HOMEWORK #5 - MA 504

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Chapter 3, problem 5. For any two real sequences $\{a_n\}, \{b_n\}$, prove that

$$\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n,$$

provided the sum on the right is not of the form $\infty - \infty$.

Solution.
First assume that the sum on the right hand side is not of the form $\infty + \infty$ or $-\infty - \infty$, otherwise we clearly have the equality.
Assume that one of the terms of the sum of the right hand side is ∞. Then without loss of generality consider $\limsup_{n\to\infty} a_n = \infty$ and $\limsup_{n\to\infty} b_n = b$. So $\limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n = \infty$ and the inequality is trivially satisfied.
Now assume $\limsup_{n\to\infty} a_n = -\infty$, and $\limsup_{n\to\infty} b_n = b \in (-\infty, \infty)$. Let $x \in \mathbb{R}$ be such that $a_{n_k} \to x$ for some subsequence $\{a_{n_k}\}$ of $\{a_n\}$. By definition 3.16, $x \le \limsup_{n\to\infty} a_n$. Therefore one must have that $x = -\infty$. Since $\limsup_{n\to\infty} b_n = b \in (-\infty, \infty)$, it follows from theorem 3.17(b) that $\{b_n\}$ is bounded, so one sees that $\limsup_{n\to\infty}(a_n + b_n) = -\infty$ and the inequality is clearly sastified.
Finally assume that $\limsup_{n\to\infty} a_n = a \in (-\infty, \infty)$ and $\limsup_{n\to\infty} b_n = b \in (-\infty, \infty)$.
Let $y \in \mathbb{R}$ be such that $a_{n_k} + b_{n_k} \to y$ for some subsequence $\{a_{n_k} + b_{n_k}\}$ of $\{a_n + b_n\}$.
Since $\{a_{n_k}\}$ and $\{b_{n_k}\}$ are bounded, by theorem 2.42 (Weierstrass) there exists convergent subsequences $\{a_{n_{k_j}}\}$ and $\{b_{n_{k_j}}\}$ such that $a_{n_{k_j}} \to s$, $b_{n_{k_j}} \to t$. Then

$$y = \lim_{k\to\infty}(a_{n_k} + b_{n_k}) = \lim_{j\to\infty}(a_{n_{k_j}} + b_{n_{k_j}}) = s + t,$$

and we have

$$y = s + t \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Since y is an arbitrary real number with the property defined in definition 3.16, we have that

$$\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Chapter 3, problem 20. Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X, and some subsequence $\{p_{n_i}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to p.

Solution.
Since $\{p_n\}$ is a Cauchy sequence and $\{p_{n_i}\}$ converges to a point $p \in X$, given $\epsilon > 0$ there exists N_1, N_2 such that

$$d(p_n, p_m) < \frac{\epsilon}{2}, \quad n, m \geq N_1;$$

$$d(p_{n_i}, p) < \frac{\epsilon}{2}, \quad i \geq N_2.$$

Assume that $n_i \leq n_{i+1}$. So

$$d(p_n, p) \leq d(p_n, p_{n_i}) + d(p_{n_i}, p) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

if $n, n_i \geq N = \max\{N_1, n_{N_2}\}$. Since $\epsilon > 0$ is arbitrary, $p_n \to p$.

Chapter 3, problem 21. Prove the following analoge of Theorem 3.10(b): If $\{E_n\}$ is a sequence of closed nonempty and bounded sets in a complete metric space X, if $E_n \supset E_{n+1}$, and if

$$\lim_{n\to\infty} \text{diam } E_n = 0,$$

then $\cap_1^\infty E_n$ consists of exactly one point.

Solution.
First we see that if there exists a N such that E_N contains only one point, say $E_N = \{p\}$, then since $E_n \supset E_{n+1}$, $E_n = E_N = \{p\}$ for all $n \geq N$ and $\cap_1^\infty E_n = \{p\}$.
Now assume that every E_n contains at least two points. Then

$$\text{diam } E_n = \sup\{d(p, q) : p, q \in E_n\} > 0 \quad \forall n.$$

Since

$$\lim_{n\to\infty} \text{diam } E_n = 0,$$

given $\epsilon > 0$, there exists N_k such that diam $E_n < \epsilon$ for $n \geq N$. So $d(p, q) < \epsilon$ for all $p, q \in E_n$, $n \geq N$. For each n choose $p_n \in E_n$. Then, since $E_n \supset E_m$ if $m \geq n$, we have

$$d(p_n, p_m) < \epsilon \quad m \geq n \geq N.$$

Therefore, since $\epsilon > 0$ is arbitrary, $\{p_n\}$ is a Cauchy sequence in X, which is a complete metric space, so $\{p_n\}$ converges. Say $p_n \to p \in X$. We have that $p \in \cap_1^\infty E_n$. Indeed, p is a limit point of $\cap_1^\infty E_n$, which is an intersection of closed sets, so $\cap_1^\infty E_n$ is closed and then $p \in \cap_1^\infty E_n$.
Then $\cap_1^\infty E_n$ is nonempty and contains at least one point. Now let us show that $\cap_1^\infty E_n$ does not contain more than one point. Suppose that there exist $p, q \in \cap_1^\infty E_n$. Then, say for any $0 < \epsilon < d(p, q)$,

$$\text{diam } E_n \geq \text{diam } \cap_1^\infty E_n \geq d(p, q) > \epsilon, \quad \forall n.$$

But this contradicts $\lim_{n\to\infty} \operatorname{diam} E_n = 0$. Therefore $\cap_1^\infty E_n$ consists of exactly one point.

Chapter 3, problem 23. Suppose that $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that the sequence $\{d(p_n, q_n)\}$ converges.

Solution.
Since $\{p_n\}$ and $\{q_n\}$ are Cauchy, given $\epsilon > 0$, there exists N_1 and N_2 such that

$$d(p_n, p_m) < \frac{\epsilon}{2}, \quad n, m \geq N_1;$$
$$d(q_n, q_m) < \frac{\epsilon}{2}, \quad n, m \geq N_2.$$

By the triangle inequality, for any m, n,

$$d(p_n, q_n) \leq d(p_n, q_m) + d(q_m, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n).$$

So if $n, m \geq N = \max\{N_1, N_2\}$, we have

$$|d(p_n, q_n) - d(p_m, q_m)| < d(p_n, p_m) + d(q_m, q_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $\{d(p_n, q_m)\} \subset \mathbb{R}$ is a Cauchy sequence, and it follows from theorem 3.11(c) that $\{d(p_n, q_m)\}$ coverges.