Inequalities of Hardy–Sobolev Type in Carnot–Carathéodory Spaces

Donatella Danielli, Nicola Garofalo, and Nguyen Cong Phuc

Abstract We consider various types of Hardy–Sobolev inequalities on a Carnot–Carathéodory space (Ω, d) associated to a system of smooth vector fields $X = \{X_1, X_2, \ldots, X_m\}$ on $\mathbb{R}^n$ satisfying the Hörmander finite rank condition $rank\ Lie[X_1, \ldots, X_m] \equiv n$. One of our main concerns is the trace inequality

$$\int_{\Omega} |\varphi(x)|^p V(x) dx \leqslant C \int_{\Omega} |X\varphi|^p dx, \qquad \varphi \in C_0^{\infty}(\Omega),$$

where V is a general weight, i.e., a nonnegative locally integrable function on Ω, and $1 < p < +\infty$. Under sharp geometric assumptions on the domain $\Omega \subset \mathbb{R}^n$ that can be measured equivalently in terms of subelliptic capacities or Hausdorff contents, we establish various forms of Hardy–Sobolev type inequalities.

1 Introduction

A celebrated inequality of S.L. Sobolev [49] states that for any $1 < p < n$ there exists a constant $S(n, p) > 0$ such that for every function $\varphi \in C_0^{\infty}(\mathbb{R}^n)$

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V. Maz'ya (ed.), *Sobolev Spaces in Mathematics I*, *International Mathematical Series*.
doi: 10.1007/978-0-387-85648-3,

$$\left(\int_{\mathbb{R}^n} |\varphi|^{\frac{np}{n-p}} dx\right)^{\frac{n-p}{np}} \leqslant S(n,p) \left(\int_{\mathbb{R}^n} |D\varphi|^p dx\right)^{\frac{1}{p}}. \tag{1.1}$$

Such an inequality admits the following extension (see [8]). For $0 \leqslant s \leqslant p$ define the critical exponent relative to s as follows:

$$p^*(s) = p\,\frac{n-s}{n-p}.$$

Then for every $\varphi \in C_0^\infty(\mathbb{R}^n)$ one has

$$\left(\int_{\mathbb{R}^n} \frac{|\varphi|^{p^*(s)}}{|x|^s} dx\right)^{\frac{1}{p^*(s)}} \leqslant \left(\frac{p}{n-p}\right)^{\frac{s}{p^*(s)}} S(n,p)^{\frac{n(p-s)}{p(n-s)}} \left(\int_{\mathbb{R}^n} |D\varphi|^p dx\right)^{\frac{1}{p}}. \tag{1.2}$$

In particular, when $s = 0$, then (1.2) is just the Sobolev embedding (1.1), whereas for $s = p$ we obtain the Hardy inequality

$$\int_{\mathbb{R}^n} \frac{|\varphi|^p}{|x|^p} dx \leqslant \left(\frac{p}{n-p}\right)^p \int_{\mathbb{R}^n} |D\varphi|^p dx. \tag{1.3}$$

The constant $\left(\dfrac{p}{n-p}\right)^p$ on the right-hand side of (1.3) is sharp. If one is not interested in the best constant, then (1.2), and hence (1.3), follows immediately by combining the generalized Hölder inequality for weak L^p spaces in [32] with the Sobolev embedding (1.1), after having observed that $|\cdot|^{-s} \in L^{\frac{n}{s},\infty}(\mathbb{R}^n)$ (the weak $L^{\frac{n}{s}}$ space).

Inequalities of Hardy–Sobolev type play a fundamental role in analysis, geometry, and mathematical physics, and there exists a vast literature concerning them. Recently, there has been a growing interest in such inequalities in connection with the study of linear and nonlinear partial differential equations of subelliptic type and related problems in CR and sub-Riemannian geometry. In this context, it is also of interest to study the situation in which the whole space is replaced by a bounded domain Ω and, instead of a one point singularity such as in (1.2), (1.3), one has the distance from a lower dimensional set. We will be particularly interested in the case in which such a set is the boundary $\partial\Omega$ of the ground domain.

In this paper, we consider various types of Hardy–Sobolev inequalities on a Carnot–Carathéodory space (Ω, d) associated to a system of smooth vector fields $X = \{X_1, X_2, \dots, X_m\}$ on $\mathbb{R}^n$ satisfying the Hörmander finite rank condition [31]

$$rank\ Lie[X_1, \dots, X_m] \equiv n. \tag{1.4}$$

Here, Ω is a connected, (Euclidean) bounded open set in $\mathbb{R}^n$, and d is the Carnot–Carathéodory (CC hereafter) metric generated by X. For instance, a situation of special geometric interest is that when the ambient manifold is a nilpotent Lie group whose Lie algebra admits a stratification of finite step $r \geqslant 1$ (see [18, 20, 53]. These groups are called Carnot groups of step r. When $r > 1$ such groups are non-Abelian, whereas when $r = 1$ one essentially has Euclidean $\mathbb{R}^n$ with its standard translations and dilations.

For a function $\varphi \in C^1(\Omega)$ we indicate with $X\varphi = (X_1\varphi, \dots, X_m\varphi)$ its "gradient" with respect to the system X. One of our main concerns is the trace inequality

$$\int\limits_{\Omega} |\varphi(x)|^p V(x) dx \leqslant C \int\limits_{\Omega} |X\varphi|^p dx, \qquad \varphi \in C_0^\infty(\Omega), \tag{1.5}$$

where V is a general weight, i.e., a nonnegative locally integrable function on Ω, and $1 < p < +\infty$. This includes Hardy inequalities of the form

$$\int\limits_{\Omega} \frac{|\varphi(x)|^p}{\delta(x)^p} dx \leqslant C \int\limits_{\Omega} |X\varphi|^p dx, \tag{1.6}$$

and

$$\int\limits_{\Omega} \frac{|\varphi(x)|^p}{d(x, x_0)^p} dx \leqslant C \int\limits_{\Omega} |X\varphi|^p dx, \tag{1.7}$$

as well as the mixed form

$$\int\limits_{\Omega} \frac{|\varphi(x)|^p}{\delta(x)^{p-\gamma} d(x, x_0)^\gamma} dx \leqslant C \int\limits_{\Omega} |X\varphi|^p dx. \tag{1.8}$$

In (1.6), we denote by $\delta(x) = \inf\{d(x, y) : y \in \partial\Omega\}$ the CC distance of x from the boundary of Ω. In (1.7), we denote by x_0 a fixed point in Ω, whereas in (1.8) we have $0 \leqslant \gamma \leqslant p$.

Our approach to the inequalities (1.6)-(1.8) is based on results on subelliptic capacitary and Fefferman–Phong type inequalities in [13], Whitney decompositions, and the so-called pointwise Hardy inequality

$$|\varphi(x)| \leqslant C\delta(x) \Big(\sup_{0 < r \leqslant 4\delta(x)} \frac{1}{|B(x, r)|} \int\limits_{B(x,r)} |X\varphi|^q dy \Big)^{\frac{1}{q}}, \tag{1.9}$$

where $1 < q < p$. In (1.9), $B(x, r)$ denotes the CC ball centered at x of radius r.

We use the ideas in [25] and [37] to show that (1.9) is essentially equivalent to several conditions on the geometry of the boundary of Ω, one of which is the uniform (X, p)-fatness of $\mathbb{R}^n \setminus \Omega$, a generalization of that of uniform p-fatness introduced in [38] in the Euclidean setting (see Definition 3.2

below). The inequality (1.9) is also equivalent to other thickness conditions of $\mathbb{R}^n \setminus \Omega$ measured in terms of a certain Hausdorff content which is introduced in Definition 3.5. For the precise statement of these results we refer to Theorem 3.9.

We stress here that the class of uniformly (X, p)-fat domains is quite rich. For instance, when $\mathbf{G}$ is a Carnot group of step $r = 2$, then every (Euclidean) $C^{1,1}$ domain is uniformly (X, p)-fat for every $p > 1$ (see [7, 43]). On the other hand, one would think that the Carnot–Carathéodory balls should share this property, but it was shown in [7] that this is not the case since even in the simplest setting of the Heisenberg group these sets fail to be regular for the Dirichlet problem for the relevant sub-Laplacian.

We now discuss our results concerning the trace inequality (1.5). In the Euclidean setting, a necessary and sufficient condition on V was found by Maz'ya [40] in 1962 (see also [41, Theorem 2.5.2]), i.e., the inequality (1.5) with the standard Euclidean metric induced by $X = \{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$ holds if and only if

$$\sup_{\substack{K \subset \Omega \\ K \text{ compact}}} \frac{\int_K V(x)dx}{\operatorname{cap}_p(K, \Omega)} < +\infty, \tag{1.10}$$

where $\operatorname{cap}_p(K, \Omega)$ is the (X, p)-capacity K defined by

$$\operatorname{cap}_p(K, \Omega) = \inf \left\{ \int_\Omega |Xu|^p dx : u \in C_0^\infty(\Omega), u \geqslant 1 \text{ on } K \right\}.$$

Maz'ya's result was generalized to the subelliptic setting by the first named author in [13]. However, although Corollary 5.9 in [13] implies that $V \in L^{\frac{Q}{p},\infty}(\Omega)$ is sufficient for (1.5), which is the case of an isolated singularity as in (1.7), the Hardy inequality (1.6) could not be deduced directly from it since $\delta(\cdot)^{-p} \notin L^{\frac{Q}{p},\infty}(\Omega)$. Here, $1 < p < Q$, where Q is the local homogeneous dimension of Ω (see Sect. 2). On the other hand, in the Euclidean setting the Hardy inequality (1.6) was established in [1], [38] and [51] (see also [42] and [3] for other settings) under the assumption that $\mathbb{R}^n \setminus \Omega$ is uniformly p-fat.

In this paper, we combine a "localized" version of (1.10) and the uniform (X, p)-fatness of $\mathbb{R}^n \setminus \Omega$ to allow the treatment of weights V with singularities which are distributed both inside and on the boundary of Ω. More specifically, we show that if $\mathbb{R}^n \setminus \Omega$ is uniformly (X, p)-fat, then the inequality (1.5) holds if and only if

$$\sup_{B \in \mathcal{W}} \sup_{\substack{K \subset 2B \\ K \text{ compact}}} \frac{\int_K V(x)dx}{\operatorname{cap}_p(K, \Omega)} < +\infty,$$

where $\mathcal{W} = \{B_j\}$ is a Whitney decomposition of Ω as in Lemma 4.2 below (see Theorem 4.3). In the Euclidean setting, this idea was introduced in [26]. Moreover, a localized version of Fefferman–Phong condition

$$\sup_{B\in\mathcal{W}} \sup_{\substack{x\in 2B \\ 0<r<\operatorname{diam}(B)}} \int_{B(x,r)} V(y)^s dy \leqslant C \frac{|B(x,r)|}{r^{sp}}$$

for some $s > 1$, is also shown to be sufficient for (1.5) (see Theorem 4.5).

With these general results in hands, in Corollaries 4.6 and 4.7 we deduce the Hardy type inequalities (1.6), (1.7), and (1.8) for domains Ω whose complements are uniformly (X,p)-fat. Note that in (1.7) and (1.8) one has to restrict the range of p to $1 < p < Q(x_0)$, where $Q(x_0)$ is the homogeneous dimension at x_0 with respect to the system X (see Sect. 2). It is worth mentioning that in the Euclidean setting inequalities of the form (1.8) were obtained in [16], but only for more regular domains, say, $C^{1,\alpha}$ domains or domains that satisfy a uniform exterior sphere condition. In closing we mention that our results are of a purely metrical character and that, similarly to [13], they can be easily generalized to the case in which the vector fields are merely Lipschitz continuous and they satisfy the conditions in [23].

2 Preliminaries

Let $X = \{X_1, \dots, X_m\}$ be a system of C^∞ vector fields in $\mathbb{R}^n$, $n \geqslant 3$, satisfying the Hörmander finite rank condition (1.4). For any two points $x, y \in \mathbb{R}^n$ a piecewise C^1 curve $\gamma(t) : [0,T] \to \mathbb{R}^n$ is said to be *sub-unitary* with respect to the system of vector fields X if for every $\xi \in \mathbb{R}^n$ and $t \in (0,T)$ for which $\gamma'(t)$ exists one has

$$(\gamma'(t)\cdot\xi)^2 \leqslant \sum_{i=1}^{m} (X_i(\gamma(t))\cdot\xi)^2.$$

We note explicitly that the above inequality forces $\gamma'(t)$ to belong to the span of $\{X_1(\gamma(t)), \dots, X_m(\gamma(t))\}$. The sub-unit length of γ is by definition $l_s(\gamma) = T$. Given $x, y \in \mathbb{R}^n$, denote by $\mathcal{S}_\Omega(x,y)$ the collection of all sub-unitary $\gamma : [0,T] \to \Omega$ which join x to y. The accessibility theorem of Chow and Rashevsky (see [46] and [9]) states that, given a connected open set $\Omega \subset \mathbb{R}^n$, for every $x, y \in \Omega$ there exists $\gamma \in \mathcal{S}_\Omega(x,y)$. As a consequence, if we pose

$$d_\Omega(x,y) = \inf\{l_s(\gamma) \mid \gamma \in \mathcal{S}_\Omega(x,y)\},$$

we obtain a distance on Ω, called the *Carnot–Carathéodory* (CC) *distance on* Ω, associated with the system X. When $\Omega = \mathbb{R}^n$, we write $d(x,y)$ instead of $d_{\mathbb{R}^n}(x,y)$. It is clear that $d(x,y) \leqslant d_\Omega(x,y)$, $x, y \in \Omega$, for every connected open set $\Omega \subset \mathbb{R}^n$. In [44], it was proved that for every connected $\Omega \subset\subset \mathbb{R}^n$ there exist $C, \varepsilon > 0$ such that

$$C\ |x-y|\ \leqslant d_\Omega(x,y)\ \leqslant C^{-1}\ |x-y|^\varepsilon, \qquad x,y \in \Omega. \tag{2.1}$$

This gives $d(x,y)\ \leqslant C^{-1}|x-y|^\varepsilon$, $x,y \in \Omega$, and therefore

$$i : (\mathbb{R}^n, |\cdot|) \to (\mathbb{R}^n, d) \qquad \textit{is continuous.}$$

It is easy to see that also the continuity of the opposite inclusion holds [23], hence the metric and the Euclidean topology are compatible. In particular, the compact sets with respect to either topology are the same.

For $x \in \mathbb{R}^n$ and $r > 0$ we let $B(x,r) = \{y \in \mathbb{R}^n \mid d(x,y) < r\}$. The basic properties of these balls were established by Nagel, Stein and Wainger in their seminal paper [44]. Denote by $Y_1, \ldots, Y_l$ the collection of the X_j's and of those commutators which are needed to generate $\mathbb{R}^n$. A formal "degree" is assigned to each Y_i, namely the corresponding order of the commutator. If $I = (i_1, \ldots, i_n), 1 \leqslant i_j \leqslant l$, is an n-tuple of integers, following [44] we let $d(I) = \sum_{j=1}^{n} deg(Y_{i_j})$, and $a_I(x) = \det\ (Y_{i_1}, \ldots, Y_{i_n})$. The *Nagel–Stein–Wainger polynomial* is defined by

$$\Lambda(x,r)\ =\ \sum_I\ |a_I(x)|\ r^{d(I)}, \qquad r > 0. \tag{2.2}$$

For a given compact set $K \subset \mathbb{R}^n$ we denote by

$$Q = \sup\{d(I) : |a_I(x)| \neq 0,\ x \in K\} \tag{2.3}$$

the *local homogeneous dimension* of K with respect to the system X and by

$$Q(x) = \inf\{d(I) : |a_I(x)| \neq 0\} \tag{2.4}$$

the *homogeneous dimension* at x with respect to X. It is obvious that $3 \leqslant n \leqslant Q(x) \leqslant Q$. It is immediate that for every $x \in K$, and every $r > 0$, one has

$$t^Q \Lambda(x,r) \leqslant \Lambda(x,tr) \leqslant t^{Q(x)} \Lambda(x,r) \tag{2.5}$$

for any $0 \leqslant t \leqslant 1$, and thus

$$Q(x) \leqslant \frac{r\Lambda'(x,r)}{\Lambda(x,r)} \leqslant Q. \tag{2.6}$$

For a simple example consider in $\mathbb{R}^3$ the system

$$X\ =\ \{X_1, X_2, X_3\}\ =\ \left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, x_1\frac{\partial}{\partial x_3}\right\}.$$

It is easy to see that $l = 4$ and

$$\{Y_1, Y_2, Y_3, Y_4\} \ = \ \{X_1, X_2, X_3, [X_1, X_3]\}.$$

Moreover, $Q(x) = 3$ for all $x \neq 0$, whereas for any compact set K containing the origin $Q(0) = Q = 4$.

The following fundamental result is due to Nagel, Stein, and Wainger [44]: *For every compact set $K \subset \mathbb{R}^n$ there exist constants $C, R_0 > 0$ such that for any $x \in K$ and $0 < r \leqslant R_0$ one has*

$$C\Lambda(x, r) \leqslant |B(x, r)| \leqslant C^{-1}\Lambda(x, r). \tag{2.7}$$

As a consequence, there exists C_0 such that for any $x \in K$ and $0 < r < s \leqslant R_0$ we have

$$C_0 \left(\frac{r}{s}\right)^Q \leqslant \frac{|B(x, r)|}{|B(x, s)|}. \tag{2.8}$$

Henceforth, the numbers C_0 and R_0 above will be referred to as the *local parameters* of K with respect to the system X. If E is any (Euclidean) bounded set in $\mathbb{R}^n$, then the local parameters of E are defined as those of $\overline{E}$. We mention explicitly that the number R_0 is always chosen in such a way that the closed metric balls $\overline{B}(x, R)$, with $x \in K$ and $0 < R \leqslant R_0$, are compact (see [23, 24]). This choice is motivated by the fact that in a CC space the closed metric balls of large radii are not necessarily compact. For instance, if one considers the Hörmander vector field on $\mathbb{R}$ given by $X_1 = (1 + x^2)\frac{d}{dx}$, then for any $R \geqslant \pi/2$ one has $B(0, R) = \mathbb{R}$ (see [23]).

Given an open set $\Omega \subset \mathbb{R}^n$, and $1 \leqslant p \leqslant \infty$, we denote by $S^{1,p}(\Omega)$, the subelliptic Sobolev space associated with the system X is defined by

$$S^{1,p}(\Omega) = \{u \in L^p(\Omega) : X_i u \in L^p(\Omega), i = 1, \ldots, m\},$$

where $X_i u$ is understood in the distributional sense, i.e.,

$$\langle X_i u, \varphi \rangle = \int\limits_{\Omega} u X_i^* \varphi dx$$

for every $\varphi \in C_0^\infty(\Omega)$. Here, X_i^* denotes the formal adjoint of X_i. Endowed with the norm

$$\|u\|_{S^{1,p}(\Omega)} = \left(\int\limits_{\Omega} (|u|^p + |Xu|^p) dx \right)^{\frac{1}{p}}, \tag{2.9}$$

$S^{1,p}(\Omega)$ is a Banach space which admits $C^\infty(\Omega) \cap S^{1,p}(\Omega)$ as a dense subset (see [23, 21]). The local version of $S^{1,p}(\Omega)$ is denoted by $S^{1,p}_{\mathrm{loc}}(\Omega)$, whereas the completion of $C_0^\infty(\Omega)$ under the norm in (2.9) is denoted by $S_0^{1,p}(\Omega)$.

A fundamental result in [47] shows that for any bounded open set $\Omega \subset \mathbb{R}^n$ the space $S_0^{1,p}(\Omega)$ embeds into a standard fractional Sobolev space $W_0^{s,p}(\Omega)$, where $s = 1/r$ and r is the largest number of commutators which are needed to generate the Lie algebra over $\overline{\Omega}$. Since, on the other hand, we have classically $W_0^{s,p}(\Omega) \subset L^p(\Omega)$, we obtain the following Poincaré inequality:

$$\int\limits_{\Omega} |\varphi|^p \, dx \;\leqslant\; C(\Omega) \int\limits_{\Omega} |X\varphi|^p \, dx \;, \quad \varphi \in S_0^{1,p}(\Omega). \tag{2.10}$$

Another fundamental result which plays a pervasive role in this paper is the following global Poincaré inequality on metric balls due to Jerison [33]. Henceforth, given a measurable set $E \subset \mathbb{R}^n$, the notation φ_E indicates the average of φ over E with respect to Lebesgue measure.

Theorem 2.1. *Let $K \subset \mathbb{R}^n$ be a compact set with local parameters C_0 and R_0. For any $1 \leqslant p < \infty$ there exists $C = C(C_0, p) > 0$ such that for any $x \in K$ and every $0 < r \leqslant R_0$ one has for all $\varphi \in S^{1,p}(B(x,r))$*

$$\int\limits_{B(x,r)} |\varphi - \varphi_{B(x,r)}|^p dy \;\leqslant\; C\, r^p \int\limits_{B(x,r)} |X\varphi|^p dy. \tag{2.11}$$

We also need the following basic result on the existence of cut-off functions in metric balls (see [24] and also [21]). Given a set $\Omega \subset \mathbb{R}^n$, we indicate with $C_d^{0,1}(\Omega)$ the collection of functions $\varphi \in C(\Omega)$ for which there exists $L \geqslant 0$ such that

$$|\varphi(x) - \varphi(y)| \;\leqslant\; L\, d(x,y), \quad x, y \in \Omega.$$

We recall that, thanks to the Rademacher–Stepanov type theorem proved in [24, 21], if Ω is metrically bounded, then any function in $C_d^{0,1}(\Omega)$ belongs to the space $S^{1,\infty}(\Omega)$. This is true, in particular, when Ω is a metric ball.

Theorem 2.2. *Let $K \subset \mathbb{R}^n$ be a compact set with local parameters C_0 and R_0. For every $0 < s < t < R_0$ there exists $\varphi \in C_d^{0,1}(\mathbb{R}^n)$, $0 \leqslant \varphi \leqslant 1$, such that*

(i) *$\varphi \equiv 1$ on $B(x,s)$ and $\varphi \equiv 0$ outside $B(x,t)$,*

(ii) *$|X\varphi| \leqslant \frac{C}{t-s}$ for a.e. $x \in \mathbb{R}^n$,*

for some $C > 0$ depending on C_0. Furthermore, we have $\varphi \in S^{1,p}(\mathbb{R}^n)$ for every $1 \leqslant p < \infty$.

A *condenser* is a couple (K, Ω), where Ω is open and $K \subset \Omega$ is compact. The *subelliptic p-capacity* of (K, Ω) is defined by

$$\mathrm{cap}_p(K, \Omega) = \inf \left\{ \int\limits_{\Omega} |X\varphi|^p dx : \varphi \in C_d^{0,1}(\mathbb{R}^n), \mathrm{supp}\ \ \varphi \subset \Omega, \varphi \geqslant 1 \text{ on } K \right\}.$$

As usual, it can be extended to arbitrary sets $E \subset \Omega$ by letting

$$\operatorname{cap}_p(E,\Omega) = \inf_{\substack{G\subset\Omega \text{ open}\\ E\subset G}} \ \sup_{\substack{K\subset G\\ K \text{ compact}}} \operatorname{cap}_p(K,\Omega).$$

It was proved in [12] that the subelliptic p-capacity of a metric "annular" condenser has the following two-sided estimate which will be used extensively in the paper. Given a compact set $K \subset \mathbb{R}^n$ with local parameters C_0 and R_0, and homogeneous dimension Q, for any $1 < p < \infty$ there exist $C_1, C_2 > 0$ depending only on C_0 and p such that

$$C_1 \frac{|B(x,r)|}{r^p} \ \leqslant \ \operatorname{cap}_p(B(x,r), B(x,2r)) \ \leqslant \ C_2 \ \frac{|B(x,r)|}{r^p} \tag{2.12}$$

for all $x \in K$, and $0 < r \leqslant R_0/2$.

The subelliptic p-Laplacian associated to the system X is the quasilinear operator defined by

$$\mathcal{L}_p[u] = -\sum_{i=1}^{m} X_i^*(|Xu|^{p-2} X_i u).$$

A weak solution $u \in S^{1,p}_{\text{loc}}(\Omega)$ to the equation $\mathcal{L}_p[u] = 0$ is said to be $\mathcal{L}_p$-*harmonic* in Ω. It is well-known that every $\mathcal{L}_p$-harmonic function in Ω has a Hölder continuous representative (see [4]). This means that, if C_0 and R_0 are the local parameters of Ω, then there exist $0 < \alpha < 1$ and $C > 0$, depending on C_0 and p, such that for every $0 < R \leqslant R_0$ for which $B_{4R}(x_0) \subset \Omega$ one has

$$|u(x) - u(y)| \ \leqslant \ C \left(\frac{d(x,y)}{R} \right)^{\alpha} \left(\frac{1}{|B_{2R}(x_0)|} \int\limits_{B_{2R}(x_0)} |u|^p dx \right)^{1/p}. \tag{2.13}$$

Given a bounded open set $\Omega \subset \mathbb{R}^n$ and $1 < p < \infty$, the Dirichlet problem for Ω and $\mathcal{L}_p$ consists in finding, for every given $\varphi \in S^{1,p}(\Omega) \cap C(\overline{\Omega})$, a function $u \in S^{1,p}(\Omega)$ such that

$$\mathcal{L}_p[u] \ = \ 0 \quad \text{in } \Omega, \qquad u - \varphi \in S^{1,p}_0(\Omega). \tag{2.14}$$

Such a problem admits a unique solution (see [12]). A point $x_0 \in \partial\Omega$ is called *regular* if for every $\varphi \in S^{1,p}(\Omega) \cap C(\overline{\Omega})$,one has $\lim\limits_{x\to x_0} u(x) = \varphi(x_0)$. If every $x_0 \in \partial\Omega$ is regular, then we say that Ω is *regular*. We need the following basic Wiener type estimate proved in [12].

Theorem 2.3. *Given a bounded open set $\Omega \subset \mathbb{R}^n$ with local parameters C_0 and R_0, let $\varphi \in S^{1,p}(\Omega) \cap C(\overline{\Omega})$. Consider the* (*unique*) *solution u to the Dirichlet problem* (2.14)*. There exists $C = C(p, C_0) > 0$ such that for given $x_0 \in \partial\Omega$ and $0 < r < R \leqslant R_0/3$ one has with $\Omega^c = \mathbb{R}^n \setminus \Omega$*

$$
\begin{aligned}
\operatorname{osc}\{u, \Omega \cap B(x_0, r)\} &\leqslant \operatorname{osc}\{\varphi, \partial\Omega \cap \overline{B}(x_0, 2R)\} \\
&+ \operatorname{osc}(\varphi, \partial\Omega)\ \exp\left\{ -\ C \int_r^R \left[\frac{\operatorname{cap}_p\ (\Omega^c \cap \overline{B}(x_0, t), B(x_0, 2t))}{\operatorname{cap}_p\ (\overline{B}(x_0, t), B(x_0, 2t))} \right]\ \frac{dt}{t} \right\}.
\end{aligned}
$$

Remark 2.4. It is clear from Theorem 2.3 that if Ω is *thin at* $x_0 \in \partial\Omega$, i.e., if one has

$$
\liminf_{t \to 0^+} \frac{\operatorname{cap}_p\ (\Omega^c \cap \overline{B}(x_0, t), B(x_0, 2t))}{\operatorname{cap}_p\ (\overline{B}(x_0, t), B(x_0, 2t))}\ >\ 0\ ,
$$

then x_0 is regular for the Dirichlet problem (2.14).

A lower semicontinuous function $u : \Omega \to (-\infty, \infty]$ such that $u \not\equiv +\infty$ is called $\mathcal{L}_p$-*superharmonic* in Ω if for all open sets D such that $\overline{D} \subset \Omega$, and all $\mathcal{L}_p$-harmonic functions $h \in C(\overline{D})$ the inequality $h \leqslant u$ on ∂D implies $h \leqslant u$ in D. Similarly to what is done in the classical case in [30], one can associate with each $\mathcal{L}_p$-superharmonic function u in Ω a nonnegative (not necessarily finite) Radon measure $\mu[u]$ such that $-\mathcal{L}_p[u] = \mu[u]$. This means that

$$
\int_\Omega |Xu|^{p-2} Xu \cdot X\varphi\ dx\ =\ \int_\Omega \varphi\ d\mu[u]
$$

for all $\varphi \in C_0^\infty(\Omega)$. Here, Xu is defined a.e. by

$$
Xu = \lim_{k \to \infty} X(\min\{u, k\}).
$$

It is known that if either $u \in L^\infty(\Omega)$ or $u \in S^{1,r}_{\text{loc}}(\Omega)$ for some $r \geqslant 1$, then Xu coincides with the regular distributional derivatives. In general, we have $Xu \in L^s_{\text{loc}}(\Omega)$ for $0 < s < \frac{Q(p-1)}{Q-1}$ (see, for example, [50] and [30]).

We need the following basic pointwise estimates for $\mathcal{L}_p$-superharmonic functions. This result was first established by Kilpeläinen and Malý [35] in the elliptic case and extended to the setting of CC metrics by Trudinger and Wang [50]. For a generalization to more general metric spaces we refer the reader to [3]. We recall that for given $1 < p < \infty$ the p-*Wolff's potential* of a Radon measure μ on a metric ball $B(x, R)$ is defined by

$$
\mathbf{W}_p^R \mu(x)\ =\ \int_0^R \left[\frac{\mu(B(x,t))}{t^{-p}|B(x,t)|} \right]^{\frac{1}{p-1}} \frac{dt}{t}. \tag{2.15}
$$

Theorem 2.5. *Let $K \subset \mathbb{R}^n$ be a compact set with relative local parameters C_0 and R_0. If $x \in K$ and $R \leqslant R_0/2$, let $u \geqslant 0$ be $\mathcal{L}_p$-superharmonic in $B(x, 2R)$ with associated measure $\mu = -\mathcal{L}_p[u]$. There exist positive constants C_1 and C_2 depending only on p and C_0 such that*

$$C_1 \mathbf{W}_p^R \mu(x) \leqslant u(x) \leqslant C_2 \left\{ \mathbf{W}_p^{2R} \mu(x) + \inf_{B(x,R)} u \right\}.$$

3 Pointwise Hardy Inequalities

We begin this section by generalizing a Sobolev type inequality that, in the Euclidean setting, was found by Maz'ya [41, Chapt. 10].

Lemma 3.1. *Let $K \subset \mathbb{R}^n$ be a compact set with local parameters C_0 and R_0. For $x \in K$ and $r \leqslant R_0/2$ we set $B = B(x,r)$. Given $1 \leqslant q < \infty$, there exists a constant $C > 0$ depending only on C_0 and q such that for all $\varphi \in C^\infty(2B)$*

$$|\varphi_B| \leqslant C \left(\frac{1}{\operatorname{cap}_q(\{\varphi = 0\} \cap \overline{B}, 2B)} \int\limits_{2B} |X\varphi|^q dx \right)^{\frac{1}{q}}. \tag{3.1}$$

Proof. We may assume that $\varphi_B \neq 0$; otherwise, there is nothing to prove. Let $\eta \in C_d^{0,1}(\mathbb{R}^n)$, $0 \leqslant \eta \leqslant 1$, supp $\eta \subset 2B$, $\eta = 1$ on $\overline{B}$ and $|X\eta| \leqslant \frac{C}{r}$ be a cut-off function as in Theorem 2.2. Define $\varphi = \eta(\varphi_B - \varphi)/\varphi_B$. Then $\varphi \in C_d^{0,1}(\mathbb{R}^n)$, supp $\varphi \subset 2B$, and $\varphi = 1$ on $\{\varphi = 0\} \cap \overline{B}$. It thus follows that

$$\begin{aligned} \operatorname{cap}_q(\{\varphi = 0\} \cap \overline{B}, 2B) &\leqslant \int\limits_{2B} |X\varphi|^q dx \\ &\leqslant |\varphi_B|^{-q} \int\limits_{2B} |X\eta|^q |\varphi - \varphi_B|^q dx + |\varphi_B|^{-q} \int\limits_{2B} |X\varphi|^q dx \\ &\leqslant C|\varphi_B|^{-q} r^{-q} \int\limits_{2B} |\varphi - \varphi_B|^q dx + |\varphi_B|^{-q} \int\limits_{2B} |X\varphi|^q dx. \end{aligned} \tag{3.2}$$

On the other hand, by Theorem 2.1 and (2.8), we infer

$$\begin{aligned} \int\limits_{2B} |\varphi - \varphi_B|^q dx &\leqslant C \int\limits_{2B} |\varphi - \varphi_{2B}|^q dx + C \int\limits_{2B} |\varphi_B - \varphi_{2B}|^q dx \\ &\leqslant Cr^q \int\limits_{2B} |X\varphi|^q dx + C \int\limits_{2B} |\varphi - \varphi_{2B}|^q dx \\ &\leqslant Cr^q \int\limits_{2B} |X\varphi|^q dx. \end{aligned}$$

Inserting the latter inequality in (3.2), we find

$$\operatorname{cap}_q(\{\varphi = 0\} \cap \overline{B}, 2B) \leqslant C|\varphi_B|^{-q} \int\limits_{2B} |X\varphi|^q dx,$$

which gives the desired inequality (3.1). □

We now introduce the notion of uniform (X, p)-fatness. As Theorem 3.9 below proves, such a notion turns out to be equivalent to a pointwise Hardy inequality and to a uniform thickness property expressed in terms of the Hausdorff content.

Definition 3.2. We say that a set $E \subset \mathbb{R}^n$ is *uniformly* (X, p)*-fat* with constants $c_0, r_0 > 0$ if

$$\operatorname{cap}_p(E \cap \overline{B}(x, r), B(x, 2r)) \;\geqslant\; c_0 \operatorname{cap}_p(\overline{B}(x, r), B(x, 2r))$$

for all $x \in \partial E$ and for all $0 < r \leqslant r_0$.

The potential theoretic relevance of Definition 3.2 is underscored in Remark 2.4. From the latter it follows that if $\mathbb{R}^n \setminus \Omega$ is uniformly (X, p)-fat, then for every $x_0 \in \partial\Omega$ one has for every $\varphi \in S^{1,p}(\Omega) \cap C(\overline{\Omega})$

$$\operatorname{osc}\{u, \Omega \cap B(x_0, r)\} \;\leqslant \operatorname{osc}\{\varphi, \partial\Omega \cap \overline{B}(x_0, 2R)\}$$

and, therefore, Ω is regular for the Dirichlet problem for the subelliptic p-Laplacian $\mathcal{L}_p$.

Uniformly (X, p)-fat sets enjoy the following self-improvement property which was discovered in [38] in the Euclidean setting. Such a property holds also in the setting of weighted Sobolev spaces and degenerate elliptic equations [42]. The proof in [42] uses the Wolff potential and works also in the general setting of metric spaces [3]. For the sake of completeness, we include its details here.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with local parameters C_0 and R_0. There exists a constant $0 < r_0 \leqslant R_0/100$ such that whenever $\mathbb{R}^n \setminus \Omega$ is uniformly (X, p)-fat with constants c_0 and r_0, then it is also uniformly (X, q)-fat for some $q < p$ with constants c_1 and r_0.*

Proof. Let $\operatorname{dist}(x, \Omega) = \inf\{d(x, y) : y \in \Omega\}$. Denote by $U \subset \mathbb{R}^n$ the compact set

$$U = \{x \in \mathbb{R}^n : \operatorname{dist}(x, \Omega) \leqslant R_0\},$$

with local parameters C_1, R_1. We show that if $\mathbb{R}^n \setminus \Omega$ is uniformly (X, p)-fat with constants c_0 and $r_0 = \min\{R_0, R_1\}/100$, then it is also uniformly (X, q)-fat for some $q < p$ with constants c_1 and r_0. To this end, we fix $x_0 \in \partial\Omega$ and $0 < R \leqslant r_0$. Following [38], we first claim that there e xists a compact set $K \subset (\mathbb{R}^n \setminus \Omega) \cap \overline{B}(x_0, R)$ containing x_0 such that K is uniformly (X, p)-fat with constants $c_1 > 0$ and R. Indeed, let $E_1 = (\mathbb{R}^n \setminus \Omega) \cap B(x_0, \frac{R}{2})$, and inductively let

$$E_k = (\mathbb{R}^n \setminus \Omega) \cap \left(\bigcup_{x \in E_{k-1}} B(x, \tfrac{R}{2^k}) \right), \quad k \in \mathbb{N}.$$

Then it is easy to see that K can be taken as the closure of $\cup_k E_k$.

Let now $B = B(x_0, R)$. Denote by $\widehat{P}_K$ the potential of K in $2B$, i.e., $\widehat{P}_K$ is the lower semicontinuous regularization

$$\widehat{P}_K(x) = \lim_{r \to 0} \inf_{B_r(x)} P_K,$$

where P_K is defined by

$$P_K = \inf\{u : u \text{ is } \mathcal{L}_p\text{-superharmonic in } 2B, \text{ and } u \geqslant \chi_K\}.$$

Let $\mu = -\mathcal{L}_p[\widehat{P}_K]$. Then $\operatorname{supp} \mu \subset \partial K$ and

$$\mu(K) = \operatorname{cap}_p(K, 2B). \tag{3.3}$$

Moreover, $\widehat{P}_K = P_K$ except for a set of zero capacity $\operatorname{cap}_p(\cdot, 2B)$ (see [50]). Hence $\widehat{P}_K$ is the unique solution in $S_0^{1,p}(2B)$ to the Dirichlet problem

$$\mathcal{L}_p[u] = 0 \quad \text{in} \quad 2B \setminus K, \qquad u - f \in S_0^{1,p}(2B \setminus K)$$

for any $f \in C_0^\infty(2B)$ such that $f \equiv 1$ on K. Thus, by Theorem 2.3 and the (X,p)-fatness of K, there are constants $C > 0$ and $\alpha > 0$ independent of R such that

$$\operatorname{osc}(\widehat{P}_K, B(x,r)) \leqslant C R^{-\alpha} r^\alpha \tag{3.4}$$

for all $x \in \partial K$ and $0 < r \leqslant R/2$. From the lower Wolff potential estimate in Theorem 2.5 we have

$$\begin{aligned} \left[\frac{\mu(B(x,r))}{r^{-p}|B(x,r)|} \right]^{\frac{1}{p-1}} &\leqslant C\, \mathbf{W}_p^{2r} \mu(x) \leqslant C \left(\widehat{P}_K(x) - \inf_{B(x,4r)} \widehat{P}_K \right) \\ &\leqslant C \operatorname{osc}(\widehat{P}_K, B(x,4r)). \end{aligned}$$

Thus, from (3.4) it follows that

$$\mu(B(x,r)) \leqslant C R^{-\alpha(p-1)} r^{\alpha(p-1)-p} |B(x,r)| \tag{3.5}$$

for all $x \in \partial K$ and $0 < r \leqslant R/8$. Moreover, since $\operatorname{supp}\ \mu \subset \partial K$, we see from the doubling property (2.8) that (3.5) holds also for all $x \in B(x_0, 2R)$ and $0 < r \leqslant R/16$. In fact, it then holds for all $R/16 < r \leqslant 3R$ as well since, again by (2.8), the ball $B(x,r)$ can be covered by a fixed finite number of balls of radius $R/16$.

We next pick $q \in \mathbb{R}$ such that $p - \alpha(p-1) < q < p$ and define a measure $\nu = R^{p-q}\mu$. From (3.5) it follows that for all $x \in B(x_0, 2R)$,

$$\mathbf{W}_q^{3R}\nu(x) \leqslant CR^{\frac{p-q-\alpha(p-1)}{q-1}} \int\limits_0^{3R} r^{\frac{q-p+\alpha(p-1)}{q-1}} \frac{dr}{r} \leqslant M, \tag{3.6}$$

where M is independent of R. Thus, by [2, Lemma 3.3], ν belongs to the dual space of $S_0^{1,q}(2B)$ and there is a unique solution $v \in S_0^{1,q}(2B)$ to the problem

$$\begin{aligned} -\mathcal{L}_q[v] &= \nu \quad \text{in} \quad 2B \\ v &= 0 \quad \text{on} \quad \partial(2B). \end{aligned} \tag{3.7}$$

We now claim that

$$v(x) \leqslant c \tag{3.8}$$

for all $x \in 2B$ and for a constant c independent of R. To this end, it is enough to show (3.8) only for $x \in \overline{B}$ since v is $\mathcal{L}_q$-harmonic in $2B \setminus \overline{B}$ and $v = 0$ on $\partial(2B)$. Fix now $x \in \overline{B}$. By Theorem 2.5, we have

$$v(x) \leqslant C \left\{ \mathbf{W}_q^{3R}\nu(x) + \inf_{B(x,R/4)} v \right\}. \tag{3.9}$$

To bound the term $\inf\limits_{B(x,R/4)} v$ in (3.9), we first use $\min\{v,k\}$, $k > 0$, as a test function in (3.7) to obtain

$$\begin{aligned} \int\limits_{2B} |X(\min\{v,k\})|^q dx &= \int\limits_{2B} |Xv|^{q-2} Xv \cdot X(\min\{v,k\}) dx \\ &= \int\limits_{2B} \min\{v,k\} d\nu \;\leqslant\; k\, \nu(K). \end{aligned} \tag{3.10}$$

Consequently,

$$\operatorname{cap}_q(\{v \geqslant k\}, 2B) \leqslant \int\limits_{2B} |X(\min\{v,k\}/k)|^q dx \leqslant k^{1-q} \nu(K) \tag{3.11}$$

for any $k > 0$. The inequality (3.11) with $k = \inf\limits_{B(x,R/4)} v$ then gives

$$\begin{aligned} R^{-q}|B(x,R)| &\leqslant C \operatorname{cap}_q(B(x,R/4), B(x,4R)) \\ &\leqslant C \operatorname{cap}_q(\{v \geqslant k\}, 2B) \\ &\leqslant C k^{1-q} \nu(K), \end{aligned}$$

which yields the estimate

$$\inf_{B(x,R/4)} v \leqslant C \left(\frac{\nu(K)}{R^{-q}|B(x,R)|} \right)^{\frac{1}{q-1}}. \tag{3.12}$$

Combining (3.6), (3.9), and (3.12). we obtain (3.8), thus proving the claim. Note that for any $\varphi \in C_0^\infty(2B)$ such that $\varphi \geqslant \chi_K$, by the Hölder inequality and by applying (3.10) with $k = c$, we have

$$\begin{aligned}\nu(K) &\leqslant \int\limits_{2B} \varphi d\nu = \int\limits_{\Omega} |Xv|^{q-2} Xv \cdot X\varphi dx \\ &\leqslant \Bigg(\int\limits_{2B} |Xv|^q dx\Bigg)^{\frac{q-1}{q}} \Bigg(\int\limits_{2B} |X\varphi|^q dx\Bigg)^{\frac{1}{q}} \\ &\leqslant [c\,\nu(K)]^{\frac{q-1}{q}} \Bigg(\int\limits_{2B} |X\varphi|^q dx\Bigg)^{\frac{1}{q}}.\end{aligned}$$

Thus, minimizing over such functions φ, we obtain

$$\nu(K) \leqslant c^{q-1} \operatorname{cap}_q(K, 2B).$$

The latter inequality and (2.12) give

$$\begin{aligned}\operatorname{cap}_q((\mathbb{R}^n \setminus \Omega) \cap \overline{B}, 2B) \geqslant \operatorname{cap}_q(K, 2B) &\geqslant C\,\nu(K) = CR^{p-q}\mu(K) \\ &= CR^{p-q}\operatorname{cap}_p(K, 2B) \geqslant CR^{p-q}\operatorname{cap}_p(\overline{B}, 2B) \\ &\geqslant CR^{-q}|B| \geqslant C\operatorname{cap}_q(\overline{B}, 2B)\end{aligned}$$

by (3.3) and the uniform (X,p)-fatness of K. This proves that $\mathbb{R}^n \setminus \Omega$ is uniformly (X,q)-fat, thus completing the proof of the theorem. □

In what follows, given $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, we denote by $\mathcal{M}_R$, $0 < R < \infty$, the truncated centered Hardy–Littlewood maximal function of f defined by

$$\mathcal{M}_R(f)(x) = \sup_{0<r\leqslant R} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y)|dy, \qquad x \in \mathbb{R}^n.$$

We note explicitly that if $R_1 < R_2$, then $\mathcal{M}_{R_1}(f)(x) \leqslant \mathcal{M}_{R_2}(f)(x)$. The first consequence of the self-improvement property of uniformly (X,p)-fat set is the following pointwise Hardy inequality which generalizes a result originally found by Hajłasz [25] in the Euclidean setting.

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with local parameters C_0 and R_0. Suppose that $\mathbb{R}^n \setminus \Omega$ is uniformly (X,p)-fat with constants c_0 and r_0, where $0 < r_0 \leqslant R_0/100$ is as in Theorem 3.3. There exist $1 < q < p$ and a constant $C > 0$, both depending on C_0 and p, such that the inequality*

$$|u(x)| \leqslant C\delta(x)\Big(\mathcal{M}_{4\delta(x)}(|\nabla u|^q)(x)\Big)^{\frac{1}{q}} \tag{3.13}$$

holds for all $x \in \Omega$ with $\delta(x) < r_0$ and all compactly supported $u \in C_d^{0,1}(\Omega)$.

Proof. For $x \in \Omega$ with $\delta(x) < r_0$ we let $B = B(\overline{x}, \delta(x))$, where $\overline{x} \in \partial\Omega$ is chosen so that $|x - \overline{x}| = \delta(x)$. By the fatness assumption and Theorem 3.3, there exists $1 < q < p$ such that

$$\mathrm{cap}_{1,q}(\overline{B} \cap (\mathbb{R}^n \setminus \Omega), 2B) \geqslant C|B|\delta(x)^{-q}.$$

Thus, by Lemma 3.1 above and Theorem 1.1 in [6],

$$\begin{aligned} u(x) &\leqslant |u(x) - u_B| + |u_B| \\ &\leqslant C\int_{2B} |Xu(y)| \frac{d(x,y)}{|B(x,d(x,y))|} dy + C\Big(\frac{\int_{2B} |Xu|^q dx}{|B|\delta(x)^{-q}}\Big)^{\frac{1}{q}}. \end{aligned} \tag{3.14}$$

Note that by the doubling property (2.8),

$$\begin{aligned} &\int_{2B} |Xu(y)| \frac{d(x,y)}{|B(x,d(x,y))|} dy \\ &\leqslant \int_{B(x,4\delta(x))} |Xu(y)| \frac{d(x,y)}{|B(x,d(x,y))|} dy \\ &= \sum_{k=0}^{\infty} \int_{B(x,2^{-k}4\delta(x)) \setminus B(x,2^{-k-1}4\delta(x))} |Xu(y)| \frac{d(x,y)}{|B(x,d(x,y))|} dy \\ &\leqslant C\sum_{k=0}^{\infty} \frac{2^{-k}4\delta(x)}{|B(x,2^{-k}4\delta(x))|} \int_{B(x,2^{-k}4\delta(x))} |Xu(y)| dy \\ &\leqslant C\delta(x)\mathcal{M}_{4\delta(x)}(|Xu|)(x). \end{aligned} \tag{3.15}$$

Also,

$$\begin{aligned} \Big(\frac{\int_{2B} |Xu|^q dx}{|B|\delta(x)^{-q}}\Big)^{\frac{1}{q}} &\leqslant C\delta(x)\Big(\frac{\int_{B(x,4\delta(x))} |Xu|^q dx}{|B(x,4\delta(x))|}\Big)^{\frac{1}{q}} \\ &\leqslant C\delta(x)\Big(\mathcal{M}_{4\delta(x)}(|Xu|^q)(x)\Big)^{\frac{1}{q}}. \end{aligned} \tag{3.16}$$

From (3.14), (3.15), (3.16) and the Hölder inequality we now obtain

$$u(x) \leqslant C\delta(x)\Big(\mathcal{M}_{4\delta(x)}(|Xu|^q)(x)\Big)^{\frac{1}{q}},$$

which completes the proof of the theorem. $\square$

As it turns out, the pointwise Hardy inequality (3.13) is in fact equivalent to certain geometric conditions on the boundary of Ω that can be measured in terms of a Hausdorff content. We introduce the relevant definition.

Definition 3.5. Let $s \in \mathbb{R}$, $r > 0$ and $E \subset \mathbb{R}^n$. The (X, s, r)-*Hausdorff content* of E is the number

$$\widetilde{\mathcal{H}}_r^s(E) = \inf \sum_j r_j^s |B_j|,$$

where the infimum is taken over all coverings of E by balls $B_j = B(x_j, r_j)$ such that $x_j \in E$ and $r_j \leqslant r$.

We next follow the idea in [37] to prove the following important consequence of the pointwise Hardy inequality (3.13).

Theorem 3.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with local parameters C_0 and R_0. Suppose that there exist $r_0 \leqslant R_0/100$, $q > 0$, and a constant $C > 0$ such that the inequality*

$$|u(x)| \leqslant C\delta(x)\Big(\mathcal{M}_{4\delta(x)}(|\nabla u|^q)(x)\Big)^{\frac{1}{q}} \tag{3.17}$$

holds for all $x \in \Omega$ with $\delta(x) < r_0$ and all compactly supported $u \in C_d^{0,1}(\Omega)$. There exists $C_1 > 0$ such that the inequality

$$\widetilde{\mathcal{H}}_{\delta(x)}^{-q}(\overline{B}(x, 2\delta(x)) \cap \partial\Omega) \geqslant C_1 \delta(x)^{-q} |B(x, \delta(x))| \tag{3.18}$$

holds for all $x \in \Omega$ with $\delta(x) < r_0$.

Proof. We argue by contradiction and suppose that (3.18) fails. We can thus find a sequence $\{x_k\}_{k=1}^{\infty} \subset \Omega$ with $\delta(x_k) < r_0$ such that

$$\widetilde{\mathcal{H}}_{\delta(x)/4}^{-q}(\overline{B}(x_k, 5\delta(x_k)) \cap \partial\Omega) < k^{-1} \delta(x_k)^{-q} |B(x_k, \delta(x_k))|.$$

Here, we used the fact that, by the continuity of the distance function δ and the doubling property (2.8), the inequality (3.18), which holds for all $x \in \Omega$ with $\delta(x) < r_0$, is equivalent to the validity of

$$\widetilde{\mathcal{H}}_{\delta(x)/4}^{-q}(\overline{B}(x, 5\delta(x)) \cap \partial\Omega) \geqslant C_2 \delta(x)^{-q} |B(x, \delta(x))|$$

for all $x \in \Omega$ with $\delta(x) < r_0$ and for a constant $C_2 > 0$. By compactness, we can now find a finite covering $\{B_i\}_{i=1}^N$, $B_i = B(z_i, r_i)$ with $z_i \in \overline{B}(x_k, 5\delta(x_k)) \cap \partial\Omega$ and $0 < r_i < \delta(x_k)/4$, such that

$$\overline{B}(x_k, 5\delta(x_k)) \cap \partial\Omega \ \subset \ \bigcup_{i=1}^N B_i \tag{3.19}$$

and

$$\sum_{i=1}^{N} r_i^{-q}|B_i| < k^{-1}\delta(x_k)^{-q}|B(x_k,\delta(x_k))|. \tag{3.20}$$

Next, for each $k \in \mathbf{N}$ we define a function φ_k by

$$\varphi_k(x) = \min\{1, \min_{1\leqslant i\leqslant N} r_i^{-1}\mathrm{dist}(x, 2B_i)\}$$

and let $\phi_k \in C_d^{0,1}(B(x_k, 5\delta(x_k)))$ be such that $0 \leqslant \phi_k \leqslant 1$ and $\phi_k \equiv 1$ on $B(x_k, 4\delta(x_k))$. Clearly, the function $u_k = \varphi_k\phi_k$ belongs to $C_d^{0,1}(\Omega)$ and, in view of (3.19), it has compact support. Moreover, $u_k(x_k) = 1$ since from the fact that $z_i \in \partial\Omega$ we have

$$d(x_k, z_i) \geqslant \delta(x_k) > 4r_i \tag{3.21}$$

for all $1 \leqslant i \leqslant N$. Also, since $\varphi_k(x) = 1$ for $x \notin \bigcup_{i=1}^{N} 3\overline{B}_i$ and $\varphi_k(x) = 0$ for $x \in \bigcup_{i=1}^{N} 2\overline{B}_i$, it is easy to see that

$$\mathrm{supp}\ (|Xu_k|) \cap B(x_k, 4\delta(x_k)) \subset \bigcup_{i=1}^{N}(3\overline{B}_i \setminus 2B_i)$$

and that for a.e. $y \in B(x_k, 4\delta(x_k))$ we have

$$|Xu_k(y)|^q \leqslant \sum_{i=1}^{N} r_i^{-q}\chi_{3\overline{B}_i\setminus 2B_i}(y). \tag{3.22}$$

Hence, using (3.21) and (3.22), we can calculate

$$\begin{aligned}
&\mathcal{M}_{4\delta(x_k)}(|Xu_k|^q)(x_k) \\
&\leqslant C \sup_{\frac{1}{4}\delta(x_k)\leqslant r\leqslant 4\delta(x_k)} \frac{1}{|B(x_k,r)|}\int\limits_{B(x_k,r)} |Xu_k(y)|^q dy \\
&\leqslant C\frac{1}{|B(x_k,\delta(x_k))|}\int\limits_{B(x_k,4\delta(x_k))} |Xu_k(y)|^q dy \\
&\leqslant C\frac{1}{|B(x_k,\delta(x_k))|}\sum_{i=1}^{N}|3\overline{B_i}\setminus 2B_i| r_i^{-q}
\end{aligned}$$

$$\leqslant C \frac{1}{|B(x_k, \delta(x_k))|} \sum_{i=1}^{N} |B_i| r_i^{-q}. \tag{3.23}$$

From (3.20) and (3.23) we obtain

$$\delta(x_k)^q \mathcal{M}_{4\delta(x_k)}(|Xu_k|^q)(x_k) \leqslant Ck^{-1}.$$

Since $u_k = 1$ for any k, this implies that the pointwise Hardy inequality (3.17) fails to hold with a uniform constant for all compactly supported $u \in C_d^{0,1}(\Omega)$. This contradiction completes the proof of the theorem. □

As in [37], from (3.18) we can also obtain the following thickness condition on $\mathbb{R}^n \setminus \Omega$.

Theorem 3.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with local parameters C_0 and R_0. Suppose that there exist $r_0 \leqslant R_0/100$, $q > 0$, and a constant $C > 0$ such that the inequality*

$$\widetilde{\mathcal{H}}^{-q}_{\delta(x)}(\overline{B}(x, 2\delta(x)) \cap \partial\Omega) \geqslant C\delta(x)^{-q}|B(x, \delta(x))| \tag{3.24}$$

holds for all $x \in \Omega$ with $\delta(x) < r_0$. Then there exists $C_1 > 0$ such that

$$\widetilde{\mathcal{H}}^{-q}_{r}(B(w, r) \cap (\mathbb{R}^n \setminus \Omega)) \geqslant C_1 r^{-q}|B(w, r)| \tag{3.25}$$

for all $w \in \partial\Omega$ and $0 < r < r_0$.

Proof. Let $w \in \partial\Omega$ and $0 < r < r_0$. If

$$|B(w, \tfrac{r}{2}) \cap (\mathbb{R}^n \setminus \Omega)| \geqslant \tfrac{1}{2}|B(w, \tfrac{r}{2})|,$$

then it is easy to see that (3.25) holds with $C_1 = 2^{-Q}C_0/2$. Thus, we may assume that

$$|B(w, \tfrac{r}{2}) \cap \Omega| \geqslant \tfrac{1}{2}|B(w, \tfrac{r}{2})|,$$

which, by (2.8), gives

$$|B(w, \tfrac{r}{2}) \cap \Omega| \ \geqslant\ 2^{-Q}C_0\ |B(w, r)|/2. \tag{3.26}$$

Now, to prove (3.25), it is enough to show that

$$\widetilde{\mathcal{H}}^{-q}_{r}(B(w, r) \cap \partial\Omega) \geqslant C_1 r^{-q}|B(w, r)|. \tag{3.27}$$

To this end, let $\{B_i\}_{i=1}^{\infty}$, $B_i = B(z_i, r_i)$ with $z_i \in \partial\Omega$ and $0 < r_i \leqslant r$ be a covering of $B(w, r) \cap \partial\Omega$. Then if

$$\sum_i |B_i| \geqslant (2^{-Q}C_0)^2|B(w, r)|/4,$$

it follows that (3.27) holds with $C_1 = \frac{1}{4}(2^{-Q}C_0)^2$. Hence we are left with considering only the case

$$\sum_i |B_i| < (2^{-Q}C_0)^2|B(w,r)|/4. \tag{3.28}$$

Using (2.8), (3.26), and (3.28), we can now estimate

$$\begin{aligned}|(B(w,\tfrac{r}{2}) \cap \Omega) \setminus \bigcup_i 2B_i| &\geqslant |B(w,\tfrac{r}{2}) \cap \Omega| - 2^Q C_0^{-1} \sum_i |B_i| \\ &\geqslant 2^{-Q}C_0|B(w,r)|/2 - 2^{-Q}C_0|B(w,r)|/4 \\ &= 2^{-Q}C_0|B(w,r)|/4.\end{aligned}$$

Thus, by a covering lemma (see [52, p. 9]), we can find a sequence of pairwise disjoint balls $B(x_k, 6\delta(x_k))$ with $x_k \in (B(w,\frac{r}{2}) \cap \Omega) \setminus \bigcup\limits_i 2B_i$ such that

$$|B(w,r)| \leqslant C|(B(w,\tfrac{r}{2}) \cap \Omega) \setminus \bigcup_i 2B_i| \leqslant C \sum_k |B(x_k, 30\delta(x_k))|.$$

This, together with (2.8) and (3.24), gives

$$\begin{aligned}|B(w,r)|r^{-q} &\leqslant C \sum_k |B(x_k,\delta(x_k))|\delta(x_k)^{-q} \\ &\leqslant C \sum_k \widetilde{\mathcal{H}}^{-q}_{\delta(x_k)}(\overline{B}(x_k, 2\delta(x_k)) \cap \partial\Omega)\end{aligned} \tag{3.29}$$

since $\delta(x_k) < \frac{r}{2}$ for all k.

We next observe that we can further assume that

$$\delta(x) < \tfrac{r}{4} \text{ for all } x \in B(w,\tfrac{r}{2}) \cap \Omega. \tag{3.30}$$

In fact, if there exits $x \in B(w,\frac{r}{2}) \cap \Omega$ such that $\delta(x) \geqslant \frac{r}{4}$, then there exists $x_0 \in B(w,\frac{r}{2}) \cap \Omega$ such that $\delta(x_0) = \frac{r}{4}$ by the continuity of δ. Thus, $B(x_0, 2\delta(x_0)) \subset B(w,r)$ and, in view of the assumption (3.24), we obtain

$$\begin{aligned}\widetilde{\mathcal{H}}^{-q}_r(B(w,r) \cap \partial\Omega) &\geqslant C\widetilde{\mathcal{H}}^{-q}_{\delta(x_0)}(B(x_0, 2\delta(x_0) \cap \partial\Omega)) \\ &\geqslant C\delta(x_0)^{-q}|B(x_0,\delta(x_0))| \geqslant Cr^{-q}|B(w,r)|\ ,\end{aligned}$$

which gives (3.27). Now, the inequality (3.30), in particular, implies that

$$\overline{B}(x_k, 2\delta(x_k)) \cap \partial\Omega \subset B(w,r) \cap \partial\Omega \subset \bigcup_i B_i,$$

and hence for every k one has

$$\widetilde{\mathcal{H}}^{-q}_{2\delta(x_k)}(\overline{B}(x_k, 2\delta(x_k) \cap \partial\Omega)) \leqslant \sum_{\{i\in\mathbb{N} | B_i \cap \overline{B}(x_k, 2\delta(x_k)) \neq \varnothing\}} |B_i| r_i^{-q}. \tag{3.31}$$

Here, we used the fact that $r_i < 2\delta(x_k)$ since $x_k \notin 2B_i$. From (3.29) and (3.31), after changing the order of summation, we obtain

$$\begin{aligned} |B(w,r)|r^{-q} &\leqslant C \sum_i \sum_{\{k\in\mathbb{N} | B_i \cap \overline{B}(x_k, 2\delta(x_k)) \neq \varnothing\}} |B_i| r_i^{-q} \\ &\leqslant C \sum_i C(i) |B_i| r_i^{-q}, \end{aligned} \tag{3.32}$$

where $C(i)$ is the number of balls $\overline{B}(x_k, 2\delta(x_k))$ that intersect B_i. Note that if $B_i \cap \overline{B}(x_k, 2\delta(x_k)) \neq \varnothing$, then, since $r_i < 2\delta(x_k)$, we see that $B_i \subset B(x_k, 6\delta(x_k))$. Hence $C(i) \leqslant 1$ for all i since, by our choice, the balls $B(x_k, 6\delta(x_k))$ are pairwise disjoint. This and (3.32) give

$$|B(w,r)|r^{-q} \leqslant C \sum_i |B_i| r_i^{-q},$$

and the inequality (3.27) follows as the coverings $\{B_i\}_i$ of $B(w,r) \cap \partial\Omega$ are arbitrary. This completes the proof of the theorem. □

The thickness condition (3.25) that involves the Hausdorff content will now be shown to imply the uniform (X,p)-fatness of $\mathbb{R}^n \setminus \Omega$. To achieve this we borrow an idea from [29].

Theorem 3.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with local parameters C_0 and R_0. Suppose that there exist $r_0 \leqslant R_0/100$, $1 < q < p$, and a constant $C > 0$ such that the inequality*

$$\widetilde{\mathcal{H}}^{-q}_r(B(w,r) \cap (\mathbb{R}^n \setminus \Omega)) \geqslant C r^{-q} |B(w,r)| \tag{3.33}$$

holds for all $w \in \partial\Omega$ and $0 < r < r_0$. Then there exists $C_1 > 0$ such that the $\mathbb{R}^n \setminus \Omega$ is uniformly (X,p)-fat with constants C_1 and r_0.

Proof. Let $z \in \partial\Omega$, and let $0 < r < r_0$. We need to find a constant $C_1 > 0$ independent of z and r such that

$$\operatorname{cap}_p(K, B(z,2r)) \geqslant C_1 r^{-p} |B(z,r)|, \tag{3.34}$$

where $K = (\mathbb{R}^n \setminus \Omega) \cap \overline{B}(z,r)$. From (3.33) we have

$$\widetilde{\mathcal{H}}^{-q}_r(K) \geqslant C r^{-q} |B(z,r)|. \tag{3.35}$$

Let $\varphi \in C_0^\infty(B(z,2r))$ be such that $\varphi \geqslant 1$ on K. If there is $x_0 \in K$ such that

$$|\varphi(x_0) - \varphi_{B(x_0,4r)}| \leqslant 1/2,$$

then

$$1 \leqslant \varphi(x_0) \leqslant |\varphi(x_0) - \varphi_{B(x_0,4r)}| + |\varphi_{B(x_0,4r)}| \leqslant 1/2 + |\varphi_{B(x_0,4r)}|.$$

By Lemma 3.1, the doubling property (2.8), and (2.12), we obtain

$$1/2 \leqslant |\varphi_{B(x_0,4r)}| \leqslant C\Big(r^p |B(z,r)|^{-1} \int\limits_{B(z,2r)} |X\varphi|^p dx\Big)^{\frac{1}{p}},$$

which gives (3.34). Thus, we may assume that

$$1/2 < |\varphi(x) - \varphi_{B(x,4r)}| \quad \text{for all } x \in K.$$

Under such an assumption, using the covering argument in Theorem 5.9 in [29], the inequality (3.34) follows from (3.35) and Theorem 2.1. □

Finally, we summarize in one single theorem the results obtained in Theorems 3.4, 3.6, 3.7, and 3.8.

Theorem 3.9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with local parameters C_0 and R_0 and let $1 < p < \infty$. There exists $0 < r_0 \leqslant R_0/100$ such that the following statements are equivalent:*

(i) *The set $\mathbb{R}^n \setminus \Omega$ is uniformly (X,p)-fat with constants c_0 and r_0 for some $c_0 > 0$, i.e.,*

$$\operatorname{cap}_p((\mathbb{R}^n \setminus \Omega) \cap \overline{B}(w,r), B(w,2r)) \geqslant c_0 r^{-p} |B(w,r)|$$

for all $w \in \partial\Omega$ and $0 < r < r_0$.

(ii) *There exists $1 < q < p$ and a constant $C > 0$ such that*

$$|u(x)| \leqslant C\delta(x)\Big(\mathcal{M}_{4\delta(x)}(|\nabla u|^q)(x)\Big)^{\frac{1}{q}}$$

for all $x \in \Omega$ with $\delta(x) < r_0$ and all compactly supported $u \in C^{0,1}_d(\Omega)$.

(iii) *There exists $1 < q < p$ and a constant $C > 0$ such that*

$$\widetilde{\mathcal{H}}^{-q}_{\delta(x)}(\overline{B}(x, 2\delta(x)) \cap \partial\Omega) \geqslant C\delta(x)^{-q} |B(x,\delta(x))|$$

for all $x \in \Omega$ with $\delta(x) < r_0$.

(iv) *There exists $1 < q < p$ and a constant $C > 0$ such that*

$$\widetilde{\mathcal{H}}^{-q}_{r}(B(w,r) \cap (\mathbb{R}^n \setminus \Omega)) \geqslant Cr^{-q} |B(w,r)|$$

for all $w \in \partial\Omega$ and $0 < r < r_0$.

Remark 3.10. As an example in [37] shows, we cannot replace the set $\mathbb{R}^n \setminus \Omega$ in statement (iv) in Theorem 3.9 with the smaller set $\partial\Omega$.

4 Hardy Inequalities on Bounded Domains

Our first result in this section is the following Hardy inequality which is a consequence of Theorem 3.4 and the L^s boundedness of the Hardy–Littlewood maximal function for $s > 1$. We remark that no assumption on the smallness of the diameter of the domain is required, as opposed to the Poicaré inequality (2.11) and Sobolev inequalities established in [23].

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with local parameters C_0 and R_0. Suppose that $\mathbb{R}^n \setminus \Omega$ is uniformly (X,p)-fat with constants $c_0 > 0$ and $0 < r_0 \leqslant R_0/100$. There is a constant $C > 0$ such that for all $\varphi \in C_0^\infty(\Omega)$*

$$\int_\Omega \frac{|\varphi(x)|^p}{\delta(x)^p}\, dx \;\leqslant\; C \int_\Omega |X\varphi|^p\, dx. \tag{4.1}$$

Proof. Let $\Omega_{r_0} = \{x \in \Omega : \delta(x) \geqslant r_0\}$, and let $\varphi \in C_0^\infty(\Omega)$. By Theorem 3.4, we can find $1 < q < p$ such that

$$\begin{aligned}\int_\Omega |\varphi(x)|^p \delta(x)^{-p} dx &= \int_{\Omega_{r_0}} |\varphi(x)|^p \delta(x)^{-p} dx + \int_{\Omega \setminus \Omega_{r_0}} |\varphi(x)|^p \delta(x)^{-p} dx \\ &\leqslant r_0^{-p} \int_\Omega |\varphi(x)|^p dx + C \int_\Omega \Big(\mathcal{M}_{4r_0}(|X\varphi|^q)(x)\Big)^{\frac{p}{q}} dx \\ &\leqslant C \int_\Omega |X\varphi(x)|^p dx.\end{aligned}$$

In the last inequality above, we used the Poincaré inequality (2.10) and the boundedness property of $\mathcal{M}_{4r_0}$ on $L^s(\Omega)$, $s > 1$ (see [52]). The proof of Theorem 4.1 is then complete. □

To state Theorems 4.3 and 4.5 below, we need to fix a Whitney decomposition of Ω into balls as in the following lemma, whose construction can be found, for example, in [33] or [18].

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with local parameters C_0 and R_0. There exists a family of balls $\mathcal{W} = \{B_j\}$ with $B_j = B(x_j, r_j)$ and a constant $M > 0$ such that*

(a) $\Omega \subset \cup_j B_j$,

(b) $B(x_j, \frac{r_j}{4}) \cap B(x_k, \frac{r_k}{4}) \neq \varnothing \quad \text{for } j \neq k$,

(c) $r_j = 10^{-3} \min\{R_0/\mathrm{diam}(\Omega), 1\}\mathrm{dist}(B_j, \partial\Omega)$,

(d) $\sum_j \chi_{4B_j}(x) \leqslant M\chi_\Omega(x)$.

In (c),

$$\mathrm{diam}(\Omega) = \sup_{x,y\in\Omega} d(x,y)$$

is the diameter of Ω with respect to the CC *metric. In particular, we have $r_j \leqslant 10^{-3}R_0$.*

We can now go further in characterizing weight functions V on Ω for which the embedding

$$\int_\Omega |\varphi(x)|^p\, V(x)dx \leqslant C \int_\Omega |X\varphi|^p dx$$

holds for all $\varphi \in C_0^\infty(\Omega)$. Here, the condition on V is formulated in terms of a localized capacitary condition adapted to a Whitney decomposition of Ω. Such a condition can be simplified further in the setting of Carnot groups as we point out in Remark 4.4 below. In the Euclidean setting, it was used in [26] to characterize the solvability of multi-dimensional Riccati equations on bounded domains.

Theorem 4.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with local parameters C_0 and R_0. Let $V \geqslant 0$ be in $L^1_{\mathrm{loc}}(\Omega)$. Suppose that $\mathbb{R}^n \setminus \Omega$ is uniformly (X,p)-fat with $1 < p < Q$. Then the embedding*

$$\int_\Omega |\varphi(x)|^p\, V(x)dx \leqslant C \int_\Omega |X\varphi|^p dx, \qquad \varphi \in C_0^\infty(\Omega), \tag{4.2}$$

holds if and only if

$$\sup_{B\in\mathcal{W}} \sup_{\substack{K\subset 2B \\ K \text{ compact}}} \frac{\int_K V(x)dx}{\bigcap limits_p(K,\Omega)} \leqslant C, \tag{4.3}$$

where $\mathcal{W} = \{B_j\}$ is a Whitney decomposition of Ω as in Lemma 4.2.

Remark 4.4. In the setting of a Carnot group $\mathbf{G}$ with homogeneous dimension Q, we can replace $\mathrm{cap}_p(K,\Omega)$ by $\mathrm{cap}_p(K,\mathbf{G})$ in (4.3) since, if $B \in \mathcal{W}$ and K is a compact set in $2B$, we have

$$c\ \mathrm{cap}_p(K,\Omega) \ \leqslant \mathrm{cap}_p(K,\mathbf{G}) \ \leqslant\ \mathrm{cap}_p(K,\Omega). \tag{4.4}$$

The second inequality in (4.4) is obvious. To see the first one, let $\varphi \in C_0^\infty(\mathbf{G})$, $\varphi \geqslant 1$ on K, and choose a cut-off function $\eta \in C_0^\infty(4B)$ such that $0 \leqslant \eta \leqslant 1$, $\eta \equiv 1$ on $2B$ and $|X\eta| \leqslant \frac{C}{r_B}$, where r_B is the radius of B. Since $\varphi\eta \in C_0^\infty(\Omega)$, $\varphi\eta \geqslant 1$ on K, we have

$$\begin{aligned}\operatorname{cap}_p(K,\Omega) &\leqslant \int\limits_{\Omega} |X(\varphi\eta)|^p dg \\ &\leqslant \int\limits_{\mathbf{G}} |X\varphi|^p dg + C \int\limits_{4B\setminus 2B} \frac{|\varphi|^p}{r_B^p} dg \\ &\leqslant \int\limits_{\mathbf{G}} |X\varphi|^p dg + C \int\limits_{\mathbf{G}} \frac{|\varphi|^p}{\rho(g,g_0)^p} dg,\end{aligned}$$

where g_0 is the center of B, and we denoted by $\rho(g,g_0)$ the pseudo-distance induced on $\mathbf{G}$ by the anisotropic Folland–Stein gauge (see [18, 20]). To bound the third integral on the right-hand side of the latter inequality, we use the following Hardy type inequality:

$$\int\limits_{\mathbf{G}} \frac{\varphi^p}{\rho(g,g_0)^p}\, dg \;\leqslant\; C \int\limits_{\mathbf{G}} |X\varphi|^p\, dg, \qquad \varphi \in C_0^\infty(\mathbf{G}), \tag{4.5}$$

which is easily proved as follows. Recall the Folland-Stein Sobolev embedding (see [20])

$$\left(\int\limits_{\mathbf{G}} |\varphi|^{\frac{pQ}{Q-p}}\, dg\right)^{\frac{Q-p}{pQ}} \leqslant S_p \left(\int\limits_{\mathbf{G}} |X\varphi|^p\, dg\right)^{\frac{1}{p}}, \quad \varphi \in C_0^\infty(\mathbf{G}). \tag{4.6}$$

Observing that for every $g_0 \in \mathbf{G}$ one has $g \to \dfrac{1}{\rho(g,g_0)^p} \in L^{Q/p,\infty}(\mathbf{G})$, from the generalized Hölder inequality for weak L^p spaces due to Hunt [32] one obtains with an absolute constant $B > 0$

$$\begin{aligned}\int\limits_{\mathbf{G}} \frac{\varphi^p}{\rho(g,g_0)^p}\, dg \;&\leqslant\; B \left(\int\limits_{\mathbf{G}} |\varphi|^{\frac{pQ}{Q-p}}\, dg\right)^{\frac{Q-p}{Q}} ||\rho(\cdot,g_0)^{-p}||_{L^{Q/p,\infty}(\mathbf{G})} \\ &\leqslant\; C \int\limits_{\mathbf{G}} |X\varphi|^p\, dg,\end{aligned}$$

where in the last inequality we used (4.6). This proves (4.5). In conclusion, we find

$$\operatorname{cap}_p(K,\Omega) \leqslant C \int\limits_{\mathbf{G}} |X\varphi|^p dg,$$

which gives the first inequality in (4.4).

Proof of Theorem 4.3. That the emdedding (4.2) implies the capacitary condition (4.3) is clear. To prove the converse, let $\{\varphi_j\}$ be a Lipschitz partition of unity associated with the Whitney decomposition $\mathcal{W} = \{B_j\}$ (see [24]), i.e., $0 \leqslant \varphi_j \leqslant 1$ is Lipschitz with respect to the CC metric, supp $\varphi_j \Subset 2B_j$, $|X\varphi_j| \leqslant C/\mathrm{diam}(B_j)$, and

$$\sum_j \varphi_j(x) = \chi_\Omega(x).$$

Moreover, by property (d) in Lemma 4.2, there is a constant $C(p)$ such that

$$\left(\sum_j \varphi_j(x)\right)^p = C(p) \sum_j \varphi_j(x)^p.$$

Then for any $\varphi \in C_0^\infty(\Omega)$ we have

$$\begin{aligned}\int_\Omega |\varphi(x)|^p\, V(x)dx &\leqslant C \sum_j \int_\Omega |\varphi_j \varphi(x)|^p\, V(x)dx\\ &\leqslant C \sum_j \int_{4B_j} |X(\varphi_j\varphi)|^p dx\end{aligned}$$

by (4.3) and [13, Theorem 5.3]. Thus, from Theorem 4.1 and Lemma 4.2 we obtain

$$\begin{aligned}&\int_\Omega |\varphi(x)|^p\, V(x)dx\\ &\leqslant C \sum_j \int_{4B_j} |X\varphi|^p dx + C\sum_j [\mathrm{diam}(B_j)]^{-p} \int_{4B_j} |\varphi|^p dx\\ &\leqslant C \int_\Omega |X\varphi|^p dx + C\int_\Omega |\varphi|^p \delta^{-p}(x)dx\\ &\leqslant C \int_\Omega |X\varphi|^p dx.\end{aligned}$$

This completes the proof of the theorem. □

In view of [13, Theorem 1.6], the above proof also gives the following Fefferman–Phong type sufficiency result [17].

Theorem 4.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with local parameters C_0 and R_0. Let $V \geqslant 0$ be in $L^1_{\mathrm{loc}}(\Omega)$. Suppose that $\mathbb{R}^n \setminus \Omega$ is uniformly (X,p)-fat* *with $1 < p < Q$. Then the embedding*

$$\int_{\Omega} |\varphi(x)|^p V(x) dx \leqslant C \int_{\Omega} |X\varphi|^p dx, \qquad \varphi \in C_0^\infty(\Omega), \tag{4.7}$$

holds if for some $s > 1$, V satisfies the following localized Fefferman–Phong type condition adapted to Ω:

$$\sup_{B \in \mathcal{W}} \sup_{\substack{x \in 2B \\ 0<r<\operatorname{diam}(B)}} \int_{B(x,r)} V(y)^s dy \leqslant C \frac{|B(x,r)|}{r^{sp}} \tag{4.8}$$

where $\mathcal{W} = \{B_j\}$ is a Whitney decomposition of Ω as in Lemma 4.2.

Let $L^{s,\infty}(\Omega)$, $0 < s < \infty$, denote the weak L^s space on Ω, i.e.,

$$L^{s,\infty}(\Omega) = \left\{ f : \|f\|_{L^{s,\infty}(\Omega)} < \infty \right\},$$

where

$$\|f\|_{L^{s,\infty}(\Omega)} = \sup_{t>0} t \, |\{x \in \Omega : |f(x)| > t\}|^{\frac{1}{s}}.$$

Equivalently, one can take

$$\|f\|_{L^{s,\infty}(\Omega)} = \sup_{E \subset \Omega\colon |E|>0} |E|^{\frac{1}{s}-\frac{1}{r}} \left(\int_E |f|^r dx \right)^{\frac{1}{r}}$$

for any $0 < r < s$. For $s = \infty$ we define

$$L^{\infty,\infty}(\Omega) = L^\infty(\Omega).$$

From Theorem 4.8 we obtain the following corollary, which improves a similar result in [16, Remark 3.7] in the sense that not only does it cover the subelliptic case, but also require a milder assumption on the boundary.

Corollary 4.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with local parameters C_0 and R_0. Suppose that $\mathbb{R}^n \setminus \Omega$ is uniformly (X,p)-fat for $1 < p < Q$, where Q is the homogeneous dimension of Ω. If $v \in L^{\frac{Q}{\gamma},\infty}(\Omega)$ for some $0 \leqslant \gamma \leqslant p$, then the embedding (4.7) holds for the weight $V(x) = \delta(x)^{-p+\gamma} v(x)$.*

Proof. Let $\mathcal{W} = \{B_j\}$ is a Whitney decompositon of Ω as in Lemma 4.2. For $x \in 2B$, $B \in \mathcal{W}$, $0 < r < \operatorname{diam}(B)$, and $1 < s < \frac{Q}{\gamma}$ we have

$$\int_{B(x,r)} V(y)^s dy \leqslant C r^{-sp+s\gamma} \int_{B(x,r)} v(y)^s dy.$$

It is then easily seen from the Hölder inequality and the doubling property (2.8) that

$$\begin{aligned}\int\limits_{B(x,r)} V(y)^s dy &\leqslant Cr^{-sp}|B(x,r)|\,\|v\|^s_{L^{\frac{Q}{\gamma},\infty}(\Omega)}\left(\frac{r}{|B(x,r)|^{\frac{1}{Q}}}\right)^{s\gamma}\\ &\leqslant Cr^{-sp}|B(x,r)|\,\|v\|_{L^{\frac{Q}{\gamma},\infty}(\Omega)}.\end{aligned}$$

By Theorem 4.5, we obtain the corollary. □

The results obtained in Corollary 4.6 do not in general cover the case in which $v(x)$ has a point singularity in Ω, such as $V(x) = \delta(x)^{-p+\gamma}d(x,x_0)^{-\gamma}$, with $0 \leqslant \gamma \leqslant p$ and $1 < p < Q(x_0)$ for some $x_0 \in \Omega$, where $Q(x_0)$ is the homogeneous dimension at x_0. The reason is that it may happen that $Q(x_0) < Q$ and hence $d(\cdot,x_0)^{-\gamma} \notin L^{\frac{Q}{\gamma},\infty}(\Omega)$. However, by the upper estimate in (2.5), we still can obtain the inequality (4.7) for such weights as follows.

Corollary 4.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with local parameters C_0 and R_0. Given $x_0 \in \Omega$, suppose that $\mathbb{R}^n \setminus \Omega$ is uniformly (X,p)-fat for $1 < p < Q(x_0)$. Then for any $0 \leqslant \gamma \leqslant p$ the embedding (4.7) holds for the weight*

$$V(x) = \delta(x)^{-p+\gamma}d(x,x_0)^{-\gamma}.$$

Proof. Let $\mathcal{W} = \{B_j\}$ be a Whitney decomposition of Ω as in Lemma 4.2. For $x \in 2B$, $B \in \mathcal{W}$, $0 < r < \operatorname{diam}(B)$, and $1 < s < \frac{Q(x_0)}{\gamma}$ we have

$$\int\limits_{B(x,r)} V(y)^s dy \leqslant C\ r^{-sp+s\gamma} \int\limits_{B(x,r)} d(y,x_0)^{-\gamma s} dy. \tag{4.9}$$

Thus, if $x \notin B(x_0,2r)$, then

$$\int\limits_{B(x,r)} V(y)^s dy \leqslant C\ \frac{|B(x,r)|}{r^{sp}}$$

since for such x we have $d(y,x_0) \geqslant r$ for every $y \in B(x,r)$. On the other hand, if $x \in B(x_0,2r)$ then from (4.9) we find

$$\begin{aligned}\int\limits_{B(x,r)} V(y)^s dy &\leqslant C\ r^{s\gamma-sp} \int\limits_{B(x_0,3r)} d(y,x_0)^{-\gamma s} dy\\ &= C\ r^{s\gamma-sp}\sum_{k=0}^{\infty}\ \int\limits_{\frac{3r}{2^{k+1}}\leqslant d(y,x_0)<\frac{3r}{2^k}} d(y,x_0)^{-\gamma s} dy\end{aligned}$$

$$\leqslant C\ r^{s\gamma - sp} \sum_{k=0}^{\infty} \left(\frac{r}{2^k}\right)^{-\gamma s} \left|B(x_0, \tfrac{3r}{2^k})\right| .$$

Thus, in view of (2.5) and the doubling property (2.8), we obtain

$$\begin{aligned}\int\limits_{B(x,r)} V(y)^s dy &\leqslant C\ r^{s\gamma - sp} \sum_{k=0}^{\infty} \left(\frac{r}{2^k}\right)^{-\gamma s} \left(\frac{1}{2^k}\right)^{Q(x_0)} |B(x_0, 3r)| \\ &\leqslant C\ \frac{|B(x_0, 3r)|}{r^{sp}} \sum_{k=0}^{\infty} \left(\frac{1}{2^k}\right)^{Q(x_0) - \gamma s} \\ &\leqslant C(x_0)\ \frac{|B(x, r)|}{r^{sp}}.\end{aligned}$$

Thus, by Theorem 4.5, we obtain the corollary. □

Remark 4.8. If we have $\gamma = p$ in Corollary 4.7, then we do not need to assume $\mathbb{R}^n \setminus \Omega$ to be uniformly (X, p)-fat. In fact, to obtain the embedding (4.7) in this case, we use [13, Theorem 1.6], the Poincaré inequality (2.10), and a finite partition of unity for Ω.

5 Hardy Inequalities with Sharp Constants

In this section, we collect, without proofs, for illustrative purposes some theorems from the forthcoming article [15]. The relevant results pertain certain Hardy–Sobolev inequalities on bounded and unbounded domains with a point singularity which are included in them.

We begin by recalling that when $X = \{X_1, \dots, X_m\}$ constitutes an orthonormal basis of bracket generating vector fields in a Carnot group $\mathbf{G}$, then a fundamental solution Γ_p for $-\mathcal{L}_p$ in all of $\mathbf{G}$ was constructed in [14]. For any bounded open set $\Omega \subset \mathbb{R}^n$ one can construct a positive fundamental solution with generalized zero boundary values, i.e., a Green function, in the more general situation of a Carnot–Carathéodory space. Henceforth, for a fixed $x \in \Omega$ we denote by $\Gamma_p(x, \cdot)$ such a fundamental solution with singularity at some fixed $x \in \Omega$. This means that $\Gamma_p(x, \cdot)$ satisfies the equation

$$\int\limits_{\Omega} |X\Gamma_p(x, y)|^{p-2} < X\Gamma_p(x, y), X\varphi(y) > dy \ = \ \varphi(x) \tag{5.1}$$

for every $\varphi \in C_0^\infty(\Omega)$.

We recall the following fundamental estimate, which is Theorem 7.2 in [5]. Let $K \subset \Omega \subset \mathbb{R}^n$ be a compact set with local parameters C_0 and R_0. Given $x \in K$, and $1 < p < Q(x)$, there exists a positive constant C depending on

C_0 and p such that for any $0 < r \leqslant R_0/2$, and $y \in B(x,r)$ one has

$$C \left(\frac{d(x,y)^p}{\Lambda(x,d(x,y))}\right)^{\frac{1}{p-1}} \leqslant \Gamma_p(x,y) \leqslant C^{-1} \left(\frac{d(x,y)^p}{\Lambda(x,d(x,y))}\right)^{\frac{1}{p-1}}. \tag{5.2}$$

The estimate (5.2) generalizes that obtained by Nagel, Stein, and Wainger [44] and independently by Sanchez-Calle [48] in the case $p = 2$.

For any given $x \in K$ we fix a number $p = p(x)$ such that $1 < p < Q(x)$ and introduce the function

$$E(x,r) \stackrel{def}{=} \left(\frac{\Lambda(x,r)}{r^p}\right)^{\frac{1}{p-1}}. \tag{5.3}$$

Because of the constraint imposed on $p = p(x)$, we see that for every fixed $x \in K$ the function $r \to E(x,r)$ is strictly increasing, and thereby invertible. We denote by $F(x,\cdot) = E(x,\cdot)^{-1}$, the inverse function of $E(x,\cdot)$, so that

$$F(x,E(x,r)) = E(x,F(x,r)) = r.$$

We now define for every $x \in K$

$$\rho_x(y) = F\left(x, \frac{1}{\Gamma(x,y)}\right). \tag{5.4}$$

We emphasize that, in a Carnot group $\mathbf{G}$, one has for every $x \in \mathbf{G}$, $Q(x) \equiv Q$ the homogeneous dimension of the group, and therefore the Nagel–Stein–Wainger polynomial is, in fact, just a monomial, i.e., $\Lambda(x,r) \equiv C(\mathbf{G})r^Q$. It follows that there exists a constant $\omega(\mathbf{G}) > 0$ such that

$$E(x,r) \equiv \omega(\mathbf{G})\, r^{(Q-p)/(p-1)}. \tag{5.5}$$

Using the function $E(x,r)$ in (5.3), it should be clear that we can recast the estimate (5.2) in the following more suggestive form:

$$\frac{C}{E(x,d(x,y))} \leqslant \Gamma_p(x,y) \leqslant \frac{C^{-1}}{E(x,d(x,y))}. \tag{5.6}$$

As a consequence of (5.6) and (5.4), we obtain the following estimate: *there exist positive constants C and R_0 depending on $X_1,\dots,X_m$ and K such that for every $x \in K$ and every $0 < r \leqslant R_0$ one has for $y \in B(x,r)$*

$$C\, d(x,y) \leqslant \rho_x(y) \leqslant C^{-1}\, d(x,y). \tag{5.7}$$

We can thus think of the function ρ_x as a *regularized pseudo-distance* adapted to the nonlinear operator $\mathcal{L}_p$. We denote by

$$B_X(x,r) = \{y \in \mathbb{R}^n \mid \rho_x(y) < r\},$$

the ball centered at x with radius r with respect to the pseudo-distance ρ_x. Because of (5.7), it is clear that

$$B(x, Cr) \subset B_X(x, r) \subset B(x, C^{-1}r).$$

Our main assumption is that for any $p > 1$ the fundamental solution of the operator $\mathcal{L}_p$ satisfy the following

Hypothesis. *For any compact set $K \subset \Omega \subset \mathbb{R}^n$ there exist $C > 0$ and $R_0 > 0$ depending on K and $X_1, \dots, X_m$ such that for every $x \in \Omega$, $0 < R < R_0$ for which $B_X(x, 4R) \subset \Omega$, and a.e. $y \in B(x, R) \setminus \{x\}$ one has*

$$|X\Gamma_p(x, y)| \leqslant C^{-1} \left(\frac{d(x, y)}{\Lambda(x, d(x, y))}\right)^{\frac{1}{p-1}}. \tag{5.8}$$

We mention explicitly that, as a consequence of the results in [44] and [48], the assumption (5.8) is fulfilled when $p = 2$. For $p \neq 2$ it is also satisfied in any Carnot group of Heisenberg type $\mathbf{G}$. This follows from the results in [5], where for every $1 < p < \infty$ the following explicit fundamental solution of $-\mathcal{L}_p$ was found:

$$-\Gamma_p(g) = \begin{cases} \frac{p-1}{Q-p} \sigma_p^{-\frac{1}{p-1}} N(g)^{-\frac{Q-p}{p-1}}, & p \neq Q, \\ \\ \sigma_Q^{-\frac{1}{Q-1}} \log N(g), & p = Q, \end{cases} \tag{5.9}$$

where we denoted by $N(g) = (|x(g)|^4 + 16|y(g)|^2)^{\frac{1}{4}}$ the Kaplan gauge on $\mathbf{G}$ (see [34]), and we set $\sigma_p = Q\omega_p$ with

$$\omega_p = \int\limits_{\{g \in \mathbf{G} | N(g) < 1\}} |XN(g)|^p \, dg.$$

We note that the case $p = 2$ of (5.9) was first discovered by Folland [19] for the Heisenberg group and subsequently generalized by Kaplan [34] to groups of Heisenberg type. The conformal case $p = Q$ was also found in [28].

We stress that the hypothesis (5.8) is not the weakest one that could be made, and that to the expenses of additional technicalities, we could have chosen substantially weaker hypothesis.

We now recall the classical one-dimensional Hardy inequality [27]: let $1 < p < \infty$, $u(t) \geqslant 0$, and $\varphi(t) = \int\limits_0^t u(s)ds$. Then

$$\int\limits_0^\infty \left(\frac{\varphi(t)}{t}\right)^p dt \leqslant \left(\frac{p}{p-1}\right)^p \int\limits_0^\infty \varphi'(t)^p dt.$$

Here is our main result.

Theorem 5.1. *Given a compact set $K \subset \Omega \subset \mathbb{R}^n$, let $x \in K$ and $1 < p < Q(x)$. For any $0 < R < R_0$ such that $B_X(x, 4R) \subset \Omega$ one has for $\varphi \in S_0^{1,p}(B_X(x, R))$*

$$\int_{B_X(x,R)} |\varphi|^p \left\{ \frac{E'(x,\rho_x)}{E(x,\rho_x)} \right\}^p |X\rho_x|^p \, dy \ \leqslant\ \left(\frac{p}{p-1} \right)^p \int_{B_X(x,R)} |X\varphi|^p \, dy.$$

When $\Lambda(x, r)$ is a monomial (thus, for example, in the case of a Carnot group) the constant on the right-hand side of the above inequality is best possible.

We do not present here the proof of Theorem 5.1, but refer the reader to the forthcoming article [15]. Some comments are in order. First of all, concerning the factor $|X\rho_x|^p$ on the left-hand side of the inequality in Theorem 5.1, we emphasize that the hypothesis (5.8) implies that $X\rho_x \in L^\infty_{\mathrm{loc}}$. Secondly, as is shown in [15], one has

$$\left(\frac{Q(x)-p}{p-1} \right)^p \frac{1}{\rho_x^p} \ \leqslant\ \left\{ \frac{E'(x,\rho_x)}{E(x,\rho_x)} \right\}^p \ \leqslant\ \left(\frac{Q-p}{p-1} \right)^p \frac{1}{\rho_x^p}. \tag{5.10}$$

As a consequence of Theorem 5.1 and (5.10) we thus obtain the following

Corollary 5.2. *Under the same assumptions of Theorem 5.1, one has for $\varphi \in S_0^{1,p}(B_X(x, R))$*

$$\int_{B_X(x,R)} \frac{|\varphi|^p}{\rho_x^p} |X\rho_x|^p \, dy \ \leqslant\ \left(\frac{p}{Q(x)-p} \right)^p \int_{B_X(x,R)} |X\varphi|^p \, dy. \tag{5.11}$$

Thirdly, it is worth observing that, with the optimal constants, neither Theorem 5.1 nor Corollary 5.2 can be obtained from Corollary 4.7.

We mention in closing that for the Heisenberg group $\mathbf{H}^n$ with $p = 2$ Corollary 5.2 was first proved in [22]. The inequality (5.11) was extended to the nonlinear case $p \neq 2$ in [45]. For Carnot groups of Heisenberg type and also for some operators of Baouendi–Grushin type the inequality (5.11) was obtained in [11]. In the case $p = 2$, various weighted Hardy inequalities with optimal constants in groups of Heisenberg type were also independently established in [36]. An interesting generalization of the results in [45], along with an extension to nilpotent Lie groups with polynomial growth, was recently obtained in [39]. In this latter setting, an interesting form of the uncertainty principle connected to the case $p = 2$ of the Hardy type inequality (5.11) was established in [10]. These latter two references are not concerned however with the problem of finding the sharp constants.

Acknowledgement. The first author was supported in part by NSF CAREER Grant, DMS-0239771. The second author was supported in part by NSF Grant DMS-0701001.

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