Chapter 4, problem 14. Let $I = [0,1]$ be the closed unit interval. Suppose $f$ is a continuous map of $I$ into $I$. Prove that $f(x) = x$ for at least one $x \in I$.

Solution.
Consider the function $g(x) = f(x) - x$. Then $g$ is a continuous function of $I = [0,1]$ into $[-1,1]$, since $0 \leq f(x) \leq 1$. We have then
\[ g(0) = f(0) \geq 0, \quad g(1) = f(1) - 1 \leq 0. \]
If $g(0) = 0$ or $g(1) = 1$, the $f(0) = 0$ or $f(1) = 1$ and we are done, otherwise $g(0) > 0$ and $g(1) < 0$, therefore by the intermediate value theorem there exists $x \in [0,1]$ such that $g(x) = 0$, ie, $f(x) = x$.

Chapter 4, problem 16. Let $[x]$ be the largest integer contained in $x$, that is, $[x]$ is the integer such that $x - 1 < [x] \leq x$; and let $(x) = x - [x]$ denote the fractional part of $x$. What discontinuities do the functions $[x]$ and $(x)$ have?

Solution.
First note that since $f(x) = x$ is continuous in $\mathbb{R}$, $(x)$ is continuous in $x$ if and only if $[x]$ is continuous in $x$.
Note that
\[ [x] = n, \quad \text{for} \quad x \in [n,n+1), n \in \mathbb{N}. \]
Therefore $[x]$ is discontinuous at $x$ if and only if $x \in \mathbb{N}$, and $[n_+] = n$, $[n_-] = n - 1$, for every $n \in \mathbb{N}$. So both $[x]$ and $(x)$ only have discontinuities of first kind (simple discontinuities).

Chapter 4, problem 20. If $E$ is a nonempty subset of a metric space $X$, define the distance from $x \in X$ to $E$ by
\[ \rho_E(x) = \inf_{z \in E} d(x,z). \]
(a) Prove that $\rho_E(x) = 0$ if and only if $x \in \overline{E}$.
(b) Prove that $\rho_E$ is a uniformly continuous function on $X$, by showing that
\[ |\rho_E(x) - \rho_E(y)| \leq d(x,y) \]
for all $x \in X, y \in X$.

Solution.
(a) Suppose that $\rho_E(x) = 0$. Then for any $\epsilon > 0$, there exists $z \in E$ such that
\[ d(x,z) < \epsilon. \]
Since $\epsilon > 0$ is arbitrary, this shows that in any neighborhood of $x$, we can find $z \in E$, i.e., $x \in \overline{E}$.

Conversely, if $x \in \overline{E}$, for any $\epsilon > 0$, there exists $w \in E$ such that $w \in B_\epsilon(x) = \{y \in X : |x - y| < \epsilon\}$, i.e.,

$$
\rho_E(x) = \inf_{z \in E} d(x, z) \leq d(w, x) < \epsilon.
$$

Since $\epsilon > 0$ is arbitrary, $\rho_E(x) = 0$.

(b) We have that for any $x \in X$, $y \in X$, $z \in E$, by the triangle inequality

$$
\rho_E(x) \leq d(x, z) \leq d(x, y) + d(y, z).
$$

Since $z \in E$ is arbitrary, in the right hand side we can take the inf over $z \in E$ and get

$$
\rho_E(x) \leq d(x, y) + \rho_E(y).
$$

Similarly one can show

$$
\rho_E(y) \leq d(x, y) + \rho_E(x),
$$

and hence

$$
|\rho_E(x) - \rho_E(y)| \leq d(x, y).
$$

**Chapter 4, problem 21.** Suppose $K$ and $F$ are disjoint sets in a metric space $X$, $K$ is compact, $F$ is closed. Prove that there exists $\delta > 0$ such that $d(p, q) > \delta$ if $p \in K, q \in F$.

Show that the conclusion may fail for two disjoint closed sets if neither is compact.

**Solution.**

Let us show first that the function $\rho_F$ defined in problem 20 is a continuous positive function. Given $\epsilon > 0$, let $\delta = \epsilon$, then by the previous problem

$$
|\rho_F(x) - \rho_F(y)| \leq d(x, y) < \delta = \epsilon.
$$

Therefore $\rho_F$ is continuous and since the metric $d$ is a nonnegative function we see that $\rho_F(x) \geq 0$, for any $x \in X$.

If we assume the contrary of the statement given above, then for any $n \in \mathbb{N}$, there exists $p_n \in K, q_n \in F$ such that

$$
d(p_n, q_n) \leq \frac{1}{n},
$$

this, in particular, implies

$$
\rho_F(p_n) \leq \frac{1}{n},
$$

in particular, $\rho_F(p_n) \to 0$.

But since $K$ is compact and $\{p_n\} \subset K$, there exists a subsequence $\{p_{n_i}\}$ such that $p_{n_i} \to p \in K$.

Since $\rho_F$ is continuous, and $\rho_F(p_{n_i}) \to \rho_F(p)$, so $\rho_F(p)$ = 0. By the previous problem, this implies that $p \in \overline{F} = F$. A contradiction with $K$ and $F$ being disjoint.

Hence there exists $\delta > 0$ such that $d(p, q) > \delta$ if $p \in K, q \in F$. 
Now let \( X = \mathbb{R} \), \( F = \{ n + \frac{1}{n} : n \in \mathbb{N} \} \) and \( K = \mathbb{N} \). We have that for any \( \delta > 0 \), there exists \( n \in \mathbb{N} \) such that \( \frac{1}{n} < \delta \), so
\[
d \left( n, n + \frac{1}{n} \right) = \left| n - \left( n + \frac{1}{n} \right) \right| = \left| \frac{1}{n} \right| < \delta.
\]

**Chapter 5, problem 23.** A real-valued function \( f \) defined in \((a,b)\) is said to be convex if
\[
f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)
\]
whenever \( a < x < b \), \( a < y < b \), \( 0 < \lambda < 1 \). Prove that every convex function is continuous. Prove that every increasing convex function of a convex function is convex.

If \( f \) is convex in \((a,b)\) and if \( a < s < t < u < b \), show that
\[
\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.
\]

**Solution.**
Let \( f \) be a convex function in \((a,b)\). We want to show that \( f \) is continuous, ie, given \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that
\[
|f(x) - f(y)| < \varepsilon \quad \text{if } |x - y| < \delta.
\]
Given \( x \in (a,b) \), there exists \( x_1, x_2 \in (a,b) \) such that \( x_1 < x < x_2 \). Since \( f \) is convex, we have
\[
x = \left( \frac{x - x_1}{x_2 - x_1} \right) x_2 + \left[ 1 - \left( \frac{x - x_1}{x_2 - x_1} \right) \right] x_1, \quad 0 < \frac{x - x_1}{x_2 - x_1} < 1,
\]
so
\[
f(x) \leq \left( \frac{x - x_1}{x_2 - x_1} \right) f(x_2) + \left[ 1 - \left( \frac{x - x_1}{x_2 - x_1} \right) \right] f(x_1).
\]
We can rewrite the equation above and get
\[
\frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1},
\]
or
\[
(x_2-x_1)(f(x)-f(x_1)) \leq (x-x_1)(f(x_2)-f(x_1)) \Rightarrow (x_2-x)f(x_2)-f(x_1)(x_2-x) \leq (f(x_2)-f(x))(x_2-x_1)
\]
\[
\Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_2) - f(x)}{x_2 - x}.
\]
Therefore for any \( x, y \in [x_1, x_2] \), assume without loss of generality \( x > y \), and since \((a,b)\) is open there exist \( x_0, x_3 \in (a,b) \) such that \( x_0 < x_1 < x_2 < x_3 \),
\[
\frac{f(x) - f(y)}{x - y} \leq \frac{f(x_3) - f(y)}{x_3 - y} \leq \frac{f(x_3) - f(x)}{x_3 - x_2},
\]
and
\[
\frac{f(x) - f(y)}{x - y} \geq \frac{f(x) - f(x_0)}{x - x_0} \leq \frac{f(x_1) - f(x_0)}{x_1 - x_0},
\]
So given $\epsilon > 0$, $x \in [x_1, x_2]$, let $\delta = \min\{\frac{\epsilon}{C}, \frac{x_2 - x_1}{2}\} > 0$, then for any $y \in (x - \delta, x + \delta) \subset [x_1, x_2]$,

$$|f(x) - f(y)| \leq C|x - y| \leq C\frac{\epsilon}{C} = \epsilon.$$ 

Hence $f$ is continuous on $x$ and since $x \in (a, b)$ is arbitrary, $f$ is continuous in $(a, b)$.

Note that we showed above that if $a < s < t < u < b$, then

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$ 

Let $h = g \circ f$, ie, $h(x) = g(f(x))$, where $g$ is an increasing function and $f$ is a convex function. We have for $x, y \in (a, b), 0 < \lambda < 1$,

$$h(\lambda x + (1 - \lambda)y) = g(f(\lambda x + (1 - \lambda)y)) \leq g(\lambda f(x) + (1 - \lambda)f(y)).$$

Assume without loss of generality $f(x) \geq f(y)$ and $y > x$, otherwise consider $\lambda y + (1 - \lambda)x$ instead of $\lambda x + (1 - \lambda)y$. So $\lambda f(x) + (1 - \lambda)f(y) \leq f(x)$, and since $g$ is increasing

$$h(\lambda x + (1 - \lambda)y) - \lambda g(f(x)) \leq \lambda g(\lambda f(x) + (1 - \lambda)f(y)) + (1 - \lambda)g(\lambda f(x) + (1 - \lambda)f(y)) - \lambda g(x) \leq (1 - \lambda)g(\lambda f(x) + (1 - \lambda)f(y)) \leq (1 - \lambda)g(f(y)),$$

ie,

$$h(\lambda x + (1 - \lambda)y) \leq g(f(x)) + 1 - \lambda)g(f(y)).$$

**Chapter 5, problem 1.** Let $f$ be defined for all real $x$, and suppose that

$$|f(x) - f(y)| \leq (x - y)^2$$

for all real $x$ and $y$. Prove that $f$ is constant.

**Solution.**

We see that, $x \neq y$,

$$|f(x) - f(y)| \leq (x - y)^2 = |x - y|^2 \Rightarrow \frac{|f(x) - f(y)|}{|x - y|} \leq |x - y|.$$ 

Therefore

$$\lim_{y \to x} \frac{|f(x) - f(y)|}{|x - y|} = 0.$$ 

Hence $f$ is differentiable at every $x \in \mathbb{R}$ and $f'(x) = 0$. By theorem 5.11 $f$ must be constant.

**Chapter 5, problem 2.** Suppose $f'(x) > 0$ in $(a, b)$. Prove that $f$ is strictly increasing in $(a, b)$, and let $g$ be its inverse function. Prove that $g$ is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)} \quad (a < x < b).$$
Solution.
Let $x, y \in (a, b)$, $x < y$. Then by the Mean Value Theorem, there exists $c \in (x, y)$ such that
\[ f(y) - f(x) = (y - x)f'(c) > 0. \]
Therefore $f(y) > f(x)$ and $f$ is increasing. Hence the inverse of $f$ is well defined, since $f$ is injective, and let $g$ be its inverse function, i.e., $g(f(x)) = f(g(x)) = x$.

Given $y \in f(a, b)$, let $\{y_n\} \subset f(a, b)$ be such that $y_n \to y$, $y_n \neq y$. Then for each $n$, there exists $x_n \in (a, b)$ such that $f(x_n) = y_n$, and let $x \in (a, b)$ such that $f(x) = y$.

By theorem 4.17, $g$ is continuous, so $x_n \to x$ (note $x_n \neq x$, since $f$ is injective and $y_n = f(x_n) \neq y = f(x)$). We have
\[
\left| \frac{g(y) - g(y_n)}{y - y_n} - \frac{1}{f'(x)} \right| = \left| \frac{g(f(x)) - g(f(x_n))}{f(x) - f(x_n)} - \frac{1}{f'(x)} \right| = \left| \frac{x - x_n}{f(x) - f(x_n)} - \frac{1}{f'(x)} \right|
\]
Since $x_n \to x$ and $f$ is differentiable at $x$, there exists $N_1$ such that
\[
|f(x) - f(x_n)| > \frac{f'(x)}{2}|x - x_n| \quad n \geq N_1,
\]
and given $\epsilon > 0$, there exists $N_2$ such that
\[
\left| \frac{f(x) - f(x_n)}{x - x_n} - f'(x) \right| < \epsilon \quad n \geq N_2.
\]

Let $N = \max\{N_1, N_2\}$. We have
\[
\left| \frac{g(y) - g(y_n)}{y - y_n} - \frac{1}{f'(x)} \right| \leq \frac{2}{(f'(x))^2|x - x_n|}|(x - x_n)f'(x) - (f(x) - f(x_n))| \leq \frac{2}{(f'(x))^2|x - x_n|} \epsilon |x - x_n| < \frac{2\epsilon}{f'(x)}.
\]
Since $\epsilon > 0$ and $\{x_n\}, x_n \to x$, are arbitrary, we have that $g$ is differentiable at any $y \in f(a, b), y = f(x)$ and
\[
g'(f(x)) = \frac{1}{f'(x)}.
\]