Math. Z. 226, 147–154 (1997)

Mathematische Zeitschrift

Subelliptic mollifiers and a basic pointwise estimate of Poincaré type

Luca Capogna*, **Donatella Danielli**, **Nicola Garofalo****

Department of Mathematics, Purdue University, West Lafayette IN 47907

Received 1 September 1995; in final form 13 September 1995

1 Introduction

The aim of this paper is to prove a pointwise estimate which plays a fundamental role in the analysis of both linear and nonlinear partial differential equations arising from a system of non-commuting vector fields $X = \{X_1, \ldots, X_m\}$. Although our result holds in a larger setting, for the unity of presentation we confine ourselves to the specific case of smooth vector fields $X_1, \ldots X_m$ satisfying Hörmander's finite rank condition [H]: $Rank(Lie[X_1, \ldots, X_m])(x) = n$ for every $x \in \mathbb{R}^n$. Let $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+$ be the Carnot-Carathéodory metric associated to $X_1, \ldots, X_m$, and for $x_0 \in \mathbb{R}^n$ and $R > 0$ set $B = B_d(x_0, R) = \{y \in \mathbb{R}^n | d(x_0, y) < R\}$. For $a > 0$ the symbol aB will denote the concentric ball $B_d(x_0, aR)$. For a given function u we let $Xu = (X_1 u, \ldots, X_m u)$ and $|Xu|^2 = \sum_{j=1}^m (X_j u)^2$. Throughout the paper we will use the standard notation $u_B = \frac{1}{|B|} \int_B u(y) dy$.

Our main result is

Theorem 1.1. *Let $U \subset\subset \mathbb{R}^n$, and $x_0 \in U$. There exist C, $R_0 > 0$ and $a > 1$ depending only on U and the vector fields $X_1, \ldots, X_m$ such that for any $u \in C^1(aB)$, $R \le R_0$, and $x \in B$ one has*

$$|u(x) - u_B| \le C \int_{aB} |Xu(y)| \frac{d(x,y)}{|B_d(x, d(x,y))|} dy \ . \tag{1.1}$$

Theorem 1.1 has several remarkable applications. Recently Nhieu and one of us [GN] have proved that (1.1) and the doubling condition (1.4) below, imply a sharp subelliptic Sobolev embedding. More precisely one has the following:

Theorem 1.2. *(see [GN]) Let $U \subset\subset \mathbb{R}^n$ be fixed. Then there exists $R_0 > 0$ and $C > 0$ such that for any $x_0 \in U, B = B_d(x_0, R)$ and $0 < R \le R_0$, one has for $u \in C^1(B)$*

* Supported by a grant of the Purdue Research Foundation
** Supported by the NSF, Grant DMS-9404358

$$\left(\frac{1}{|B|}\int_B |u-u_B|^{\frac{Q}{Q-1}}dx\right)^{\frac{Q-1}{Q}} \le C\,\frac{R}{|B|}\int_B |Xu|dx\ ,$$

where Q is the homogeneous dimension of the family $X_1,\ldots,X_m$ in U (see below).

It is worth emphasizing that, in opposition to (1.1), the inequality in Theorem 1.2 involves integration over the same ball B in both left- and right-hand side.

Another important consequence of Theorem 1.1 is that Theorem 1.2 is equivalent to a relative isoperimetric inequality for sets of finite (generalized) perimeter (for the relevant definitions see also [CDG1]):

Theorem 1.3. *(see [GN]) Let $E\subset\mathbb{R}^n$ be an X-Caccioppoli set. Then, with U, Q, C, R_0 and B as in Theorem 1.2 one has*

$$\min(|E\cap B|,\ |E^C\cap B|)^{\frac{Q-1}{Q}} \le CR|B|^{-\frac{1}{Q}}P_X(E;B)\ ,$$

where $P_X(E,B)$ is the X-perimeter of E relative to B.

For other applications of Theorem 1.1 see for instance [He], [HeHo] and [Gr].

In this paper we will show that Theorem 1.1 follows in an elementary way only from the basic properties (a), (b) and (c) listed below.

Let $U\subset\subset\mathbb{R}^n$. There exist constants $Q,R,R_0>0$, depending only on U and $X_1,\ldots,X_m$, such that for any $x\in U$ and $0<R\le R_0$:

(a) One has

$$C\,\Lambda(x,R)\le |B_d(x,R)|\le C^{-1}\Lambda(x,R)\ , \tag{1.2}$$

where $\Lambda(x,R)$ is a polynomial in r with positive coefficients satisfying for $0<\lambda<1$

$$C\lambda^Q\Lambda(x,R)\le\Lambda(x,\lambda R)\le C^{-1}\lambda^{Q(x)}\Lambda(x,R)\ , \tag{1.3}$$

with $Q\ge Q(x)\ge n\ge 3$. The number Q is called *the homogeneous dimension of* $X_1,\ldots,X_m$ relative to U. The estimate (1.2) implies

$$|B_d(x,2R)|\le C|B_d(x,R)|\ . \tag{1.4}$$

(b) For any $u\in C^1(2B)$

$$\int_B |u(y)-u_B|dy\le CR\int_{2B}|Xu(y)|dy\ . \tag{1.5}$$

(c) Denote by $\Gamma(x,y)$ the fundamental solution with pole in x of the operator $\mathscr{L}=\sum_{j=1}^m X_j^*X_j$. Then one has

$$C\,\frac{d(x,y)^2}{|B_d(x,d(x,y))|}\le\Gamma(x,y)\le C^{-1}\frac{d(x,y)^2}{|B_d(x,d(x,y))|} \tag{1.6}$$

and

$$|X\Gamma(x,y)|\le C^{-1}\frac{d(x,y)}{|B_d(x,d(x,y))|}\ . \tag{1.7}$$

Properties (a), (b) and (c) were established in a series fo fundamental papers. For (a) see [NSW], (b) is contained in [J], (c) was independently obtained in [SC] and [NSW].

Remark 1. Given a system of Lipschitz continuous vector fields $X = \{X_1, \dots, X_m\}$ one can still define the associated Carnot-Carathéodory metric. In this more general setting properties (a), (b) and (c) can be thought of as a set of hypothesis, under which Theorem 1.1 holds.

The key to our approach consists in working with a family of "balls" different from the Carnot-Carathéodory balls, namely some appropriately rescaled level sets of the fundamental solution $\Gamma(x, y)$ defined in (c). To be more precise let

$$E(x, r) = r^{-2}\Lambda(x, r) \tag{1.8}$$

and observe that since $Q(x) \geq n \geq 3$, $E(x, r)$ is a strictly increasing function of r. Denote by

$$F(x, \cdot) = E^{-1}(x, \cdot) \tag{1.9}$$

its inverse function and define for $x \in \mathbb{R}^n$ and $R > 0$

$$B(x, r) = \left\{ y \in \mathbb{R}^n | \Gamma(x, y) > \frac{1}{E(x, R)} \right\} .$$

By virtue of (1.4) and (c), one can prove that if U, C and R are defined as in (a), (b) and (c), then

$$B_d(x, a^{-1}R) \subset B(x, R) \subset B_d(x, aR) , \tag{1.10}$$

with $a > 0$ depending on U and $X_1, \dots, X_m$. The latter allows, in many cases, to substitute the metric balls with the smooth sets $B(x, R)$ and obtain equivalent results. In particular, we are going to prove Theorem 1.1 for the sets $B(x, R)$, observing that the original version with $B_d(x, R)$ follows directly from (1.10) (with different constants). Also, (a) and (c) imply

$$Cd(x, y) \leq F(x, \Gamma^{-1}(x, y)) \leq C^{-1}d(x, y) . \tag{1.11}$$

An advantage in using the level sets of the fundamental solution consists in the existence of a representation formula analogous to the classical one, supported on such sets. In fact, from (1.4), (c) and the co-area formula [F] one can deduce (see [CGL]) that for any $x \in U$, $0 < s < R_0$ and $u \in C^1(B(x, 2R_0))$

$$\int_{\partial B(x,s)} u(y) \frac{|X_y \Gamma(x, y)|^2}{|D_y \Gamma(x, y)|} dH_{n-1} = u(x) + \int_{B(x,s)} < Xu(y), X_y \Gamma(x, y) > dy \ . \tag{1.12}$$

In this paper we assume (a), (b) and (c), together with their consequences (1.10)–(1.12) as a starting point. From there we reach the conclusion through an elementary argument involving a family of mollifiers tailored on the "subelliptic geometry" attached to $\mathscr{L}$ which were introduced in [CDG2] to characterize domains supporting the Poincaré inequality or the compact embedding. These mollifiers turn out to be a powerful tool in the analysis of the operator $\mathscr{L}$. In particular, applications to boundary problems will be given in forthcoming work.

Remark 2. In the setting of nilpotent, stratified Lie groups Theorem 1.1 was established in [L] with a different approach. Even in this restricted context our proof is much simpler.

Remark 3. We stress that Theorem 1.1 holds globally on a stratified, nilpotent Lie group. More in general, thanks to the works [Gu], [V] and [MS], the assumptions (a), (b) an (c) in this paper are fulfilled in any connected Lie group of polynomial volume growth. Therefore, Theorem 1.1. continues to hold globally in that setting.

Remark 4. After this paper was completed we learned that a completely different proof of Theorem 1.1 has independently been given by Franchi, Lu and Wheeden [FLW1]. Compared to our approach, however, the proof in [FLW1] is considerably more complicated and longer.

2 Proof of the main theorem

Following [CDG2] for $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ we define a family of *mollifiers*

$$J_R u(x) = \int_{\mathbb{R}^n} u(y) K_R(x,y) dy, \quad R > 0 , \tag{2.1}$$

where

$$K_R(x,y) = f_R(F(x, \Gamma^{-1}(x,y)) \frac{|X_y \Gamma(x,y)|^2}{|\Gamma(x,y)|^2} F'(x, \Gamma^{-1}(x,y)) . \tag{2.2}$$

In (2.1) we have let $F'(x,s) = \frac{d}{ds} F(x,s)$, $f \in C_0^\infty(1,2)$, $\int_{\mathbb{R}} f(s) ds = 1$ and $f_R(s) = R^{-1} f(sR^{-1})$.

Remark 5. We explicitly observe that for any $x \in \mathbb{R}^n$, *supp* $K_R(x,\cdot) \subset B(x,2R) \setminus B(x,R)$.

The properties of J_R have been extensively studied in [CDG2]. In the present paper only the following ones are needed

Theorem 2.1. *Let $U \subset\subset \mathbb{R}^n$. There exist C, $R_0 > 0$ depending only on U and $X_1, \ldots, X_m$ such that for every $x \in U$, $0 < R \le R_0$ one has*

– *i)* $J_R 1 = \int_{\mathbb{R}^n} K_R(x,y) dy = 1$.
– *ii) For $u \in C^1(B(x,2R))$ one has*

$$J_R u(x) = u(x) + \int_0^\infty f_R(s) \int_{B(x,s)} < Xu(y), X_y \Gamma(x,y) > dyds . \tag{2.3}$$

– *iii)* $\sup_{y \in B(x,2R) \setminus B(x,R)} K_R(x,y) \le C |B(x,R)|^{-1}$.

Proof of (i). Letting $u = 1$ in (1.12) one has

$$\int_{\partial B(x,s)} \frac{|X_y \Gamma(x,y)|^2}{|D_y \Gamma(x,y)|} dH_{n-1}(y) = 1 \ . \tag{2.4}$$

By the co-area formula (see [F]) and (2.2) it follows that

$$\int_{\mathbb{R}^n} K_R(x,y)dy = \int_0^\infty \int_{\Gamma(x,y)=t} f_R(F(x,\Gamma^{-1}(x,y)) \frac{|X_y \Gamma(x,y)|^2}{\Gamma^2(x,y)}$$
$$F'(x,\Gamma^{-1}(x,y))|D_y \Gamma(x,y)|^{-1} dH_{n-1}(y)dt \ . \tag{2.5}$$

If we set $t = E(x,\tau)^{-1}$, then $\tau = F(x,t^{-1})$ and $dt = -\frac{E'(x,\tau)}{E(x,\tau)^2}d\tau$, where E, F are as in (1.8), (1.9) and $E'(x,s) = \frac{d}{ds}E(x,s)$. By this change of variable (2.5) gives

$$\int_{\mathbb{R}^n} K_R(x,y)dy = \int_0^\infty \int_{\partial B(x,\tau)} f_R(\tau) \frac{|X_y \Gamma(x,y)|^2}{D_y \Gamma(x,y)|}$$
$$E(x,\tau)^2 F'(x,E(x,\tau)) \frac{E'(x,\tau)}{E(x,\tau)^2} dH_{n-1}(y)d\tau \ . \tag{2.6}$$

Since $F'(x,E(x,\tau)) = E'(x,\tau)^{-1}$, (2.6) yields

$$\int_{\mathbb{R}^n} K_R(x,y)dy = \int_0^\infty f_R(\tau) \int_{\partial B(x,\tau)} \frac{|X_y \Gamma(x,y)|^2}{|D_y \Gamma(x,y)|} dH_{n-1}(y)d\tau \ . \tag{2.7}$$

The latter and (2.4) allow to conclude the proof of (i). □

Proof of (ii). Using the co-area formula and the same change of variables that led to (2.6) one has for $x \in U$

$$J_R u(x) = \int_R^{2R} \left(\int_{\partial B(x,s)} u(y) \frac{|X_y \Gamma(x,y)|^2}{|D_y \Gamma(x,y)|} dH_{n-1}(y) \right) f_R(s)ds \ . \tag{2.8}$$

The conclusion now follows from (2.8) after multiplying (1.12) by $f_R(s)$ and integrating in $s \in (0,\infty)$.

Proof of (iii). From (1.6)-(1.7) and (2.2) we have for $y \in B(x,2R) \setminus B(x,R)$,

$$\begin{aligned} K_R(x,y) &\leq \frac{C}{R} \frac{|F'(x,\Gamma^{-1}(x,y))||X\Gamma(x,y)|^2}{\Gamma^2(x,y)} \\ &= CR^{-1} \frac{|X\Gamma(x,y)|^2}{\Gamma^2(x,y)} \frac{1}{|E'(x,F(x,\Gamma^{-1}(x,y)))|} \ , \end{aligned} \tag{2.9}$$

where in the last equality we have used the inverse function theorem. Recalling that $\Lambda(x,r)$ is a polynomial function in r with positive coefficients, from (1.3) and (1.8) we obtain

$$n \leq \frac{sE'(x,s)}{E(x,s)} \leq Q \ .$$

Inserting this information in (2.9) and using (1.6), (1.7) and (1.11) we conclude

$$K_R(x,y) \le CR^{-1}\left(\frac{|F(x,\Gamma^{-1}(x,y))|}{|E(x,F(x,\Gamma^{-1}(x,y)))|}\right)\frac{|X\Gamma(x,y)|^2}{\Gamma^2(x,y)} \le C\,|B(x,d(x,y))|^{-1}\,,$$

that implies

$$K_R(x,y) \le C\,|B(x,R)|^{-1}\,. \tag{2.10}$$

□

The following corollary of Theorem 2.1, (ii) plays a crucial role.

Corollary 2.1. *Let $U \subset\subset \mathbb{R}^n$ and $x_0 \in U$. There exist constants C, $R_0 > 0$ depending on U and the vector fields $X_1,\dots,X_m$ such that for any $4R < R_0$, $x \in B(x_0,R)$ and $u \in C^1(B(x_0,3R))$*

$$|J_R u(x) - u(x)| \le C\int_{B(x_0,3R)} |Xu(y)|\frac{d(x,y)}{|B(x,d(x,y))|}dy\,.$$

Proof.. The inequality follows immediately from Theorem 2.1, (ii) and the estimates of the fundamental solution (1.7). □

We are now ready to give the

Proof of Theorem 1.1. Denote $B = B(x_0,R)$.

By (ii) in Theorem 2.1 one has

$$|u(x) - u_{3B}| \le |u(x) - J_R u(x)| + |J_R(u - u_{3B})(x)|\,. \tag{2.11}$$

From (iii) in Theorem 2.1 and Remark 5 we estimate

$$\begin{aligned} |J_R(u-u_{3B})(x)| &\le \int_{B(x,2R)\setminus B(x,R)} K_R(x,y)|u-u_{3B}|(y)dy \\ &\le C\,|B(x,R)|^{-1}\int_{B(x,2R)} |u-u_{3B}|(y)dy\,. \end{aligned} \tag{2.12}$$

By (2.2) and property (b) one infers

$$\begin{aligned} |J_R(u-u_{3B})(x)| &\le C\,|B(x,R)|^{-1}\int_{3B} |u-u_{3B}|(y)dy \\ &\le CR|B(x,R)|^{-1}\int_{6B} |Xu(y)dy\,. \end{aligned} \tag{2.13}$$

Finally, observe that (1.3) implies for $y \in 6B$, $y \neq x$

$$\begin{aligned} |B(x,R)| &= |B\left(x,\frac{R}{d(x,y)}d(x,y)\right)| \ge C\left(\frac{R}{d(x,y)}\right)^n |B(x,d(x,y))| \\ &\ge C\left(\frac{R}{d(x,y)}\right)^{n-1} R\frac{B(x,d(x,y))}{d(x,y)}\,. \end{aligned} \tag{2.14}$$

This gives

$$\frac{R}{|B(x,R)|} \leq C \frac{d(x,y)}{|B(x,d(x,y))|} . \tag{2.15}$$

As a consequence of (2.11), (2.13), (2.15) and Corollary 2.1 we have that for any $x \in B$

$$|u(x) - u_{3B}| \leq C \int_{6B} |Xu(y)| \frac{d(x,y)}{B(x,d(x,y))|} dy .$$

The proof is now concluded by using (1.10), the triangle inequality, property (b) and (2.15). □

Remark 6. We would like to comment on the above argument. The estimate of the piece $|J_R(u - u_{3B})|(x)$ in (2.11) does not use assumption (c). We could have employed as a mollifier any J_R with a kernel $K_R(x,y)$ satisfying (2.10). Thus, in particular,

$$J_R u(x) = \frac{1}{|B_d(x,R)|} \int_{B_d(x,R)} u(y) dy , \tag{2.16}$$

for which

$$K_R(x,y) = \frac{1}{|B_d(x,R)|} \chi_{B_d(x,R)}(y) ,$$

works (here χ_E denotes the characteristic function of the set E).

In the estimate of the piece $|u(x) - J_R u(x)|$ in (2.11), instead, we have used the special structure of the mollifier which is reflected by the important formula (2.3) above.

Note: *The result of this paper was part of a lecture delivered by the third named author at the conference on "Integral Inequalities and Nonlinear Variational Problems" held in Ischia, Italy, June 1–3, 1995.*

In September 1995 R. Wheeden has kindly communicated to us an argument, based on (2.14), which in the estimate of the first addend in (2.11), allows to use any mollifier with a kernel satisfying (2.10), see [FLW2]. As a consequence, Theorem 1.1 holds under assumptions (a) and (b) alone.

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