HOMEWORK #8 - MA 504

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Chapter 5, problem 5. Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.

Solution.
Since $f'(x) \to 0$ as $x \to +\infty$, for every $\epsilon > 0$, there exists $M > 0$ such that

$$|f'(x)| < \epsilon, \quad x \geq M.$$

By the Mean Value Theorem (thm 5.10), for every x there exists c such that $x < c < x+1$ and

$$f(x+1) - f(x) = f'(c).$$

Hence

$$|g(x)| = |f(x+1) - f(x)| = |f'(c)| < \epsilon, \quad x \geq M.$$

Since $\epsilon > 0$ is arbitrary, we see that $g(x) \to 0$ as $x \to +\infty$.

Chapter 5, problem 6. Suppose
(a) f is continuous for $x \geq 0$,
(b) $f'(x)$ exists for $x > 0$,
(c) $f(0) = 0$,
(d) f' is monotonically increasing.
Put

$$g(x) = \frac{f(x)}{x} \quad (x > 0)$$

and prove that g is monotonically increasing.

Solution.
First of all, by theorem 5.3 (c), g is differentiable at every $x > 0$.
By theorem 5.10 (Mean Value Theorem), for every $x > 0$, there exists c_x, $0 < c_x < x$, such that

$$f(x) = f(x) - f(0) = xf'(c_x).$$

Hence $g(x) = f'(c_x)$.
Now assume that there exist $x > y$ such that $c_x < c_y$. Then, since f' is monotonically increasing, $g(y) = f'(c_y) > f'(c_x) = g(x)$. By the Mean Value theorem, there exists c, $y < c < x$ such that

$$f(x) - f(y) = (x-y)f'(c)$$
$$\Rightarrow (x-y)f'(c_y) > xf'(c_x) - yf'(c_y) = f(x) - f(y) = (x-y)f'(c)$$
$$\Rightarrow f'(c_y) > f'(c),$$

but $c_y < y < c$ and f' is monotonically increasing. So we got a contradiction, $c_y \leq c_x$, and therefore $g(y) \leq g(x)$.

Chapter 5, problem 9. Let f be a continuous real function on $\mathbb{R}$, of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 3$ as $x \to 0$. Does it follow that $f'(0)$ exists?

Solution.
Answer: SIM! (YES!).
Indeed, we have that $f(x) - f(0)$ and $g(x) = x$ are real and differentiable in $(0, 1)$, and $g'(x) = 1 \neq 0$ for all $x \in (0, 1)$, and

$$\frac{f'(x)}{g'(x)} = f'(x) \to 3 \quad \text{as } x \to 0^+,$$

$$f(x) - f(0) \to 0 \text{ and } g(x) \to 0 \text{ as } x \to 0,$$

then by L'Hôpital's rule

$$\frac{f(x) - f(0)}{x} \to 3 \text{ as } x \to 0^+.$$

Similarly one can show that

$$\frac{f(x) - f(0)}{x} \to 3 \text{ as } x \to 0^-.$$

Hence $f'(0)$ exists and $f'(0) = 3$.

Chapter 5, problem 14. Let f be a differentiable real function defined in (a, b). Prove that f is convex if and only if f' is monotonically increasing. Assume that $f''(x)$ exists for every $x \in (a, b)$, and prove that f is convex if and only if $f''(x) \geq 0$ for all $x \in (a, b)$.

Solution.
Assume that f is convex. Let $x > y$. We have that

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} = \lim_{t \to x^+} \frac{f(t) - f(x)}{t - x},$$

$$f'(y) = \lim_{s \to y} \frac{f(y) - f(s)}{y - s} = \lim_{s \to y^-} \frac{f(y) - f(s)}{y - s}.$$

By Problem 23 (Chapter 4), we have for $s < y < x < t$

$$\frac{f(t) - f(x)}{t - x} \geq \frac{f(t) - f(s)}{t - s} \geq \frac{f(y) - f(s)}{y - s},$$

hence

$$f'(x) = \lim_{t \to x^+} \frac{f(t) - f(x)}{t - x} \geq \lim_{s \to y^-} \frac{f(y) - f(s)}{y - s} = f'(y).$$

So f' is monotonically increasing.
Now if f' is monotonically increasing, then by the Mean Value Theorem, for every $x, y \in (a,b)$, $x > y$, there exists $c \in (y,x)$ such that

$$\frac{f(x)-f(y)}{x-y} = f'(c).$$

Since f' is monotonically increasing, we have

$$f'(y) \leq \frac{f(x)-f(y)}{x-y} = f'(c) \leq f'(x).$$

Hence for every $y < z < x$,

$$f'(y) \leq \frac{f(z)-f(y)}{z-y} \leq f'(z) \leq \frac{f(x)-f(z)}{x-z} \leq f'(x).$$

So if $z = \lambda x + (1-\lambda)y, \lambda \in (0,1)$, we have

$$(\lambda x + (1-\lambda)y - y)(f(x) - f(\lambda x + (1-\lambda)y) \geq (x - \lambda x - (1-\lambda)y)(f(\lambda x + (1-\lambda)y) - f(y)$$
$$\Rightarrow \lambda(x-y)(f(x) - f(\lambda x + (1-\lambda)y)) \geq (1-\lambda)(x-y)(f(\lambda x + (1-\lambda)y) - f(y))$$
$$\Rightarrow \lambda(f(x) - f(\lambda x + (1-\lambda)y)) \geq (1-\lambda)(f(\lambda x + (1-\lambda)y) - f(y))$$
$$\Rightarrow \lambda f(x) + (1-\lambda)f(y) \geq f(\lambda x + (1-\lambda)y).$$

Therefore f is convex.
Now note that if $f''(x)$ exists for every $x \in (a,b)$, the f' is continuous in (a,b). So if $f''(x) < 0$ for some $x \in (a,b)$, then there exists an $\epsilon > 0$ such that

$$f''(x) < -\epsilon.$$

Now, we have that there exists an $\delta > 0$ such that

$$-\frac{\epsilon}{2} < \frac{f'(x)-f'(t)}{x-t} - f''(x) < \frac{\epsilon}{2}, \quad |x-t| < \delta.$$

So, in particular, for $t < x, |x-t| < \delta$,

$$\frac{f'(x)-f'(t)}{x-t} < f''(x) + \frac{\epsilon}{2} < -\epsilon + \frac{\epsilon}{2} = -\frac{\epsilon}{2} < 0$$
$$\Rightarrow f'(x) - f'(t) < 0 \Rightarrow f'(x) < f'(t),$$

a contradiction with f' being monotonically increasing.
Therefore $f''(x) \geq 0$.
Now if $f''(x) \geq 0$ for every $x \in (a,b)$, then by theorem 5.11, f' is monotonically increasing in (a,b), so as we showed previously, f is convex.

Chapter 5, problem 26. Suppose f is differentiable on $[a,b]$, $f(a) = 0$, and there is a real number A such that $|f'(x)| \leq A|f(x)|$ on $[a,b]$. Prove that $f(x) = 0$ for all $x \in [a,b]$.

Solution.
Since f is differentiable on $[a,b]$, then f is continuous on $[a,b]$, so f is bounded on $[a,b]$.

Also, since $|f'(x)| \leq A|f(x)|$ on $[a,b]$, then f' is also bounded on $[a,b]$. Therefore we can take, for a fixed $x_0 \in [a,b]$,

$$M_0 = \sup_{x\in[a,x_0]} |f(x)|, \quad M_1 = \sup_{x\in[a,x_0]} ||f'(x)|.$$

So, for any x, $a \leq x \leq x_0$, as a consequence of the Mean Value Theorem

$$|f(x)| = |f(x) - f(a)| \leq M_1(x-a) \leq M_1(x_0 - a) \leq A(x_0-a)M_0.$$

Hence $M_0 = 0$ if $A(x_0 - a) < 1$, otherwise $N_0 = A(x_0 - a)M_0 < M_0$ and $N_0 \geq |f(x)|$ for all $x \in [a, x_0]$, so $N_0 \geq M_0$, a contradiction.
Therefore $f(x) = 0$ for all $x \in \left[a, a + \dfrac{1}{A}\right)$, assuming $a + \dfrac{1}{A} \leq b$ and $A > 0$, otherwise $f'(x) = 0$ (since $|f'(x)| \leq A|f(x)|$) for all $x \in [a,b]$, so f would be constant and since $f(a) = 0$, we have then $f(x) = 0$ for all $x \in [a,b]$. By continuity it follows that $f\left(a + \frac{1}{A}\right) = 0$.
Similarly one can show that $f(x) = 0$ for all $x \in \left[a + \dfrac{1}{A}, \min\left\{a + \dfrac{2}{A}, b\right\}\right]$, so $f(x) = 0$ for all $x \in \left[a, \min\left\{a + \dfrac{2}{A}, b\right\}\right]$. We can continue this process and get that $f(x) = 0$ for all $x \in \left[a, \min\left\{a + \dfrac{n}{A}, b\right\}\right]$. But there exists $n \in \mathbb{N}$ such that $\frac{n}{A} \geq b$, so we get $f(x) = 0$ for all $x \in [a,b]$.

Chapter 5, problem 27. Let ϕ be a real function defined on a rectangle R in the plane, given by $a \leq x \leq b$, $\alpha \leq y \leq \beta$. A solution of the initial-value problem

$$y' = \phi(x,y), \quad y(a) = c \quad (\alpha \leq c \leq \beta)$$

is, by definition, a differentiable function f on $[a,b]$ such that $f(a) = c$, $\alpha \leq f(x) \leq \beta$, and

$$f'(x) = \phi(x, f(x)) \quad (a \leq x \leq b).$$

Prove that such a problem has at most one solution if there exists a constant A such that

$$|\phi(x,y_2) - \phi(x,y_1)| \leq A|y_2 - y_1|$$

whenever $(x,y_1) \in R$ and $(x,y_2) \in R$.

Solution.
Assume that there exist two differentiable functions f_1 and f_2 on $[a,b]$ such that $f_1(a) = f_2(a) = c$, $\alpha \leq f_1(x), f_2(x) \leq \beta$, and

$$f_1'(x) = \phi(x, f_1(x)) \quad f_2'(x) = \phi(x, f_2(x)),$$

where there exists a constant A such that

$$|\phi(x,y_2) - \phi(x,y_1)| \leq A|y_2 - y_1|$$

whenever $(x,y_1) \in R$ and $(x,y_2) \in R$.

Let $f = f_1 - f_2$. Then f f is differentiable on $[a,b]$, $f(a) = 0$, and there is a real number

A such that $|f'(x)| \leq A|f(x)|$ on $[a, b]$. So by the previous problem (problem 26), $f(x) = 0$ for all $x \in [a, b]$, ie, $f_1 = f_2$ as we wanted to show.