On the pointwise jump condition at the free boundary in the 1-phase Stefan problem *

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Abstract

In this paper we obtain the jump (or Rankine-Hugoniot) condition at the interphase for solutions in the sense of distributions to the one phase Stefan problem $u_t = \Delta(u-1)_+$. We do this by approximating the free boundary with level sets, and using methods from the theory of bounded variation functions. We show that the spatial component of the normal derivative of the solution has a trace at the free boundary that is picked up in a natural sense. The jump condition is then obtained from the equality of the n-density of two different disintegrations of the free boundary measure. This is done under an additional condition on the n-density of this measure. In the last section we show that this condition is optimal, in the sense that its satisfaction depends on the geometry of the initial data.

1 Introduction

The purpose of this paper is to obtain the jump condition at free boundary points for a nonnegative solution (in the sense of distributions) $u \in L^1_{\text{loc}}(\mathbb{R}^n \times (0,T))$ to the one phase Stefan problem

$$u_t - \Delta(u-1)_+ = 0. \tag{1.1}$$

From the work in [Ko], [AnKo],[Ko1], [Ko2] we know that

$$t(x) = \begin{cases} \inf\{t : u(x,t) > 1\} \\ T \text{ if } u(x,T^-) \leq 1 \end{cases}$$

is a well defined and measurable function. The free boundary

$$F = \partial(\{(u-1)_+ > 0\})$$

*Math subject classification: 35K65, 35R35 is composed of the graph G of this function $t(x)$, plus segments $\{x_0\}\times(t_0,t_1)$, $x_0 \in \mathbb{R}^n$, parallel to the t-axis. The measure

$$\lambda = \frac{\partial}{\partial t}(u-(u-1)_+) = \operatorname{div}_{x,t}(\nabla(u-1)_+, -(u-1)_+) \tag{1.2}$$

is supported on the set F, and is carried by a countably rectifiable subset S of F, see [Ko2] and [KoMo]. In addition, $\mathcal{L}^n(\{x \in \mathbb{R}^n : t(x) \text{ jumps at } x\} = 0$, and the disintegration

$$\lambda = (1-u_I(x))_+ \, \delta_{t(x)}(t)\, dx \tag{1.3}$$

holds (see [Ko2]). At $\mathcal{H}^n$-a.e. point of S, F has a measure theoretic normal and an approximate tangent plane. As usual, we will denote by F_{red} the *reduced free boundary*, i.e. the subset of points of F at which

$$\lim_{r\to 0} \frac{\mathcal{L}^{n+1}(C_r \cap \{(u-1)_+ > 0\})}{\mathcal{L}^{n+1}(C_r)} > 0,$$

$$\lim_{r\to 0} \frac{\mathcal{L}^{n+1}(C_r \setminus \{(u-1)_+ > 0\})}{\mathcal{L}^{n+1}(C_r)} > 0,$$

and a measure theoretical normal exists. Here we are using the variant of Hausdorff measure in which coverings are composed of cylinders $C_r = B_r(x_0)\times(t_0-r, t_0+r)$, and $\mathcal{L}^n$ stands for n dimensional Lebesgue measure. We recall that for any $r>0$, $\lambda(C_r) \le \mathcal{L}^n(B_r) = c_n r^n$, and that for $\mathcal{H}^n$-a.e. $(x,t) \in F_{\text{red}}$ one has that $\theta_n = \lim_{r\to 0} \frac{\lambda(C_r)}{r^n}$ exists and is positive.

Our main result shows that the classical jump condition at the free boundary for the Stefan problem (1.1)

$$-\nabla(u-1)_+(x_0, t_0+)\cdot \nu_x = (1-u_I(x))_+ \nu_t$$

holds in a suitable (and natural) density sense. Here we write ν_x and ν_t, respectively, for the components of the normal to the free boundary at (x_0,t_0) in the space and time variables. In order to state the result precisely, we need to introduce the notion of trace of the spatial gradient on the reduced free boundary.

Definition 1.1. *Let $\nu(x,t) = (\nu_x,\nu_t)(x,t)$ be the measure theoretic normal to $F_{\text{red}} \cap R$ at (x,t) pointing into the diffusive region $\{(u-1)_+ > 0\}$. We define the trace of $(\nabla(u-1)_+, 0)\cdot\nu$ at the point $(x_0,t_0) \in F_{\text{red}}$ by*

$$L(x_0,t_0) = \lim_{r\to 0} \frac{\lambda(C_r)}{Per(\{(u-1)_+ > 0\}, C_r)}, \tag{1.4}$$

where C_r is as above.

Troughout the paper we will work in a bounded open cylinder $R \subset I\!R^n \times (0,T)$ with the property that there exists $\alpha > 0$ such that

$$\alpha < \theta_n(x,t) = \lim_{s\to 0} \frac{\lambda(C_s(x,t))}{s^n} \tag{1.5}$$

for all $(x,t) \in F_{\text{red}} \cap R$. Assumption (1.5) is in some sense optimal, as we will discuss in Section 4.

Theorem 1.1. *Let ν be the measure theoretic normal to the reduced free boundary pointing into the diffusive region, and let*

$$\nu_t(x,t) = \nu(x,t) \cdot (0, \cdots, 0, 1)$$

stand for its time component. For $\mathcal{H}^n$ a.e. $(x,t) \in F_{\text{red}} \cap R$,

$$L(x,t) = \lim_{r\to 0} \frac{\lambda(C_r)}{r^n} = (1-u_I)_+(x)\nu_t(x,t). \tag{1.6}$$

The proof of Theorem 1.1 relies on approximating the free boundary by level sets and BV theory techniques. These are available to us because we work in an Euclidean setting, where all directions (including time) weigh equally in the Hausdorff measure. Our paper is close in scope to [We2], where the author studies the structure of the free boundary in the space variables for a.e. level of t. He does this instead by means of parabolic blow up techniques. In view of Section 4 of this paper, our conditions are general, since they solely depend on the initial data. In addition, no mushy region is allowed in [We2]. Other related work can be found in [ChFr], [ChFr2] in the context of divergence measure fields, and in [DeOWe] where the authors study entropy solutions for multidimensional conservation laws.

The one phase Stefan problem with a mushy zone (1.1) also bears some relation to the one phase premixed flame free boundary problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \{u > 0\}, \\ u = 0,\ |\nabla u| = 1 & \text{on } \partial\{u > 0\}. \end{cases} \tag{1.7}$$

This problem was first studied by Caffarelli and Vázquez in [CaV]. Later, the existence of solutions to (1.7) satisfying the free boundary condition $|\nabla u| = 1$ on $\partial\{u > 0\}$ in a pointwise sense has been proved, under suitable density assumptions on the set $\{u \equiv 0\}$, in [CaLeWo] and [Da]. If u is a solution to (1.1), and the initial datum u_I does not take values in the open interval $(0,1)$, then $v = (u-1)_+$ is a solution to (1.7). If instead we allow initial data u_I for (1.1) that take on values in the interval $(1-\eta, 1)$ for some $0 < \eta < 1$, we may construct data with a region $\Omega = \{u_I \equiv 1\}$ surrounded by a region $\{u_I < 1\}$. Once a region $\{u(x,t) > 1\}$ evolves so to reach $\{u_I \equiv 1\}$ at time t_0, the latter region admitting an inner tangent x-ball, then by the maximum principle the free boundary crosses Ω instantaneously, and $\Omega \times \{t_0\}$

becomes part of the free boundary F, but the measure λ will not charge the set $\Omega^0 \times \{t_0\}$.

The paper is organized as follows. In Section 2 we prove various measure theoretic properties of the set $\{(u-1)_+ > 0\}$, including the locally finiteness of its perimeter. In addition, we obtain an alternative expression of the free boundary measure λ (a variant of one obtained in [Ko2]). With this in hand, in Section 3 we prove Theorem 1.1. In Section 4 we show that condition (1.5) is actually optimal, in the sense that the size and distribution of the exceptional set depend, except on a set of $\mathcal{H}^n$ measure zero, only on the initial data.

2 Finite Perimeter of level sets near regular free boundary points

In this section we prove various measure theoretic properties of the set $\{(u-1)_+ > 0\}$. In particular we show that it has locally finite, positive perimeter. We also give an integral representation of the free boundary measure λ. We begin by recalling that since $(u-1)_+$ satisfies the heat equation in $\{(x,t) : (u-1)_+(x,t) > 0\})$, and the heat operator is hypoelliptic, $(u-1)_+ \in C^\infty(\{(x,t) : (u-1)_+(x,t) > 0\})$.

Proposition 2.1. *For any* $(x_0, t_0) \in F_{\text{red}}$,

$$\limsup_{(x,t)\to(x_0,t_0)} |\nabla (u-1)_+(x,t)| > 0$$

.

Proof. Assume $\lim_{(x,t)\to(x_0,t_0)} |\nabla (u-1)_+(x,t)| = 0$. From the continuity of $(u-1)_+$ (see [DiB], [AnKo]) we immediately obtain

$$\lim_{r\to 0} \left(\fint_{B_r(x_0)\times\{t_0-r\}} + \fint_{B_r(x_0)\times\{t_0+r\}} \right) (u-1)_+ \, dx = 0.$$

Approximate $\chi_{C_r(x_0,t_0)}$ with the increasing sequence

$$\varphi_k(x,t) = \chi_{[t_0-r, t_0+r]}(t) \; \psi_k(|x-x_0|), \tag{2.1}$$

where we choose

$$\psi_k(s) = \begin{cases} 1 & \text{in } B_{r-\frac{1}{k}}(0), \\ 0 & \text{in } (B_r(0))^c, \end{cases}$$

and $\psi_k(s)$ is continuous and linear for $r - \frac{1}{k} \le s \le r$. Then, using (1.2) and polar coordinates, we obtain

$$\begin{aligned}
\lambda(C_r) &= \lim_{k\to\infty}(\lambda,\varphi_k) \\
&= \lim_{k\to\infty} ck\int_{r-\frac{1}{k}}^{r}\int_{(\partial B_s(x_0))\times(t_0-r,t_0+r)} -\nabla(u-1)_+\cdot\nu\, d\mathcal{H}^n\, ds \\
&+\left(\int_{B_r(x_0)\times\{t_0-r\}}-\int_{B_r(x_0)\times\{t_0+r\}}\right)(u-1)_+\, dx
\end{aligned}$$

where ν stands for the inner unit normal. On the other hand, given $\epsilon>0$,

$$\lim_{k\to\infty} ck\int_{r-\frac{1}{k}}^{r}\int_{(\partial B_s(x_0))\times(t_0-r,t_0+r)} |\nabla(u-1)_+|\, d\mathcal{H}^n\, ds \leq \int_{\partial_{\mathrm{lat}}C_r}\epsilon\, d\mathcal{H}^n,$$

for all sufficiently small r, whence $\lim_{r\to 0}\frac{\lambda(C_r)}{r^n}=0$ follows. We are using the notation $\partial_{\mathrm{lat}}C_r$ for $(\partial B_s(x_0))\times(t_0-r,t_0+r)$. We have thus reached a contradiction, and the proof is complete. □

Proposition 2.2. *The set* $\{(u-1)_+>0\}$ *has finite perimeter in* R*.*

Proof. From (1.5) it follows that $\mathcal{H}^n \lfloor \partial\{(u-1)_+>0\}$ and λ are mutually absolutely continuous on R. In particular, $\partial\{(u-1)_+>0\}\cap C_r$ is $\mathcal{H}^n$-measurable, with

$$\mathcal{H}^n(\partial\{(u-1)_+>0\}\cap R)<\infty.$$

Then $\chi_{C_r\cap\{(u-1)_+>0\}}\in BV(R)$, and

$$\mathrm{Per}\,(\{(u-1)_+>0\},R)<\infty.$$

□

Proposition 2.3. *Per*$(\{(u-1)_+>0\},R)>0$.

Proof. Since by the Radon-Nikodyn theorem $R\cap\partial\{(u-1)_+>0\}$ is $\mathcal{H}^n$-measurable with $\mathcal{H}^n(R\cap\partial\{(u-1)_+>0\})<\infty$, by [Ma, 15.19] for almost all $(x_0,t_0)\in\partial\{(u-1)_+>0\}\cap R$ there exists a unique approximate tangent plane to $\partial\{(u-1)_+>0\}$ at (x_0,t_0). Let $\eta(x_0,t_0)$ be the measure theoretic normal to $\partial\{(u-1)_+>0\}$ at (x_0,t_0). Then

$$\lim_{r\to 0}\frac{\mathcal{L}^{n+1}(C_r\cap\{(u-1)_+>0\}\cap\{(x,t):(x-x_0,t-t_0)\cdot\eta(x_0,t_0)>0\})}{\mathcal{L}^{n+1}(C_r)}=\frac{1}{2}$$

and

$$\lim_{r\to 0}\frac{\mathcal{L}^{n+1}(C_r\cap\{(u-1)_+=0\}\cap\{(x,t):(x-x_0,t-t_0)\cdot\eta(x_0,t_0)<0\})}{\mathcal{L}^{n+1}(C_r)}=\frac{1}{2}.$$

It follows that for small enough r so that $C_r \subset R$,

$$\begin{aligned} \min\{&\mathcal{L}^{n+1}\left[C_r \cap \{(u-1)_+ > 0\}\right], \\ &\mathcal{L}^{n+1}\left[C_r \backslash \{(u-1)_+ > 0\}\right]\} > \frac{1}{4}\mathcal{L}^{n+1}(C_r), \end{aligned}$$

whence by the relative isoperimetric inequality,

$$\mathrm{Per}\left(\{(u-1)_+ > 0\}, C_r\right) > 0.$$

□

Lemma 2.1. *There exists a constant M_R such that*

$$\mathit{Per}\left(\{(u-1)_+ < \varepsilon\}, R\right) \leq M_R,$$

for all sufficiently small ε.

Proof. From the continuity of $(u-1)_+(x,t)$, we know that

$$\chi_{\{(u-1)_+<\varepsilon\}\cap R} \to \chi_{\{(u-1)_+=0\}\cap R}$$

a.e., and in L^1, as $\varepsilon \to 0$. Pick $\varphi \in C_0^1(R, \mathbb{R}^n)$, such that $|\varphi| \leq 1$. By the dominated convergence theorem,

$$\begin{aligned} |\lim_{\varepsilon\to 0}\int \chi_{\{(u-1)_+<\varepsilon\}} \operatorname{div}\varphi\, dx| &= |\int \chi_{\{(u-1)_+=0\}} \operatorname{div}\varphi\, dx| \\ &= |\int_{F_{\mathrm{red}}\{(u-1)_+=0\}\cap R} \varphi\cdot\nu\, d\mathcal{H}^n| \leq \mathrm{Per}\left(\{(u-1)_+ > 0\}, R\right). \end{aligned}$$

Therefore, for small enough $\varepsilon > 0$,

$$|\int \chi_{\{(u-1)_+<\varepsilon\}} \operatorname{div}\varphi\, dx| \leq \mathrm{Per}\left(\{(u-1)_+ > 0\}, R\right) + 1 = M_R.$$

For each such fixed ε, take the supremum of the left-hand side over all $\varphi \in C_0^1(R, \mathbb{R}^n)$ such that $\|\varphi\|_{L^\infty} \leq 1$ to get

$$\mathrm{Per}\left(\{(u-1)_+ < \varepsilon\}, R\right) \leq M_R$$

for all small enough ε. □

Remark 2.1. *With the same strategy one also proves that*

$$\lim_{\varepsilon\to 0}\int_{\partial\{(u-1)_+<\varepsilon\}\cap C_r} \varphi\cdot\nu\, d\mathcal{H}^n = \int_{\partial_{\mathrm{red}}\{(u-1)_+=0\}\cap C_r} \varphi\cdot\nu\, d\mathcal{H}^n,$$

for any $C_r \subset R$ and $\varphi \in C_0^1(C_r, \mathbb{R}^n)$, and that the same formula holds replacing C_r by R.

Proposition 2.4. *Let $(x_0,t_0) \in F_{\text{red}} \cap R$, and consider cylinders $C_r = C_r(x_0,t_0)$. For a. e. $0 < r < r_0$ such that $C_{r_0} \subset R$,*

$$\begin{aligned} \lambda(C_r) &= -\lim_{\varepsilon\to 0}\int_{\partial\{(u-1)_+<\varepsilon\}\cap C_r} (\nabla(u-1)_+,0)\cdot \nu_\varepsilon \, d\mathcal{H}^n \\ &= \lim_{\varepsilon\to 0}\int_{\partial\{(u-1)_+<\varepsilon\}\cap C_r} \frac{|\nabla(u-1)_+|^2}{|D(u-1)_+|}\, d\mathcal{H}^n, \end{aligned} \tag{2.2}$$

where $D(u-1)_+ = (\nabla(u-1)_+, \frac{\partial}{\partial t}(u-1)_+)$ and ν_ε is the inward normal to the set $\{(u-1)_+ < \varepsilon\}$.

Proof. First note that for any $0 < \varepsilon$,

$$\lambda(C_r) = \lambda(C_r \cap \{(u-1)_+ < \varepsilon\}),$$

since $(u-1)_+$ solves the heat equation in $C_r \cap \{(u-1)_+ > \varepsilon\}$, and using (1.2). Then

$$\begin{aligned} \lambda(C_r) = &-\lim_{\varepsilon\to 0}\int_{C_r\cap\partial\{(u-1)_+<\varepsilon\}} (\nabla(u-1)_+, -(u-1)_+)\cdot \nu \, d\mathcal{H}^n \\ &-\lim_{\varepsilon\to 0}\int_{\{(u-1)_+<\varepsilon\}\cap\partial C_r} (\nabla(u-1)_+, -(u-1)_+)\cdot \nu \, d\mathcal{H}^n \\ &= -\lim_{\varepsilon\to 0} I_1 - \lim_{\varepsilon\to 0} I_2. \end{aligned} \tag{2.3}$$

Now,

$$\begin{aligned} I_2 = &\int_{\partial_{\text{lat}} C_r\cap\{0<(u-1)_+<\varepsilon\}} \nabla(u-1)_+ \cdot \frac{x-x_0}{|x-x_0|}\, d\mathcal{H}^n \\ &-\int_{\{0<(u-1)_+<\varepsilon\}\cap(B_r(x_0)\times\{t_0+r\})} (u-1)_+ \, dx \\ &+\int_{\{0<(u-1)_+<\varepsilon\}\cap(B_r(x_0)\times\{t_0-r\})} (u-1)_+ \, dx + = I_{21} + I_{22} + I_{23}. \end{aligned}$$

By the continuity of $(u-1)_+$, $|I_{22}| \le \varepsilon\mathcal{L}^n(B_r(x_0))$ whence $I_{22} \to 0$ as $\varepsilon \to 0$. The same reasoning shows that $I_{23} \to 0$ as $\varepsilon \to 0$.

To see that $\lim_{\varepsilon\to 0} I_{21} = 0$, let us first note that $\nabla(u-1)_+$ is defined pointwise at $\mathcal{H}^n$ a.e. point of $\partial B_r(x_0)) \times (t_0 - r, t_0 + r)$: Since $\{(u-1)_+ > 0\}$ has finite perimeter in R, $\partial\{(u-1)_+ > 0\}$ is (countably) $n-$rectifiable in R. Using a general version of the coarea formula (e.g. theorem 2.93 in [AmFuP]), we see that $\mathcal{H}^n\left((\partial\{(u-1)_+ > 0\}) \cap (\partial_{\text{lat}}(C_r))\right) = 0$. But this is the set of points at which $\nabla(u-1)_+$ is not continuous.

Since $\nabla(u-1)_+ \in L^2_{\text{loc}}(\mathbb{R}^n \times (0,T))$ and again using the continuity of $(u-1)_+$,

$$I_{21} \le \int_{(\partial_{\text{lat}} C_r)\cap\{0<(u-1)_+<\varepsilon\}} |\nabla(u-1)_+|\, d\mathcal{H}^n$$

$$\leq \|\nabla(u-1)_+\|_{L^2(\partial_{\text{lat}} C_r \cap \{0<(u-1)_+<\varepsilon\})} \, [\mathcal{H}^n(\partial_{\text{lat}} C_r)]^{\frac{1}{2}} < \infty,$$

and

$$|\nabla(u-1)_+| \; \chi_{\{0<(u-1)_+<\varepsilon\}}(x,t) \to 0$$

$\mathcal{H}^n$ a.e. on $\partial_{\text{lat}} C_r$ as $\varepsilon \to 0$.

By the dominated convergence theorem it follows that

$$\lim_{\varepsilon \to 0} \int_{\partial_{\text{lat}} C_r} |\nabla(u-1)_+| \; \chi_{\{0<(u-1)_+<\varepsilon\}} \, d\mathcal{H}^n = 0.$$

for a.e. $0 < r \leq r_0$.

Next, write I_1 in (2.3) as

$$I_1 = \int_{C_r \cap \partial\{(u-1)_+<\varepsilon\}} (\nabla(u-1)_+, 0) \cdot \nu \, d\mathcal{H}^n$$

$$+ \int_{C_r \cap \partial\{(u-1)_+<\varepsilon\}} -(u-1)_+ \nu_t \, d\mathcal{H}^n = I_{11} + I_{12}$$

where $\nu_t = \nu \cdot (0, \ldots, 0, 1)$. We use Lemma 2.1 to bound I_{12}:

$$|I_{12}| \leq \varepsilon \operatorname{Per}\left(\{(u-1)_+ < \varepsilon\}, C_r\right) \leq \varepsilon \; M_r \leq \varepsilon M_R$$

whence for a.e. $r > 0$ we have

$$\lambda(C_r) = \lim_{\varepsilon \to 0} \int_{C_r \cap \partial\{(u-1)_+<\varepsilon\}} \frac{|\nabla(u-1)_+|^2}{|D(u-1)_+|} \, d\mathcal{H}^n. \tag{2.4}$$

□

3 Traces and the jump condition

In this section we show that the trace of the spatial component of the normal derivative of $(u-1)_+$ introduced in Definition 1.1 is picked up in the sense of Gauss-Green's theorem, and then prove our main result, Theorem 1.1.

We begin by observing that under assumption (1.5), by the Radon-Nikodym theorem the trace $L(x,t)$ introduced in (1.4) is defined at $\mathcal{H}^n$ a.e.

$(x,t) \in F_{\text{red}} \cap R$, and

$$\lambda(C_r) = \int_{\partial\{(u-1)_+>0\} \cap C_r} L(x,t) \, d\mathcal{H}^n. \tag{3.1}$$

Proposition 3.1. *For any $\varphi \in C_0^\infty(R)$,*

$$\int_{\partial\{(u-1)_+>0\}} L(x,t) \; \varphi(x,t) \, d\mathcal{H}^n = \lim_{\varepsilon \to 0} \int_{\partial\{(u-1)_+>\varepsilon\}} \varphi(x,t) \frac{|\nabla(u-1)_+|^2}{|D(u-1)_+|} \, d\mathcal{H}^n.$$

Proof. Recall that by (1.3)

$$(\lambda, \varphi) = \int_{\pi(\partial\{(u-1)_+>0\}\cap R)} (1 - u_I(x))_+ \varphi(x,t)\, dx,$$

where $\pi : \mathbb{R}^n \times (0,T) \to \mathbb{R}^n$ is the projection on the spatial coordinates. Using (1.2) and since $(u-1)_+$ solves the heat equation in $\{(u-1)_+ > 0\}$, we have that $(\lambda, \varphi) = (\lambda, \varphi\chi_{\{(u-1)_{<\varepsilon}\}})$ for all $\varepsilon > 0$. and therefore

$$(\lambda, \varphi) = \lim_{\varepsilon\to 0} (\lambda, \varphi\chi_{\{(u-1)_+<\varepsilon\}}) =$$

$$\lim_{\varepsilon\to 0} \int_{\{(u-1)_+<\varepsilon\}} (\nabla(u-1)_+, -(u-1)_+) \cdot D\varphi\, dx\, dt$$

$$+ \lim_{\varepsilon\to 0} \int_{\partial\{(u-1)_+<\varepsilon\}} \varphi(x,t) \frac{|\nabla(u-1)_+|^2}{|D(u-1)_+|} d\mathcal{H}^n = L_1 + L_2.$$

By the dominated convergence theorem, and using that $D(u-1)_+ \in L^2_{\text{loc}}$, we obtain that $L_1 = 0$ and thus

$$\int_{\partial\{(u-1)_+>0\}} \varphi(x,t) L(x,t)\, d\mathcal{H}^n = \lim_{\varepsilon\to 0} \int_{\partial\{(u-1)_+<\varepsilon\}} \varphi(x,t) \frac{|\nabla(u-1)_+|^2}{|D(u-1)_+|}\, d\mathcal{H}^n.$$

□

We now prove our main result.

Proof of Theorem 1.1. From (1.3), since $t(x)$ has jumps at most at a null set for $\mathcal{L}^n$, for a.e. $r > 0$ it holds that $\lambda(\partial B_r(x_0) \times (t(x_0) - r, t(x_0) + r)) = 0$ and we can write

$$\begin{aligned} \frac{\lambda(C_r)}{r^n} &= \frac{1}{r^n} \int_{\partial\{(u-1)_+>0\}\cap C_r} L(x,t)\, d\mathcal{H}^n \\ &= \frac{1}{r^n} \int_{B_r(x_0)} \chi_{\pi(F_{\text{red}}\cap C_r)}(x) (1 - u_I(x))_+\, dx. \end{aligned} \tag{3.2}$$

We want to let r tend to 0 in (3.2). Now, for $\mathcal{H}^n$ a.e. $(x_0, t_0) \in F_{\text{red}}$,

$$\lim_{r\to 0} \frac{\int_{\partial\{(u-1)_+>0\}\cap C_r} L(x,t)\, d\mathcal{H}^n}{c_n r^n} =$$

$$\lim_{r\to 0} \frac{\int_{\partial\{(u-1)_+>0\}\cap C_r} L(x,t)\, d\mathcal{H}^n}{\text{Per}\,(\{(u-1)_+ > 0\}, C_r)} \lim_{r\to 0} \frac{\text{Per}\,(\{(u-1)_+ > 0\}, C_r)}{c_n r^n} = L(x_0, t_0),$$

where we have used (1.4), proposition 2.2, DeGiorgi's rectifiability theorem (see [AmFuP]), and the notation $c_n r^n$ for $\mathcal{L}^n(B_r)$.

Now write $P(x_0, t_0)$ for the measure theoretic tangent plane to F_{red} at (x_0, t_0). To obtain the other equality, we first note that since

$$1 > \theta_n(x_0, t_0) > \alpha$$

and

$$\lim_{r \to 0} \frac{\mathcal{L}^n(\pi(P(x_0, t_0) \cap C_r(x_0, t_0)))}{\mathcal{L}^n(B_r(x_0))}$$

exists, this limit must be positive, and

$$\lim_{r \to 0} \frac{\int_{\pi(P(x_0,t_0) \cap B_r(x_0))} (1 - u_I(x))_+ \, dx}{\mathcal{L}^n(\pi(P(x_0, t_0) \cap B_r(x_0)))} = (1 - u_I(x_0))_+.$$

By Lebesgue's differentiation theorem then, for $\mathcal{H}^n$ a.e. $(x_0, t_0) \in F_{\text{red}}$,

$$L(x_0, t_0) = (1 - u_I(x))_+ \nu_t(x_0, t_0) \tag{3.3}$$

where $(\nu_x(x_0, t_0), \nu_t(x_0, t_0))$ is the outer unit normal to $\partial\{(u-1)_+ > 0\}$ at (x_0, t_0). Notice that $|\nu_t(x_0, t_0)|$ expresses the ratio of the measure of the portion of the (measure theoretic) tangent plane to F_{red} at (x_0, t_0) in C_r to $\mathcal{L}^n(B(x_0, r))$. □

Remark 3.1. *If the free boundary is Lipschitz, we explicitely observe that* (1.6) *coincides with the classical jump condition*

$$-\nabla(u-1)_+(x_0, t_0) \cdot \nu_x = (1 - u_I(x))_+ \nu_t,$$

where we write ν_x and ν_t, respectively, for the components of the normal to the free boundary at (x_0, t_0) in the space and time variables.

4 On the subset of the free boundary where $\Theta_n > 0$.

In this section we discuss the nature of the set of points $x \in \mathbb{R}^n$ such that the density of the free boundary measure vanishes at $(x, t(x))$. From the results in this section it will follow that condition (1.5) is optimal, in the sense that it will be satisfied automatically if we impose conditions on the initial datum, specifically on the regularity of $(1 - u_I)_+(x)$, and on the geometry of the boundary of the set $\{(1 - u_I(x))_+ > 0)\}$. In addition, from the finite speed advance of the free boundary and since in the "mushy" region $\{0 \le u \le 1\}^\circ$ the initial datum stays unchanged a.e., one can devise initial data that are identically 1 on sets with boundaries that have outward pointing cusps or angles, and are set a distance from each other, in such a way that they are reached by the propagation of the diffusive zone at arbitrary times. Such behavior is common in free boundaries arising in degenerate parabolic equations (see e.g. [CaWo]).

For simplicity, we state the lemmas in this ection for $t = 0$. By restating them $t = t_0$, with $0 < t_0 < T$, the results hold for $t_0 < t < T$.

As usual, we denote by u_I the absolutely continuous part (with respect to $\mathcal{L}^n$) of the initial trace. Questions about the size and geometry of singular sets of free boundaries have been addressed recently in the context of variational free boundary problems by Weiss (see [We1]), and by Caffarelli, Jerison, and Kenig (see [CaJKe]).

Lemma 4.1. *Let*

$$O = \{x \in \mathbb{R}^n : t(x) \text{ is continuous at } x, \text{ and } \exists \lim_{r\to 0} \int_{B(x)} (1 - u_I(y))_+ \, dy > 0\},$$

and

$$Z = \{x \in \mathbb{R}^n : \lim_{r\to 0} \frac{\lambda(C_r(x, t(x)))}{r^n} = 0\}.$$

Then Z is a closed set relative to O.

Remark 4.1. *We recall that $\lambda(Z \times (0, T)) = 0$ (see* [Ko2]*. In addition, in* [KoMo] *Korten and Moore remove the restriction of the free boundary measure to the complement of the set Z defined above, required in* [Ko2] *for the rectifiability of the free boundary measure λ to hold.*

Proof of Lemma 4.1 . The proof is straightforward. Let $\{x_k\}_{k\in\mathbb{N}}$ be a sequence in $O \bigcap Z$ such that $\lim_{k\to\infty} x_k = x \in O$. Because x is a point of continuity of $t(x)$,

$$\limsup_{k\to\infty} C_r(x_k, t(x_k)) = C_r(x, t(x)),$$

and

$$\limsup_{k\to\infty} \pi(C_r(x_k, t(x_k)) = \pi(C_r(x, t(x)).$$

Then,

$$\begin{aligned} \frac{\lambda(C_r(x, t(x)))}{r^n} &\le \frac{|\lambda(C_r(x, t(x)) - \lambda(C_r(x_k, t(x_k)))|}{r^n} \\ &+ \frac{\lambda(C_r(x_k, t(x_k)))}{r^n} \le \frac{\lambda(C_r(x, t(x))\Delta C_r(x_k, t(x_k)))}{r^n} \\ &+ \frac{\lambda(C_r(x_k, t(x_k)))}{r^n} = A_1 + A_2. \end{aligned}$$

Pick $\varepsilon > 0$.

$$\begin{aligned} A_1 &= \int_{\pi(C_r(x,t(x))\Delta C_r(x_k,t(x_k)))} (1 - u_I(y))_+ dy \\ &\le \mathcal{L}^n(\pi(C_r(x, t(x))\Delta C_r(x_k, t(x_k)))) < \frac{\varepsilon}{2} r^n \end{aligned}$$

if k is chosen $\le k_0$. For $k = k_0$, A_2 can be made $< \frac{\varepsilon}{z}$ by choosing r small enough. Here $A\Delta B$ stands for the symmetric difference of the sets A and B. □

Lemma 4.2. *Let* $(x_0, t_0) \in \partial\{(u-1)_+ > 0\}$ *such that there exists an inner tangent* n*-ball in* $\mathbb{R}^n \times \{t_0\}$, $B_{r+0}(y_0)$, *to* $\partial\{(u-1)_+ > 0\}$ *at* (x_0, t_0). *Then the free boundary moves strictly outward at* (x_0, t_0), *i.e.,* $(u(x_0, t) - 1)_+ > 0$ *for* $t > t_0$, *and* (x_0, t_0) *is not an interior point of a "vertical" (i.e. , parallel to the* t*-axis) segment at* x_0.

Proof. By the maximum principle, $(u(x,t)-1)_+$ is positive on $\partial B_{r_0}(y_0) \times \{t_0\}) \setminus \{(x_0, t_0)\}$. Pick a positive function $v_{t_0}(x)$, radially symmetric and radially decreasing in $B_r(y_0)$ with respect to the center y_0, such that $v_{t_0}(x) \geq u(x, t_0)$ in $B_{r_0}(y)$. Set $v_{t_0}(x) \equiv 0$ on $B_r^C(y_0)$. The solution $v(x,t)$ to (1.1) in $\mathbb{R}^n \times (t_0, T)$ generated by $v_{t_0}(x)$ is radially decreasing for all $T > t > t_0$, $v(x,t) \leq u(x,t)$ in $\mathbb{R}^n \times (t_0, T)$, and the boundary of $\{(v(x,t)-1)_+ > 0\}$ expands in outward direction with strictly positive speed. □

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