frac{pQ}{Q-p\alpha}$. The conclusion of the proof of Theorem 2.7 will follow by the $L^p$-continuity of the Hardy–Littlewood maximal operator in a space of homogeneous type, see [**CW**] or [**C**]. ∎

*Remark*: With the constant $R$ in the right-hand side replaced by the larger constant $|B_R|^{1/Q}$, Theorem 2.3 was announced in [**D2**].

## 3. Harnack Inequality and the Regularity of Solutions

Throughout this section $U \subset \mathbf{R}^n$ will denote a given bounded open set, with relative homogeneous dimension $Q = Q(U) > 0$. Moreover, we assume

$1 < p \leq Q$. We state the relevant integrability requirements on the functions $f_i$, $g_i$, $h_i$ in the structural assumptions $(S)$ for the equation (1.2):

(i) $g_2, g_3 \in L^r_{\mathrm{loc}}(U)$, with $r = \dfrac{Q}{p-1}$ if $p < Q$, and $r > \dfrac{Q}{Q-1}$ if $p = Q$;

(ii) $f_2, f_3, h_3 \in L^s_{\mathrm{loc}}(U)$, with $s > \dfrac{Q}{p}$;

(iii) $f_1 \in L^t_{\mathrm{loc}}(U)$, with $t > Q$.

Assumptions (ii) and (iii) allow to write, for some $0 < \varepsilon < 1$, $s = \dfrac{Q}{p-\varepsilon}$, $t = \dfrac{Q}{1-\varepsilon}$. From now on, the letter $\varepsilon$ will only be used with this meaning.

The main result of this section is the following

THEOREM 3.1 (HARNACK INEQUALITY). *Let $u \in S^{1,p}_{\mathrm{loc}}(U)$ be a nonnegative solution to the equation* (1.2). *Then, there exist $C > 0$ and $R_0 > 0$ such that for any $B_R = B(x,R)$, with $B(x,4R) \subset U$, and $R \leq R_0$,*

$$\operatorname*{ess\,sup}_{B_R} u \leq C \left( \operatorname*{ess\,inf}_{B_R} u + K(R) \right).$$

*Here,*

$$K(R) = \left( |B_R|^{\frac{\varepsilon}{Q}} \|f_3\|_{L^s(B_R)} + \|g_3\|_{L^r(B_R)} \right)^{\frac{1}{p-1}} + \left( |B_R|^{\frac{\varepsilon}{Q}} \|h_3\|_{L^s(B_R)} \right)^{\frac{1}{p}},$$

*with $r, s$ as in* (i), (ii) *above.*

Essential to the proof of Theorem 3.1 are the subelliptic cut–off functions found in [**CGL**]. For the sake of completeness we reproduce the proof of their existence.

LEMMA 3.2. *There exists $R_0 > 0$ such that given a metric ball $B(x,t) \subset\subset U$, with $t \leq R_0$ and $0 < s < t$, there exists a function $\psi \in C^\infty_0(B(x,t))$ such that $0 \leq \psi \leq 1$, $\psi \equiv 1$ in $B(x,s)$ and $|D_{\mathcal{L}}\psi| \leq \dfrac{C}{t-s}$. Here, $C > 0$ is a constant independent of $s$ and $t$.*

*Proof*: Let $h \in C^\infty_0([0,at))$ be such that $0 \leq h \leq 1$, $h \equiv 1$ on $\left[0, \dfrac{s}{a}\right]$, and $|h'| \le \frac{C}{t-s}$. Here, $a > 0$ is as in Lemma 2.20. Recall the definition of the function $F(x, R)$ after (2.19). Set $\psi(y) = h\left(F\left(x, \frac{1}{\Gamma(x,y)}\right)\right)$. Using Lemma 2.20 one sees that $\psi \in C_0^\infty(B(x,t))$, and $\psi \equiv 1$ on $B(x,s)$. Denoting $F'(x,R) = \frac{dF}{dR}(x,R)$, we have

$$
\begin{aligned}
(3.3)\quad |D_{\mathcal{L}}\psi(y)| &\le |h'(F(x,\Gamma(x,y)^{-1}))|\,|F'(x,\Gamma(x,y)^{-1})|\frac{|D_{\mathcal{L}}\Gamma(x,y)|}{\Gamma(x,y)^2} \\
&\le \frac{C}{t-s}\frac{1}{|E'(x,F(x,\Gamma(x,y)^{-1}))|}\frac{|D_{\mathcal{L}}\Gamma(x,y)|}{\Gamma(x,y)^2} \\
&\le \frac{C}{t-s}\frac{F(x,\Gamma(x,y)^{-1})}{E(x,F(x,\Gamma(x,y)^{-1}))}\frac{|D_{\mathcal{L}}\Gamma(x,y)|}{\Gamma(x,y)^2}.
\end{aligned}
$$

In the latter inequality we have used the fact that $E(x, R)$ is a polynomial function. Next, we observe that (1.7) and (2.2) give

$$\frac{C_1}{\Gamma(x,y)} \le E(x, d(x,y)) \le \frac{C_2}{\Gamma(x,y)}.$$

Applying $F(x,\cdot)$ to this inequality we obtain

$$C_1' d(x,y) \le F\left(x, \frac{1}{\Gamma(x,y)}\right) \le C_2' d(x,y).$$

Substituting in (3.3) and using (1.8) we conclude

$$|D_{\mathcal{L}}\psi(y)| \le \frac{C}{t-s}\frac{F(x,\Gamma(x,y)^{-1})}{d(x,y)} \le \frac{C'}{t-s}.$$

This proves Lemma 3.2. ∎

Our next result concerns the local boundedness of weak solutions of (1.2). We begin by observing that the structural assumptions $(S)$ and (i)–(iii) above imply that $(x,u,\zeta) \mapsto |A(x,u,\zeta)|$ belongs to $L_{\mathrm{loc}}^{\frac{p}{p-1}}(U)$, whereas $(x,u,\zeta) \mapsto f(x,u,\zeta)$ is in $L_{\mathrm{loc}}^{\frac{\kappa p}{\kappa p-1}}(U)$, with $\kappa = \frac{Q}{Q-p}$, if $p < Q$, and is in $L_{\mathrm{loc}}^{1+\varepsilon}(U)$ for some $\varepsilon > 0$, when $p = Q$. In view of this observation we now modify the definition of a weak solution to allow test functions $\varphi \in \overset{\circ}{S}{}^{1,p}(U)$.

We have

THEOREM 3.4. *Suppose that $u \in S^{1,p}_{\mathrm{loc}}(U)$ is a weak solution to* (1.2) *in $U$. Then, there exist $C > 0$ and $R_0 > 0$ such that for any $B_R = B(x,R)$ for which $B(x,4R) \subset U$, and $R \leq R_0$, we have*

$$\operatorname*{ess\,sup}_{B_R} |u| \leq C\left[\left(\frac{1}{|B_{2R}|}\int_{B_{2R}} |u|^p dx\right)^{\frac{1}{p}} + K(R)\right].$$

*Here, $K(R)$ is as in Theorem 3.1.*

*Proof*: Let $x \in U$. Choose $R_0 > 0$ sufficiently small so that $B_{4R_0} = B(x,4R_0) \subset U$ and Theorem 2.3 holds for $B_{4R_0}$. We observe that for a fixed $R \leq R_0$ and $K = K(R)$, the function $\overline{u} = |u| + K$ satisfies $D_{\mathcal{L}}u = D_{\mathcal{L}}\overline{u}$ a.e. in $U$. Moreover, the assumptions $(S)$ may be rewritten as follows

$$(\overline{S}) \qquad \begin{cases} |A(x,u,\zeta)| \leq c_1|\zeta|^{p-1} + \overline{g}_2|\overline{u}|^{p-1} \\ |f(x,u,\zeta)| \leq f_1|\zeta|^{p-1} + \overline{f}_2|\overline{u}|^{p-1} \\ A(x,u,\zeta)\cdot\zeta \geq |\zeta|^p - \overline{f}_2|\overline{u}|^p, \end{cases}$$

with $\overline{g}_2 = g_2 + K^{1-p}g_3$, and $\overline{f}_2 = f_2 + K^{1-p}f_3 + K^{-p}h_3$.

We observe explicitly that, with these choices, we have for any $R \leq R_0$

$$\|\overline{f}_2\|_{L^s(B_{2R})} \leq \|f_2\|_{L^s(B_{2R})} + 2|B_R|^{-\frac{s}{Q}},$$
$$\|\overline{g}_2\|_{L^r(B_{2R})} \leq \|g_2\|_{L^r(B_{2R})} + 1.$$

Following [**Se**] we define for $\ell > K$ and $q \leq 1$

$$F(t) = \begin{cases} t^q & \text{if } K \leq t \leq \ell, \\ q\ell^{q-1}t - (q-1)\ell^q, & \text{if } \ell \leq t. \end{cases}$$

Also, with $\beta = pq - p + 1$, we let

$$G(\tau) = \operatorname{sgn} \tau \left[ F(|\tau| + K)F'(|\tau| + K) - q^{\beta-1}K^{\beta} \right].$$

By the chain rule one can prove that $F(\overline{u})$, $G(u) \in S^{1,p}_{\text{loc}}(U)$. Let now $\eta \in C_0^\infty(B_{2R})$, $0 \leq \eta \leq 1$, and define $\varphi = \eta^p G(u)$. Then, $\varphi \in \overset{\circ}{S}{}^{1,p}(U)$. Setting $v = F(\overline{u})$ we obtain from (1.3) and $(\overline{S})$

(3.5)
$$\begin{aligned}
0 &= \sum_{j=1}^{m} \int_U A_j(x, u, X_1 u, \ldots, X_m u) X_j \varphi - \int_U f(x, u, X_1 u, \ldots, X_m u)\varphi \\
&\geq \int_U \Big[ \eta^p |D_{\mathcal{L}} v|^p - pc_1 v |D_{\mathcal{L}}\eta| (\eta |D_{\mathcal{L}} v|)^{p-1} - f_1(\eta v)(\eta |D_{\mathcal{L}} v|)^{p-1} \\
&\quad - (1+\beta) q^{p-1} \overline{f}_2 (\eta v)^p - p q^{p-1} \overline{g}_2 (v |D_{\mathcal{L}}\eta|)(\eta v)^{p-1} \Big].
\end{aligned}$$

Hölder's inequality yields

$$\int_U v|D_{\mathcal{L}}\eta|(\eta|D_{\mathcal{L}}v|)^{p-1} \leq \left( \int_U \eta^p |D_{\mathcal{L}} v|^p \right)^{\frac{p-1}{p}} \left( \int_U v^p |D_{\mathcal{L}}\eta|^p \right)^{\frac{1}{p}}. \tag{3.6}$$

To estimate the terms containing $f_1$, $\overline{f}_2$ in (3.5), suppose at first $1 < p < Q$ and let $\kappa = \dfrac{Q}{Q-p}$, so that $\dfrac{1}{p} - \dfrac{1}{\kappa p} = \dfrac{1}{Q}$. Since $f_1 \in L^{\frac{Q}{1-\varepsilon}}_{\text{loc}}$, $(\eta|D_{\mathcal{L}}v|)^{p-1} \in L^{p/p-1}_{\text{loc}}$ we see that the factor $\eta v$ in the product $f_1(\eta v)(\eta|D_{\mathcal{L}}v|)^{p-1}$ belong to $L^{\alpha}_{\text{loc}}$ with $\dfrac{1}{\alpha} = 1 - \dfrac{p-1}{p} - \dfrac{1-\varepsilon}{Q} = \dfrac{1}{\kappa p} + \dfrac{\varepsilon}{Q} = \dfrac{1-\varepsilon}{\kappa p} + \dfrac{\varepsilon}{\kappa p} + \dfrac{\varepsilon}{Q} = \dfrac{1-\varepsilon}{\kappa p} + \dfrac{\varepsilon}{p}$. Therefore, Hölder's inequality gives

$$\begin{aligned}
\int_U f_1(\eta v)(\eta|D_{\mathcal{L}}v|)^{p-1} &\leq \left( \int_{B_{2R}} f_1^{\frac{Q}{1-\varepsilon}} \right)^{\frac{1-\varepsilon}{Q}} \left( \int_U \eta^p |D_{\mathcal{L}} v|^p \right)^{\frac{p-1}{p}} \cdot \\
&\quad \cdot \left( \int_U (\eta v)^p \right)^{\frac{\varepsilon}{p}} \left( \int_U (\eta v)^{\kappa p} \right)^{\frac{1-\varepsilon}{\kappa p}}.
\end{aligned} \tag{3.7}$$

If $p = Q$, we let $\widetilde{p} = Q\left(1 + \dfrac{\varepsilon}{2Q}\right)^{-1}$ and observe that $\widetilde{p} < Q$. Using the fact $(\eta|D_{\mathcal{L}}v|)^{p-1} \in L^{\widetilde{p}/p-1}_{\text{loc}}$, and arguing as for (3.7), we conclude with $\widetilde{\kappa} = \dfrac{Q}{Q-\widetilde{p}}$

$$(3.8)\quad \int_U f_1(\eta v)(\eta|D_{\mathcal{L}}v|)^{p-1} \le \left(\int_{B_{2R}} f_1^{\frac{Q}{1-\epsilon}}\right)^{\frac{1-\epsilon}{Q}} \left(\int_U \eta^{\widetilde{p}}|D_{\mathcal{L}}v|^{\widetilde{p}}\right)^{\frac{p-1}{p}} \cdot \left(\int_U (\eta v)^{\widetilde{p}}\right)^{\epsilon/2\widetilde{p}} \left(\int_U (\eta v)^{\widetilde{\kappa}\widetilde{p}}\right)^{(1-\epsilon/2)/\widetilde{\kappa}\widetilde{p}}.$$

At this point we use Sobolev embedding Theorem 2.3 to estimate the last integrals in the right hand sides of (3.7), (3.8). We obtain

$$\left(\int_U (\eta v)^{\kappa p}\right)^{\frac{1-\epsilon}{\kappa p}} \le C\left(\int_U |D_{\mathcal{L}}(\eta v)|^p\right)^{\frac{1-\epsilon}{p}} \le C\left[\left(\int_U \eta^p|D_{\mathcal{L}}v|^p\right)^{\frac{1-\epsilon}{p}} + \left(\int_U v^p|D_{\mathcal{L}}\eta|^p\right)^{\frac{1-\epsilon}{p}}\right],$$

where $C = C(U) > 0$ is as in Theorem 2.3. Substitution in (3.7) yields

$$(3.9)\quad \int_U f_1(\eta v)(\eta|D_{\mathcal{L}}v|)^{p-1} \le C\left(\int_{B_{2R}} f_1^{\frac{Q}{1-\epsilon}}\right)^{\frac{1-\epsilon}{Q}} \left(\int_U \eta^p|D_{\mathcal{L}}v|^p\right)^{\frac{p-1}{p}} \cdot \left(\int_U (\eta v)^p\right)^{\frac{\epsilon}{p}} \left[\left(\int_U \eta^p|D_{\mathcal{L}}v|^p\right)^{\frac{1-\epsilon}{p}} + \left(\int_U v^p|D_{\mathcal{L}}\eta|^p\right)^{\frac{1-\epsilon}{p}}\right].$$

When $p = Q$, from (3.8) and Theorem 2.3 we obtain an estimate similar to (3.9), but with $p$ in the right hand side replaced by $\widetilde{p}$. Since $\widetilde{p} < p$, a routine application of Hölder's inequality allows to conclude that (3.9) holds also when $p = Q$. By analogous arguments we can estimate the terms containing $\overline{f}_2$, $\overline{g}_2$ in the right hand side of (3.5). One proves

$$(3.10)\quad \int_U \overline{f}_2(\eta v)^p \le C\left(\int_{B_{2R}} \overline{f}_2^{\frac{Q}{p-\epsilon}}\right)^{\frac{p-\epsilon}{Q}} \left(\int_U (\eta v)^p\right)^{\frac{\epsilon}{p}} \cdot \left[\left(\int_U \eta^p|D_{\mathcal{L}}v|^p\right)^{\frac{p-\epsilon}{p}} + \left(\int_U v^p|D_{\mathcal{L}}\eta|^p|\right)^{\frac{p-\epsilon}{p}}\right],$$

$$(3.11)\quad \int_U \overline{g}_2(v|D_{\mathcal{L}}\eta|)(\eta v)^{p-1} \le C\left(\int_U \overline{g}_2^{\frac{Q}{p-1}}\right)^{\frac{p-1}{Q}} \left(\int_U v^p|D_{\mathcal{L}}\eta|^p\right)^{\frac{1}{p}} \cdot \left[\left(\int_U v^p|D_{\mathcal{L}}\eta|^p\right)^{\frac{p-1}{p}} + \left(\int_U \eta^p|D_{\mathcal{L}}v|^p\right)^{\frac{p-1}{p}}\right].$$

Using (3.6) and (3.9)-(3.11) in (3.5), we obtain the following inequality

$$
\begin{aligned}
\|\eta D_{\mathcal{L}} v\|_p^p \le\ & p\|\eta D_{\mathcal{L}} v\|_p^{p-1}\|v D_{\mathcal{L}}\eta\|_p \\
& +\|f_1\|_t\|\eta D_{\mathcal{L}} v\|_p^{p-1}\|\eta v\|_p^{\varepsilon}(\|\eta D_{\mathcal{L}} v\|_p^{1-\varepsilon}+\|v D_{\mathcal{L}}\eta\|_p^{1-\varepsilon}) \\
& +(1+\beta)q^{p-1}\|\overline{f}_2\|_s\|\eta v\|_p^{\varepsilon}\left(\|\eta D_{\mathcal{L}} v\|_p^{p-\varepsilon}+\|v D_{\mathcal{L}}\eta\|_p^{p-\varepsilon}\right) \\
& +pq^{p-1}\|\overline{g}_2\|_r+\|v D_{\mathcal{L}}\eta\|_p(\|v D_{\mathcal{L}}\eta\|_p^{p-1}+\|\eta D_{\mathcal{L}} v\|_p^{p-1}).
\end{aligned}
$$

In the above, all norms are in the Lebesgue spaces, thus, e.g., $\|u\|_p^p = \int_U |u|^p$. The exponents $t$, $s$, $r$ for $f_1$, $\overline{f}_2$ and $\overline{g}_2$ are as in (i)–(iii). We now recall the following numerical lemma [Se]: Let $z > 0$, and suppose $z^\alpha \le \sum_{i=1}^{N} a_i z^{\beta_i}$, with $a_i > 0$, and $0 \le \beta_i < \alpha$. Then, $z \le C\sum_{i=1}^N a_i^{\gamma_i}$. Here, $\gamma_i^{-1} = \alpha - \beta_i$, and $C > 0$ only depends on $N$, $\alpha$ and $\beta_i$. Using this lemma in the above inequality we obtain after some elementary, but lengthy, computations

$$
\begin{aligned}
\|\eta D_{\mathcal{L}} v\|_p \le\ & \Bigg[\|f_1\|_t^{\frac{1}{\varepsilon}}+\varepsilon\|f_1\|_t+\left((1+\beta)q^{p-1}\|\overline{f}_2\|_s\right)^{\frac{1}{\varepsilon}} \\
& +(1+\beta)q^{p-1}\|\overline{f}_2\|_s\Bigg]\|\eta v\|_p \\
& +\Bigg[p+(1-\varepsilon)\|f_1\|_t+(p-\varepsilon)(1+\beta)q^{p-1}\|\overline{f}_2\|_s+ \\
& +(pq^{p-1}\|\overline{g}_2\|_r)^{\frac{1}{p}}+pq^{p-1}\|\overline{g}_2\|_r\Bigg]\|v D_{\mathcal{L}}\eta\|_p.
\end{aligned}
$$

Observing that $\beta q^{p-1} \le (p+1)q^p$ we finally obtain

$$
\left(\int_U \eta^p|D_{\mathcal{L}} v|^p\right)^{\frac{1}{p}} \le C q^{\frac{p}{\varepsilon}}\left[\left(\int_U (\eta v)^p\right)^{\frac{1}{p}}+\left(\int_U v^p|D_{\mathcal{L}}\eta|^p\right)^{\frac{1}{p}}\right], \tag{3.12}
$$

with a $C > 0$ which solely depends on $U$, $p$, $\|f_1\|_{L^t(U)}$, $\|f_2\|_{L^s(U)}$, $\|g_2\|_{L^r(U)}$. If $1 < p < Q$, we use the embedding Theorem 2.3, which gives, along with (3.12),

$$\left(\frac{1}{|B_{2R}|}\int_{B_{2R}}(\eta v)^{\kappa p}\right)^{\frac{1}{\kappa p}} \le Cq^{\frac{p}{\varepsilon}}R\Bigg[\left(\frac{1}{|B_{2R}|}\int_{B_{2R}}(v\eta)^{p}\right)^{\frac{1}{p}} + \left(\frac{1}{|B_{2R}|}\int_{B_{2R}}v^{p}|D_{\mathcal{L}}\eta|^{p}\right)^{\frac{1}{p}}\Bigg]. \tag{3.13}$$

We recall that $\kappa = \dfrac{Q}{Q-p}$. In the case $p = Q$, (3.13) holds with $p$ and $\kappa$ in the left hand side replaced by $\widetilde{p} = Q/(1+\varepsilon/2Q)$ and $\widetilde{\kappa} = \dfrac{Q}{Q-\widetilde{p}}$ (recall $\widetilde{p} < p$).

At this point we specialize the choice of the cut–off function $\eta$ in (3.13). For $1 \le a < b \le 2$ we choose $\eta \in C_0^\infty(B_{bR})$, with $\eta \equiv 1$ on $B_{aR}$, and $|D_{\mathcal{L}}\eta| \le \dfrac{C}{(b-a)R}$. Such a function exists in virtue of Lemma 3.2. We draw from (3.13) if $p < Q$

$$\left(\int_{B_{aR}} v^{\kappa p}\right)^{\frac{1}{\kappa p}} \le C\frac{q^{\frac{p}{\varepsilon}}|B_{2R}|^{\frac{1}{\kappa p}-\frac{1}{p}}}{(b-a)}\left(\int_{B_{bR}} v^{p}\right)^{\frac{1}{p}}. \tag{3.14}$$

If $p = Q$ we obtain instead

$$\left(\int_{B_{aR}} v^{\widetilde{\kappa}\widetilde{p}}\right)^{1/\widetilde{\kappa}\widetilde{p}} \le C\frac{q^{\frac{p}{\varepsilon}}|B_{2R}|^{1/\widetilde{\kappa}\widetilde{p}-1/p}}{(b-a)}\left(\int_{B_{bR}} v^{p}\right)^{\frac{1}{p}}, \tag{3.15}$$

where $\widetilde{p}$ and $\widetilde{\kappa}$ are as above. We now observe explicitly that, in (3.14), $|B_{2R}|^{\frac{1}{\kappa p}-\frac{1}{p}} = |B_{2R}|^{\frac{1}{Q}}$. Similarly, using the above definitions of $\widetilde{p}$, $\widetilde{\kappa}$, one recognizes that $\dfrac{1}{\widetilde{\kappa}\widetilde{p}} - \dfrac{1}{p} = \dfrac{1}{Q}\left(\dfrac{\varepsilon}{2Q} - 1\right)$. At this point we let $\ell \to \infty$ in the definition of $F$. Consequently, $v = F(\overline{u})$ tends to $\overline{u}^q$ monotonically. By the monotone convergence theorem we thereby obtain from (3.14), (3.15)

$$\left(\int_{B_{aR}} \overline{u}^{\theta q p}\right)^{\frac{1}{\theta q p}} \le C\frac{q^{\frac{p}{\varepsilon q}}|B_{2R}|^{\frac{\tau}{qQ}}}{(b-a)^{1/q}}\left(\int_{B_{bR}} \overline{u}^{qp}\right)^{\frac{1}{qp}}. \tag{3.16}$$

In (3.16) we have let $\theta = \kappa$ and $\tau = 1$, if $1 < p < Q$, whereas $\theta = \dfrac{\widetilde{\kappa}\widetilde{p}}{Q} = \dfrac{2Q}{\varepsilon}$ and $\tau = \dfrac{\varepsilon}{2Q} - 1$, if $p = Q$. At this point, by Moser's iteration technique we easily infer from (3.16)

$$\operatorname*{ess\,sup}_{B_R} \overline{u} \le C\left(\frac{1}{|B_{2R}|}\int_{B_{2R}} \overline{u}^p\right)^{\frac{1}{p}}.$$

Recalling now that $\overline{u} = |u| + K = |u| + K(R)$, the conclusion of the theorem follows. ∎

In the context of Lebesgue spaces the assumptions in (ii) and (iii) above are optimal for the local boundedness of solutions of (1.2). Nonetheless, for $1 < p < Q$, in the borderline cases $s = \frac{Q}{p}$ and $t = Q$ in (ii) and (iii) we still have a Caccioppoli type inequality (see (3.12) above) in the small, or for lower order terms with sufficiently small norms. This fact is of interest in questions of existence and regularity of solutions of equations with critical growth.

We are now ready to give the

*Proof of Theorem 3.1*: We assume without loss of generality $u \ge \alpha > 0$ in $U$. With $K = K(R)$ as in the statement of Theorem 3.1, we let $\overline{u} = u + K$. For $\eta \in C_0^\infty(B_{2R})$ we choose $\varphi = \eta^p \overline{u}^{1-p}$ as a test function in (1.3). This is possible since, in virtue of Theorem 3.4, $\overline{u}$ is bounded in $B_{2R}$. Denoting $v = \log \overline{u}$, from (1.3) and $(\overline{S})$ in the proof of Theorem 3.4, we obtain

$$\begin{aligned}(3.17)\qquad 0 = \sum_{j=1}^m \int_U & A_j(x,u,X_1u,\dots,X_mu)X_j\varphi \\ &+ \int_U f(x,u,X_1u,\dots,X_mu)\varphi \le \\ \int_U \Big[ & pc_1(\eta|D_{\mathcal{L}}v|)^{p-1}|D_{\mathcal{L}}\eta| + p\overline{g}_2\eta^{p-1}|D_{\mathcal{L}}\eta| \\ -(p-1)(\eta|D_{\mathcal{L}}v|)^p + (p-1)&\overline{f}_2\eta^p + \eta(\eta|D_{\mathcal{L}}v|)^{p-1}f_1 + \overline{f}_2\eta^p\Big].\end{aligned}$$

In what follows we assume for simplicity $1 < p < Q$. The case $p = Q$ is dealt with by easy modifications, as in the proof of Theorem 3.4.

We now specialize the choice of $\eta \in C_0^\infty(B_{2R})$ in (3.17) using Lemma 3.2. We take $\eta \equiv 1$ on $B_R$, with $|D_{\mathcal{L}}\eta| \le \dfrac{C}{R}$. With this choice, a routine use of Hölder's inequality, and of the assumptions (i)–(iii), yields with $\kappa = \dfrac{Q}{Q-p}$

(3.18)
$$\begin{aligned}\int_U \overline{g}_2\eta^{p-1}|D_{\mathcal{L}}\eta| &\le \|\overline{g}_2\|_{L^r(B_{2R})}\left(\int_{B_{2R}}\eta^{\kappa p}\right)^{\frac{p-1}{\kappa p}}\left(\int_{B_{2R}}|D_{\mathcal{L}}\eta|^p\right)^{\frac{1}{p}}\\ &\le \frac{C}{R}|B_{2R}|^{\frac{Q-p+1}{Q}}\|\overline{g}_2\|_{L^r(B_{2R})},\end{aligned}$$

(3.19)
$$\begin{aligned}\int_U \overline{f}_2\eta^p &\le \left(\int_{B_{2R}}|\overline{f}_2|^{Q/p}\right)^{\frac{p}{Q}}\left(\int_{B_{2R}}\eta^{\kappa p}\right)^{\frac{1}{\kappa}}\\ &\le |B_{2R}|^{\frac{1}{\kappa}}\|\overline{f}_2\|_{L^{Q/p}(B_{2R})},\end{aligned}$$

(3.20)
$$\begin{aligned}\int_U \eta(\eta|D_{\mathcal{L}}v|)^{p-1}f_1 &\le \left(\int_{B_{2R}}f_1^Q\right)^{\frac{1}{Q}}\left(\int_{B_{2R}}\eta^{\kappa p}\right)^{\frac{1}{\kappa p}}\left(\int_U \eta^p|D_{\mathcal{L}}v|^p\right)^{\frac{p-1}{p}}\\ &\le |B_{2R}|^{\frac{1}{\kappa p}}\|f_1\|_{L^Q(B_{2R})}\left(\int_U \eta^p|D_{\mathcal{L}}v|^p\right)^{\frac{p-1}{p}},\end{aligned}$$

and finally

(3.21)
$$\begin{aligned}\int_U(\eta|D_{\mathcal{L}}v|)^{p-1}|D_{\mathcal{L}}\eta| &\le \left(\int_U \eta^p|D_{\mathcal{L}}v|^p\right)^{\frac{p-1}{p}}\left(\int_U |D_{\mathcal{L}}\eta|^p\right)^{\frac{1}{p}}\\ &\le \frac{C}{R}|B_{2R}|^{\frac{1}{p}}\left(\int_U \eta^p|D_{\mathcal{L}}v|^p\right)^{\frac{p-1}{p}}.\end{aligned}$$

Inserting (3.18)–(3.21) in (3.17) yields

(3.22)
$$\begin{aligned}\int_U \eta^p|D_{\mathcal{L}}v|^p \le C_p\Big[&R^{-1}|B_{2R}|^{\frac{1}{p}}\left(\int_U \eta^p|D_{\mathcal{L}}v|^p\right)^{\frac{p-1}{p}}\\ &+R^{-1}|B_{2R}|^{\frac{Q-p+1}{Q}}\|\overline{g}_2\|_{L^r(B_{2R})}+|B_{2R}|^{\frac{1}{\kappa}}\|\overline{f}_2\|_{L^{Q/p}(B_{2R})}\\ &+|B_{2R}|^{\frac{1}{\kappa p}}\|f_1\|_{L^Q(B_{2R})}\left(\int_U \eta^p|D_{\mathcal{L}}v|^p\right)^{\frac{p-1}{p}}\Big].\end{aligned}$$

At this point we use the numerical lemma recalled in the proof of Theorem 3.4, obtaining

$$(3.23)\qquad \left(\int_{B_R} |D_{\mathcal{L}}v|^p\right)^{\frac{1}{p}} \leq \frac{C_p}{R}\Big[|B_{2R}|^{\frac{1}{p}} + R^{1-\frac{1}{p}}|B_{2R}|^{\frac{1}{\kappa p}+\frac{1}{pQ}}\cdot$$
$$\cdot\|\overline{g}_2\|^{\frac{1}{p}}_{L^r(B_{2R})} + R|B_{2R}|^{\frac{1}{\kappa p}}\left(\|\overline{f}_2\|^{\frac{1}{p}}_{L^{Q/p}(B_{2R})} + \|f_1\|^{\frac{1}{p}}_{L^Q(B_{2R})}\right)\Big].$$

Estimating the right hand side of (3.23) would present a serious difficulty if Proposition 2.1 were not available. In the Euclidean context the measure of a ball of radius $R$ is like $R^n$, and the proof of Theorem 3.1 can be reduced, by a rescaling, to the case $R = 1$, see [**Se**]. In the present subelliptic context, appealing to Proposition 2.1 we obtain $R \leq C|B_R|^{\frac{1}{Q}}$. This estimate, Theorem 1., and the fact $\dfrac{1}{\kappa p} + \dfrac{1}{Q} = \dfrac{1}{p}$, allow to infer from (3.23)

$$(3.24)\qquad \int_{B_R} |D_{\mathcal{L}}v|^p \leq \frac{C}{R^p}|B_R|\Big[1 + \|\overline{g}_2\|_{L^2(B_{2R})}| + \|\overline{f}_2\|_{L^{Q/p}(B_{2R})} + \|f_1\|_{L^Q(B_{2R})}\Big],$$

where $C = C(p) > 0$. Recall now that from (ii), (iii) $\overline{f}_2$, $f_1$ respectively belong to $L^s_{\text{loc}}$ and $L^t_{\text{loc}}$, with $s = \dfrac{Q}{p-\varepsilon}$, $t = \dfrac{Q}{1-\varepsilon}$, for some $0 < \varepsilon < 1$. Also, as in the proof of Theorem 3.4, we have $\|\overline{f}_2\|_{L^s(B_{2R})} \leq \|f_2\|_{L^s(B_{2R})} + 2|B_R|^{-\frac{\varepsilon}{Q}}$, $\|\overline{g}_2\|_{L^r(B_{2R})} \leq \|g_2\|_{L^r(B_{2R})} + 1$.

These estimates, together with Hölder's inequality and Theorem 1., give

$$\begin{aligned}
\|\overline{f}_2\|_{L^{Q/p}(B_{2R})} &\leq |B_{2R}|^{\frac{\varepsilon}{Q}}\|\overline{f}_2\|_{L^s(B_{2R})} \\
&\leq |B_{2R}|^{\frac{\varepsilon}{Q}}\left(\|f_2\|_{L^s(B_{2R})} + 2|B_R|^{-\frac{\varepsilon}{Q}}\right) \\
&\leq C\left(1 + |B_{2R}|^{\frac{\varepsilon}{Q}}\|f_2\|_{L^s(B_{2R})}\right).
\end{aligned}$$

Similarly,

$$\|f_1\|_{L^Q(B_{2R})} \leq |B_{2R}|^{\frac{\varepsilon}{Q}}\|f_1\|_{L^t(B_{2R})}.$$

Inserting these inequalities in (3.24) we obtain

$$
\begin{aligned}
\int_{B_R} |D_{\mathcal{L}} v|^p &\le \frac{C}{R^p}|B_R|\Big[1 + \|g_2\|_{L^r(B_{2R})} \\
&\qquad + |B_{2R}|^{\frac{\epsilon}{Q}}\Big(\|f_2\|_{L^s(B_{2R})} + \|f_1\|_{L^t(B_{2R})}\Big)\Big] \\
&\le \frac{C}{R^p}|B_R|\Big[1 + \|g_2\|_{L^r(U)} \\
&\qquad + |U|^{\frac{\epsilon}{Q}}\Big(\|f_2\|_{L^s(U)} + \|f_1\|_{L^t(U)}\Big)\Big].
\end{aligned} \tag{3.25}
$$

Jerison's Poincarè inequality **[J]** and (3.25) finally give

$$\frac{1}{|B_R|}\int_{B_R} |v - v_{B_R}|^p \le C^*,$$

where $C^* = C^*(U, \|g_2\|_{L^r(U)}, \|f_2\|_{L^s(U)}, \|f_1\|_{L^t(U)}) > 0$.

By the John–Nirenberg theorem for spaces of homogeneous type, see e.g. **[Bu]**, we infer the existence of $p_0 > 0$ and $C > 0$ such that

$$\left(\frac{1}{|B_{2R}|}\int_{B_{2R}} (\overline{u})^{p_0}\right)^{\frac{1}{p_0}} \le C\left(\frac{1}{|B_{2R}|}\int_{B_{2R}} (\overline{u})^{-p_0}\right)^{-\frac{1}{p_0}}. \tag{3.26}$$

To complete the proof we need the following estimates:

$$\operatorname*{ess\,sup}_{B_R} \overline{u} \le C\left(\frac{1}{|B_{2R}|}\int_{B_{2R}} (\overline{u})^{p_0}\right)^{\frac{1}{p_0}}, \tag{3.27}$$

$$\operatorname*{ess\,inf}_{B_R} \overline{u} \ge C\left(\frac{1}{|B_{2R}|}\int_{B_{2R}} (\overline{u})^{-p_0}\right)^{-\frac{1}{p_0}} \tag{3.28}$$

for some constant $C > 0$ independent of $u$. Inequality (3.27) is a particular case of the following

LEMMA 3.29. *Let $u > 0$ be a weak solution of (1.2), and define $\overline{u}$ as in the proofs of Theorem 3.1 and 3.4. Then, for every $\alpha > 0$ there exist $C > 0$ and $R_0 > 0$ such that, given $B_R = B(x, R)$, with $B(x, 4R) \subset U$ and $R \le R_0$, we have*

$$\operatorname*{ess\,sup}_{B_R} \overline{u} \le C \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} (\overline{u})^\alpha \right)^{\frac{1}{\alpha}}.$$

*Proof*: An easy modification of the proof of Theorem 3.4 yields a constant $C > 0$ such that, for any numbers $a$, $b$, with $\frac{1}{2} \le a < b \le 1$, we have

$$\operatorname*{ess\,sup}_{B_{aR}} \overline{u} \le \frac{C}{(b-a)^{Q/p}} \left( \frac{1}{|B_R|} \int_{B_{bR}} (\overline{u})^p \right)^{\frac{1}{p}}. \tag{3.30}$$

The proof of the lemma now follows from (3.30) and an adaptation of an idea, due to Dahlberg and Kenig. Without loss of generality we assume $0 < \alpha < p$ and define

$$J(s) = |B_R|^{-(1-\frac{p}{\alpha})} \left( \int_{B_{sR}} (\overline{u})^p \right) \left( \int_{B_R} (\overline{u})^\alpha \right)^{-\frac{p}{\alpha}}.$$

By (3.30) we obtain

$$\operatorname*{ess\,sup}_{B_{\frac{R}{2}}} \overline{u}^\alpha \le C \left( \frac{1}{|B_R|} \int_{B_{\frac{2}{3}}} (\overline{u})^p \right)^{\frac{\alpha}{p}} \le \left( \frac{C}{|B_R|} \int_{B_R} (\overline{u})^\alpha \right) J\left(\frac{2}{3}\right)^{\frac{\alpha}{p}}.$$

We are going to prove that there exists $C > 0$, independent of $R$, such that $J\left(\frac{2}{3}\right) \le C$. Assume $J\left(\frac{2}{3}\right) > 1$. By (3.30) we have for $\frac{1}{2} < s \le \frac{2}{3}$

$$\begin{aligned}
J(a) &= |B_R|^{-(1-\frac{p}{\alpha})} \left( \int_{B_{aR}} (\overline{u})^{p-\alpha} (\overline{u})^\alpha \right) \left( \int_{B_R} (\overline{u})^\alpha \right)^{-\frac{p}{\alpha}} \\
&\le \frac{C}{(b-a)^{\frac{Q(p-\alpha)}{p}}} \left( \frac{1}{|B_R|} \int_{B_{bR}} (\overline{u})^p \right)^{\frac{p-\alpha}{p}} \cdot \\
&\quad \cdot |B_R|^{-(1-\frac{p}{\alpha})} \int_{B_{aR}} (\overline{u})^\alpha \left( \int_{B_R} (\overline{u})^\alpha \right)^{-\frac{p}{\alpha}} \\
&\le C[(b-a)^{-Q} J(b)]^{\frac{p-\alpha}{p}}.
\end{aligned}$$

This yields for

$$\log J(a) \le \frac{p-\alpha}{p} [\log C - Q \log(b-a) + \log J(b)].$$

For $\theta > 1$ to be suitably chosen, we let $a = b^\theta$ in the above inequality and integrate on the interval $\left[\left(\frac{2}{3}\right)^{\frac{1}{\theta}}, 1\right]$ with respect to $\frac{db}{b}$. We arrive at

$$\frac{1}{\theta}\int_{\frac{2}{3}}^{1} \log J(\rho)\frac{d\rho}{\rho} \le C_1 + \frac{p-\alpha}{p}\int_{(\frac{2}{3})^{\frac{1}{\theta}}}^{1} \log J(\rho)\frac{d\rho}{\rho}$$
$$\le C_1 + \frac{p-\alpha}{p}\int_{\frac{2}{3}}^{1} \log J(\rho)\frac{d\rho}{\rho},$$

where in the last inequality we have used the assumption $J\left(\frac{2}{3}\right) > 1$ and the fact that $\log J(\rho)$ is increasing. Choosing now $\theta \in \left(1, \frac{p}{p-\alpha}\right)$ we conclude that $J\left(\frac{2}{3}\right) \le C_2$, for some $C_2 > 0$. This proves the lemma. ∎

To finish the proof of Theorem 3.1 we are left with proving (3.28). We choose $\beta \le 1 - p < 0$ and set $q = \frac{p+\beta-1}{p} < 0$. Observe that by varying $\beta$ on $(-\infty, 1-p]$, $q$ ranges over $(-\infty, 0]$. For $\eta \in C_0^\infty(B_{2R})$, with $0 \le \eta \le 1$, let $\varphi = \eta^p \overline{u}^\beta$. As before, $\varphi$ is an admissible test function for (1.3). From the latter, and from $(\overline{S})$ above, we obtain after some computations similar to those in (3.5)

(3.31)
$$|\beta| \int_U \eta^p |D_{\mathcal{L}} v|^p \le p|q|C_1 \int_U v|D_{\mathcal{L}}\eta|(\eta|D_{\mathcal{L}} v|)^{p-1}$$
$$+ p|q|^p \int_U \overline{g}_2 v|D_{\mathcal{L}}\eta|(v\eta)^{p-1} + |q|^p(1+|\beta|)\int_U \overline{f}_2(\eta v)^p$$
$$+ |q| \int_U f_1 \eta v(\eta|D_{\mathcal{L}} v|)^{p-1}.$$

In the above inequality $v = \overline{u}^q$. At this point one proceeds as in the estimates (3.9)–(3.11) above, arriving to an inequality analogous to (3.13). The only difference is that the factor $q^{\frac{\varepsilon}{\epsilon}}$ in the right hand side of (3.13) must be replaced by $(1+|q|)^{\frac{\varepsilon}{\epsilon}}$. Using Lemma 3.2, for $1 \le a < b \le 2$ choose $\eta \in C_0^\infty(B_{bR})$, with $\eta \equiv 1$ on $B_{aR}$, and $|D_{\mathcal{L}}\eta| \le \frac{C}{(b-a)R}$. We finally obtain from (3.31)

$$\left(\int_{B_{aR}} v^{\kappa p}\right)^{\frac{1}{\kappa p}} \le \frac{C(1+|q|)^{\frac{p}{\varepsilon}}}{(b-a)} |B_{2R}|^{\frac{1}{\kappa p}-\frac{1}{p}}(b-a)\left(\int_{B_{bR}} v^{p}\right)^{\frac{1}{p}}. \tag{3.32}$$

Recall that $\kappa = \dfrac{Q}{Q-p}$, so that $\dfrac{1}{\kappa p} - \dfrac{1}{p} = -\dfrac{1}{Q}$. Since $v = \overline{u}^q$, taking $q$-rooths in (3.32), and keeping in mind that $q < 0$, we have

$$\left(\int_{B_{aR}} \overline{u}^{\kappa pq}\right)^{\frac{1}{\kappa pq}} \ge \frac{C(1+|q|)^{\frac{p}{\varepsilon q}}}{|B_{2R}|^{\frac{1}{qQ}}(t-s)^{\frac{1}{q}}} \left(\int_{B_{bR}} \overline{u}^{pq}\right)^{\frac{1}{pq}}. \tag{3.33}$$

From (3.33), (3.28) follows by Moser's iteration procedure. We omit the standard details. The proof of Theorem 3.1 is thus completed. ∎

As a consequence of Theorem 3.1 we prove that weak solutions of (1.2) are, in fact, Hölder continuous with respect to the $(X_1, \ldots, X_m)$–control distance. For this we need to strengthen assumption (i) as follows: $g_2, g_3 \in L^r_{\mathrm{loc}}$ with $r > \dfrac{Q}{p-1}$ for $1 < p \le Q$. In the sequel, we write $r = \dfrac{Q}{p-1-\varepsilon}$, where $\varepsilon \in (0,1)$ is the same as in the proof of Theorem 3.4. With this assumption and by (ii), Hölder's inequality implies

(3.34)

$$K(R) \le \left(|B_R|^{\frac{\varepsilon}{Qp}}\|f_3\|_{L^s(B_R)} + |B_R|^{\frac{\varepsilon}{Qp}}\|g_3\|_{L^r(B_R)}\right)^{\frac{1}{p-1}} + \left(|B_R|^{\frac{\varepsilon}{Qp}}\|h_3\|_{L^s(B_R)}\right)^{\frac{1}{p}}.$$

It is at this point that we use for the first time a bound from above of $|B_R|$. We recall that, from (2.2), $|B(x,R)| \le C_2\Lambda(x,R)$. Since the degree of the polynomial function $\Lambda(x,\cdot)$ is $\ge n$, we have $\Lambda(x,tR) \le t^n\Lambda(x,R)$. These considerations yield: $|B_R| = |B(x,R)| \le CR^n$, for every $x \in U$ and $R \le R_0$. We obtain from (3.34) $K(R) \le CR^\gamma$, for some $\gamma > 0$, where $C = C(U, \|f_3\|_{L^s(U)}, \|g_3\|_{L^r(U)}, \|h_3\|_{L^s(U)}) > 0$. We are now ready to state the following

THEOREM 3.35. *Let $u \in S^{1,p}_{\mathrm{loc}}(U)$ be a weak solution to* (1.2), *and suppose that* ess $\sup_U |u| = M < \infty$. *Then, there exist* $C > 0$ *and* $0 < \alpha < 1$, *depending on $U$ and $M$, such that*

$$\operatorname*{ess\,sup}_{x,y\in U} \frac{|u(x)-u(y)|}{d(x,y)^\alpha} \le C.$$

By means of Theorem 3.1, and of the important observation $K(R) \le CR^\gamma$, this result follows by a step by step imitation of a by now classical argument. We omit the details.

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