HOMEWORK #8 - MA 504

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Chapter 4, problem 14. Let $I = [0,1]$ be the closed unit interval. Suppose f is a continuous map of I into I. Prove that $f(x) = x$ for at least one $x \in I$.

Solution.
Consider the function $g(x) = f(x) - x$. Then g is a continuous function of $I = [0,1]$ into $[-1,1]$, since $0 \leq f(x) \leq 1$. We have then

$$g(0) = f(0) \geq 0, \quad g(1) = f(1) - 1 \leq 0.$$

If $g(0) = 0$ or $g(1) = 1$, the $f(0) = 0$ or $f(1) = 1$ and we are done, otherwise $g(0) > 0$ and $g(1) < 0$, therefore by the intermediate value theorem there exists $x \in [0,1]$ such that $g(x) = 0$, ie, $f(x) = x$.

Chapter 4, problem 16. Let $[x]$ be the largest integer contained in x, that is, $[x]$ is the integer such that $x - 1 < [x] \leq x$; and let $(x) = x - [x]$ denote the fractional part of x. What discontinuities do the functions $[x]$ and (x) have?

Solution.
First note that since $f(x) = x$ is continuous in $\mathbb{R}$, (x) is continuous in x if and only if $[x]$ is continuous in x.
Note that

$$[x] = n, \quad \text{for } x \in [n, n+1), n \in \mathbb{N}.$$

Therefore $[x]$ is discontinuous at x if and only if $x \in \mathbb{N}$, and $[n_+] = n$, $[n_-] = n - 1$, for every $n \in \mathbb{N}$. So both $[x]$ and (x) only have discontinuities of first kind (simple discontinuities).

Chapter 4, problem 20. If E is a nonempty subset of a metric space X, define the distance from $x \in X$ to E by

$$\rho_E(x) = \inf_{z \in E} d(x,z).$$

(a) Prove that $\rho_E(x) = 0$ if and only if $x \in \overline{E}$.
(b) Prove that ρ_E is a uniformly continuous function on X, by showing that

$$|\rho_E(x) - \rho_E(y)| \leq d(x,y)$$

for all $x \in X, y \in X$.

Solution.
(a) Suppose that $\rho_E(x) = 0$. Then for any $\epsilon > 0$, there exists $z \in E$ such that

$$d(x,z) < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, this shows that in any neighborhood of x, we can find $z \in E$, ie, $x \in \overline{E}$.
Conversely, if $x \in \overline{E}$, for any $\epsilon > 0$, there exists $w \in E$ such that $w \in B_\epsilon(x) = \{y \in X : |x - y| < \epsilon\}$, ie,

$$\rho_E(x) = \inf_{z \in E} d(x, z) \leq d(w, x) < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, $\rho_E(x) = 0$.

(b) We have that for any $x \in X$, $y \in X$, $z \in E$, by the triangle inequality

$$\rho_E(x) \leq d(x, z) \leq d(x, y) + d(y, z).$$

Since $z \in E$ is arbitrary, in the right hand side we can take the inf over $z \in E$ and get

$$\rho_E(x) \leq d(x, y) + \rho_E(y).$$

Similarly one can show

$$\rho_E(y) \leq d(x, y) + \rho_E(x),$$

and hence

$$|\rho_E(x) - \rho_E(y)| \leq d(x, y).$$

Chapter 4, problem 21. Suppose K and F are disjoint sets in a metric space X, K is compact, F is closed. Prove that there exists $\delta > 0$ such that $d(p, q) > \delta$ if $p \in K, q \in F$. Show that the clonclusion may fail for two disjoint closed sets if neither is compact.

Solution.
Let us show first that the function ρ_F defined in problem 20 is a continuous positive function. Given $\epsilon > 0$, let $\delta = \epsilon$, then by the previous problem

$$|\rho_F(x) - \rho_F(y)| \leq d(x, y) < \delta = \epsilon.$$

Therefore ρ_F is continuous and since the metric d is a nonnegative function we see that $\rho_F(x) \geq 0$, for any $x \in X$.
If the assume the contrary of the statement given above, then for any $n \in \mathbb{N}$, there exists $p_n \in K, q_n \in F$ such that

$$d(p_n, q_n) \leq \frac{1}{n},$$

this, in particular, implies

$$\rho_F(p_n) \leq \frac{1}{n},$$

in particular, $\rho_F(p_n) \to 0$.
But since K is compact and $\{p_n\} \subset K$, there exists a subsequence $\{p_{n_i}\}$ such that $p_{n_i} \to p \in K$.
Since ρ_F is continuous, and $\rho_F(p_n) \to \rho_F(p)$, so $\rho_F(p) = 0$. By the previous problem, this implies that $p \in \overline{F} = F$. A contradiction with K and F being disjoint.
Hence there exists $\delta > 0$ such that $d(p, q) > \delta$ if $p \in K, q \in F$.

Now let $X = \mathbb{R}$, $F = \left\{n + \frac{1}{n} : n \in \mathbb{N}\right\}$ and $K = \mathbb{N}$. We have that for any $\delta > 0$, there exists $n \in \mathbb{N}$ such that $\dfrac{1}{n} < \delta$, so

$$d\left(n, n + \frac{1}{n}\right) = \left|n - \left(n + \frac{1}{n}\right)\right| = \left|\frac{1}{n}\right| < \delta.$$

Chapter 5, problem 23. A real-valued function f defined in (a, b) is said to be convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

whenever $a < x < b$, $a < y < b$, $0 < \lambda < 1$. Prove that every convex function is continuous. Prove that every increasing convex function of a convex function is convex.
If f is convex in (a, b) and if $a < s < t < u < b$, show that

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

Solution.
Let f be a convex function in (a, b). We want to show that f is continuous, ie, given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \quad \text{if } |x - y| < \delta.$$

Given $x \in (a, b)$, there exists $x_1, x_2 \in (a, b)$ such that $x_1 < x < x_2$. Since f is convex, we have

$$x = \left(\frac{x - x_1}{x_2 - x_1}\right) x_2 + \left[1 - \left(\frac{x - x_1}{x_2 - x_1}\right)\right] x_1, \quad 0 < \frac{x - x_1}{x_2 - x_1} < 1,$$

so

$$f(x) \leq \left(\frac{x - x_1}{x_2 - x_1}\right) f(x_2) + \left[1 - \left(\frac{x - x_1}{x_2 - x_1}\right)\right] f(x_1).$$

We can rewrite the equation above and get

$$\frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

or

$$(x_2 - x_1)(f(x) - f(x_1)) \leq (x - x_1)(f(x_2) - f(x_1)) \Rightarrow (x_2 - x)f(x_2) - f(x_1)(x_2 - x) \leq (f(x_2) - f(x))(x_2 - x_1)$$

$$\Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_2) - f(x)}{x_2 - x}.$$

Therefore for any $x, y \in [x_1, x_2]$, assume without loss of generality $x > y$, and since (a, b) is open there exist $x_0, x_3 \in (a, b)$ such that $x_0 < x_1 < x_2 < x_3$,

$$\frac{f(x) - f(y)}{x - y} \leq \frac{f(x_3) - f(y)}{x_3 - y} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2},$$

and

$$\frac{f(x) - f(y)}{x - y} \geq \frac{f(x) - f(x_0)}{x - x_0} \leq \frac{f(x_1) - f(x_0)}{x_1 - x_0},$$

ie,

$$\frac{|f(x)-f(y)|}{|x-y|} \le C = \max\left\{\frac{|f(x_3)-f(x_2)|}{x_3-x_2}, \frac{|f(x_1)-f(x_0)|}{x_1-x_0}\right\}.$$

So given $\epsilon > 0$, $x \in [x_1, x_2]$, let $\delta = \min\{\frac{\epsilon}{C}, \frac{x_2-x_1}{2}\} > 0$, then for any $y \in (x-\delta, x+\delta) \subset [x_1, x_2]$,

$$|f(x)-f(y)| \le C|x-y| \le C\frac{\epsilon}{C} = \epsilon.$$

Hence f is continuous on x and since $x \in (a,b)$ is arbitrary, f is continuous in (a,b).
Note that we showed above that if $a < s < t < u < b$, then

$$\frac{f(t)-f(s)}{t-s} \le \frac{f(u)-f(s)}{u-s} \le \frac{f(u)-f(t)}{u-t}.$$

Let $h = g \circ f$, ie, $h(x) = g(f(x))$, where g is an increasing function and f is a convex function. We have for $x, y \in (a,b), 0 < \lambda < 1$,

$$h(\lambda x + (1-\lambda)y) = g(f(\lambda x + (1-\lambda)y)) \le g(\lambda f(x) + (1-\lambda)f(y)).$$

Assume without loss of generality $f(x) \ge f(y)$ and $y > x$, otherwise consider $\lambda y + (1-\lambda)x$ instead of $\lambda x + (1-\lambda)y$. So $\lambda f(x) + (1-\lambda)f(y) \le f(x)$, and since g is increasing

$$h(\lambda x+(1-\lambda)y)-\lambda g(f(x)) \le \lambda g(\lambda f(x)+(1-\lambda)f(y))+(1-\lambda)g(\lambda f(x)+(1-\lambda)f(y))-\lambda g(x) \le$$

$$\le (1-\lambda)g(\lambda f(x)+(1-\lambda)f(y)) \le (1-\lambda)g(f(y)),$$

ie,

$$h(\lambda x + (1-\lambda)y) \le g(f(x)) + 1 - \lambda)g(f(y)).$$

Chapter 5, problem 1. Let f be defined for all real x, and suppose that

$$|f(x)-f(y)| \le (x-y)^2$$

for all real x and y. Prove that f is constant.

Solution.
We see that, $x \neq y$,

$$|f(x)-f(y)| \le (x-y)^2 = |x-y|^2 \Rightarrow \frac{|f(x)-f(y)|}{|x-y|} \le |x-y|.$$

Therefore

$$\lim_{y\to x}\frac{|f(x)-f(y)|}{|x-y|} = 0.$$

Hence f is differentiable at every $x \in \mathbb{R}$ and $f'(x) = 0$. By theorem 5.11 f must be constant.

Chapter 5, problem 2. Suppose $f'(x) > 0$ in (a,b). Prove that f is strictly increasing in (a,b), and let g be its inverse function. Prove that g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)} \quad (a < x < b).$$

Solution.
Let $x, y \in (a,b)$, $x < y$. Then by the Mean Value Theorem, there exists $c \in (x,y)$ such that
$$f(y) - f(x) = (y-x)f'(c) > 0.$$
Therefore $f(y) > f(x)$ and f is increasing. Hence the inverse of f is well defined, since f is injective, and let g be its inverse function, ie, $g(f(x)) = f(g(x)) = x$.
Given $y \in f(a,b)$, let $\{y_n\} \subset f(a,b)$ be such that $y_n \to y$, $y_n \neq y$. Then for each n, there exists $x_n \in (a,b)$ such that $f(x_n) = y_n$, and let $x \in (a,b)$ such that $f(x) = y$.
By theorem 4.17, g is continuous, so $x_n \to x$ (note $x_n \neq x$, since f is injective and $y_n = f(x_n) \neq y = f(x)$). We have
$$\left|\frac{g(y) - g(y_n)}{y - y_n} - \frac{1}{f'(x)}\right| = \left|\frac{g(f(x)) - g(f(x_n))}{f(x) - f(x_n)} - \frac{1}{f'(x)}\right| = \left|\frac{x - x_n}{f(x) - f(x_n)} - \frac{1}{f'(x)}\right|$$
Since $x_n \to x$ and f is differentiable at x, there exists N_1 such that
$$|f(x) - f(x_n)| > \frac{f'(x)}{2}|x - x_n| \quad n \geq N_1,$$
and given $\epsilon > 0$, there exists N_2 such that
$$\left|\frac{f(x) - f(x_n)}{x - x_n} - f'(x)\right| < \epsilon \quad n \geq N_2.$$
Let $N = \max\{N_1, N_2\}$. We have
$$\left|\frac{g(y) - g(y_n)}{y - y_n} - \frac{1}{f'(x)}\right| = \left|\frac{x - x_n}{f(x) - f(x_n)} - \frac{1}{f'(x)}\right| = \frac{1}{|f(x) - f(x_n)| f'(x)}\,|(x - x_n)f'(x) - (f(x) - f(x_n)|$$
$$\leq \frac{2}{(f'(x))^2|x - x_n|}|(x - x_n)f'(x) - (f(x) - f(x_n))| \leq \frac{2}{(f'(x))^2|x - x_n|}\epsilon|x - x_n| < \frac{2\epsilon}{f'(x)}.$$
Since $\epsilon > 0$ and $\{x_n\}, x_n \to x$, are arbitrary, we have that g is differentiable at any $\acute{y} \in f(a,b), y = f(x)$ and
$$g'(f(x)) = \frac{1}{f'(x)}.$$