A Singular Perturbation Problem for the p-Laplace Operator

D. DANIELLI, A. PETROSYAN & H. SHAHGHOLIAN

ABSTRACT. In this paper we initiate the study of the nonlinear one phase singular perturbation problem

$$\operatorname{div}(|\nabla u^{\varepsilon}|^{p-2}\nabla u^{\varepsilon}) = \beta_{\varepsilon}(u^{\varepsilon}), \quad (1 < p < \infty)$$

in a domain Ω of $\mathbb{R}^N$. We prove uniform Lipschitz regularity of uniformly bounded solutions. Once this is done we can pass to the limit to obtain a solution to the stationary case of a combustion problem with a nonlinearity of power type. (The case $p = 2$ has been considered earlier by several authors.)

1. INTRODUCTION

Our objective in this paper is to study the singular perturbation problem

$$(P_{\varepsilon}) \qquad \Delta_p u^{\varepsilon} = \beta_{\varepsilon}(u^{\varepsilon}), \quad u^{\varepsilon} \geq 0$$

in a domain Ω of $\mathbb{R}^N$. Here, for $1 < p < \infty$, Δ_p denotes the p-Laplace operator, i.e., $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. We recall that a solution to (P_{ε}) is a function $u^{\varepsilon} \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$ such that

$$\int_{\Omega} |\nabla u^{\varepsilon}|^{p-2}\nabla u^{\varepsilon} \cdot \nabla \varphi \, dx = -\int_{\Omega} \varphi \beta_{\varepsilon}(u^{\varepsilon})\, dx \tag{1.1}$$

for all $\varphi \in C_0^{\infty}(\Omega)$. We require β_{ε} to be $\operatorname{Lip}(\mathbb{R})$ and to satisfy

$$0 \leq \beta_{\varepsilon} \leq \frac{A}{\varepsilon}\chi_{(0,\varepsilon)} \quad \text{and} \quad \int_0^{\varepsilon} \beta_{\varepsilon}(s)\, ds = M \tag{1.2}$$

for positive constants A and M. In particular, these conditions are fulfilled when the functions β_{ε} are constructed from a single nonnegative Lipschitz function β

Indiana University Mathematics Journal ©, Vol. 52, No. 2 (2003)

supported in $[0, 1]$ by setting

$$\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta\left(\frac{s}{\varepsilon}\right). \tag{1.3}$$

Although our analysis applies to a general type of operators, as the ones considered in [17] of the form $\operatorname{div}(\mathcal{A}(x, u, \nabla u))$, for simplicity and clarity of the arguments we focus on the specific form of the p-Laplacian Δ_p.

The motivation of the study in this paper comes from the applications to the one-phase case of the combustion problem, appearing in the description of laminar flames as an asymptotic limit for high activation energy, that corresponds to the limit as $\varepsilon \to 0$ in (P_ε). For the case $p = 2$ there is an extensive study of the problem and more or less a complete resolution of it; see [2], [15] for the elliptic case and [8], [3], [6], [7], [4], [9] for the parabolic one. However, the nonlinear case, addressed here, has never been considered earlier. This might partly depend on the lack of an established theory for the p-Laplace operator, and partly on the fact that some of the earlier techniques fail in the absence of linearity.

We show that, in a sense, the limit of (P_ε) as $\varepsilon \to 0$ is a free boundary problem

$$\begin{cases} \Delta_p u = 0 & \text{in } \{u > 0\}, \\ |\nabla u| = c & \text{on } \partial\{u > 0\} \cap \Omega, \end{cases} \tag{P}$$

with $c = ((p/(p-1))M)^{1/p}$. Namely, the main result of this paper (Theorem 4.3 in Section 4) asserts that the uniform limits u of u^ε have the asymptotic development

$$u(x) = \left(\frac{p}{p-1}M\right)^{1/p} \langle x - x_0, \eta\rangle^+ + o(|x - x_0|),$$

near $x_0 \in \partial\{u > 0\}$, provided $\partial\{u > 0\}$ admits a measure theoretic normal and u is not degenerate at x_0 (see Definitions 4.1-4.2).

The free boundary problem (P) for the p-Laplacian was studied earlier under certain geometric (convexity) assumptions, by different techniques; see e.g. [1] by Acker and Meyer and a series of papers [11]-[13] by Henrot and Shahgholian.

To prove the main theorem (Theorem 4.3) we need a uniform bound (Theorem 2.1) for the gradient of the solutions, in order to have some stability of the problem as one passes to the limit. This type of uniform bounds on the gradient usually constitutes the basics of the analysis to follow, and it is by no means an obvious generalization of earlier results. Indeed, it needs to be pointed out that one of the main difficulties in the consideration of operators that do not admit linearization, as it is for the p-Laplacian, appears in the deduction of the uniform gradient bound, which has its own independent interest. In this part of our analysis we apply techniques that have been recently developed for related free boundary problems, see [14] and [5].

2. The Uniform Gradient Bound for Solutions

In this section we prove that the solutions u^ε of the singular perturbation problem (P_ε) are locally uniformly Lipschitz. Our main theorem in this section is the following.

Theorem 2.1. *Let u^ε be a nonnegative solution of (P_ε) in a domain Ω of $\mathbb{R}^N$ with β_ε satisfying (1.2) and such that $\|u^\varepsilon\|_{L^\infty(\Omega)} \le L$. Then for every compact $K \Subset \Omega$ there is a constant $C = C(N, p, A, L, \operatorname{dist}(K, \partial\Omega))$ independent of ε such that*

$$\|\nabla u^\varepsilon\|_{L^\infty(K)} \le C.$$

It is also noteworthy that as far as the proof of Theorem 2.1 goes, one can relax the conditions on β_ε. An important observation is that the same technique to follow shows that, in the case of two-phase problems (see [6]-[7]), one may deduce gradient bound for the non-negative part of the solution if one already knows that the negative part of the solution is Lipschitz. In [3], L. Caffarelli applied this idea in combination with the monotonicity formulas to deduce gradient bound for the solution of the two-phase singular perturbation problem for the Laplacian; see also [4]. In the absence of the monotonicity formula we are not able to prove a similar result as that in [3]. It is apparent that some new technique is to be developed to handle the sign change in the case of the p-Laplacian or any other nonlinear case. This remains an open and tantalizing problem.

The proof of Theorem 2.1 will be based on the following lemma.

Lemma 2.2. *Let v be a bounded nonnegative solution of*

$$0 \le \Delta_p v \le A\chi_{\{0<v<1\}}$$

in the unit ball B_1 of $\mathbb{R}^N$, with $v(0) \le 1$. Then there is a constant $C = C(N, p, A)$ such that

$$\|v\|_{L^\infty(B_{1/4})} \le C.$$

Remark 2.3. We explicitly observe that $v \in C^{1,\alpha}(\Omega)$ for some $\alpha > 0$, thanks to the results in [17]. In the case when $\Delta_p v = \beta(v)$ with $|\beta(s)| \le c_0|s|^p + c_1|s|^{p-1}$ for some constants c_0 and c_1, the conclusion of the Lemma 2.2 follows directly from Serrin's Harnack inequality for nonhomogeneous quasilinear operators, see [16]. Our proof, however, uses only Harnack inequality for homogeneous operators and is based on compactness rather than energy methods, which allows to generalize it to a broad range of operators.

Proof. Indeed, assume the contrary. Then there exists a sequence of functions $\{v_k\}$, $k = 1, 2, \dots$, satisfying the assumptions of the lemma and such that

$$\max_{\bar{B}_{1/4}} v_k(x) > \frac{4}{3}k.$$

Consider the sets

$$\Omega_k = \{x \in B_1 \mid v_k(x) > 1\} \quad \text{and} \quad \Gamma_k = \partial\Omega_k \cap B_1.$$

Note that v_k is p-harmonic in Ω_k. Let now $\delta_k(x) = \operatorname{dist}(x, B_1 \setminus \Omega_k)$ and define

$$\mathcal{O}_k = \left\{x \in B_1 \mid \delta_k(x) \le \frac{1}{3}(1 - |x|)\right\} \supset B_1 \setminus \Omega_k.$$

Observe that $\bar{B}_{1/4} \subset \mathcal{O}_k$. In particular

$$\mathrm{m}_k := \sup_{\mathcal{O}_k}(1 - |x|)v_k(x) \ge \frac{3}{4} \max_{\bar{B}_{1/4}} v_k(x) > k.$$

Since $v_k(x)$ is bounded (for fixed k), we will have $(1-|x|)v_k(x) \to 0$ as $|x| \to 1$, and therefore m_k will be attained at some point $x_k \in \mathcal{O}_k$:

$$(1 - |x_k|)v_k(x_k) = \max_{\mathcal{O}_k}(1 - |x|)v_k(x). \tag{2.1}$$

Clearly,

$$v_k(x_k) = \frac{\mathrm{m}_k}{1 - |x_k|} \ge \mathrm{m}_k > k.$$

Since $x_k \in \mathcal{O}_k$, by the definition we will have

$$\delta_k := \delta_k(x_k) \le \frac{1}{3}(1 - |x_k|). \tag{2.2}$$

Let now $y_k \in \Gamma_k$ be a point where $\delta_k = \operatorname{dist}(x_k, \Gamma_k)$ is realized, so that

$$|y_k - x_k| = \delta_k. \tag{2.3}$$

Then we will have two inclusions, $B_{2\delta_k}(y_k) \subset B_1$ and $B_{\delta_k/2}(y_k) \subset \mathcal{O}_k$, both consequences of (2.2)-(2.3). In particular, for $z \in B_{\delta_k/2}(y_k)$ the following inequality holds

$$(1 - |z|) \ge (1 - |x_k|) - |x_k - z| \ge (1 - |x_k|) - \frac{3}{2}\delta_k \ge \frac{1}{2}(1 - |x_k|).$$

This, in conjunction with (2.1), implies that

$$\max_{\bar{B}_{\delta_k/2}(y_k)} v_k \le 2v_k(x_k).$$

Next, since $B_{\delta_k}(x_k) \subset \Omega_k$, v_k satisfies $\Delta_p v_k = 0$ in $B_{\delta_k}(x_k)$. By the Harnack inequality for p-harmonic functions there is a constant $c = c(N, p) > 0$ such that

$$\min_{\bar{B}_{3\delta_k/4}(x_k)} v_k \ge c v_k(x_k).$$

In particular,

$$\max_{\bar{B}_{\delta_k/4}(y_k)} v_k \geq c v_k(x_k).$$

Further, define

$$w_k(x) = \frac{v_k(y_k + \delta_k x)}{v_k(x_k)} \quad \text{for } x \in B_2.$$

Summarizing the properties of v_k above, we see that w_k satisfies the following system

$$\begin{cases} 0 \leq \Delta_p w_k \leq \dfrac{A(\delta_k)^p}{k^{p-1}} & \text{in } B_2, \\ \max_{\bar{B}_{1/2}} w_k \leq 2, \quad \max_{\bar{B}_{1/4}} w_k \geq c > 0, \\ w_k \geq 0, \quad w_k(0) \leq \dfrac{1}{k}. \end{cases}$$

Therefore, from a priori estimates, we can conclude that a subsequence of $\{w_k\}$ will converge in $C^{1,\alpha}$ norm on every compact subset of $B_{1/2}$ to a function w_0 that satisfies

$$\begin{cases} \Delta_p w_0 = 0 \quad \text{in } B_{1/2}, \\ \max_{\bar{B}_{1/4}} w_0 \geq c > 0, \\ w_0 \geq 0, \quad w_0(0) = 0. \end{cases}$$

This, however, contradicts the strong maximum principle for p-harmonic functions. The lemma is proved. $\square$

Proof of Theorem 2.1. We start with the observation that it is enough to prove the theorem in the case when $\Omega = B_1$, $K = \bar{B}_{1/8}$, and under the assumptions $u^\varepsilon(0) = \varepsilon$ and $\|u^\varepsilon\|_{L^\infty(B_1)} = 1$. Denote

$$\Omega^\varepsilon = \{x \in B_1 \mid u^\varepsilon > \varepsilon\} \quad \text{and} \quad \Gamma^\varepsilon = \partial\Omega^\varepsilon \cap B_1.$$

Step 1. Prove that there is a constant $C = C(N, p, A)$ such that

$$|\nabla u^\varepsilon(x)| \leq C, \quad \text{for } x \in \bar{B}_{1/2} \setminus \Omega^\varepsilon. \tag{2.4}$$

Indeed, take a point $x_0 \in \bar{B}_{1/2}$ with $u^\varepsilon(x_0) \leq \varepsilon$ and consider a function

$$v^\varepsilon(x) = \frac{u^\varepsilon(x_0 + \varepsilon x)}{\varepsilon}.$$

Direct computation shows that v^ε satisfies

$$0 \leq \Delta_p v^\varepsilon \leq A\chi_{\{0<v^\varepsilon<1\}}.$$

All the assumptions of Lemma 2.2 are thus fulfilled for $v = v^\varepsilon$ and we can conclude

$$\max_{\bar{B}_{1/4}} v^\varepsilon \le C(N,p,A). \tag{2.5}$$

From interior gradient estimates we obtain

$$|\nabla u^\varepsilon(x_0)| = |\nabla v^\varepsilon(0)| \le C_1(N,p,A) \max_{\bar{B}_{1/4}} v^\varepsilon \le C_2(N,p,A)$$

for all $x_0 \in \bar{B}_{1/2} \setminus \Omega^\varepsilon$. Hence (2.4) is proved.

Step 2. Prove that

$$|u^\varepsilon(x)| \le \varepsilon + C(N,p,A)\operatorname{dist}(x, B_1 \setminus \Omega^\varepsilon) \quad \text{for } x \in \bar{B}_{1/4} \cap \Omega^\varepsilon. \tag{2.6}$$

Indeed, for $x_0 \in \bar{B}_{1/4} \cap \Omega^\varepsilon$ denote

$$\mathrm{m}_0 = u^\varepsilon(x_0) - \varepsilon \quad \text{and} \quad \delta_0 = \operatorname{dist}(x_0, B_1 \setminus \Omega^\varepsilon)$$

Notice that, since $0 \in \Gamma^\varepsilon$, $\delta_0 \le \frac{1}{4}$. We want to prove that

$$\mathrm{m}_0 \le C(N,p,A)\delta_0. \tag{2.7}$$

Since $B_{\delta_0}(x_0)$ is contained in Ω^ε, $u^\varepsilon - \varepsilon$ will be nonnegative and p-harmonic there. We can thus apply the Harnack inequality to conclude

$$\min_{B_{\delta_0/2}(x_0)} (u^\varepsilon(x) - \varepsilon) \ge c_1 \mathrm{m}_0$$

for $c_1 = c_1(N,p) > 0$. Next, consider the p-capacitary potential $\varphi(x)$ of the ring $B_1 \setminus \bar{B}_{1/2}$ which satisfies

$$\Delta_p \varphi(x) = 0 \quad \text{in } B_1 \setminus \bar{B}_{1/2},\ \varphi\big|_{\partial B_1} = 0, \text{ and } \varphi\big|_{\partial B_{1/2}} = 1.$$

The function φ will be spherically symmetric with $|\nabla\varphi| = c_0 = c_0(N,p) > 0$ on ∂B_1. Define

$$\psi(x) = c_1 \mathrm{m}_0 \varphi\left(\frac{x - x_0}{\delta_0}\right) \quad \text{for } x \in \bar{B}_{\delta_0}(x_0) \setminus \bar{B}_{\delta_0/2}(x_0).$$

From the comparison principle for p-harmonic functions we will have

$$\psi(x) \le u^\varepsilon(x) - \varepsilon \quad \text{for } x \in \bar{B}_{\delta_0}(x_0) \setminus \bar{B}_{\delta_0/2}(x_0). \tag{2.8}$$

Take $y_0 \in \partial B_{\delta_0}(x_0) \cap \Gamma_\varepsilon$. Then $y_0 \in \bar{B}_{1/2}$, and

$$\psi(y_0) = u^\varepsilon(y_0) - \varepsilon = 0. \tag{2.9}$$

We infer from (2.8), (2.9), and (2.4) that $|\nabla \psi(y_0)| \leq |\nabla u^\varepsilon(y_0)| \leq c_2(N, p, A)$. Observe now that $|\nabla \psi(y_0)| = c_1 \mathrm{m}_0 c_0 / \delta_0$ and therefore we obtain

$$\mathrm{m}_0 \leq \frac{c_2}{c_0 c_1} \delta_0.$$

Thus, inequalities (2.7) and (2.6) are proved.

Step 3. Prove that

$$|\nabla u^\varepsilon(x)| \leq C(n, p, A) \quad \text{for } x \in \bar{B}_{1/8} \cap \Omega^\varepsilon. \tag{2.10}$$

Indeed, let $x_0 \in \bar{B}_{1/8} \cap \Omega^\varepsilon$, $\delta_0 = \operatorname{dist}(x_0, B_1 \setminus \Omega^\varepsilon)$ and define

$$w(x) = \frac{u^\varepsilon(x_0 + \delta_0 x) - \varepsilon}{\delta_0} \quad \text{for } x \in B_1.$$

Then, from the inclusion $B_{\delta_0}(x_0) \subset \Omega^\varepsilon$ and inequality (2.6) we will have

$$0 \leq w \leq C(N, p, A) \quad \text{in } B_1.$$

Since also w is p-harmonic in B_1, from the interior gradient estimates we obtain

$$|\nabla u^\varepsilon(x_0)| = |\nabla w(0)| \leq C_1(N, p, A),$$

which proves (2.10).

Now the theorem follows from (2.4) and (2.10). □

3. Passage to the Limit: $\varepsilon_j \searrow 0$

From now on we will assume that functions β_ε in (P_ε) satisfy (1.2)-(1.3).

This section embodies the main technical tools that one needs to establish the main theorem (Theorem 4.3).

Lemma 3.1. *Let $\{u^\varepsilon\}$ be a uniformly bounded family of solutions to (P_ε). Then for every sequence $\varepsilon_j \to 0$ there exists a subsequence $\varepsilon'_j \to 0$ and $u \in \mathrm{Lip}(\Omega)$ such that:*

(i) *$u^{\varepsilon'_j} \to u$ uniformly on compact subsets of Ω;*
(ii) *$\Delta_p u = 0$ in $\Omega \setminus \partial\{u > 0\}$;*
(iii) *$\nabla u^{\varepsilon'_j} \to \nabla u$ in $L^p_{\mathrm{loc}}(\Omega)$.*

Proof. Part (i) follows by Theorem 2.1 and a standard compactness argument. Let now $E \Subset \{u > 0\}$ be open. Then $u \geq c > 0$ in E. By the uniform convergence, we will have $u^{\varepsilon'_j} > c/2$ in E for small ε'_j. Hence, if also $\varepsilon'_j < c/2$, $u^{\varepsilon'_j}$ will be p-harmonic in E. This implies that u is p-harmonic in E and since E was arbitrary, (ii) follows.

Finally, we prove (iii). Let ψ be a nonnegative $C_0^\infty(\Omega)$ function, and $\delta > 0$. Take $(u-\delta)^+\psi$ as a test function. Since $\Delta_p u = 0$ in the positivity set of u, integrating by parts, we obtain

$$\int_{\{u>\delta\}} |\nabla u|^p \psi = -\int_{\{u>\delta\}} |\nabla u|^{p-2}\nabla u \cdot \nabla\psi\, u + \delta\int_{\{u>\delta\}} |\nabla u|^{p-2}\nabla u \cdot \nabla\psi.$$

Letting $\delta \to 0$ we find

$$\int_{\{u>0\}} |\nabla u|^p \psi = -\int_{\{u>0\}} |\nabla u|^{p-2}\nabla u \cdot \nabla\psi\, u. \tag{3.1}$$

On the other hand, the observation $\beta_\varepsilon(u^\varepsilon)u^\varepsilon \geq 0$ yields

$$\int_\Omega |\nabla u^\varepsilon|^p \psi \leq -\int_\Omega |\nabla u^\varepsilon|^{p-2}\nabla u^\varepsilon \cdot \nabla\psi\, u^\varepsilon. \tag{3.2}$$

Using the uniform convergence of u^ε to u and the weak convergence of $|\nabla u^\varepsilon|^{p-2}\nabla u^\varepsilon$ to $|\nabla u|^{p-2}\nabla u$ in $L^{p/(p-1)}_{\mathrm{loc}}(\Omega)$, we infer from (3.1) and (3.2) that

$$\limsup_{j\to\infty} \int |\nabla u^{\varepsilon_j}|^p \psi \leq \int |\nabla u|^p \psi. \tag{3.3}$$

Since $\nabla u^{\varepsilon_j} \rightharpoonup \nabla u$ in $L^p_{\mathrm{loc}}(\Omega)$, we have

$$\int |\nabla u|^p \psi \leq \liminf_{j\to\infty} \int |\nabla u^{\varepsilon_j}|^p \psi. \tag{3.4}$$

It follows from (3.3), (3.4), and a simple compactness argument that $\nabla u^{\varepsilon_j} \to \nabla u$ in $L^p_{\mathrm{loc}}(\Omega)$.

The conclusion of part (iii) is proved, and so is the lemma. □

We now prove that limit solutions are solutions to the free boundary problem in a very weak sense.

Proposition 3.2. *Let $\{u^{\varepsilon_j}\}$ be a family of solutions to (P_{ε_j}). Assume that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω as $\varepsilon_j \to 0$. Then there exists a locally finite measure μ supported on the free boundary $\Omega \cap \partial\{u > 0\}$ such that $\beta_{\varepsilon_j}(u^{\varepsilon_j}) \rightharpoonup \mu$ in Ω. In particular, $\Delta_p u = \mu$ in Ω, i.e.,*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\, dx = -\int_\Omega \varphi\, d\mu \tag{3.5}$$

for all $\varphi \in C_0^\infty(\Omega)$.

Proof. By definition of weak solutions to (P_ε), if $\varphi \in C_0^\infty(\Omega)$, one has

$$\int_\Omega |\nabla u^\varepsilon|^{p-2} \nabla u^\varepsilon \cdot \nabla \varphi = -\int_\Omega \beta_\varepsilon(u^\varepsilon)\varphi. \tag{3.6}$$

Since $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω, by Lemma 3.1 we know $\nabla u^{\varepsilon_j} \to \nabla u$ in $L^p_{\mathrm{loc}}(\Omega)$, and so the left-hand side of (3.6) converges to the left-hand side of (3.5). Now let $F \subset \Omega$ be compact, and take $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$, $\varphi \equiv 1$ in F. The sequence $\{\int_\Omega \beta_{\varepsilon_j}(u^{\varepsilon_j})\varphi\, dx\}$ is convergent, and therefore it is bounded. Hence

$$\int_F \beta_{\varepsilon_j}(u^{\varepsilon_j})\, dx \leq \int_\Omega \beta_{\varepsilon_j}(u^{\varepsilon_j})\varphi\, dx \leq C(\varphi).$$

This implies that there exists a locally finite measure μ such that, passing to a subsequence (still denoted by ε_j) if necessary, $\beta_{\varepsilon_j}(u^{\varepsilon_j}) \to \mu$ as measures in Ω. Passing to the limit in (3.6), we get (3.5). Moreover, since $\Delta_p u = 0$ in $\Omega \setminus \partial\{u > 0\}$ by Lemma 3.1, we conclude that μ is supported in $\Omega \cap \partial\{u > 0\}$. The proof is thus complete. $\square$

Lemma 3.3. *Let* $\{u^{\varepsilon_j}\}$ *be a family of solutions to* (P_{ε_j}) *in* Ω *such that* $u^{\varepsilon_j} \to u$ *uniformly on compact subsets of* Ω *and* $\varepsilon_j \to 0$ *as* $j \to \infty$. *Let* $x_0, x_n \in \Omega \cap \partial\{u > 0\}$ *be such that* $x_n \to x_0$ *as* $n \to \infty$. *Let* $\lambda_n \to 0$, $u_{\lambda_n}(x) = (1/\lambda_n)u(x_n + \lambda_n x)$, *and* $(u^{\varepsilon_j})_{\lambda_n}(x) = (1/\lambda_n)u^{\varepsilon_j}(x_n + \lambda_n x)$. *Suppose that* $u_{\lambda_n} \to U$ *as* $n \to \infty$ *uniformly on compact sets of* $\mathbb{R}^N$. *Then, there exists* $j(n) \to \infty$ *such that for every* $j_n \geq j(n)$ *there holds that* $\varepsilon_{j_n}/\lambda_n \to 0$, *and*

(i) $(u^{\varepsilon_{j_n}})_{\lambda_n} \to U$ *uniformly on compact sets of* $\mathbb{R}^N$;
(ii) $\nabla (u^{\varepsilon_{j_n}})_{\lambda_n} \to \nabla U$ *in* $L^p_{\mathrm{loc}}(\mathbb{R}^N)$;
(iii) $\nabla u_{\lambda_n} \to \nabla U$ *in* $L^p_{\mathrm{loc}}(\mathbb{R}^N)$.

Proof. The proof is along the lines of the one of Lemma 3.2 in [6]. We discuss here only the relevant modifications. For simplicity we assume $x_n = x_0$. Proceeding as in the cited reference, one can show that (i) holds. The functions $(u^{\varepsilon_{j_n}})_{\lambda_n}$ are solutions to

$$\Delta_p (u^{\varepsilon_{j_n}})_{\lambda_n} = \beta_{\varepsilon_{j_n}/\lambda_n}((u^{\varepsilon_{j_n}})_{\lambda_n})$$

in B_k, where k is a fixed positive number. By Lemma 3.1 there exists a subsequence, still denoted by j_n, such that $\nabla (u^{\varepsilon_{j_n}})_{\lambda_n} \to \nabla U$ in $L^p(B_k)$. Then also (ii) holds. In order to prove (iii), let $\delta > 0$ and consider

$$\|\nabla u_{\lambda_n} - \nabla U\| \leq \|\nabla u_{\lambda_n} - \nabla (u^{\varepsilon_j})_{\lambda_n}\| + \|\nabla (u^{\varepsilon_j})_{\lambda_n} - \nabla U\| = I + II,$$

where all the norms are in $L^p(B_k)$. By (ii) we already know that $II < \delta$ if $j \geq j_n$ and n is sufficiently large. Moreover, by virtue of Lemma 3.1 it holds

$$\begin{aligned} I^p &= \int_{B_k} |\nabla u - \nabla u^{\varepsilon_j}|^p (x_0 + \lambda_n x)\, dx \\ &= \frac{1}{\lambda_n^N} \int_{B_{\lambda_n k}(x_0)} |\nabla u - \nabla u^{\varepsilon_j}|^p (x)\, dx < \delta^p \end{aligned}$$

if j and n are sufficiently large. This proves (iii). ◻

We now turn our attention to the special case when the limit function u is one-dimensional.

Proposition 3.4. *Let $x_0 \in \Omega$, and let u^{ε_k} be solutions to*

$$\Delta_p u^{\varepsilon_k} = \beta_{\varepsilon_k}(u^{\varepsilon_k})$$

in Ω. If u^{ε_k} converge to $\alpha(x - x_0)_1^+$ uniformly on compact subsets of Ω, with $\alpha \in \mathbb{R}$, and $\varepsilon_k \to 0$ as $k \to \infty$, then

$$0 \leq \alpha \leq \left(\frac{p}{p-1} M\right)^{1/p}.$$

Proof. Without loss of generality, we assume $x_0 = 0$. Since $u^{\varepsilon_k} \geq 0$, we readily have $\alpha \geq 0$. Next, let $\psi \in C_0^\infty(\Omega)$. Choosing $u_{x_1}^{\varepsilon_k} \psi$ as a test function in the weak formulation of $\Delta_p u^{\varepsilon_k} = \beta_{\varepsilon_k}(u^{\varepsilon_k})$ (see Remark 3.5 below) and integrating by parts, we obtain

$$-\frac{1}{p} \int_\Omega |\nabla u^{\varepsilon_k}|^p \psi_{x_1} + \int_\Omega |\nabla u^{\varepsilon_k}|^{p-2} u_{x_1}^{\varepsilon_k} \nabla u^{\varepsilon_k} \cdot \nabla \psi = \int_\Omega B_{\varepsilon_k}(u^{\varepsilon_k}) \psi_{x_1}. \tag{3.7}$$

Here, $B_{\varepsilon_k}(s) = \int_0^s \beta_{\varepsilon_k}(\tau)\, d\tau$. Since $0 \leq B_{\varepsilon_k}(s) \leq M$, there exists $M(x) \in L^\infty(\Omega)$, $0 \leq M(x) \leq M$, such that on a subsequence (still denoted by ε_k) $B_{\varepsilon_k}(u^{\varepsilon_k}) \to M(x)$ $*$-weakly in $L^\infty(\Omega)$. If $y \in \Omega \cap \{x_1 > 0\}$, then $u^{\varepsilon_k} \geq \alpha y_1/2$ in a neighborhood of y for k sufficiently large. Hence, if $u^{\varepsilon_k}(x) \geq \varepsilon_k$ we have, by (1.2),

$$B_{\varepsilon_k}(u^{\varepsilon_k})(x) = \int_0^{u^{\varepsilon_k}(x)/\varepsilon_k} \beta(s)\, ds = M.$$

Moreover, using Proposition 3.2, it is immediate to recognize that $\nabla B_{\varepsilon_k}(u^{\varepsilon_k}) = \beta_{\varepsilon_k}(u^{\varepsilon_k}) \nabla u^{\varepsilon_k} \to 0$ in $L^1_{\mathrm{loc}}(\Omega \cap \{x_1 < 0\})$. Hence $M(x) = \bar{M} \in [0, M]$ in $\Omega \cap \{x_1 < 0\}$. Passing to the limit in (3.7) yields

$$\frac{p-1}{p} \alpha^p \int_{\{x_1 > 0\}} \psi_{x_1}\, dx = M \int_{\{x_1 > 0\}} \psi_{x_1}\, dx + \bar{M} \int_{\{x_1 < 0\}} \psi_{x_1}\, dx,$$

and integrating by parts we find

$$\frac{p-1}{p}\alpha^p \int_{\{x_1=0\}} \psi \, dx' = M \int_{\{x_1=0\}} \psi \, dx' - \bar{M} \int_{\{x_1=0\}} \psi \, dx'.$$

The arbitrariness of $\psi \in C_0^\infty(\Omega)$ allows to conclude $((p-1)/p)\alpha^p = M - \bar{M} \le M$, since $\bar{M} \ge 0$. Hence $\alpha^p \le (p/(p-1))M$, and the proof is complete. □

Remark 3.5. We recall that the weak solution u of the equation $\Delta_p u = f$ in Ω with bounded f has a representative in $W^{2,2}_{\mathrm{loc}}(\Omega)$ if $2 \le p < \infty$, and in $W^{2,p}_{\mathrm{loc}}(\Omega)$ for $1 < p \le 2$, see e.g. [17]. This, in conjunction with local L^∞ bounds on $|\nabla u^{\varepsilon_k}|$ in Ω, justifies the integration by parts in the proof of Proposition 3.4 above.

Proposition 3.6. *Let $x_0 \in \Omega$, and let u^{ε_k} be solutions to*

$$\Delta_p u^{\varepsilon_k} = \beta_{\varepsilon_k}(u^{\varepsilon_k})$$

in Ω. If u^{ε_k} converge to $\alpha(x-x_0)_1^+ + \gamma(x-x_0)_1^-$ uniformly on compact subsets of Ω, with α, $\gamma > 0$ and $\varepsilon_k \to 0$ as $k \to \infty$, then

$$\alpha = \gamma \le \left(\frac{p}{p-1}M\right)^{1/p}.$$

Proof. Without loss of generality we assume $x_0 = 0$. As in Proposition 3.4, u^{ε_k} satisfies (3.7), and it is immediate to recognize that $B_{\varepsilon_k}(u^{\varepsilon_k}) \to M$ in $L^1_{\mathrm{loc}}(\Omega)$. Passing to the limit in (3.7), and integrating by parts in the resulting equation, we find that $\alpha = \gamma$.

Now we assume that $\alpha > ((p/(p-1))M)^{1/p}$ and show that this leads to a contradiction.

Step 1. Let $\mathcal{R}_2 := \{x = (x_1, x') \in \mathbb{R}^N : |x_1| < 2,\ |x'| < 2\}$. Without loss of generality, we may assume $\mathcal{R}_2 \Subset \Omega$. First of all, we construct a family $\{v^{\varepsilon_j}\}$ of solutions to (P_{ε_j}) in $\mathcal{R}_2$ with the property

$$v^{\varepsilon_j}(x_1, x') = v^{\varepsilon_j}(-x_1, x') \quad \text{in } \mathcal{R}_2, \tag{3.8}$$

and such that $v^{\varepsilon_j} \to u$ uniformly on compact subsets of $\mathcal{R}_2$, where $u(x) = \alpha|x_1|$. To this end, we let $b_{\varepsilon_j} = \sup_{\mathcal{R}_2} |u^{\varepsilon_j} - u|$ and v^{ε_j} be the minimal solution (i.e., minimum of all supersolutions) to (P_{ε_j}) in $\mathcal{R}_2$ with boundary values $v^{\varepsilon_j} = u - b_{\varepsilon_j}$ on $\partial\mathcal{R}_2$. By virtue of Lemma 3.1, there exists $v \in \mathrm{Lip}_{\mathrm{loc}}(\mathcal{R}_2)$ such that, on a subsequence, $v^{\varepsilon_j} \to v$ uniformly on compact subsets of $\mathcal{R}_2$. From the minimality of v^{ε_j} it is immediate to recognize that $u \ge v$.

To prove the reverse inequality, we consider $w \in C^2(\mathbb{R})$, satisfying

$$(|w'|^{p-2}w')' = \beta(w) \text{ in } \mathbb{R}, \quad w(0) = 1, \quad w'(0) = \alpha,$$

and let

$$w^{\varepsilon_j}(x_1) = \varepsilon_j w\left(\frac{x_1}{\varepsilon_j} - \frac{b_{\varepsilon_j}}{\gamma\varepsilon_j} + \bar{s}\right).$$

Here $\bar{s} < 0$ is a constant, determined as in [6, Proposition 5.3], such that

$$w(s) = \begin{cases} 1 + \alpha s, & s \geq 0, \\ \gamma(s - \bar{s}), & s \leq \bar{s}. \end{cases}$$

Here α and γ are related by $\alpha^p - \gamma^p = (p/(p-1))M$ and therefore $\gamma > 0$ under the assumption $\alpha > ((p/(p-1))M)^{1/p}$. Moreover, we have $w'(x_1) \geq \gamma$ for all $x_1 \in \mathbb{R}$. Observe also that $w^{\varepsilon_j} \leq u - b_{\varepsilon_j} = v^{\varepsilon_j}$ on $\partial\mathcal{R}_2$. From Lemma 3.7 below, it follows that $w^{\varepsilon_j} \leq v^{\varepsilon_j}$ in $\mathcal{R}_2$. Let us point out that such an implication is not immediate since, in general, there is no comparison principle for $\Delta_p - \beta$. As a consequence, $u \leq v$ in $\mathcal{R}_2 \cap \{x_1 > 0\}$. Using (3.8), finally, we see that $u \leq v$ in $\mathcal{R}_2$, and the construction is complete.

Step 2. Let $\mathcal{R}^+ = \{x \mid 0 < x_1 < 1,\ |x'| < 1\}$. Then using the weak formulation of (P_{ε_j}) in $\mathcal{R}^+$ we have

$$\begin{aligned} E_j &= \iint_{\mathcal{R}^+} \frac{\partial}{\partial x_1}\left(\frac{1}{p}|\nabla v^{\varepsilon_j}|^p\right) dx = \iint_{\mathcal{R}^+} |\nabla v^{\varepsilon_j}|^{p-2}\nabla v^{\varepsilon_j} \cdot \nabla v^{\varepsilon_j}_{x_1}\, dx \\ &= \iint_{\mathcal{R}^+} \operatorname{div}(|\nabla v^{\varepsilon_j}|^{p-2}\nabla v^{\varepsilon_j} v^{\varepsilon_j}_{x_1})\, dx - \iint_{\mathcal{R}^+} \beta_{\varepsilon_j}(v^{\varepsilon_j}) v^{\varepsilon_j}_{x_1}\, dx = F_j - G_j. \end{aligned}$$

Using the divergence theorem and that $v^{\varepsilon_j}_{x_1}(0, x') = 0$ (from symmetry in the x_1 variable) we find that

$$F_j = \int_{\partial\mathcal{R}^+ \cap \{x_1 = 1\}} |\nabla v^{\varepsilon_j}|^{p-2}(v^{\varepsilon_j}_{x_1})^2\, dx' + \int_{\partial\mathcal{R}^+ \cap \{|x'| = 1\}} |\nabla v^{\varepsilon_j}|^{p-2} v^{\varepsilon_j}_n v^{\varepsilon_j}_{x_1}\, dS,$$

where $v^{\varepsilon_j}_n$ is the exterior normal derivative of v^{ε_j} on $\partial\mathcal{R}^+ \cap \{|x'| = 1\}$. From the convergence $v^{\varepsilon_j} \to u = \alpha x_1^+ + \alpha x_1^-$ in $\mathcal{R}_2$ and Lemma 3.1 it follows (at least for a subsequence) that

$$\nabla v^{\varepsilon_j}_{x_1} \to \alpha \mathbf{e}_1 \quad \text{pointwise a.e. in } \mathcal{R}_2^+ = \mathcal{R}_2 \cap \{x_1 > 0\}.$$

Since $|\nabla v^{\varepsilon_j}|$ are uniformly bounded, from the dominated convergence theorem we deduce that

$$\lim_{j \to \infty} F_j = \int_{\partial\mathcal{R}^+ \cap \{x_1 = 1\}} \alpha^p\, dx'. \tag{3.9}$$

On the other hand

$$E_j + G_j = \iint_{\mathcal{R}^+} \frac{\partial}{\partial} x_1 \left(\frac{1}{p} |\nabla v^{\varepsilon_j}|^p + B_{\varepsilon_j}(v^{\varepsilon_j}) \right) dx$$
$$\leq \int_{\partial\mathcal{R}^+ \cap \{x_1 = 1\}} \left(\frac{1}{p} |\nabla v^{\varepsilon_j}|^p + B_{\varepsilon_j}(v^{\varepsilon_j}) \right) dx'.$$

Using again that $v^{\varepsilon_j} \to u = \alpha x_1^+ + \alpha x_1^-$ uniformly on compact subsets of $\mathcal{R}_2$, we have $|\nabla v^{\varepsilon_j}| \to \alpha$ uniformly and $B_{\varepsilon_j}(v^{\varepsilon_j}) = M$ on $\partial\mathcal{R}^+ \cap \{x_1 = 1\}$, and therefore

$$\limsup_{j\to\infty} (E_j + G_j) \leq \int_{\mathcal{R}^+ \cap \{x_1 = 1\}} \left(\frac{1}{p} \alpha^p + M \right) dx'. \tag{3.10}$$

Combining (3.9) and (3.10) we obtain

$$\alpha^p \leq \frac{1}{p} \alpha^p + M$$

or equivalently

$$\alpha \leq \left(\frac{p}{p-1} M \right)^{1/p},$$

which is a contradiction. The proof is thus complete. □

Lemma 3.7. *Let $w^\varepsilon(x_1)$ be a strictly increasing solution of*

$$(|w^{\varepsilon\prime}|^{p-2} w^{\varepsilon\prime})' = \beta_\varepsilon(w^\varepsilon)$$

on $\mathbb{R}$, and $v^\varepsilon(x)$ be a solution of $\Delta_p v^\varepsilon = \beta_\varepsilon(v^\varepsilon)$ in $\mathcal{R} = \{x = (x_1, x') \mid a < x_1 < b,\ |x'| < r\}$, continuous up to $\partial\mathcal{R}$. Then the following comparison principle holds: if $v^\varepsilon(x) \geq w^\varepsilon(x_1)$ for all $x \in \partial\mathcal{R}$, then $v^\varepsilon(x) \geq w^\varepsilon(x_1)$ for all $x \in \mathcal{R}$.

Proof. Without loss of generality we assume that $w^\varepsilon(0) = 0$. Since w^ε is strictly increasing, we can find $\tau > \max\{|a|, |b|\}$ such that

$$w^\varepsilon(x_1 - \tau) < v^\varepsilon(x) \quad \text{on } \bar{\mathcal{R}}.$$

For $\eta > 0$ sufficiently small define $w^{\varepsilon,\eta}(x_1) := w^\varepsilon(\varphi_\eta(x_1 - c_\eta))$, where $\varphi_\eta(s) = s + \eta s^2$ and $c_\eta > 0$ is the smallest constant such that $\varphi_\eta(s - c_\eta) \leq s$ on $[-2\tau, 2\tau]$. By the construction we readily have $w^{\varepsilon,\eta} \leq w^\varepsilon$ on $[-2\tau, 2\tau]$ and the straightforward computation shows that

$$\Delta_p w^{\varepsilon,\eta} = (\varphi_\eta')^p (\Delta_p w^\varepsilon)(\varphi_\eta) + ((w^\varepsilon)'(\varphi_\eta))^{p-1} \Delta_p \varphi_\eta$$
$$> (\varphi_\eta')^p \beta_\varepsilon(w^{\varepsilon,\eta}) \geq \beta_\varepsilon(w^{\varepsilon,\eta})$$

on $[-2\tau, 2\tau]$. The first inequality follows from the observation that $\Delta_p \varphi_\eta > 0$ on $[-2\tau, 2\tau]$ (for small $\eta > 0$) and the second inequality follows from the fact that $x_1 \le c_\eta$ implies $\beta_\varepsilon(w^{\varepsilon,\eta}) = 0$ and that for $x_1 \ge c_\eta$ we have $\varphi'_\eta \ge 1$.

Summarizing the construction above, we see that, for small $\eta > 0$, the function $w^{\varepsilon,\eta}$ is strictly increasing on the interval $[-2\tau, 2\tau]$, and satisfies $\Delta_p w^{\varepsilon,\eta} > \beta_\varepsilon(w^{\varepsilon,\eta})$ and $w^{\varepsilon,\eta} \le w^\varepsilon$. Moreover, as $\eta \to 0$, $w^{\varepsilon,\eta}$ converges uniformly to w^ε.

Let now $\tau^* \ge 0$ be the smallest constant with the property

$$w^{\varepsilon,\eta}(x_1 - \tau^*) \le v^\varepsilon(x) \quad \text{on } \bar{\mathcal{R}}.$$

Evidently, $\tau_* < \tau$, and in fact, we claim that $\tau^* = 0$. Indeed, the minimality of τ^* implies that there is a point $x^* \in \bar{\mathcal{R}}$ such that $w^{\varepsilon,\eta}(x_1^* - \tau^*) = v^\varepsilon(x^*)$. If $\tau^* > 0$, we have $w^{\varepsilon,\eta}(\cdot - \tau^*) < w^{\varepsilon,\eta} \le w^\varepsilon \le v^\varepsilon$ on $\partial\mathcal{R}$, and hence x^* is an interior point of $\mathcal{R}$. At this point we observe that the gradient of $w^{\varepsilon,\eta}(x_1 - \tau^*)$ is non-degenerate. We can thus apply the strong comparison principle for the p-Laplacian to obtain a contradiction, since at x^* we have $\Delta_p w^{\varepsilon,\eta}(x_1^* - \tau^*) > \Delta_p v^\varepsilon(x^*)$. This shows that $\tau^* = 0$, and in particular that $w^{\varepsilon,\eta} \le v^\varepsilon$ on $\bar{\mathcal{R}}$. Letting $\eta \to 0$, we conclude the proof of the lemma. $\square$

4. ASYMPTOTIC BEHAVIOR OF LIMIT SOLUTIONS

In this section we prove the asymptotic development of solutions to (P_ε). We begin with the relevant definitions.

Definition 4.1. A unit vector $\eta \in \mathbb{R}^N$ is said to be the *inward unit normal in the measure theoretic sense* to the free boundary $\partial\{u > 0\}$ at a point $x_0 \in \partial\{u > 0\}$ if

$$\lim_{r\to 0} \frac{1}{r^N} \int_{B_r(x_0)} |\chi_{\{u>0\}} - \chi_{\{x \mid \langle x - x_0, \eta\rangle > 0\}}| \, dx = 0. \tag{4.1}$$

Definition 4.2. Let v be a continuous function in a domain $\Omega \subset \mathbb{R}^N$. We say that v is *non-degenerate* at a point $x_0 \in \Omega \cap \{v = 0\}$ if there exist c, $r_0 > 0$ such that

$$\frac{1}{r^N} \int_{B_r(x_0)} v \, dx \ge cr \quad \text{for any } r \in (0, r_0).$$

The main result in this section is the following.

Theorem 4.3. *Let u^{ε_j} be solutions to (P_{ε_j}) in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω and $\varepsilon_j \to 0$. Let $x_0 \in \Omega \cap \partial\{u > 0\}$ be such that $\partial\{u > 0\}$ has an inward unit normal η in the measure theoretic sense at x_0, and suppose that u is non-degenerate at x_0. Under these assumptions, we have*

$$u(x) = \left(\frac{p}{p-1} M\right)^{1/p} \langle x - x_0, \eta\rangle^+ + o(|x - x_0|).$$

The proof of Theorem 4.3 relies heavily on the following result.

Theorem 4.4. *Let u^{ε_j} be a solution to (P_{ε_j}) in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω as $\varepsilon_j \to 0$. Then*

$$\limsup_{x \to x_0} |\nabla u(x)| \le \left(\frac{p}{p-1} M \right)^{1/p}.$$

Proof. Let $\alpha = \limsup_{x \to x_0} |\nabla u(x)|$. Since $u \in \mathrm{Lip}(\Omega)$, clearly $\alpha < \infty$. If $\alpha = 0$, there is nothing to prove, so we may assume $\alpha > 0$. There exists a sequence $x_n \to x_0$ such that $|\nabla u(x_n)| \to \alpha$ and $u(x_n) > 0$. Let $z_n \in \Omega \cap \partial\{u > 0\}$ be such that $d_n = |z_n - x_n| = \mathrm{dist}(x_n, \partial\{u > 0\})$. Define $u_{d_n}(x) = (1/d_n) u(z_n + d_n x)$. Since $u \in \mathrm{Lip}(\Omega)$ and $u_{d_n}(0) = 0$ for every n, $\{u_{d_n}\}$ is uniformly bounded on compact subsets of $\mathbb{R}^N$, and therefore for a subsequence (still denoted by d_n) $u_{d_n} \to u_0$ uniformly on compact subsets of $\mathbb{R}^N$, where $u_0 \in \mathrm{Lip}(\mathbb{R}^N)$.

Now, set $\bar{x}_n = (x_n - z_n)/d_n \in \partial B_1$. We may choose the subsequence d_n so that $\bar{x}_n \to \bar{x} \in \partial B_1$. Then u_0 is p-harmonic and nonnegative in $B_1(\bar{x})$. Consider now the sequence

$$\nu_n = \frac{\nabla u_{d_n}(\bar{x}_n)}{|\nabla u_{d_n}(\bar{x}_n)|} = \frac{\nabla u(x_n)}{|\nabla u(x_n)|}.$$

Passing to a subsequence, we assume, without loss of generality, that $\nu_n \to \mathbf{e}_1$. At this point we observe that $B_{2/3}(\bar{x}) \subset B_1(\bar{x}_n)$ for n sufficiently large, and therefore u_{d_n} is p-harmonic in $B_{2/3}(\bar{x})$. By interior gradient estimates, $u_{d_n} \to u_0$ in $C^{1,\sigma}(B_{1/2}(\bar{x}))$ for some $\sigma > 0$. This suffices to show that $\nabla u_{d_n} \to \nabla u_0$ uniformly in $B_{1/3}(\bar{x})$ and thus, as a consequence, $|\nabla u(x_n)| \to \partial_{x_1} u_0(\bar{x})$. In particular, $\partial_{x_1} u_0(\bar{x}) = \alpha$.

Next, it is easy to show that $|\nabla u_0| \le \alpha$ in $\mathbb{R}^N$. Indeed, let $R > 1$, $\delta > 0$. Then there exists $\tau_0 > 0$ such that $|\nabla u(x)| \le \alpha + \delta$ for any $x \in B_{\tau_0 R}(x_0)$. Observe now that $|x_n - x_0| < \tau_0 R/2$ and $d_n < \tau_0/2$ imply $B_{d_n R}(x_n) \subset B_{\tau_0 R}(x_0)$, and therefore $|\nabla u_{d_n}(x)| \le \alpha + \delta$ in B_R for n large enough. In particular, $\nabla u_{d_n} \to \nabla u_0$ $*$-weakly in $L^\infty(B_R)$ and thus $|\nabla u_0| \le \alpha + \delta$ in B_R. Since δ and R are arbitrary, we conclude

$$|\nabla u_0| \le \alpha \quad \text{in } \mathbb{R}^N.$$

Let $w = \partial_{x_1} u_0$, which is a weak solution to the equation

$$\partial_{x_j} \left(\theta_{ij}(\nabla u_0) \partial_{x_i} w \right) = 0 \quad \text{in } B_1(\bar{x}),$$

with $\theta_{ij}(\xi) = (p-2)|\xi|^{p-4} \xi_i \xi_j + \delta_{ij} |\xi|^{p-2} \xi_{ij}$. We also know that $w \le \alpha$ in $B_1(\bar{x})$, $w(\bar{x}) = \alpha$. By the maximum principle (recall that $\partial_{x_1} u_0(\bar{x}) = \alpha > 0$), $w \equiv \alpha$ in $B_1(\bar{x})$ and so $u_0(x) = \alpha(x_1 - y_1)$ in $B_1(\bar{x})$ for some $y \in \mathbb{R}^N$. It is not difficult to recognize that

$$u_0(x) = \alpha(x_1 - y_1) \quad \text{in } \{x_1 \ge y_1\}.$$

We now apply Lemma A.1 from the Appendix to u_0 in $\{x_1 - y_1 < 0\}$ and obtain

$$u_0(x) = \gamma(x-y)_1^- + o(|x-y|) \quad \text{in } \{x_1 - y_1 < 0\}$$

for some $\gamma \geq 0$. Define, for $\lambda > 0$, $(u_0)_\lambda(x) = (1/\lambda)u_0(\lambda x + y)$. There exist a sequence $\lambda_n \to 0$ and a function $u_{00} \in \mathrm{Lip}(\mathbb{R}^N)$ such that $(u_0)_{\lambda_n} \to u_{00}$ uniformly on compact sets of $\mathbb{R}^N$. We have $u_{00}(x) = \alpha x_1^+ + \gamma x_1^-$ in $\mathbb{R}^N$. By Lemma 3.3, there exists a sequence $\varepsilon_j^0 \to 0$ such that $u^{\varepsilon_j^0}$ is a solution to $(P_{\varepsilon_j^0})$ and $u^{\varepsilon_j^0} \to u_0$ uniformly on compact subsets of B_1. Applying Lemma 3.3 again (the second "blowup") we find a sequence $\varepsilon_j^{00} \to 0$ and solutions $u^{\varepsilon_j^{00}}$ to $(P_{\varepsilon_j^{00}})$ converging uniformly on compact subsets of B_1 to $u_{00}(x) = \alpha x_1^+ + \gamma x_1^-$. Finally, we may apply Proposition 3.4 or Proposition 3.6, depending on whether $\gamma = 0$ or $\gamma > 0$, to conclude $\alpha \leq ((p/(p-1))M)^{1/p}$. □

We are now ready to prove Theorem 4.3.

Proof of Theorem 4.3. Without loss of generality we may assume $x_0 = 0$ and $\eta = \mathbf{e}_1$. Define, for $\lambda > 0$,

$$u_\lambda(x) = \frac{1}{\lambda}u(\lambda x),$$

and let $\rho > 0$ be such that $B_\rho \Subset \Omega$. Since $u_\lambda \in \mathrm{Lip}(B_{\rho/\lambda})$ uniformly in λ, and $u_\lambda(0) = 0$, there exist a subsequence $\lambda_j \to 0$ and a function $U \in \mathrm{Lip}(\mathbb{R}^N)$ such that $u_{\lambda_j} \to U$ uniformly on compact subsets of $\mathbb{R}^N$. From Lemmas 3.1 and 3.3 it follows that u_λ is p-harmonic in its positivity set $\{u_\lambda > 0\}$. Next, rescaling (4.1) we see that, for every fixed $k > 0$

$$|\{u_\lambda > 0\} \cap \{x_1 < 0\} \cap B_k| \to 0 \quad \text{as } \lambda \to 0.$$

Hence, U is nonnegative in $\{x_1 > 0\}$, p-harmonic in $\{U > 0\}$, and vanishes on $\{x_1 \leq 0\}$. By Lemma A.1, there exists $\alpha \geq 0$ such that

$$U(x) = \alpha x_1^+ + o(|x|) \quad \text{in } \{x_1 > 0\}.$$

By virtue of Lemma 3.3, we can find a sequence $\varepsilon_j' \to 0$ and solutions $u^{\varepsilon_j'}$ to $(P_{\varepsilon_j'})$ such that $u^{\varepsilon_j'} \to U$ uniformly on compact sets of $\mathbb{R}^N$ as $j \to \infty$.

On the other hand, if we define $U_\lambda(x) = (1/\lambda)U(\lambda x)$, $U_\lambda \to \alpha x_1^+$ uniformly on compact subsets of $\mathbb{R}^N$ as $\lambda \to 0$. Applying Lemma 3.3 the second time, we find now a sequence $\sigma_j \to 0$ and solutions u^{σ_j} to (P_{σ_j}) such that $u^{\sigma_j} \to \alpha x_1^+$ uniformly on compact subsets of $\mathbb{R}^N$, and

$$\nabla u^{\sigma_j} \to \alpha \chi_{\{x_1 > 0\}} \mathbf{e}_1 \quad \text{in } L^p_{\mathrm{loc}}(\mathbb{R}^N), \tag{4.2}$$

by virtue of Lemma 3.1.

Further, we proceed as in the proof of Proposition 3.4. Let $\psi \in C_0^\infty(\mathbb{R}^N)$ and choose $u_{x_1}^{\sigma_j}\psi$ as a test function in the weak formulation of $\Delta_p u^{\sigma_j} = \beta_{\sigma_j}(u^{\sigma_j})$. Then we find that $B_{\sigma_j}(u^{\sigma_j}) \to M\chi_{\{x_1>0\}} + \bar{M}\chi_{\{x_1<0\}}$ $*$-weakly in $L^\infty(\Omega)$, and $((p-1)/p)\alpha^p = M - \bar{M}$. We now claim that either $\bar{M} = 0$ or $\bar{M} = M$. To this end, let $K \Subset \{x_1 < 0\}$. Then for any $\varepsilon > 0$ there exists $0 < \delta < 1$ such that

$$
\begin{aligned}
|K \cap \{\varepsilon < B_{\sigma_j}(u^{\sigma_j}) < M - \varepsilon\}| &\leq \left| K \cap \left\{ \delta < \frac{u^{\sigma_j}}{\sigma_j} < 1 - \delta \right\} \right| \\
&\leq \left| K \cap \left\{ \beta_{\sigma_j}(u^{\sigma_j}) \geq \frac{a}{\sigma_j} \right\} \right| \to 0
\end{aligned}
$$

as $j \to \infty$, where $a = \inf_{[\delta,1-\delta]} \beta > 0$, since $\beta_{\sigma_j}(u^{\sigma_j}) \to 0$ in $L^1(K)$ by virtue of Proposition 3.2. At this point we observe that, thanks to a simple compactness argument, the latter fact also implies that the convergence of $B_{\sigma_j}(u^{\sigma_j})$ to $\bar{M}$ is actually in $L^1_{\mathrm{loc}}(\{x_1 < 0\})$. We may thus conclude

$$
|K \cap \{\varepsilon < \bar{M} < M - \varepsilon\}| = 0
$$

for every $\varepsilon > 0$ and $K \Subset \{x_1 < 0\}$. Hence, the claim is proved.

Let us see now that $\alpha > 0$. By virtue of the non-degeneracy assumption on u at 0, for every $r > 0$ and j sufficiently large,

$$
\frac{1}{r^N} \int_{B_r} u_{\lambda_j} \geq cr,
$$

and passing to the limit as $j \to \infty$,

$$
\frac{1}{r^N} \int_{B_r} U \geq cr.
$$

Clearly, this forces $\alpha > 0$, and as a consequence, $\bar{M} = 0$ and $\alpha = ((p/(p-1))M)^{1/p}$. We have thus shown that

$$
U(x) = \begin{cases} \left(\dfrac{p}{p-1} M \right)^{1/p} x_1 + o(|x|), & x_1 > 0; \\ 0, & x_1 \leq 0. \end{cases} \tag{4.3}
$$

Next, it follows from Theorem 4.4 that

$$
|\nabla U| \leq \left(\frac{p}{p-1} M \right)^{1/p} \quad \text{in } \mathbb{R}^N.
$$

At this point it suffices to observe that $U \equiv 0$ on $\{x_1 = 0\}$ to conclude that $U \le ((p/(p-1))M)^{1/p} x_1$ in $\{x_1 > 0\}$. Applying Hopf's boundary principle we see that necessarily

$$U(x) = \left(\frac{p}{p-1}M\right)^{1/p} x_1 \quad \text{in } \{x_1 > 0\}.$$

As an immediate consequence we finally obtain

$$u(x) = \left(\frac{p}{p-1}M\right)^{1/p} x_1^+ + o(|x|),$$

and the proof is complete. □

APPENDIX A.

Here we prove an analogue of Corollary A.1 in [6] on asymptotic development of p-harmonic functions near flat boundary points, by a slight modification of the proof of Lemma A.1 in [6].

Lemma A.1. *Let $U \in \operatorname{Lip}(\bar{B}_1^+)$ and assume that U is nonnegative in B_1^+, p-harmonic in $\{U > 0\}$ and vanishes on $\partial B_1^+ \cap \{x_1 = 0\}$. Then in B_1^+, U has the asymptotic development*

$$U(x) = \alpha x_1 + o(|x|)$$

with $\alpha \ge 0$.

Proof. Let $\ell_k := \inf\{l \mid U(x) \le l x_1 \text{ in } B_{2^{-k}}^+\}$. Since ℓ_k is a nonincreasing sequence of nonnegative finite numbers, there exists $\alpha = \lim_{k\to\infty} \ell_k$. Then

$$U(x) \le \alpha x_1 + o(|x|) \quad \text{in } B_1^+.$$

If $\alpha = 0$, the conclusion of the lemma will follow. Assume therefore that $\alpha > 0$. Then there exists a sequence $x^k \in B_1^+$ with $r_k = |x^k| \to 0$ such that

$$U(x^k) \le \alpha x_1^k - \delta_0 r_k,$$

for some $\delta_0 > 0$. Note that x^k will belong to the cone $C = \{|x| \le (\alpha/\delta_0)x_1\}$, and thus we can assume that $\nu_k := x^k/r_k$ converges to some $\nu_0 \in \partial B_1^+ \cap C$. Next, let

$$U^k(x) := \frac{U(r_k x)}{r_k}.$$

Since $\{U^k\}$ are uniformly Lipschitz, we may assume that U_k converges uniformly on $\bar{B}_1^+$ to a nonnegative function V. Then from the construction we will have $V(x) \le \alpha x_1$ in B_1^+, and in addition

$$V(x) \le \alpha x_1 - \frac{\delta_0}{2} \quad \text{and} \quad U^k(x) \le \ell_k x_1 - \frac{\delta_0}{2} \quad \text{on } \partial B_1^+ \cap B_\varepsilon(\nu_0)$$

for a sufficiently small $\varepsilon > 0$ and large k.

Let now w be a p-harmonic function in B_1^+ with smooth boundary values, such that

$$\begin{aligned} w &= x_1 && \text{on } \partial B_1^+ \setminus B_{\varepsilon/2}(\nu_0), \\ w &= x_1 - \frac{\delta_0}{4\alpha} && \text{on } \partial B_1^+ \cap B_{\varepsilon/4}(\nu_0), \\ x_1 - \frac{\delta_0}{4\alpha} &\le w \le x_1 && \text{on } \partial B_1^+ \cap B_{\varepsilon/2}(\nu_0). \end{aligned}$$

Then w vanishes on the flat boundary $\{x_1 = 0\} \cap B_1$, it is $C^{1,\sigma}$ up to $\{x_1 = 0\} \cap B_1$, and by the Hopf boundary principle

$$w(x) \le (1-\mu)x_1 \quad \text{in } B_\gamma^+$$

for some small μ, γ.

Now, from the comparison principle we will have $U^k \le \ell_k w$ in B_1^+ for large k, and consequently $U^k \le \ell_k(1-\mu)x_1$ in B_γ^+. This implies $\alpha \le (1-\mu)\alpha$, which contradicts the assumption that $\alpha > 0$. The lemma is proved. □

REFERENCES

[1] A. ACKER & R. MEYER, *A free boundary problem for the p-Laplacian: uniqueness, convexity, and successive approximation of solutions*, Electron. J. Differential Equations (1995), No. 08, approx. 20 pp. (electronic).

[2] H. BERESTYCKI, L.A. CAFFARELLI & L. NIRENBERG, *Uniform estimates for regularization of free boundary problems*, In: Analysis and Partial Differential Equations (C. Sadosky, ed.), Lecture Notes in Pure and Applied Mathematics, Volume 122, Marcel Dekker, New York, 1988, pp.567-619.

[3] L.A. CAFFARELLI, *Uniform Lipschitz regularity of a singular perturbation problem*, Differential Integral Equations **8** (1995), 1585-1590.

[4] L.A. CAFFARELLI & C.E. KENIG, *Gradient estimates for variable coefficient parabolic equations and singular perturbation problems*, Amer. J. Math. **120** (1998), 391-439.

[5] L.A. CAFFARELLI, L. KARP & H. SHAHGHOLIAN, *Regularity of a free boundary with application to the Pompeiu problem*, Ann. of Math. (2) **151** (2000), 269-292.

[6] L.A. CAFFARELLI, C. LEDERMAN & N. WOLANSKI, *Uniform estimates and limits for a two phase parabolic singular perturbation problem*, Indiana Univ. Math. J. **46** (1997), 453-489.

[7] ———, *Pointwise and viscosity solutions for the limit of a two phase parabolic singular perturbation problem*, Indiana Univ. Math. J. **46** (1997), 719-740.

[8] L.A. CAFFARELLI & J.L. VAZQUEZ, *A free boundary problem for the heat equation arising in flame propagation*, Trans. Amer. Math. Soc., **347** (1995), 411-441.

[9] D. DANIELLI, *A singular perturbation approach to a two phase parabolic free boundary problem arising in flame propagation*, Preprint.

[10] J. HEINONEN, T. KILPELÄINEN & O. MARTIO, *Nonlinear potential theory of degenerate elliptic equations*, The Clarendon Press Oxford University Press, New York, 1993, Oxford Science Publications.

[11] A. HENROT & H. SHAHGHOLIAN, *Existence of classical solutions to a free boundary problem for the p-Laplace operator I. The exterior convex case*, J. Reine Angew. Math. **521** (2000), 85-97.

[12] ______, *Existence of classical solutions to a free boundary problem for the p-Laplace operator II. The interior convex case*, Indiana Univ. Math. J. **49** (2000), 311-323.

[13] ______, *The one-phase free boundary problem for the p-laplacian with non-constant bernoulli boundary condition*, Trans. Amer. Math. Soc., to appear.

[14] L. KARP, T. KILPELÄINEN, A. PETROSYAN & H. SHAHGHOLIAN, *On the porosity of free boundaries in degenerate variational inequalities*, J. Differential Equations **164** (2000), 110-117.

[15] C. LEDERMAN & N. WOLANSKI, *Viscosity solutions and regularity of the free boundary for the limit of an elliptic two phase singular perturbation problem*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **27** (1998), 253-288.

[16] J. SERRIN, *Local behavior of solutions of quasi-linear equations*, Acta Math. **111** (1964), 247-302.

[17] P. TOLKSDORF, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations **51** (1984), 126-150.

D. DANIELLI:
Department of Mathematics
Purdue University
West Lafayette, IN 47907, U. S. A.
E-MAIL: danielli@math.purdue.edu

A. PETROSYAN:
Department of Mathematics
University of Texas at Austin
Austin, TX 78712, U. S. A.
E-MAIL: arshak@math.utexas.edu

H. SHAHGHOLIAN:
Department of Mathematics
Royal Institute of Technology
100 44 Stockholm, SWEDEN.
E-MAIL: henriks@math.kth.se

ACKNOWLEDGMENT: H. Shahgholian was supported by the Swedish Natural Science Research Council. This work started when all three authors were attending the special PDE year at the Mittag-Leffler Institute in 1999-2000. The authors also thank Luis Caffarelli for valuable suggestions.

KEY WORDS AND PHRASES: singular perturbation problem; free boundary problem; p-Laplace operator; uniform gradient bounds.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 35R35; 35J60.
Received: *April 3rd, 2001; revised: September 10th, 2002.*