Chapter 6, problem 4. If \( f(x) = 0 \) for all irrational \( x \), \( f(x) = 1 \) for all rational \( x \), prove that \( f \not\in \mathbb{R} \) on \([a, b]\) for any \( a < b \).

Solution.
Let \( a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b \) be a partition of \([a, b]\), call it \( P \). We have, by the density of the rationals (respectively the irrationals) on \( \mathbb{R} \),

\[
1 = M_i = \sup f(x) \quad (x_{i-1} \leq x \leq x_i),
\]

\[
0 = m_i = \inf f(x) \quad (x_{i-1} \leq x \leq x_i),
\]

so

\[
U(P, f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = b - a,
\]

\[
L(P, f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = 0.
\]

Since \( P \) is an arbitrary partition of \([a, b]\), we have

\[
\int_{a}^{b} f dx = b - a > 0 = \int_{a}^{b} f(x) dx.
\]

Hence \( f \not\in \mathbb{R} \) on \([a, b], a < b \).

Chapter 6, problem 5. Suppose \( f \) is a bounded real function on \([a, b]\), and \( f^2 \in \mathbb{R} \) on \([a, b]\). Does it follow that \( f \in \mathbb{R} \) on \([a, b]\)? Does the answer change if we assume that \( f^3 \in \mathbb{R} \)?

Solution.
Answer to the first question: NÃO! (NO!).

Indeed, let \( f(x) = 1 \) for all irrational \( x \), \( f(x) = -1 \) for all rational \( x \). Similarly as we showed in the previous problem, one can show that

\[
\int_{a}^{b} f dx = b - a > 0 = \int_{a}^{b} f(x) dx.
\]

Hence \( f \not\in \mathbb{R} \), but \( f^2(x) = 1 \) for all \( x \), so \( f^2 \in \mathbb{R} \) on \([a, b]\).

Answer to the second question: SIM! (YES!).

We have that \( \phi(x) = \sqrt{x} \) is continuous on \([a, b]\) for any \( a, b \in \mathbb{R} \), so by theorem 6.11 if \( f^3 \in \mathbb{R} \), then \( h = \phi \circ f^3 \in \mathbb{R} \), where \( h(x) = \phi(f^3(x)) = f(x) \), ie, \( h = f \).
Chapter 6, problem 10. Let $p$ and $q$ be positive real numbers such that
\[
\frac{1}{p} + \frac{1}{q} = 1.
\]
Prove the following statements.
(a) If $u \geq 0$ and $v \geq 0$, then
\[
uv \leq \frac{u^p}{p} + \frac{v^q}{q}.
\]
Equality holds if and only if $u^p = v^q$.
(b) If $f \in \mathcal{R}(\alpha)$, $g \in \mathcal{R}(\alpha)$, $f \geq 0$, $g \geq 0$, and
\[
\int_a^b f^p \, d\alpha = 1 = \int_a^b g^q \, d\alpha,
\]
then
\[
\int_a^b fg \, d\alpha \leq 1.
\]
(c) If $f$ and $g$ are complex functions in $\mathcal{R}(\alpha)$, then
\[
\left| \int_a^b fg \, d\alpha \right| \leq \left\{ \int_a^b |f|^p \, d\alpha \right\}^{1/p} \left\{ \int_a^b |g|^q \, d\alpha \right\}^{1/q}.
\]
(d) Show that Hölder’s inequality is also true for the “improper” integrals described in Exercises 7 and 8.

Solution.
First of all, assume $u, v > 0$, otherwise the inequality is trivial. We see that by making the substitution $\tilde{u} = u^p > 0$ and $\tilde{v} = v^q > 0$ the inequality that we want to show is equivalent to the following inequality
\[
\left( \frac{\tilde{u}}{\tilde{v}} \right)^{1/p} \leq \frac{1}{p} \left( \frac{\tilde{u}}{\tilde{v}} \right) + \frac{1}{q}.
\]
Now if we make the substitution $z = \frac{\tilde{u}}{\tilde{v}}$ and assume without loss of generality $\tilde{u} \geq \tilde{v}$, so $z \geq 1$, it suffices to show
\[
z^{1/p} \leq \frac{z}{p} + \frac{1}{q},
\]
whenever $z \geq 1$, and equality holds if and only if $z = 1$.
Now the previous inequality is equivalent, by making $x = z^{1/p} \geq 1$ to
\[
0 \leq \frac{x^p}{p} - x + \frac{1}{q}.
\]
Let $f(x) = \frac{x^p}{p} - x + \frac{1}{q}$. We have $f(1) = 0$ and $f'(x) = x^{p-1} > 1 > 0$, 

whenever $x > 1$. So $f$ is a strictly increasing function on $(1, +\infty)$. In particular,

$$f(x) = \frac{x^p}{p} - x + \frac{1}{q} > f(1) = 0,$$

whenever $x > 1$, as we wanted to show.

(b) If $f \in \mathcal{R}(\alpha)$, $g \in \mathcal{R}(\alpha)$, $f \geq 0$, $g \geq 0$, and

$$\int_a^b f^p \, d\alpha = 1 = \int_a^b g^q \, d\alpha,$$

then it follows from the previous item that

$$\int_a^b fg \, d\alpha \leq \int_a^b \left( \frac{f^p}{p} + \frac{g^q}{q} \right) \, d\alpha = \frac{1}{p} + \frac{1}{q} = 1.$$

(c) If $f$ and $g$ are complex functions in $\mathcal{R}(\alpha)$. Assume without loss of generality that

$$\int_a^b |f|^p \, d\alpha > 0, \quad \int_a^b |g|^q \, d\alpha > 0.$$

Then let

$$\tilde{f} = \frac{|f|}{(\int_a^b |f|^p \, d\alpha)^{1/p}}, \quad \tilde{g} = \frac{|g|}{(\int_a^b |g|^q \, d\alpha)^{1/q}}.$$

We have $\tilde{f} \in \mathcal{R}(\alpha)$, $\tilde{g} \in \mathcal{R}(\alpha)$, $\tilde{f} \geq 0$, $\tilde{g} \geq 0$, and

$$\int_a^b \tilde{f}^p \, d\alpha = 1 = \int_a^b \tilde{g}^q \, d\alpha,$$

so it follows from the previous item that

$$\int_a^b \tilde{f} \tilde{g} \, d\alpha \leq 1,$$

which implies

$$\left| \int_a^b fg \, d\alpha \right| \leq \int_a^b |f||g| \, d\alpha \leq \left\{ \int_a^b |f|^p \, d\alpha \right\}^{1/p} \left\{ \int_a^b |g|^q \, d\alpha \right\}^{1/q}.$$

(d) By the definitions given in problems 7 and 8, the result follows trivially.

**Chapter 6, problem 11.** Let $\alpha$ be a fixed increasing function on $[a, b]$. For $u \in \mathcal{R}(\alpha)$, define

$$\|u\|_2 = \left\{ \int_a^b |u|^2 \, d\alpha \right\}^{1/2}.$$

Suppose that $f, g, h \in \mathcal{R}(\alpha)$, and prove the triangle inequality

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2$$

as a consequence of the Schwartz inequality, as in the proof of Theorem 1.37.
Solution.
We have
\[
\|f - h\|_2^2 = \int_a^b |f - h|^2 \, d\alpha \leq \int_a^b (|f - g| + |g - h|)^2 \, d\alpha
\]
\[
\Rightarrow \|f - h\|_2^2 \leq \int_a^b (|f - g|^2 + 2|f - g||g - h| + |g - h|^2) \, d\alpha = \|f - g\|_2^2 + 2\int_a^b |f - g||g - h| \, d\alpha + \|g - h\|_2^2.
\]
but it follows from the Schwartz inequality that
\[
\int_a^b |f - g||g - h| \, d\alpha \leq \left\{\int_a^b |f - g|^2 \, d\alpha\right\}^{1/2} + \left\{\int_a^b |g - h|^2 \, d\alpha\right\}^{1/2}.
\]
So
\[
\|f - h\|_2^2 \leq \|f - g\|_2^2 + 2\|f - g\|_2\|g - h\|_2 + \|g - h\|_2^2 = (\|f - g\|_2 + \|g - h\|_2)^2.
\]

**Chapter 7, problem 2.** If \(\{f_n\}\) and \(\{g_n\}\) converge uniformly on a set \(E\), prove that \(\{f_n + g_n\}\) converges uniformly on \(E\). If, in addition, \(\{f_n\}\) and \(\{g_n\}\) are sequences of bounded functions, prove that \(f_n g_n\) converges uniformly on \(E\).

Solution.
Assume \(f_n \to f\) uniformly and \(g_n \to g\) uniformly. Then given \(\epsilon > 0\), there exists \(N_1\) and \(N_2\) such that
\[
|f_n(x) - f(x)| < \epsilon \quad \forall n \geq N_1, x \in E,
\]
\[
|g_n(x) - g(x)| < \epsilon \quad \forall n \geq N_2, x \in E.
\]
So
\[
|f_n(x) + g_n(x) - f(x) - g(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \epsilon + \epsilon = 2\epsilon, \quad \forall n \geq N, x \in E,
\]
where \(N = \max\{N_1, N_2\}\). Hence \((f_n + g_n) \to (f + g)\) uniformly.
Now assume that there exists \(M_n\) and \(K_n\) such that
\[
|f_n(x)| \leq M_n \quad \forall x \in E,
\]
\[
|g_n(x)| \leq K_n \quad \forall x \in E.
\]
Then first we see that, since \(f_n \to f\) uniformly and \(g_n \to g\) uniformly, there exists \(N_1\) and \(N_2\) such that
\[
|f_n(x) - f_m(x)| < 1 \quad \forall n, m \geq N_1, x \in E,
\]
\[
|g_n(x) - g_m(x)| < 1 \quad \forall n, m \geq N_2, x \in E.
\]
Let \(N = \max\{N_1, N_2\}\). We have
\[
|f_n(x)| \leq |f_N(x)| + |f_n(x) - f_N(x)| \leq M_N + 1, n \geq N,
\]
so
\[
|f_n(x)| \leq M = \max\{M_1, M_2, ..., M_{N-1}, M_N + 1\}, \quad \forall n, x \in E.
\]
Similarly, one can show
\[
|g_n(x)| \leq K = \max\{K_1, K_2, ..., K_{N-1}, K_N + 1\}, \quad \forall n, x \in E.
\]
In particular, $|f(x)| \leq M$ and $|g(x)| \leq K$ for all $x \in E$.
Now we have

$$|f_n(x)g_n(x) - f(x)g(x)| = |f_n(x)(g_n(x) - g(x)) + g(x)(f_n(x) - f(x))|$$

$$\Rightarrow |f_n(x)g_n(x) - f(x)g(x)| \leq |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)|$$

$$\Rightarrow |f_n(x)g_n(x) - f(x)g(x)| \leq M|g_n(x) - g(x)| + K|f_n(x) - f(x)|,$$

but since $M, K < \infty$ and $|g_n - g| \to 0$ and $|f_n - f| \to 0$ uniformly, it follows from the inequality above that $|f_n g_n - fg| \to 0$ uniformly, i.e., $f_n g_n \to fg$ uniformly.

**Chapter 7, problem 5.** Let

$$f_n(x) = \begin{cases} 0 & (x < \frac{1}{n+1}) , \\ \sin^2 \frac{\pi}{x} & \left(\frac{1}{n+1} \leq x \leq \frac{1}{n}\right) , \\ 0 & \left(\frac{1}{n} < x\right) . \end{cases}$$

Show that $\{f_n\}$ converges to a continuous function, but not uniformly. Use the series $\sum f_n$ to show that absolute convergence, even for all $x$, does not imply uniform convergence.

**Solution.**

Clearly we see that $f_n(x) \to 0$ for all $x$, since for any $x > 0$ there exists $N$ such that

$$\frac{1}{n} \leq \frac{1}{N} < x, \quad n \geq N.$$

If $x \leq 0$, then $f_n(x) = 0$ for all $n$.

Now given $\epsilon > 0$, let $x_n = \frac{2}{2n+1}$. Then we see that

$$\frac{1}{n+1} \leq \frac{2}{2n+1} \leq \frac{1}{n},$$

so

$$f_n(x_n) = \sin^2 \left(\frac{(2n+1)\pi}{2}\right) = 1, \quad \forall n.$$

Therefore, since $n$ is arbitrary, $\{f_n\}$ does not converge uniformly to 0.

We see that if $x \geq 1$ or $x \leq 0$, then $f_n(x) = 0$ for all $n$, and if $0 < x < 1$, there exists at most two $n$'s such that $\frac{1}{n+1} \leq x \leq \frac{1}{n}$, in this case $x = \frac{1}{k}$ for some $k \in \mathbb{N}$.

Hence trivially we see that $\sum f_n$ is convergent, in particular absolute convergent since $f_n \geq 0$. But as we saw previously $f_n$ does not converge uniformly.