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Sub-Riemannian calculus and monotonicity of the perimeter for graphical strips

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Abstract We consider the class of minimal surfaces given by the graphical strips $\mathcal{S}$ in the Heisenberg group $\mathbb{H}^1$ and we prove that for points p along the center of $\mathbb{H}^1$ the quantity $\frac{\sigma_H(\mathcal{S}\cap B(p,r))}{r^{Q-1}}$ is monotone increasing. Here, Q is the homogeneous dimension of $\mathbb{H}^1$. We also prove that these minimal surfaces have maximum volume growth at infinity.

Keywords Minimal surfaces · H-mean curvature · Integration by parts · First and second variation · Monotonicity of the H-perimeter

1 Introduction

In recent years the study of surfaces of constant horizontal mean curvature $\mathcal{H}$ (to be defined below) in sub-Riemannian spaces has seen an explosion of interest. Similarly to the classical situation, this interest has provided a strong stimulus for the development of a corresponding

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geometric measure theory. For a partial account of such surge of activity the reader should consult [1–7,9–20,22,25–27,29–32,34–50,53–62,64,65].

In this context, the Heisenberg group $\mathbb{H}^n$ occupies a central position, especially in connection with the sub-Riemannian Bernstein and isoperimetric problems. We recall that $\mathbb{H}^n$ is the stratified nilpotent Lie group whose (real) underlying manifold is $\mathbb{R}^{2n+1}$ with the non-Abelian group law inherited by the complex product in $\mathbb{C}^{n+1}$

$$(x, y, t) \cdot (x', y', t') = \left(x + x', y + y', t + t' + \frac{1}{2}(\langle x, y'\rangle - \langle x', y\rangle)\right).$$

If we set $p = (x, y, t)$, $p' = (x', y', t') \in \mathbb{R}^{2n+1}$, define the left-translation map by $L_p(p') = p \circ p'$, and we indicate with L_p^* its differential, then the Lie algebra of all left-invariant vector fields in $\mathbb{H}^n$ is spanned by the $2n + 1$ vector fields

$$X_i = L_p^*(\partial_{x_i}) = \partial_{x_i} - \frac{y_i}{2}\partial_t, \quad X_{n+i} = L_p^*(\partial_{y_i}) = \partial_{y_i} + \frac{x_i}{2}\partial_t, \quad T = L_p^*(\partial_t) = \partial_t,$$

where $i = 1, \dots, n$. We note the important commutation relations $[X_i, X_{n+j}] = T$, $i, j = 1, \dots, n$. They guarantee that the vector fields $X_1, \dots, X_{2n}$ suffice to generate the whole Lie algebra, and therefore the Heisenberg group is a stratified nilpotent Lie group of step two, see [8,33,66]. Such group is in fact the basic model of such sub-Riemannian manifolds, and it plays in this context much the same role played by $\mathbb{R}^n$ in Riemannian geometry. The first Heisenberg group $\mathbb{H}^1$ is obtained when $n = 1$. If we indicate with $p = (x, y, t) \in \mathbb{R}^3$ a generic point of its underlying manifold, then the generators of its (real) Lie algebra are the two vector fields

$$X_1 = L_p^*(\partial_x) = \partial_x - \frac{y}{2}\partial_t, \quad X_2 = L_p^*(\partial_y) = \partial_y + \frac{x}{2}\partial_t,$$

and we clearly have $[X_1, X_2] = T = L_p^*(\partial_t) = \partial_t$.

To introduce the results in this paper we recall that one of the most fundamental properties of classical minimal surfaces $\mathcal{S} \subset \mathbb{R}^m$ is the following well-known monotonicity theorem, see [52], and also [21,51,63].

Theorem 1.1 *Let $\mathcal{S} \subset \mathbb{R}^m$ be a C^2 hypersurface, with H being its mean curvature, then for every fixed $p \in \mathcal{S}$ the function*

$$r \to \frac{H_{m-1}(\mathcal{S} \cap B_e(p, r))}{r^{m-1}} + \int_0^r \frac{m-1}{t^{m-1}} \int_{\mathcal{S} \cap B_e(p,t)} |H| \, dH_{m-1} \, dt, \tag{1.1}$$

is non-decreasing. In particular, if $\mathcal{S}$ is minimal, i.e., if $H \equiv 0$, then

$$r \to \frac{H_{m-1}(\mathcal{S} \cap B_e(p, r))}{r^{m-1}} \tag{1.2}$$

is non-decreasing.

In (1.1), (1.2) we have denoted by H_{m-1} the $(m-1)$-dimensional Hausdorff measure in $\mathbb{R}^m$. Theorem 1.1 has many deep implications. It says, in particular, that minimal hypersurfaces have maximum volume growth at infinity, i.e., there exists $c_m > 0$ such that $H_{m-1}(S \cap B(p, r)) \geq c_m r^{m-1}$ as $r \to \infty$.

In this paper we are interested in related growth properties of the sub-Riemannian volume on a minimal surface in $\mathbb{H}^1$. By minimal we mean a C^2 oriented surface $\mathcal{S} \subset \mathbb{H}^1$ such that its horizontal mean curvature $\mathcal{H}$ vanishes identically on $\mathcal{S}$, see Definition 2.2 below. The sub-Riemannian volume instead is the so-called horizontal perimeter, see Definition 2.3 below. We should say right upfront that, despite the efforts of several workers, the monotonic character of the sub-Riemannian volume on a general minimal hypersurface in $\mathbb{H}^n$ continues to represent a fundamental open question. As far as we are aware, our main result, Theorem 1.3 below, is the first sub-Riemannian monotonicity theorem.

The main obstacle so far has been represented by finding an appropriate substitute of some basic properties such as, for instance, the following elementary, yet fundamental fact from Riemannian geometry. Consider in $\mathbb{R}^m$ the radial vector field $\zeta(x) = \sum_{i=1}^m x_i \partial_{x_i}$, then on *any* C^2 hypersurface $\mathcal{S} \subset \mathbb{R}^m$, one has

$$\mathrm{div}_{\mathcal{S}}\zeta \ \equiv \ m - 1, \tag{1.3}$$

where we have indicated with $\mathrm{div}_{\mathcal{S}}$ the Riemannian divergence on $\mathcal{S}$. The elementary identity (1.3) has many deep implications, and one could safely claim that behind most fundamental results from the classical theory of minimal surfaces there is (1.3). For instance, Theorem 1.1 and the Sobolev inequalities on minimal surfaces [52] are consequences (highly non-trivial, of course) of (1.3). The number $m-1$ in the right-hand side of (1.3) is dimensionally correct since the standard volume form σ on a hypersurface in $\mathbb{R}^m$ scales according to the rule

$$\sigma(\delta_\lambda(E)) = \lambda^{m-1}\sigma(E), \quad E \subset \mathcal{S},$$

where $\delta_\lambda(x) = \lambda x$ represent the isotropic dilations in $\mathbb{R}^m$.

In sub-Riemannian geometry, however, the correct dimension is dictated by the non-isotropic dilations of the ambient non-Abelian group, and this seemingly natural fact becomes a source of great complications. For instance, given a C^1 hypersurface $\mathcal{S} \subset \mathbb{H}^n$, and indicating with σ_H the horizontal perimeter on $\mathcal{S}$ (for its definition we refer the reader to Sect. 2), then one has

$$\sigma_H(\delta_\lambda(E)) \ = \ \lambda^{Q-1}\sigma_H(E), \quad E \subset \mathcal{S}, \tag{1.4}$$

where $\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t)$ indicates the non-isotropic dilations in $\mathbb{H}^n$ associated with the grading of its Lie algebra. Here, the number $Q = 2n + 2$ represents the homogeneous dimension of $\mathbb{H}^n$ associated with the dilations $\{\delta_\lambda\}_{\lambda>0}$. Thus for instance, when $n = 1$, we have $Q = 4$.

Guided by the analogy with (1.3) one would like to find a horizontal vector field ζ in $\mathbb{H}^n$ whose sub-Riemannian divergence on $\mathcal{S}$ (to be precisely defined below) satisfy the equation

$$\mathrm{div}_{H,\mathcal{S}}\zeta = Q - 1. \tag{1.5}$$

Such attempt would not possibly work however, for several reasons which are all connected to one another. First of all, the integration by parts formula in which one would like to use such a ζ contains a corrective term which is produced by the above mentioned non trivial commutation relations which connect the generators of the Lie algebra of $\mathbb{H}^n$. Secondly, one should not forget that not only the radial vector field ζ satisfies (1.3), but it also possess the equally important property that

$$\sup_{x\in\mathcal{S}\cap B(0,r)} |\langle\zeta(x), \nabla^{\mathcal{S}}|x|\rangle| \ \le \ r, \tag{1.6}$$

where $\nabla^{\mathcal{S}}$ indicates the Riemannian gradient on $\mathcal{S}$. Because of these obstructions, there has been no progress so far on the question of the monotonic character of sub-Riemannian minimal surfaces.

One of the main contributions of the present paper is a monotonicity formula for an interesting class of minimal surfaces in $\mathbb{H}^1$, the so-called *graphical strips*. Such surfaces were introduced in the work [28], where they played a crucial role in the solution of the sub-Riemannian Bernstein problem in $\mathbb{H}^1$.

Our main result hinges on the discovery that, despite the original evidence against it, for such class of surfaces the generator of the non-isotropic group dilations in $\mathbb{H}^1$ provides a valid replacement of the radial vector field in $\mathbb{R}^m$. This sentence must, however, be suitably interpreted, in the sense that things do not work so simply. What we mean by this is that the horizontal integration by parts formulas from [25] (see also [23]) which constitute the sub-Riemannian counterpart of the classical integration by parts formulas on hypersurfaces (for these, see, e.g., [21,51,63]), do not suffice. They need to be appropriately intertwined with a twisted vertical integration by parts formula also discovered in [25]. Both such formulas have played a pervasive role in the establishment of a general second variation formula for the horizontal perimeter. For other independent works on integration by parts and variation formulas one should also see [15,44,53,65]. To state our main result we recall the relevant definition.

Definition 1.2 We say that $\mathcal{S} \subset \mathbb{H}^1$ is a *graphical strip* if there exist an interval $I \subset \mathbb{R}$, and $G \in C^2(I)$, with $G' \geq 0$ on I, such that, after possibly a left-translation and a rotation about the t-axis, then either

$$\mathcal{S} = \{(x, y, t) \in \mathbb{H}^1 \mid (y, t) \in \mathbb{R} \times I, x = yG(t)\}, \tag{1.7}$$

or

$$\mathcal{S} = \{(x, y, t) \in \mathbb{H}^1 \mid (x, t) \in \mathbb{R} \times I, y = -xG(t)\}. \tag{1.8}$$

If there exists $J \subset I$ such that $G' > 0$ on J, then we call $\mathcal{S}$ a *strict graphical strip*.

When the interval I can be taken to be the whole real line, then we call $\mathcal{S}$ an entire graphical strip (strict, if $G' > 0$ on some $J \subset \mathbb{R}$). Examples of entire strict graphical strips are the surfaces

$$x = y(\alpha t + \beta), \quad \alpha > 0, \quad (y, t) \in \mathbb{R}^2,$$

and

$$x = y \tan \tanh t, \quad (y, t) \in \mathbb{R}^2.$$

In Theorem 1.5 in [28] it was proved that every graphical strip is a minimal surface in $\mathbb{H}^1$. The relevance of these minimal surfaces lies in some results from [28], which we recall in Proposition 4.1, Theorems 4.2 and 4.3 below. The main result of this paper is the following theorem.

Theorem 1.3 *Let $\mathcal{S} \subset \mathbb{H}^1$ be a graphical strip, and denote by σ_H the sub-Riemannian volume form, or horizontal perimeter, on $\mathcal{S}$. For every $p_0 = (0, 0, t_0) \in \mathcal{S}$ the function*

$$r \rightarrow \frac{\sigma_H(\mathcal{S} \cap B(p_0, r))}{r^{Q-1}}, \quad r > 0,$$

is monotone non-decreasing. Moreover, there exists a universal constant $\omega > 0$, independent of the point p_0 and of the particular graphical strip $\mathcal{S}$, such that

$$\sigma_H(\mathcal{S} \cap B(p_0, r)) \geq \omega r^{Q-1}, \quad \textit{for every } r > 0.$$

As a consequence, graphical strips have maximum volume growth at infinity along the center of $\mathbb{H}^1$.

In the statement of Theorem 1.3 we have denoted by $B(p_0, r) = \{p \in \mathbb{H}^n \mid d(p, p_0) < r\}$, where $d(p, p_0) = N(p_0^{-1} p)$ represents the gauge distance on $\mathbb{H}^n$ defined via the Koranyi-Folland gauge function $N(p) = (|z|^4 + 16t^2)^{1/4}$, $p = (z, t) \in \mathbb{H}^n$.

The proof of Theorem 1.3 is inspired to the ideas set forth in the beautiful paper [52], except that, as we have said, we need some new ideas to bypass the obstacles posed by the sub-Riemannian setting.

A description of the content of the paper is as follows. In Sect. 2 we introduce the relevant geometric setup, and we recall the main integration by parts theorems from [25] which constitute the backbone of the paper. In Sect. 3 we combine such results with a suitable adaptation of the ideas in [52] to establish some general growth results for hypersurfaces in $\mathbb{H}^n$. A basic new fact is the identity (3.12) in Proposition 3.5 which represents the appropriate sub-Riemannian analogue of (1.3). Combining it with the integration by parts we obtain the growth Theorem 3.6, which concludes Sect. 3. Finally, Sect. 4 is devoted to proving Theorem 1.3.

2 Sub-Riemannian calculus on hypersurfaces

In this section we introduce the relevant notation and recall some basic integration by parts formulas involving the tangential horizontal gradient on a hypersurface, and the horizontal mean curvature of the latter, which are special case of some general formulas discovered in [25]. Such formulas are reminiscent of the classical one, and in fact they encompass the latter. However, an important difference is that the ordinary volume form on the hypersurface $\mathcal{S}$ is replaced by the horizontal perimeter measure $d\sigma_H$. Furthermore, they contain additional terms which are due to the non-trivial commutation relations, which is reflected in the lack of torsion freeness of the horizontal connection on $\mathcal{S}$. Such term prevents the corresponding horizontal Laplace-Beltrami operator from being formally self-adjoint in $L^2(\mathcal{S}, d\sigma_H)$ in general.

We next recall some basic concepts from the sub-Riemannian geometry of a C^2 hypersurface $\mathcal{S} \subset \mathbb{H}^n$. For a detailed account we refer the reader to [25]. We consider the Riemannian manifold $\mathbb{H}^n$ equipped with the left-invariant metric tensor with respect to which $X_1, \dots, X_{2n}$ is an orthonormal basis, the corresponding Levi-Civita connection ∇ on $\mathbb{H}^n$, and the horizontal Levi-Civita connection ∇^H. We assume that $\mathcal{S}$ is oriented and denote by $\boldsymbol{\nu}$ the Riemannian Gauss map on $\mathcal{S}$. We define the so-called *angle function* on $\mathcal{S}$ as follows

$$W \;=\; |\boldsymbol{N}^H| \;=\; \sqrt{\sum_{j=1}^{2n} \langle \boldsymbol{\nu}, X_j \rangle^2}. \tag{2.1}$$

The *characteristic set* of $\mathcal{S}$, hereafter denoted by $\Sigma_{\mathcal{S}}$, is the compact subset of $\mathcal{S}$ where the continuous function W vanishes

$$\Sigma(\mathcal{S}) = \{p \in \mathcal{S} \mid W(p) = 0\}. \tag{2.2}$$

The next definition plays a basic role in sub-Riemannian geometry.

Definition 2.1 We define a *horizontal normal* on $\mathcal{S}$ as follows

$$\boldsymbol{N}^H \;=\; \sum_{j=1}^{2n} \langle \boldsymbol{\nu}, X_j \rangle X_j, \tag{2.3}$$

so that $W = |\boldsymbol{N}^H|$. The *horizontal Gauss map* $\boldsymbol{\nu}^H$ on $\mathcal{S}$ is defined by

$$\boldsymbol{\nu}^H = \frac{\boldsymbol{N}^H}{|\boldsymbol{N}^H|}, \quad \text{on } \mathcal{S}\setminus\Sigma(\mathcal{S}). \tag{2.4}$$

Henceforth, we set $\langle \boldsymbol{\nu}^H, X_i\rangle = \overline{p}_i$, $\langle \boldsymbol{\nu}^H, X_{n+i}\rangle = \overline{q}_i$, $i = 1, \ldots, n$, so that

$$\overline{p}_1^2 + \cdots + \overline{p}_n^2 + \overline{q}_1^2 + \cdots + \overline{q}_n^2 = 1.$$

We note that $\boldsymbol{N}^H$ is the projection of the Riemannian Gauss map on $\mathcal{S}$ onto the horizontal subbundle $H\mathbb{H}^n \subset T\mathbb{H}^n$. Such projection vanishes only at characteristic points, and this is why the horizontal Gauss map is not defined on $\Sigma(\mathcal{S})$. The following definition is taken from [25], but the reader should also see [43] for a related notion in the more general setting of vertically rigid spaces.

In what follows we will indicate with

$$HT\mathcal{S} \overset{\text{def}}{=} T\mathcal{S} \cap H\mathbb{H}^n$$

the so-called horizontal tangent bundle of $\mathcal{S}$. Let $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{2n-1}\}$ denote an orthonormal basis of the horizontal tangent bundle $HT\mathcal{S}$.

Definition 2.2 The *horizontal* or H-*mean curvature* at a point $p_0 \in \mathcal{S}\setminus\Sigma(\mathcal{S})$ is defined as

$$\mathcal{H} = \sum_{i=1}^{2n-1} \langle \nabla^H_{\boldsymbol{e}_i} \boldsymbol{e}_i, \boldsymbol{\nu}^H\rangle.$$

If instead $p_0 \in \Sigma(\mathcal{S})$, then we define $\mathcal{H}(p_0) = \lim_{p\to p_0} \mathcal{H}(p)$, provided that the limit exists and is finite.

Given an open set $\Omega \subset \mathbb{H}^n$ denote by

$$\mathcal{F}(\Omega) = \left\{ \phi = \sum_{j=1}^{2n} \phi_j X_j \in C_0^1(\Omega, H\mathbb{H}^n) \mid ||\phi||_\infty = \sup_\Omega \left(\sum_{j=1}^{2n} \phi_j^2 \right)^{1/2} \le 1 \right\}.$$

Given $\phi = \sum_{j=1}^{2n} \phi_j X_j \in C_0^1(\Omega, H\mathbb{H}^n)$, we let $\mathrm{div}_H \phi = \sum_{j=1}^{2n} X_j \phi_j$. The H-perimeter of a measurable set $E \subset \mathbb{H}^n$ with respect to Ω was defined in [11] as

$$P_H(E; \Omega) = \sup_{\phi\in\mathcal{F}(\Omega)} \int_{E\cap\Omega} \mathrm{div}_H \phi \, dg.$$

If E is a bounded open set of class C^1, then the divergence theorem gives

$$\begin{aligned} P_H(E; \Omega) &= \sup_{\phi\in\mathcal{F}(\Omega)} \int_{\partial E\cap\Omega} \sum_{j=1}^{2n} \langle \boldsymbol{\nu}, X_j\rangle \phi_j \, d\sigma = \sup_{\phi\in\mathcal{F}(\Omega)} \int_{\partial E\cap\Omega} \langle \boldsymbol{N}^H, \phi\rangle d\sigma \\ &= \int_{\partial E\cap\Omega} |\boldsymbol{N}^H| d\sigma, \end{aligned}$$

where $d\sigma$ is the Riemannian surface measure on ∂E. It is clear from this formula that the measure on ∂E, defined by

$$\sigma_H(\partial E \cap \Omega) \overset{\text{def}}{=} P_H(E; \Omega)$$

on the open sets of ∂E, is absolutely continuous with respect to σ, and its density is represented by the angle function W of ∂E. We formalize this observation in the following definition.

Definition 2.3 Given a C^2 non-characteristic hypersurface $\mathcal{S} \subset \mathbb{H}^n$, with angle function W as in (2.1), we will denote by

$$d\sigma_H = |\boldsymbol{N}^H| d\sigma = W d\sigma, \tag{2.5}$$

the H-perimeter measure supported on $\mathcal{S}$.

Definition 2.4 Let $\mathcal{S} \subset \mathbb{H}^n$ be a non-characteristic, C^2 hypersurface, then we define the *horizontal connection* on $\mathcal{S}$ as follows. Let ∇^H denote the horizontal Levi-Civita connection in $\mathbb{H}^n$. For every $X, Y \in C^1(\mathcal{S}; HT\mathcal{S})$ we define

$$\nabla^{H,\mathcal{S}}_X Y \; = \; \nabla^H_{\overline{X}} \overline{Y} \; - \; \langle \nabla^H_{\overline{X}} \overline{Y}, \boldsymbol{\nu}^H \rangle \boldsymbol{\nu}^H,$$

where $\overline{X}, \overline{Y}$ are any two horizontal vector fields on $\mathbb{H}^n$ such that $\overline{X} = X, \overline{Y} = Y$ on $\mathcal{S}$.

One can check that Definition 2.4 is well-posed, i.e., it is independent of the extensions $\overline{X}, \overline{Y}$ of the vector fields X, Y.

Proposition 2.5 *For every $X, Y \in C^1(\mathcal{S}; HT\mathcal{S})$ one has*

$$\nabla^{H,\mathcal{S}}_X Y \; - \; \nabla^{H,\mathcal{S}}_Y X \; = \; [X, Y]^H \; - \; \langle [X, Y]^H, \boldsymbol{\nu}^H \rangle \boldsymbol{\nu}^H.$$

In the latter identity the notation $[X, Y]^H$ indicates the projection of the vector field $[X, Y]$ onto the horizontal bundle $H\mathbb{H}^n$. It is clear from this proposition that the horizontal connection $\nabla^{H,\mathcal{S}}$ on $\mathcal{S}$ is not necessarily torsion free. This depends on the fact that it is not true in general that, if $X, Y \in C^1(\mathcal{S}; HT\mathcal{S})$, then $[X, Y]^H \in C^1(\mathcal{S}; HT\mathcal{S})$. In the special case of the first Heisenberg group $\mathbb{H}^1$ this fact is true, and we have the following result, see Proposition 7.3 in [25].

Proposition 2.6 *Given a C^2 non-characteristic surface $\mathcal{S} \subset \mathbb{H}^1$, one has $[X, Y]^H \in HT\mathcal{S}$ for every $X, Y \in C^1(\mathcal{S}; HT\mathcal{S})$, and therefore the horizontal connection on $\mathcal{S}$ is torsion free.*

Definition 2.7 Let $\mathcal{S}$ be as in Definition 2.4. Consider a function $u \in C^1(\mathcal{S})$. We define the *tangential horizontal gradient* of u as follows

$$\nabla^{H,\mathcal{S}} u \; \overset{\text{def}}{=} \; \nabla^H \overline{u} \; - \; \langle \nabla^H \overline{u}, \boldsymbol{\nu}^H \rangle \; \boldsymbol{\nu}^H,$$

where $\overline{u} \in C^1(\boldsymbol{G})$ is such that $\overline{u} = u$ on $\mathcal{S}$.

We are now ready to state the integration by parts formulas from [25] which constitute the backbone of this paper.

Theorem 2.8 (Horizontal integration by parts formula) *Consider a C^2 oriented hypersurface $\mathcal{S} \subset \mathbb{H}^n$. If $u \in C^1_0(\mathcal{S} \setminus \Sigma_{\mathcal{S}})$, then we have*

$$\int_{\mathcal{S}} \nabla^{H,\mathcal{S}}_i u \, d\sigma_H \; = \; \int_{\mathcal{S}} u \, \left\{ \mathcal{H} \nu^H_i \; - \; \boldsymbol{c}^{H,\mathcal{S}}_i \right\} d\sigma_H, \quad i = 1, \ldots, 2n, \tag{2.6}$$

where the C^1 vector field $\boldsymbol{c}^{H,\mathcal{S}} = \sum_{i=1}^m \boldsymbol{c}^{H,\mathcal{S}}_i X_i$ is given by

$$\boldsymbol{c}^{H,\mathcal{S}} \; = \; \overline{\omega} \, (\boldsymbol{\nu}^H)^\perp \; = \; \overline{\omega} \, \left(\overline{q}_1 X_1 + \cdots + \overline{q}_n X_n - \overline{p}_1 X_{n+1} - \cdots - \overline{p}_n X_{2n} \right). \tag{2.7}$$

As a consequence, $\boldsymbol{c}^{H,\mathcal{S}}$ *is perpendicular to the horizontal Gauss map* $\boldsymbol{\nu}^H$*, i.e., one has*

$$\langle \boldsymbol{c}^{H,\mathcal{S}}, \boldsymbol{\nu}^H \rangle = 0, \tag{2.8}$$

and therefore $\boldsymbol{c}^{H,\mathcal{S}} \in C^1(\mathcal{S}\setminus\Sigma_{\mathcal{S}}, HT\mathcal{S})$.

Remark 2.9 We note explicitly that in view of (2.7) we can re-write (2.6) as follows

$$\int_{\mathcal{S}} \nabla^{H,\mathcal{S}} u \, d\sigma_H = \int_{\mathcal{S}} u \left\{\mathcal{H}\, \boldsymbol{\nu}^H - \overline{\omega}\, (\boldsymbol{\nu}^H)^{\perp}\right\} d\sigma_H. \tag{2.9}$$

We have the following notable consequences of Theorem 2.8.

Theorem 2.10 *Let* $\mathcal{S} \subset \mathbb{H}^n$ *be a* C^2 *oriented hypersurface, with characteristic set* $\Sigma_{\mathcal{S}}$*. If* $\zeta \in C_0^1(\mathcal{S}\setminus\Sigma_{\mathcal{S}}, HT\mathcal{S})$*, then we have*

$$\int_{\mathcal{S}} \left\{\mathrm{div}_{H,\mathcal{S}}\zeta + \langle \boldsymbol{c}^{H,\mathcal{S}}, \zeta\rangle\right\} d\sigma_H = \int_{\mathcal{S}} \mathcal{H}\, \langle \zeta, \boldsymbol{\nu}^H\rangle \, d\sigma_H, \tag{2.10}$$

where we have let

$$\mathrm{div}_{H,\mathcal{S}}\zeta = \sum_{i=1}^{2n} \nabla_i^{H,\mathcal{S}} \zeta_i.$$

We next recall a different integration by parts formula which involves differentiation along a special combination of the vector fields $\boldsymbol{\nu}^H$ and T.

Theorem 2.11 (Vertical integration by parts formula) *Let* $\mathcal{S} \subset \mathbb{H}^n$ *be a* C^2 *oriented hypersurface. For every* $f \in C^1(\mathcal{S})$*,* $g \in C_0^1(\mathcal{S}\setminus\Sigma(\mathcal{S}))$*, one has*

$$\int_{\mathcal{S}} f\, (T - \overline{\omega} Y) g \, d\sigma_H = - \int_{\mathcal{S}} g\, (T - \overline{\omega} Y) f \, d\sigma_H + \int_{\mathcal{S}} f g \overline{\omega}\, \mathcal{H} \, d\sigma_H, \tag{2.11}$$

where we have let $Yf = \langle \nabla f, \boldsymbol{\nu}^H\rangle$.

3 Growth formulas for the horizontal perimeter on hypersurfaces

In this section we establish some preparatory results which constitute sub-Riemannian versions of some basic growth lemmas on hypersurfaces which in the Riemannian case were first found in [52]. Throughout the section we will work in $\mathbb{H}^n$, for arbitrary $n \in \mathbb{N}$. In what follows, we consider functions $\rho \in C^1(\mathbb{H}^n)$ and $\lambda \in C^1(\mathbb{R})$, to be determined later.

Lemma 3.1 *Consider a horizontal vector field* $\zeta = \sum_{i=1}^{2n} \zeta_i X_i \in C^1(\mathbb{H}^n, H\mathbb{H}^n)$*. Let* $\mathcal{S} \subset \mathbb{H}^n$ *be a* C^2 *hypersurface with empty characteristic locus* $\Sigma(\mathcal{S})$*. Suppose that the level sets of* ρ *are compact, and let* λ *be non-decreasing, with* $\lambda(t) \equiv 0$ *for* $t \le 0$*. Given* $\psi \in C^1(\mathcal{S})$*, for every* $r > 0$ *we have*

$$\int_{\mathcal{S}} \left\{\mathrm{div}_{H,\mathcal{S}}\zeta + \langle \boldsymbol{c}^{H,\mathcal{S}}, \zeta\rangle\right\} \lambda(r-\rho)\, \psi \, d\sigma_H - \int_{\mathcal{S}} \lambda'(r-\rho)\, \psi \langle \zeta, \nabla^{H,\mathcal{S}}\rho\rangle \, d\sigma_H$$
$$\le \int_{\mathcal{S}} \lambda(r-\rho)\, |\zeta| \left\{|\mathcal{H}|\, \psi + |\nabla^{H,\mathcal{S}}\psi|\right\} d\sigma_H. \tag{3.1}$$

In particular, choosing $\psi \equiv 1$ *we obtain from* (3.1)

$$\int_{\mathcal{S}} \left\{ \operatorname{div}_{H,\mathcal{S}} \zeta + \langle \boldsymbol{c}^{H,\mathcal{S}}, \zeta \rangle \right\} \lambda(r-\rho)\, d\sigma_H - \int_{\mathcal{S}} \lambda'(r-\rho)\langle \zeta, \nabla^{H,\mathcal{S}} \rho \rangle\, d\sigma_H \leq \int_{\mathcal{S}} \lambda(r-\rho)\, |\zeta|\, |\mathcal{H}|\, d\sigma_H. \tag{3.2}$$

Proof For a fixed $r > 0$ we define

$$u = \zeta_i\, \lambda(r-\rho)\, \psi, \tag{3.3}$$

where $\lambda : \mathbb{R} \to \mathbb{R}$ is non-decreasing, and $\lambda \equiv 0$ for $t \leq 0$. We have

$$\sum_{i=1}^{m} \nabla_i^{H,\mathcal{S}} u = \left(\operatorname{div}_{H,\mathcal{S}} \zeta\right) \lambda(r-\rho)\, \psi + \lambda(r-\rho)\langle \zeta, \nabla^{H,\mathcal{S}} \psi \rangle - \lambda'(r-\rho)\, \psi \langle \zeta, \nabla^{H,\mathcal{S}} \rho \rangle. \tag{3.4}$$

We now integrate (3.4) on $\mathcal{S}$ with respect to the measure σ_H. Applying (2.6) in Theorem 2.8 we obtain

$$\int_{\mathcal{S}} \operatorname{div}_{H,\mathcal{S}} \zeta\, \lambda(r-\rho)\, \psi\, d\sigma_H + \int_{\mathcal{S}} \lambda(r-\rho)\langle \zeta, \nabla^{H,\mathcal{S}} \psi \rangle d\sigma_H - \int_{\mathcal{S}} \lambda'(r-\rho)\, \psi \langle \zeta, \nabla^{H,\mathcal{S}} \rho \rangle d\sigma_H + \int_{\mathcal{S}} \lambda(r-\rho)\, \psi \langle \boldsymbol{c}^{H,\mathcal{S}}, \zeta \rangle d\sigma_H = \int_{\mathcal{S}} \mathcal{H}\, \lambda(r-\rho)\, \psi \langle \zeta, \boldsymbol{\nu}^H \rangle d\sigma_H. \tag{3.5}$$

From the identity (3.5) we easily obtain (3.1). □

We next use the formula (2.11) in Theorem 2.11 with the choice $g(p) = \lambda(r - \rho(p))$ to obtain the following result.

Lemma 3.2 *Let* $\mathcal{S} \subset \mathbb{H}^n$ *be a* C^2 *hypersurface with empty characteristic locus. Suppose that the level sets of* ρ *are compact, and let* λ *be non-decreasing, with* $\lambda(t) \equiv 0$ *for* $t \leq 0$*. For every* $r > 0$ *we have for any* $f \in C^1(\mathcal{S})$

$$\int_{\mathcal{S}} \lambda(r-\rho)\, (T - \overline{\omega} Y) f\, d\sigma_H = \int_{\mathcal{S}} f\, \lambda'(r-\rho)(T - \overline{\omega} Y)\rho\, d\sigma_H + \int_{\mathcal{S}} \lambda(r-\rho) f\, \overline{\omega}\, \mathcal{H}\, d\sigma_H. \tag{3.6}$$

At this point we combine (3.2) in Lemma 3.1 with (3.6) in Lemma 3.2, obtaining the following basic result.

Theorem 3.3 *Let* $\mathcal{S} \subset \mathbb{H}^n$ *be a* C^2 *hypersurface with empty characteristic locus. Consider a horizontal vector field* $\zeta = \sum_{i=1}^{2n} \zeta_i X_i \in C^1(\mathbb{H}^n, H\mathbb{H}^n)$*. Suppose that the level sets of* ρ

are compact, and let λ *be non-decreasing, with* $\lambda(t) \equiv 0$ *for* $t \leq 0$. *For every* $r > 0$ *we have for any* $f \in C^1(\mathcal{S})$

$$\int_{\mathcal{S}} \left\{ \operatorname{div}_{H,\mathcal{S}} \zeta + \langle \boldsymbol{c}^{H,\mathcal{S}}, \zeta \rangle + (T - \overline{\omega} Y) f \right\} \lambda(r - \rho)\, d\sigma_H - \int_{\mathcal{S}} \lambda'(r - \rho) \left\{ \langle \zeta, \nabla^{H,\mathcal{S}} \rho \rangle + f(T - \overline{\omega} Y)\rho \right\} d\sigma_H \leq \int_{\mathcal{S}} \lambda(r - \rho) \{ |\zeta| + |f| \overline{\omega} \}\ |\mathcal{H}|\, d\sigma_H. \tag{3.7}$$

In particular, if $\mathcal{S}$ *is minimal, we obtain from* (3.7)

$$\int_{\mathcal{S}} \left\{ \operatorname{div}_{H,\mathcal{S}} \zeta + \langle \boldsymbol{c}^{H,\mathcal{S}}, \zeta \rangle + (T - \overline{\omega} Y) f \right\} \lambda(r - \rho)\, d\sigma_H - \int_{\mathcal{S}} \lambda'(r - \rho) \left\{ \langle \zeta, \nabla^{H,\mathcal{S}} \rho \rangle + f(T - \overline{\omega} Y)\rho \right\} d\sigma_H \leq\ 0. \tag{3.8}$$

We now turn to the fundamental question of the choice of the horizontal vector field ζ and of the function f in Theorem 3.3. With this objective in mind we introduce the following definition.

Definition 3.4 Let $p_0 \in \mathbb{H}^n$, then the generator of the non-isotropic dilations $\{\delta_\lambda\}_{\lambda > 0}$ centered at p_0 is defined by

$$Z_{p_0} f(p) = \sum_{i=1}^{n} (x_i - x_{0,i}) X_i + (y_i - y_{0,i}) X_{n+i} + [2(t - t_0) + (\langle x, y_0 \rangle - \langle x_0, y \rangle)] T.$$

Definition 3.4 is motivated by the following considerations. Let $F \in C^1(\mathbb{H}^n)$, then

$$Z_{p_0} F(p) \overset{\text{def}}{=} \frac{d}{d\lambda} F(p_0 \delta_\lambda (p_0^{-1} p)) \Big|_{\lambda = 1}.$$

Now

$$p_0 \delta_\lambda (p_0^{-1} p) = \Bigg(x_0 + \lambda (x - x_0),\, y + \lambda (y - y_0),\ t_0 + \lambda^2 \left(t - t_0 + \frac{1}{2} (\langle x, y_0 \rangle - \langle x_0, y \rangle) \right) + \frac{\lambda}{2} (\langle x_0, y - y_0 \rangle - \langle y_0, x - x_0 \rangle) \Bigg).$$

A simple calculation now gives

$$\frac{d}{d\lambda} F(p_0 \delta_\lambda (p_0^{-1} p)) \Big|_{\lambda = 1} = \sum_{i=1}^{n} (x_i - x_{0,i}) \frac{\partial F}{\partial x_i}(p) + (y_i - y_{0,i}) \frac{\partial F}{\partial y_i}(p) + [2(t - t_0) + \frac{1}{2} (\langle x, y_0 \rangle - \langle x_0, y \rangle)] T F(p). \tag{3.9}$$

If in (3.9) we now use the fact that

$$\frac{\partial F}{\partial x_i}(p) = X_i F(p) + \frac{y_i}{2} TF(p), \quad \frac{\partial F}{\partial y_i}(p) = X_{n+i} F(p) - \frac{x_i}{2} TF(p),$$

we easily obtain the formula in Definition 3.4.

Guided by Definition 3.4, we now choose the horizontal vector field ζ and the function f in Theorem 3.3 as follows

$$\zeta(p) = \sum_{i=1}^{n} \left((x_i - x_{0,i})X_i + (y_i - y_{0,i})X_{n+i}\right), \tag{3.10}$$

$$f(p) = 2(t - t_0) + \langle x, y_0\rangle - \langle x_0, y\rangle. \tag{3.11}$$

With these choices, we next establish a remarkable identity which should be considered as the sub-Riemannian counterpart of the above recalled (1.3). In what follows, similarly to formula (2.11) above, we will use the notation $Yf = \langle \nabla f, \boldsymbol{\nu}^H\rangle$.

Proposition 3.5 *Fix a point $p_0 = (x_0, y_0, t_0) \in \mathbb{H}^n$ and consider the horizontal vector field $\zeta \in C^\infty(\mathbb{H}^n, H\mathbb{H}^n)$ given by (3.10), and the function $f \in C^1(\mathbb{H}^n)$ in (3.11), then on any C^2 non-characteristic hypersurface $\mathcal{S} \subset \mathbb{H}^n$ (or on any hypersurface $\mathcal{S}$, but away from its characteristic set $\Sigma(\mathcal{S})$) one has the identity*

$$\mathrm{div}_{H,\mathcal{S}}\, \zeta + \langle \boldsymbol{c}^{H,\mathcal{S}}, \zeta\rangle + (T - \overline{\omega} Y) f \ \equiv\ Q - 1. \tag{3.12}$$

Proof We begin by observing that with $\zeta = \sum_{i=1}^n (\zeta_i X_i + \zeta_{n+i} X_{n+i})$ one has

$$\nabla_i^{H,\mathcal{S}} \zeta_i = 1 - \overline{p}_i^2, \quad \nabla_{n+i}^{H,\mathcal{S}} \zeta_{n+i} = 1 - \overline{q}_i^2, \quad i = 1, \dots, n.$$

Therefore,

$$\mathrm{div}_{H,\mathcal{S}} \zeta \ =\ \sum_{i=1}^{n} \left(\nabla_i^{H,\mathcal{S}} \zeta_i + \nabla_{n+i}^{H,\mathcal{S}} \zeta_{n+i}\right) \ =\ 2n - \sum_{i=1}^{n} (\overline{p}_i^2 + \overline{q}_i^2) \ \equiv\ 2n - 1 \ =\ Q - 3. \tag{3.13}$$

We now have from (2.7)

$$\boldsymbol{c}^{H,\mathcal{S}} \ =\ \overline{\omega}(\overline{q}_1 X_1 + \cdots + \overline{q}_n X_n - \overline{p}_1 X_{n+1} - \cdots - \overline{p}_n X_{2n}),$$

and therefore

$$\langle \boldsymbol{c}^{H,\mathcal{S}}, \zeta\rangle \ =\ \overline{\omega}\langle z, (\boldsymbol{\nu}^H)^\perp\rangle - \overline{\omega}\langle z_0, (\boldsymbol{\nu}^H)^\perp\rangle,$$

where, abusing the notation, we have set $z = \sum_{i=1}^n x_i X_i + y_i X_{n+i}$, $z_0 = \sum_{i=1}^n x_{0,i} X_i + y_{0,i} X_{n+i}$. On the other hand, since $Yt = \frac{1}{2}(x_1\overline{q}_1 + \cdots + x_n\overline{q}_n - y_1\overline{p}_1 - \cdots - y_n\overline{p}_n)$, we have

$$(T - \overline{\omega}Y)(2(t - t_0)) = 2Tt - 2\overline{\omega}Yt = 2 - \overline{\omega}\langle z, (\boldsymbol{\nu}^H)^\perp\rangle.$$

We also have

$$(T - \overline{\omega}Y)(\langle x, y_0\rangle - \langle x_0, y\rangle) = -\overline{\omega}Y(\langle x, y_0\rangle - \langle x_0, y\rangle) = \overline{\omega}\langle z_0, (\boldsymbol{\nu}^H)^\perp\rangle,$$

and so

$$\langle \boldsymbol{c}^{H,\mathcal{S}}, \zeta\rangle + (T - \overline{\omega}Y) f \ \equiv\ 2. \tag{3.14}$$

Combining (3.14) with (3.13) we obtain (3.12). □

If we now combine (3.8) in Theorem 3.3 with Proposition 3.5, we obtain the following basic result.

Theorem 3.6 *Let $\mathcal{S} \subset \mathbb{H}^n$ be a non-characteristic minimal surface, then with ζ as in (3.10) and f as in (3.11), one has for any $p_0 = (x_0, y_0, t_0) \in \mathbb{H}^n$*

$$\begin{aligned}(Q-1) \int_{\mathcal{S}} \lambda(r-\rho)\, d\sigma_H \\ - \int_{\mathcal{S}} \lambda'(r-\rho) \left\{ \langle \zeta, \nabla^{H,\mathcal{S}} \rho \rangle + f(T - \overline{\omega} Y)\rho \right\} d\sigma_H \\ \leq\ 0. \end{aligned} \tag{3.15}$$

4 Monotonicity for graphical strips

In this section as an interesting consequence of Theorem 3.6 we prove Theorem 1.3. To provide the reader with some motivation for the class of *graphical strips* we begin by recalling a result which is part of Theorem 1.5 in [28].

Proposition 4.1 *Every C^2 graphical strip is a minimal surface in $\mathbb{H}^1$ with empty characteristic locus.*

One fundamental aspect of graphical strips is represented by the following Theorem 4.2, which constitutes one of the two central results in [28]. In order to state it we mention that by $\mathcal{V}^H_{II}(\mathcal{S}; \mathcal{X})$ we denote the second variation of the H-perimeter measure σ_H with respect to a deformation of $\mathcal{S}$ in the direction of the vector field $\mathcal{X}$. A minimal surface $\mathcal{S}$ with empty characteristic locus is called *stable* if $\mathcal{V}^H_{II}(\mathcal{S}; \mathcal{X}) \geq 0$ for every compactly supported $\mathcal{X} = aX_1 + bX_2 + kT$. Otherwise, it is called unstable. We note that, since thanks to Proposition 4.1 every graphical strip has empty characteristic locus, the horizontal Gauss map $\boldsymbol{\nu}^H$ of such a surface is globally defined.

Theorem 4.2 *Let $\mathcal{S}$ be a C^2 strict graphical strip, then $\mathcal{S}$ is unstable. In fact, there exists a continuum of $h \in C^2_0(\mathcal{S})$, for which $\mathcal{V}^H_{II}(\mathcal{S}; h\boldsymbol{\nu}^H) < 0$.*

In connection with the stability assumption in Theorem 4.2 we mention that it represents a new phenomenon with respect to the classical case. In fact, thanks to the strict convexity of the area functional $\mathcal{A}(f) = \int_\Omega \sqrt{1 + |\nabla f|^2} dx$, any critical point of the latter (and therefore any minimal graph) is automatically stable, see for instance [21]. In the sub-Riemannian setting this is no longer true. This central aspect of the problem was first discovered in [26] where it was shown that the non-planar minimal surface $\mathcal{S} = \{(x, y, t) \in \mathbb{H}^1 \mid x = yt\}$ is unstable (recall that such surface is a strict graphical strip).

The following theorem constitutes the second main result in [28]. It underscores the central relevance of graphical strips in the study of minimal graphs in $\mathbb{H}^1$.

Theorem 4.3 *Let $\mathcal{S} \subset \mathbb{H}^1$ be a minimal entire graph of class C^2, with empty characteristic locus, and that is not itself a vertical plane*

$$\Pi_0 = \{(x, y, t) \in \mathbb{H}^1 \mid ax + by = \gamma_0\}, \tag{4.1}$$

then there exists a strict graphical strip $\mathcal{S}_0 \subset \mathcal{S}$.

By combining Theorems 4.2 and 4.3 the following solution of the sub-Riemannian Bernstein problem was obtained in [28].

Theorem 4.4 (of Bernstein type) *In $\mathbb{H}^1$ the only C^2 stable entire minimal graphs, with empty characteristic locus, are the vertical planes* (4.1).

After these preliminaries we turn to the proof of Theorem 1.3. In what follows we denote by $B(p_0, r) = \{p \in \mathbb{H}^n \mid d(p, p_0) < r\}$, where $d(p, p_0) = N(p_0^{-1} p)$ represents the gauge distance on $\mathbb{H}^n$ defined via the Koranyi-Folland gauge function $N(p) = (|z|^4 + 16t^2)^{1/4}$, $p = (z, t) \in \mathbb{H}^n$. Using this function we now specialize the choice of the function ρ in Theorem 3.6 by letting $\rho(p) = N(p_0^{-1} p)$. Of course, this is not the only possible choice of ρ, but at the moment we will not further investigate this question since we plan to return to it in a future study.

Notice that we can write

$$\rho(p) = \left[\left((x - x_0)^2 + (y - y_0)^2\right)^2 + 4\left(2(t - t_0) + (x y_0 - x_0 y)\right)^2 \right]^{1/4}. \tag{4.2}$$

A simple calculation gives

$$X_1\rho = \rho^{-3}\left[(x - x_0)|z - z_0|^2 - 2(y - y_0)\left(2(t - t_0) + (x y_0 - x_0 y)\right)\right], \tag{4.3}$$

$$X_2\rho = \rho^{-3}\left[(y - y_0)|z - z_0|^2 + 2(x - x_0)\left(2(t - t_0) + (x y_0 - x_0 y)\right)\right], \tag{4.4}$$

$$T\rho = \rho^{-3} 4\left[2(t - t_0) + (x y_0 - x_0 y)\right]. \tag{4.5}$$

From (4.3), (4.4), (4.5) we obtain with ζ and f as in (3.10), (3.11) respectively,

$$\langle \zeta, \nabla^H \rho \rangle + f T\rho \;=\; \rho. \tag{4.6}$$

On the other hand, we have from the expression of the horizontal covariant derivative on $\mathcal{S}$

$$\langle \zeta, \nabla^{H,\mathcal{S}} \rho \rangle \;=\; \langle \zeta, \nabla^H \rho \rangle - \langle \nabla^H \rho, \boldsymbol{\nu}^H \rangle \langle \zeta, \boldsymbol{\nu}^H \rangle.$$

Using (4.6) we find

$$\begin{aligned} \langle \zeta, \nabla^{H,\mathcal{S}} \rho \rangle + f(T - \overline{\omega} Y)\rho &= \langle \zeta, \nabla^H \rho \rangle + f T\rho \\ &\quad - \langle \nabla^H \rho, \boldsymbol{\nu}^H \rangle \langle \zeta, \boldsymbol{\nu}^H \rangle - \overline{\omega} f Y\rho \\ &= \rho - \langle \nabla^H \rho, \boldsymbol{\nu}^H \rangle \left(\langle \zeta, \boldsymbol{\nu}^H \rangle + \overline{\omega} f \right), \end{aligned} \tag{4.7}$$

where in the last equality we have used the fact that $Y\rho = \langle \nabla^H \rho, \boldsymbol{\nu}^H \rangle$.

The next result provides a fundamental estimate. It is at this point that we use the special structural assumption that $\mathcal{S}$ be a graphical strip in $\mathbb{H}^1$.

Lemma 4.5 *Let $\mathcal{S} \subset \mathbb{H}^1$ be a C^2 graphical strip. Let $p_0 = (0, 0, t_0) \in \mathcal{S}$, then with ζ as in* (3.10) *and f as in* (3.11)*, one has*

$$\sup_{\mathcal{S} \cap B(p_0, r)} \left| \langle \zeta, \nabla^{H,\mathcal{S}} \rho \rangle + f(T - \overline{\omega} Y)\rho \right| \;\le\; r.$$

Proof In view of (4.7), proving the lemma is equivalent to showing

$$\sup_{\mathcal{S} \cap B(p_0, r)} \left| \rho - \langle \nabla^H \rho, \boldsymbol{\nu}^H \rangle \left(\langle \zeta, \boldsymbol{\nu}^H \rangle + \overline{\omega} f \right) \right| \;\le\; r.$$

Without loss of generality we assume that

$$\mathcal{S} = \{(x, y, t) \in \mathbb{H}^1 \mid (y, t) \in \mathbb{R} \times I, x = yG(t)\},$$

for some $G \in C^2(I)$, such that $G'(t) \geq 0$ for every $t \in I$. We next recall some calculations from [28]. It is obvious from the definition that $\mathcal{S}$ is a C^2 graph over the (y, t)-plane. We can use the global defining function

$$\phi(x, y, t) \;=\; x - yG(t), \tag{4.8}$$

and assume that $\mathcal{S}$ is oriented in such a way that a non-unit Riemannian normal on $\mathcal{S}$ be given by $\boldsymbol{N} = \nabla\phi = (X_1\phi)X_1 + (X_2\phi)X_2 + (T\phi)T$. We thus find

$$p = X_1\phi = 1 + \frac{y^2}{2}G'(t), \quad q = X_2\phi = -G(t) - \frac{xy}{2}G'(t), \quad \omega \;= T\phi = -yG'(t). \tag{4.9}$$

Incidentally, since $p \geq 1 > 0$, we see from (4.9) that $\Sigma(\mathcal{S}) = \varnothing$.

From now on, to simplify the notation, we will omit the variable t in all expressions involving $G(t)$, $G'(t)$. The second equation in (4.9) becomes on $\mathcal{S}$

$$q \;=\; -\, G\left(1 \;+\; \frac{y^2}{2}G'\right). \tag{4.10}$$

We thus find on $\mathcal{S}$

$$W \;=\; \sqrt{p^2 + q^2} \;=\; \sqrt{1 + G^2}\left(1 \;+\; \frac{y^2}{2}G'\right). \tag{4.11}$$

The equations (4.9), (4.10) and (4.11) give on $\mathcal{S}$

$$\overline{p} = \frac{1}{\sqrt{1 + G^2}}, \quad \overline{q} = -\frac{G}{\sqrt{1 + G^2}}, \quad \overline{\omega} = -\frac{yG'}{\sqrt{1 + G^2}\left(1 \;+\; \frac{y^2}{2}G'\right)}. \tag{4.12}$$

We thus have on $\mathcal{S}$

$$x\overline{q} - y\overline{p} = -\left\{\frac{yG^2}{\sqrt{1 + G^2}} + \frac{y}{\sqrt{1 + G^2}}\right\} = -y\sqrt{1 + G^2}, \tag{4.13}$$

and also

$$x\overline{p} + y\overline{q} \;=\; \frac{y\,G(t)}{\sqrt{1 + G(t)^2}} - \frac{y\,G(t)}{\sqrt{1 + G(t)^2}} \;=\; 0. \tag{4.14}$$

On the other hand, if $p_0 = (x_0, y_0, t_0) \in \mathcal{S}$, we must have $x_0 = y_0 G(t_0)$, and therefore

$$x_0\overline{q} - y_0\overline{p} = -y_0\left\{\frac{G(t_0)G}{\sqrt{1 + G^2}} + \frac{1}{\sqrt{1 + G^2}}\right\} = -y_0\frac{1 + G(t_0)G}{\sqrt{1 + G^2}}, \tag{4.15}$$

and also

$$x_0\overline{p} + y_0\overline{q} \;=\; -y_0\frac{G - G(t_0)}{\sqrt{1 + G^2}}. \tag{4.16}$$

We also have on $\mathcal{S}$

$$xy_0 - x_0 y \;=\; y_0 y(G - G(t_0)). \tag{4.17}$$

Combining (4.14) and (4.16) we find

$$\langle \zeta, \boldsymbol{\nu}^H \rangle \;=\; (x - x_0)\overline{p} + (y - y_0)\overline{q} \;=\; y_0\frac{G - G(t_0)}{\sqrt{1 + G^2}}. \tag{4.18}$$

From (4.12), (4.17) we have

$$\overline{\omega}(2(t-t_0)+(xy_0-x_0y)) = -2y_0\frac{\frac{y^2}{2}G'(G-G(t_0))}{\sqrt{1+G^2}\left(1\,+\,\frac{y^2}{2}G'\right)} - \frac{2(t-t_0)yG'}{\sqrt{1+G^2}\left(1\,+\,\frac{y^2}{2}G'\right)}. \tag{4.19}$$

Combining (4.18) and (4.19) we find

$$\begin{aligned}\langle\zeta,\boldsymbol{\nu}^H\rangle + f\overline{\omega} &= y_0\frac{G-G(t_0)}{\sqrt{1+G^2}}\\ &\quad -2y_0\frac{\frac{y^2}{2}G'(G-G(t_0))}{\sqrt{1+G^2}\left(1\,+\,\frac{y^2}{2}G'\right)} - \frac{2(t-t_0)yG'}{\sqrt{1+G^2}\left(1\,+\,\frac{y^2}{2}G'\right)}.\end{aligned} \tag{4.20}$$

When $x_0=y_0=0$, and therefore $p_0=(0,0,t_0)$, we obtain from (4.20)

$$\langle\zeta,\boldsymbol{\nu}^H\rangle + f\overline{\omega} = -\frac{2(t-t_0)yG'}{\sqrt{1+G^2}\left(1\,+\,\frac{y^2}{2}G'\right)}. \tag{4.21}$$

Next, we observe that we have on $\mathcal{S}$

$$|z|^2 = y^2(1+G^2),\quad x|z|^2 = y^3G(1+G^2),\quad y|z|^2 = y^3(1+G^2).$$

If we use these formulas in (4.3), (4.4), in combination with (4.12), we obtain

$$\langle\nabla^H\rho,\boldsymbol{\nu}^H\rangle = -\frac{4y(t-t_0)(1+G^2)}{\rho^3\sqrt{1+G^2}}. \tag{4.22}$$

Combining equations (4.21), (4.22) we find

$$\langle\nabla^H\rho,\boldsymbol{\nu}^H\rangle\left(\langle\zeta,\boldsymbol{\nu}^H\rangle + f\overline{\omega}\right) = \frac{16(t-t_0)^2\frac{y^2}{2}G'}{\rho^3\left(1\,+\,\frac{y^2}{2}G'\right)}. \tag{4.23}$$

Since on $\mathcal{S}$ we have

$$\rho^4 = (x^2+y^2)^2 + 16(t-t_0)^2 = y^4(1+G^2)^2 + 16(t-t_0)^2,$$

from this equation and from (4.23) it is at this point easy to check that on $\mathcal{S}$ one has

$$\rho - \langle\nabla^H\rho,\boldsymbol{\nu}^H\rangle\left(\langle\zeta,\boldsymbol{\nu}^H\rangle + \overline{\omega}f\right) \geq 0.$$

Since from (4.23) again we see that $\langle\nabla^H\rho,\boldsymbol{\nu}^H\rangle\left(\langle\zeta,\boldsymbol{\nu}^H\rangle + f\overline{\omega}\right)\geq 0$, we finally obtain

$$\left|\rho - \langle\nabla^H\rho,\boldsymbol{\nu}^H\rangle\left(\langle\zeta,\boldsymbol{\nu}^H\rangle + \overline{\omega}f\right)\right| = \rho - \langle\nabla^H\rho,\boldsymbol{\nu}^H\rangle\left(\langle\zeta,\boldsymbol{\nu}^H\rangle + \overline{\omega}f\right) \leq \rho,$$

which, in particular, proves the lemma. □

We can now prove the main result in this section.

Proof of Theorem 1.3 We define

$$\mathcal{P}(r) = \int_{\mathcal{S}} \lambda(r-\rho) d\sigma_H. \tag{4.24}$$

We easily find

$$\frac{d}{dr}\left(\frac{\mathcal{P}(r)}{r^{Q-1}}\right) = \frac{1}{r^Q}\left(r\mathcal{P}'(r) - (Q-1)\mathcal{P}(r)\right).$$

We next recall that for any $p_0 = (x_0, y_0, t_0) \in \mathbb{H}^n$ one has from (3.15),

$$(Q-1)\mathcal{P}(r) - \int_{\mathcal{S}} \lambda'(r-\rho)\left\{\langle \zeta, \nabla^{H,\mathcal{S}}\rho\rangle + f(T-\overline{\omega}Y)\rho\right\} d\sigma_H \le 0, \tag{4.25}$$

where ζ is as in (3.10) and f as in (3.11).

At this point the crucial Lemma 4.5 enters the picture. In it we have proved that on the set $B(p_0, r) = \{\rho < r\}$ one has

$$\left|\langle \zeta, \nabla^{H,\mathcal{S}}\rho\rangle + f(T-\overline{\omega}Y)\rho\right| \le r. \tag{4.26}$$

Then from (4.26), the fact that $\lambda'(r-\rho) \ge 0$ and from (4.25) we can conclude that

$$\frac{d}{dr}\left(\frac{\mathcal{P}(r)}{r^{Q-1}}\right) = \frac{1}{r^Q}\left(r\mathcal{P}'(r) - (Q-1)\mathcal{P}(r)\right) \ge 0.$$

We now fix $0 < r_1 < r_2 < \infty$ and integrate the latter inequality on the interval (r_1, r_2) obtaining

$$\begin{aligned} 0 &\le \int_{r_1}^{r_2} \frac{d}{dr}\left(\frac{\mathcal{P}(r)}{r^{Q-1}}\right) dr = \frac{\mathcal{P}(r_2)}{r_2^{Q-1}} - \frac{\mathcal{P}(r_1)}{r_1^{Q-1}} \\ &= \frac{1}{r_2^{Q-1}} \int_{\mathcal{S}} \lambda(r_2-\rho) d\sigma_H - \frac{1}{r_1^{Q-1}} \int_{\mathcal{S}} \lambda(r_1-\rho) d\sigma_H. \end{aligned} \tag{4.27}$$

At this point we fix arbitrarily $0 < \epsilon < r_1$, and choose a non-decreasing $0 \le \lambda(s) \le 1$, with $\lambda \equiv 0$ if $s \le 0$, $\lambda \equiv 1$ if $s \ge \epsilon$. With this choice we obtain from (4.27)

$$\begin{aligned} 0 &\le \frac{1}{r_2^{Q-1}} \int_{\mathcal{S}\cap B(p_0,r_2)} \lambda(r_2-\rho) d\sigma_H - \frac{1}{r_1^{Q-1}} \int_{\mathcal{S}\cap B(p_0,r_1-\epsilon)} \lambda(r_1-\rho) d\sigma_H \\ &\quad - \frac{1}{r_1^{Q-1}} \int_{\mathcal{S}\cap[B(p_0,r_1)\setminus B(p_0,r_1-\epsilon)]} \lambda(r_1-\rho) d\sigma_H \\ &\le \frac{\sigma_H(\mathcal{S}\cap B(p_0,r_2))}{r_2^{Q-1}} - \frac{\sigma_H(\mathcal{S}\cap B(p_0,r_1-\epsilon))}{r_1^{Q-1}}. \end{aligned} \tag{4.28}$$

Letting $\epsilon \to 0$ we reach the conclusion

$$\frac{\sigma_H(\mathcal{S}\cap B(p_0,r_1))}{r_1^{Q-1}} \le \frac{\sigma_H(\mathcal{S}\cap B(p_0,r_2))}{r_2^{Q-1}}.$$

$\square$

According to Theorem 1.3 the limit

$$\lim_{r\to 0^+} \frac{\sigma_H(\mathcal{S}\cap B(p_0,r))}{r^{Q-1}}$$

exists. In the next proposition we show that such limit is actually positive and independent of the point $p_0 \in \mathcal{S}$ or of the function $G(t)$ which defines the graphical strip $\mathcal{S}$.

Proposition 4.6 *Let $\mathcal{S}$ be a graphical strip, that is,*

$$\mathcal{S} = \{(x,y,t) \mid x = y\,G(t)\} \quad \textit{where } G\in C^1(\mathbb{R}),\quad G'(t)\ge 0 \textit{ for all } t\in\mathbb{R},$$

then for every $p_0=(0,0,t_0)\in\mathcal{S}$ we have

$$\lim_{r\to 0^+} \frac{\sigma_H(\mathcal{S}\cap B(p_0,r))}{r^3} = \int_0^1 (1-\tau^2)^{\frac14}\,d\tau > 0.$$

Note that this limit is independent of $G(t)$ or of the point $p_0=(0,0,t_0)\in\mathcal{S}$.

Proof Let ϕ be as in (4.8). We then have

$$\mathcal{S}\cap B(p_0,r) = \{(x,y,t)\in\mathbb{H}^1 \mid x = y\,G(t),\; y^4(1+G(t)^2)^2 + 16\,(t-t_0)^2 < r^4\}.$$

$$|X\phi| = \left(1+\frac{y^2}{2}G'(t)\right)\sqrt{1+G(t)^2}.$$

Hence

$$\frac{\sigma_H(\mathcal{S}\cap B(p_0,r))}{r^3} = \frac{1}{r^3}\int_{\mathcal{S}\cap B(p_0,r)} \frac{|X\phi|}{|\nabla\phi|}\,d\sigma$$

$$= \frac{1}{r^3}\int_{\{(y,t)\,|\,y^4(1+G(t)^2)^2+16(t-t_0)^2<r^4\}} \left(1+\frac{y^2}{2}G'(t)\right)\sqrt{1+G(t)^2}\,dy\,dt$$

$$= \frac{1}{r^3}\int_{t_0-\frac{r^2}{4}}^{t_0+\frac{r^2}{4}} \sqrt{1+G(t)^2}\left(\int_{-\frac{(r^4-16(t-t_0)^2)^{\frac14}}{\sqrt{1+G(t)^2}}}^{\frac{(r^4-16(t-t_0)^2)^{\frac14}}{\sqrt{1+G(t)^2}}} \left(1+\frac{y^2}{2}G'(t)\right)dy\right)dt$$

$$= \frac{2}{r^3}\int_{t_0-\frac{r^2}{4}}^{t_0+\frac{r^2}{4}} \sqrt{1+G(t)^2}\left\{\frac{(r^4-16(t-t_0)^2)^{\frac14}}{\sqrt{1+G(t)^2}} + \frac{G'(t)}{6}\frac{(r^4-16(t-t_0)^2)^{\frac34}}{(1+G(t)^2)^{\frac32}}\right\}dt$$

$$= \frac{2}{r^3}\int_{t_0-\frac{r^2}{4}}^{t_0+\frac{r^2}{4}} (r^4-16(t-t_0)^2)^{\frac14} + \frac{G'(t)}{6}\frac{(r^4-16(t-t_0)^2)^{\frac34}}{1+G(t)^2}\,dt$$

$$= \frac{2}{r^3}\int_{t_0-\frac{r^2}{4}}^{t_0+\frac{r^2}{4}} (r^4-16(t-t_0)^2)^{\frac14}\,dt + \frac{2}{r^3}\int_{t_0-\frac{r^2}{4}}^{t_0+\frac{r^2}{4}} \frac{G'(t)}{6(1+G(t)^2)}(r^4-16(t-t_0)^2)^{\frac34}\,dt. \quad (4.29)$$

To continue we make the change of variable $t - t_0 = \frac{r^2}{4}\tau$ and analyze the following two terms.

$$\frac{2}{r^3}\int_{t_0-\frac{r^2}{4}}^{t_0+\frac{r^2}{4}} (r^4 - 16(t-t_0)^2)^{\frac{1}{4}}\, dt = \frac{2}{r^3}\int_{-1}^{1} r\,(1-\tau^2)^{\frac{1}{4}}\,\frac{r^2}{4}\, d\tau = \int_0^1 (1-\tau^2)^{\frac{1}{4}}\, d\tau \tag{4.30}$$

$$\begin{aligned}&\frac{2}{r^3}\int_{t_0-\frac{r^2}{4}}^{t_0+\frac{r^2}{4}} \frac{G'(t)}{6(1+G(t)^2)}\,(r^4-16(t-t_0)^2)^{\frac{3}{4}}\, dt\\ &= \frac{1}{3r^3}\int_{-1}^{1} \frac{G'(t_0+r^2\tau/4)}{1+G(t_0+r^2\,\tau/4)^2}\, r^3\,(1-\tau^2)^{\frac{3}{4}}\,\frac{r^2}{4}\, d\tau\\ &= \frac{r^2}{12}\int_{-1}^{1} \frac{G'(t_0+r^2\tau/4)}{1+G(t_0+r^2\,\tau/4)^2}\,(1-\tau^2)^{\frac{3}{4}}\, d\tau.\end{aligned} \tag{4.31}$$

Using (4.30) and (4.31) in (4.29) and Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}&\lim_{r\to 0^+}\frac{\sigma_H(\mathcal{S}\cap B(p_0,r))}{r^3}\\ &= \lim_{r\to 0^+}\int_0^1 (1-\tau^2)^{\frac{1}{4}}\, d\tau \;+\; \lim_{r\to 0^+}\frac{r^2}{12}\int_{-1}^{1}\frac{G'(t_0+r^2\tau/4)}{1+G(t_0+r^2\,\tau/4)^2}\,(1-\tau^2)^{\frac{3}{4}}\, d\tau\\ &= \int_0^1 (1-\tau^2)^{\frac{1}{4}}\, d\tau \;+\; \left(\lim_{r\to 0^+}\frac{r^2}{12}\right)\int_{-1}^{1}(1-\tau^2)^{\frac{3}{4}}\,\lim_{r\to 0^+}\frac{G'(t_0+r^2\tau/4)}{1+G(t_0+r^2\,\tau/4)^2}\, d\tau\\ &= \int_0^1 (1-\tau^2)^{\frac{1}{4}}\, d\tau.\end{aligned} \tag{4.32}$$

□

Using (4.29), (4.30) and (4.31) we can also compute and obtain

$$\begin{aligned}&\lim_{r\to\infty}\frac{\sigma_H(\mathcal{S}\cap B(p_0,r))}{r^3}\\ &= \int_0^1 (1-\tau^2)^{\frac{1}{4}}\, d\tau \;+\; \lim_{r\to\infty}\int_{-1}^{1}\frac{r^2\,G'(t_0+r^2\tau/4)}{12(1+G(t_0+r^2\,\tau/4)^2)}\,(1-\tau^2)^{\frac{3}{4}}\;\, d\tau.\end{aligned} \tag{4.33}$$

Of course, the above limit may or may not be finite.

At this point, combining Theorem 1.3 and Proposition 4.6 we obtain the maximum sub-Riemannian volume growth of graphical strips at infinity.

Corollary 4.7 *Let $\mathcal{S} \subset \mathbb{H}^1$ be a graphical strip, then for every $p_0 = (0, 0, t_0) \in \mathcal{S}$, and every $r > 0$ one has*

$$\sigma_H(\mathcal{S} \cap B(p_0, r)) \geq \omega r^{Q-1},$$

where we have set $\omega = \int_0^1 (1 - \tau^2)^{\frac{1}{4}}\, d\tau$.

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