Potential Analysis **11:** 387–413, 1999.

A Fefferman–Phong Type Inequality and Applications to Quasilinear Subelliptic Equations

DONATELLA DANIELLI*
Department of Mathematics, Purdue University, West Lafayette, IN 47907.
(e-mail: dxd@math.purdue.edu)

(Received: 31 July 1997; accepted: 11 June 1998)

Abstract. We establish a nonlocal generalization of a well-known inequality by C. Fefferman and D. H. Phong

$$\int_{\mathbb{R}^n} Vu^2\,\mathrm{d}x \leqslant C\int_{\mathbb{R}^n} |Du|^2\,\mathrm{d}x,$$

for $u \in C_0^\infty(\mathbb{R}^n)$ and V belonging to the Morrey space $M_n^{s,2s}$ with $1 < s \leqslant n/2$, when the gradient in the right-hand side is replaced by the energy associated to an arbitrary system of Lipschitz continuous vector fields. Accordingly, the multiplier V is taken in an appropriate Morrey space defined using the Carnot–Carathéodory metric generated by the vector fields.

As an application, we prove the Harnack inequality and the Hölder continuity of solutions for a wide class of second order quasilinear subelliptic equations.

Mathematics Subject Classification: 35H05.

Key words: Fefferman–Phong inequality, Morrey spaces, subelliptic equations, Harnack inequality.

1. Introduction

A theorem of C. Fefferman and D. H. Phong [F] states that a sufficient condition for a measurable $V \geqslant 0$ to satisfy the embedding

$$\int_{\mathbb{R}^n} Vu^2\,\mathrm{d}x \leqslant C\int_{\mathbb{R}^n} |Du|^2\,\mathrm{d}x, \quad u \in C_0^\infty(\mathbb{R}^n), \tag{1.1}$$

is membership of V in the Morrey space $M_n^{s,2s}$, where $1 < s \leqslant n/2$. Here, for $1 \leqslant s < \infty$ and $\lambda > 0$, the Morrey space $M_n^{s,\lambda}$ is defined as the collection of all V's in $L^s_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\sup_{\substack{x\in\mathbb{R}^n\\ r>0}} \left(\frac{1}{r^{n-\lambda}}\int_{|x-y|<r} V^s\,\mathrm{d}y\right)^{1/s} < \infty. \tag{1.2}$$

* Supported in part by a grant of the Purdue Research Foundation, and subsequently by a C.N.R. Fellowship.

We observe that when $s = n/2$, then $M_n^{s,2s} = L^{n/2}(\mathbb{R}^n)$, and (1.1) simply follows from the Sobolev embedding, after an application of Hölder inequality. We also note that $M_n^{s,2s} \hookrightarrow M_n^{1,2}$, so in general (1.1) would follow from an analogous result for the class $M_n^{1,2}$. The latter, however, is false. This can be seen by considering the Radon measure $\mathrm{d}\mu = V\,\mathrm{d}x$, for $V \in M_n^{1,2}$. Should (1.1) be true, such measure would define an element of the dual space $W^{1,2}(\mathbb{R}^n)^*$. But this would contradict the necessary and sufficient condition due to Hedberg and Wolff [HW], see also [Z, Thm 4.7.5]. It is important to note that, if the Schrödinger operator $H = -\Delta - V$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$, then its positivity is equivalent to (1.1) (via integration by parts). Furthermore, Fefferman and Phong have shown [F] that (1.1) is useful to obtain eigenvalue estimates for H. The proof of (1.1) in [F] rested on a variant of Lusin area integral and on related maximal function inequalities.

In this paper we establish a nonlocal generalization of Fefferman and Phong's inequality (1.1) when the standard gradient in the right-hand side is replaced by the energy associated to an arbitrary system of Lipschitz vector fields. Accordingly, the multiplier V will be taken in an appropriate Morrey space defined using the Carnot–Carathéodory metric associated to the vector fields. Our main point is to show that, even in such generality, Fefferman and Phong's embedding can be obtained from two basic facts:

(1) The size estimates of the metric balls;

(2) A weak Poincaré type inequality.

The interest of such general setting arises from the fact that it includes, for example, the case of C^∞ systems of Hörmander type and the general subelliptic operators studied in [OR], [FSC], since by the results in [PS] the factorization matrix of a smooth positive semi-definite matrix has in general at most Lipschitz continuous entries. It is important to stress the nonlocal nature of our result. By this we mean that the relevant functional class does not involve functions with compact support, as it is for (1.1). Moreover, our estimates hold for a class of domains which is essentially as large as possible, see the discussion below. As a consequence, some of the statements are new even in the standard Euclidean context.

We also would like to mention that the presence of a differentiable structure is not really crucial for us. Therefore, appropriately reformulated, the main theorems go over to the interesting purely metric context studied in [Ha].

In order to state our results we introduce the relevant assumptions. We consider in $\mathbb{R}^n$ a system $X = \{X_1, \ldots, X_m\}$ of vector fields having real valued, locally Lipschitz coefficients. We write

$$X_j = \sum_{k=1}^{n} b_{jk} \frac{\partial}{\partial x_k}, \quad j = 1, \ldots, m,$$

and denote by $X_j^* = -\sum_{k=1}^n (\partial/\partial x_k)(b_{jk}\cdot)$ the formal adjoint of X_j. For a given function u, we let $Xu = (X_1u, \dots, X_mu)$, and set $|Xu| = (\sum_{j=1}^m (X_ju)^2)^{1/2}$. For $1 \leqslant p < \infty$ and an open set $\Omega \subset \mathbb{R}^n$ we let

$$\mathcal{L}^{1,p}(\Omega) = \{u \in L^p(\Omega) \mid X_ju \in L^p(\Omega),\ j = 1, \dots, m\}.$$

Here the derivative along the vector fields X_ju, $j = 1, \dots, m$, are understood in the distributional sense. Endowed with the norm

$$\|u\|_{\mathcal{L}^{1,p}(\Omega)} = \left(\int_\Omega (|u|^p + |Xu|^p)\, \mathrm{d}x\right)^{1/p},$$

$\mathcal{L}^{1,p}(\Omega)$ becomes a Banach space. We also define the space $\overset{\circ}{S}{}^{1,p}(\Omega)$ as the completion of the set

$$\left\{u \in \mathrm{Lip}_0(\Omega) \mid \|u\|_{\mathcal{L}^{1,p}(\Omega)} < \infty\right\},$$

with respect to the norm $\|\cdot\|_{\mathcal{L}^{1,p}(\Omega)}$.

A piecewise C^1 curve $\gamma\colon [0, T] \to \mathbb{R}^n$ is called *sub-unitary* (with respect to the system X) if, whenever it exists, $\gamma'(t)$ is sub-unit according to [FP]. Given $x, y \in \mathbb{R}^n$, denote $S(x, y)$ the collection of all sub-unitary curves connecting x to y. The sub-unitary length of γ is defined to be $l_s(\gamma) = T$. Throughout the paper *we assume* $S(x, y) \neq \emptyset$. Then

$$d(x, y) = \inf\{l_s(\gamma) \mid \gamma \in S(x, y)\}$$

defines a distance, usually called the *control*, or *Carnot–Carathéodory, distance* generated by X. We will denote $B = B(x, R) = \{y \in \mathbb{R}^n \mid d(x, y) < R\}$ the metric ball and, whenever convenient, simply write aB for $B(x, aR)$. The Euclidean distance will be denoted $d_e(x, y) = |x - y|$, and the relative metric balls with $B_e(x, R)$. We shall also suppose that *metric balls are open in the Euclidean topology of* $\mathbb{R}^n$. A very useful consequence of such assumption is that, in fact, the metric topology and the Euclidean topology are compatible, so that, in particular, compact sets in either topology coincide. We stress that, even when the system X is C^∞, it is not always true, in general, that the inclusion $i\colon (\mathbb{R}^n, d_e) \to (\mathbb{R}^n, d)$ be continuous.

We now introduce the relevant quantitative assumptions.

HYPOTHESIS. For any (Euclidean) bounded set $U \subset \mathbb{R}^n$ there exist numbers C_1, C_2, $R_0 > 0$ and $a \geqslant 1$ such that for $x \in U$ and $0 < R < R_0$ one has

$$|B(x, 2R)| \leqslant C_1 |B(x, R)|, \tag{1.3}$$

$$\sup_{\lambda>0}\left[\lambda|\{y\in B \mid |u(y)-u_B|>\lambda\}|\right]\leqslant C_2 R\int_{aB}|Xu|\,\mathrm{d}y$$
$$\text{for every } u\in C^1(aB). \tag{1.4}$$

In (1.4) we have let $u_B = 1/|B|\int_B u\,\mathrm{d}y$. Given a set $U\subset\mathbb{R}^n$ as in the Hypothesis, with relative parameters C_1, R_0, we let

$$Q=\log_2 C_1,$$

and call this number the *local homogeneous dimension* of U. The number Q plays in our results the same role played by the Euclidean dimension in (1.1). In fact, when $X=\{\partial/\partial x_1,\dots,\partial/\partial x_n\}$ is the standard basis in $\mathbb{R}^n$, then (1.3) holds (with equality) with $C_1=2^n$, and therefore $Q=n$. In this case (1.3) is fulfilled with $R_0=\infty$, whereas (1.4) trivially follows from the Poincaré inequality.

Other more important examples are as follows:

(a) Let G be a stratified nilpotent Lie group of step r and let $g=V^1\oplus\dots\oplus V^r$ be a stratification of its Lie algebra. If $X=\{X_1,\dots,X_m\}$ is a basis of V^1, then (1.3) holds with $C_1=2^Q$, Q being the homogeneous dimension of G (see [FoS]).

(b) More in general, if X is a system of C^∞ vector fields satisfying Hörmander's finite rank condition [H]: rank Lie $[X_1,\dots,X_m]\equiv n$, then (1.3) follows from the fundamental work of Nagel, Stein, and Wainger [NSW].

(c) An interesting family of nonsmooth vector fields for which our assumptions hold is provided by the Baouendi–Grushin systems modelled on

$$X_j=\frac{\partial}{\partial x_j},\quad j=1,\dots,k,$$
$$X_{k+j}=|x|^\alpha\frac{\partial}{\partial y_j},\quad j=1,\dots,n-k, \tag{1.5}$$

where we have written a generic point in $\mathbb{R}^n$ as (x,y), with $x\in\mathbb{R}^k$, $y\in\mathbb{R}^{n-k}$. These vector fields, and the associated second order pde's, were introduced in [B] and subsequently studied in [G]. Later on, Franchi and Lanconelli [FL1], [FL2], [FL3] developed the relevant De Giorgi–Moser theory for a class of vector fields modelled on (1.5), see also the subsequent works [FSe], [Fr], [FGW]. In particular, (1.3) follows from the results in [FL1]. So far, we have not discussed (1.4). A proof of the latter (in a version which involves the strong L^1-norm of the function u in the left-hand side) that covers the cases (a) and (b) simultaneously is due to D. Jerison [J]. In the case (c), on the other hand, the condition (1.4) has been proved in [FL2].

Before stating our main result we need to introduce the relevant Morrey spaces. Let Ω be a bounded open set. Given $1 \leqslant p < \infty$, a function $u \in L^p_{\mathrm{loc}}(\Omega)$ is said to belong to the Morrey space $M_X^{p,\lambda}(\Omega)$, with $\lambda > 0$, if

$$\|u\|_{M_X^{p,\lambda}(\Omega)} = \sup_{\substack{x\in\Omega\\ 0<r<d_0}} \left(\frac{r^\lambda}{|\Omega \cap B(x,r)|}\int_{\Omega\cap B(x,r)} |u|^p \, \mathrm{d}y\right)^{1/p} < \infty,$$

where $d_0 = \min(\mathrm{diam}(\Omega), R_0)$. Here, $\mathrm{diam}(\Omega) = \sup_{x\in\Omega} d(x,y)$. We note that the continuity of the inclusion $i\colon (\mathbb{R}^n, d_e) \to (\mathbb{R}^n, d)$ implies that any (Euclidean) bounded set Ω is also d-bounded and therefore $\mathrm{diam}(\Omega) < \infty$. We have the following

THEOREM 1.6. *Let $U \subset \mathbb{R}^n$ be a bounded set with relative homogeneous dimension Q, and let $R_0 > 0$ be as in the Hypothesis. Assume $1 < p < Q$, $1 < s \leqslant Q/p$, $\beta \geqslant 1$ and $V \in M_X^{s,ps}(\beta B)$, where $B = B(x_0, R)$, $x_0 \in U$, $0 < R < R_0$, and $\beta B \subset U$. Then, there exists $C = C(U, X) > 0$ such that for any $u \in \mathcal{L}^{1,p}(\beta B)$ one has*

$$\int_B |u(x) - u_B|^p |V(x)| \, \mathrm{d}x \leqslant C \|V\|_{M_X^{s,ps}(\beta B)} \int_{\beta B} |Xu|^p \, \mathrm{d}x, \tag{1.7}$$

where $u_B = 1/(|B|)\int_B u \, \mathrm{d}y$. If instead $u \in \overset{\circ}{S}{}^{1,p}(B)$, then

$$\int_B |u(x)|^p |V(x)| \, \mathrm{d}x \leqslant C \|V\|_{M_X^{s,ps}(B)} \int_B |Xu|^p \, \mathrm{d}x. \tag{1.8}$$

REMARK. It is clear that when $X = \{\partial/\partial x_1, \dots, \partial/\partial x_n\}$ in $\mathbb{R}^n$, then $M_X^{p,\lambda}(\Omega) = M_n^{p,\lambda}(\Omega)$ and Theorem 1.6 gives back (1.1) when $p = 2$. In this context, the case $p \neq 2$ was proved in [CF] with a clever approach completely different from that in [F]. A crucial role in [CF] is played by the pointwise gradient estimates of the solution to the Dirichlet problem for $V \in M_n^{s,ps}$

$$\begin{cases} \Delta u = V \text{ in } B_e = B_e(x_0, R), \\ u\,|_{\partial B_e} = 0. \end{cases}$$

Such estimates, in our context, would be possible if pointwise bounds of the subelliptic gradient of the Green's function $G(x, y)$ of

$$\mathcal{L} = \sum_{j=1}^m X_j^* X_j$$

were available. What we are alluding to is an estimate of the type

$$|XG(x,y)| \leqslant C \frac{d(x,y)}{|B(x, d(x,y))|}.$$

We emphasize that such bounds are known to hold when the system X is C^∞ and of Hörmander type (see [NSW], [SC]), and it is thus possible to adapt the ideas in [CF] to obtain a very short proof of Theorem 1.6. Under our very weak assumptions, however, they may plainly be false. Pointwise estimates such as

$$C\frac{d(x,y)^2}{|B(x,d(x,y))|} \leqslant G(x,y) \leqslant C^{-1}\frac{d(x,y)^2}{|B(x,d(x,y))|}$$

would not suffice for the approach in [CF]. We have thus adopted some of the ideas in [FS], [Sa], and [KS], but with some simplifications based on the paper [He] and on the use of the atomic decomposition for tent spaces in [S2]. Another important tool is the recent work [FW].

In the second part of the paper we present some applications of Theorem 1.6 to the regularity of solutions to quasilinear subelliptic equations of the type

$$\sum_{j=1}^{m} X_j^* A_j(x,u,Xu) = f(x,u,Xu). \tag{1.9}$$

The latter are modelled on the p-sublaplacian

$$\sum_{j=1}^{m} X_j^*(|Xu|^{p-2}X_j u) = 0, \tag{1.10}$$

first introduced in [CDG1]. For C^∞ systems of Hörmander type, a complete study of the local properties of weak solutions to (1.9) was developed in [CDG1], [CDG3]. There the authors were able to generalize the theory established by Serrin in his famous papers [Se1], [Se2]. The structural assumptions in [CDG1], [CDG3] required membership of the 'lower order' coefficients in appropriate L^p spaces which are optimal for the applications of the sharp subelliptic Sobolev embedding established in [D2], [CDG1], [L1], [L2]. Earlier (non optimal) results in this framework were obtained in [Xu].

By using Theorem 1.6 we can sharpen the results in [CDG1], [CDG3] by allowing the 'lower order' terms in the relevant structural assumptions to belong to appropriate Morrey spaces. The ensuing Harnack inequality and Hölder continuity for weak solutions of (1.9) are presented in Section 4.

In closing, we remark that the inequality (1.2) is one of several criteria for (1.1) to hold.

In 1962, Maz'ya [M1], see also [M3, Thm 2.5.2], obtained the following necessary and sufficient condition for (1.1)

$$\sup_{\substack{K\subset\mathbb{R}^n\\ K\text{ compact}}} \frac{\int_K V(x)\,\mathrm{d}x}{\operatorname{cap} K} < \infty, \tag{1.11}$$

which, combined with the inequality

$$\operatorname{cap} K \geqslant c(n)|K|_n^{(n-2)/n},$$

implies the sufficient condition

$$\sup_{\substack{K\subset\mathbb{R}^n\\ K \text{ compact}}} \frac{\int_K V(x)\,\mathrm{d}x}{|K|_n^{(n-2)/n}} < \infty. \tag{1.12}$$

These criteria can be generalized to the present context, as we show in Section 5.

Next, we recall the definition of the Kato class K^n. A measurable $V \geqslant 0$ is said to belong to K^n ($n \geqslant 3$) if

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \int_{|x-y|<r} \frac{V(y)}{|x-y|^{n-2}}\,\mathrm{d}y = 0.$$

Schechter [Sc] proved that if $V \in K^n$ (in fact, K^n_{loc} would suffice), then a local version of (1.1) holds. One can easily see that $M_n^{1,\lambda} \subset K_n$ for $0 < \lambda < 2$, but this inclusion does not suffice to recover Fefferman and Phong's result. More recently, several authors have obtained stronger results than the one in [F], see [CWW], [ChW], [KS], [MV]. However, we chose as the subject of extension the Fefferman–Phong criterion and not, for example, its improvement

$$\sup_{\substack{x\in\mathbb{R}^n\\ r>0}} \frac{1}{r^{n-2}} \int_{|x-y|<r} (I_1V(y))^{2s}\,\mathrm{d}y < \infty,$$

(see [MV]) because we feel it is more flexible for the applications to partial differential equations. The Harnack inequality and the ensuing Hölder continuity of solutions to (1.9) constituted, in fact, our main motivation to study the embeddings (1.7) and (1.8).

2. Fractional Integration in Spaces of Homogeneous Type and Morrey Spaces

This section is devoted to establishing a basic ingredient in the proof of Theorem 1.6. We stress right away that our results here do not depend on any differentiable structure and, in fact, hold in any space of homogeneous type according to [CW]. To avoid the introduction of new notations and definitions we will however keep the discussion to the context of $\mathbb{R}^n$ with Lebesgue measure $\mathrm{d}x$, and leave it to the interested reader to provide the needed modifications of our statements. Hereafter, we fix a bounded set $U \subset \mathbb{R}^n$ and denote by C_1, R_0 its doubling constants as in (1.3). The number $Q = \log_2 C_1$ will indicate the relative homogeneous dimension. For $x_0 \in U$, $0 < R < R_0$ and $B = B(x_0, R)$ we introduce the *operator of fractional integration* of order $\alpha \in (0, Q)$

$$I_a f(x) = \int_B |f(y)| \frac{d(x,y)^\alpha}{|B(x, d(x,y))|}\,\mathrm{d}y, \quad x \in B.$$

The goal of this section is to establish the following.

THEOREM 2.1. *Let* $1 < p < Q$, $1 < s \leqslant Q/p$, *and* $V \in M_X^{s,ps}(B)$, $V \geqslant 0$. *Then, there exists* $C > 0$, *depending only on* C_1, *such that for any* $f \in L^p(B)$ *one has*

$$\int_B (I_1 f)^p V\,\mathrm{d}x \leqslant C \|V\|_{M_X^{s,ps}(B)} \int_B |f|^p\,\mathrm{d}x.$$

The proof of Theorem 2.1 will be accomplished in several steps. We start with a simple, yet useful, lemma which generalizes a result in [He]. We need to introduce the maximal function and a variant of it. If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then the *fractional maximal function* of f of order $\alpha \in [0, Q]$ is

$$M_\alpha f(x) = \sup_{r>0} \frac{r^\alpha}{|B(x,r)|} \int_{B(x,r)} |f(y)|\,\mathrm{d}y.$$

When $\alpha = 0$ we obtain the Hardy–Littlewood maximal function in a space of homogeneous type, see [CW] or [C], and simply write Mf, instead of $M_0 f$.

LEMMA 2.2. *For any* $0 < \alpha < Q$ *and* $0 < \theta < 1$ *one has for some* $C = C(C_1, \theta, \alpha) > 0$

$$I_{\alpha\theta} f(x) \leqslant C M_\alpha f(x)^\theta M f(x)^{1-\theta}.$$

Proof. For $0 < \delta < R$ we have for $x \in B$

$$\begin{aligned} I_{\alpha\theta} f(x) &= \int_B |f(y)| \frac{d(x,y)^{\alpha\theta}}{|B(x, d(x,y))|}\,\mathrm{d}y \\ &\leqslant \int_{B(x,\delta)} |f(y)| \frac{d(x,y)^{\alpha\theta}}{|B(x, d(x,y))|}\,\mathrm{d}y \\ &\quad + \int_{B(x,2R)\setminus B(x,\delta)} |f(y)| \frac{d(x,y)^{\alpha\theta}}{|B(x, d(x,y))|}\,\mathrm{d}y = I + II. \end{aligned}$$

Here, we are supposing that f has been set to be zero outside B. Using (1.3) we obtain

$$I = \sum_{k=0}^{\infty} \int_{B(x,2^{-k}\delta)\setminus B(x,2^{-(k+1)}\delta)} |f(y)| \frac{d(x,y)^{\alpha\theta}}{|B(x, d(x,y))|}\,\mathrm{d}y$$

$$\leqslant C\sum_{k=0}^{\infty}\frac{(2^{-k}\delta)^{\alpha\theta}}{|B(x,2^{-k}\delta)|}\int_{B(x,2^{-k}\delta)}|f(y)|\,\mathrm{d}y$$

$$\leqslant CMf(x)\delta^{\alpha\theta}.$$

On the other hand, again by (1.3) we have for $k_0 \simeq \log_2(R/\delta)$

$$II = \sum_{k=0}^{k_0}\int_{B(x,2^{k+1}\delta)\setminus B(x,2^k\delta)}|f(y)|\frac{d(x,y)^{\alpha\theta}}{|B(x,d(x,y))|}\,\mathrm{d}y$$

$$\leqslant C\sum_{k=0}^{k_0}\frac{(2^{k+1}\delta)^{\alpha\theta}}{|B(x,2^{k+1}\delta)|}\int_{B(x,2^{k+1}\delta)}|f(y)|\,\mathrm{d}y$$

$$\leqslant CM_\alpha f(x)\delta^{-\alpha(1-\theta)}.$$

In conclusion, we obtain

$$I_{\alpha\theta}f(x) \leqslant CMf(x)\delta^{\alpha\theta} + CM_\alpha f(x)\delta^{-\alpha(1-\theta)}.$$

Minimizing with respect to $\delta > 0$ yields the conclusion. □

To state our next result consider a nonnegative Borel measure on $\mathbb{R}^n$, $\mathrm{d}\mu$. For $0 \leqslant \alpha \leqslant Q$ its fractional maximal function of order α is given by

$$M_\alpha\mu(x) = \sup_{r>0}\frac{r^\alpha}{|B(x,r)|}\mu(B(x,r)),$$

where for any Borel set E we set $\mu(E) = \int_E \mathrm{d}\mu$.

LEMMA 2.3. *Let* $1 < p < Q$, $0 < \alpha \leqslant Q/p$. *Suppose that* μ *is a nonnegative Borel measure satisfying for some* $M > 0$

$$M_{\alpha p}\mu(x) \leqslant M, \quad x \in \mathbb{R}^n.$$

Then, there exists $C > 0$, *depending only on* C_1 *in* (1.3), *such that*

$$\int_B M_\alpha f^p\,\mathrm{d}\mu \leqslant CM\int_B |f|^p\,\mathrm{d}x,$$

for every $f \in L^p(B)$.

Proof. It is based on the atomic decomposition for tent spaces given in Sections 2 and 3 of Chapter II in [S2]. In Section 3.2 Stein considers the problem of determining those unbounded positive functions $\varphi(x,t)$ on $\mathbb{R}^{n+1}_+$ for which

$$\int_{\mathbb{R}^n}\sup_{0<t<\infty}[\varphi(x,t)F(x,t)]^p\,\mathrm{d}x \leqslant A\int_{\mathbb{R}^n}|f(x)|^p\,\mathrm{d}x,$$

where $F(x,t) = 1/|B(x,t)| \int_{B(x,t)} |f| \, \mathrm{d}y$. More in general, given a nonnegative Borel measure $\mathrm{d}\mu$ on $\mathbb{R}^n$ one can study the validity of an analogous inequality when the integral in the left-hand side is taken with respect to $\mathrm{d}\mu$, i.e.

$$\int_{\mathbb{R}^n} \sup_{0<t<\infty} [\varphi(x,t)F(x,t)]^p \, \mathrm{d}\mu(x) \leqslant A \int_{\mathbb{R}^n} |f(x)|^p \, \mathrm{d}x. \tag{2.4}$$

Testing (2.4) against $f = \chi_{B(x_0,r)}$ one obtains from (2.4)

$$\int_{B(x_0,r)} \sup_{0<t<r} [\varphi(x,t)F(x,t)]^p \, \mathrm{d}\mu(x) \leqslant A|B(x_0,r)|. \tag{2.5}$$

Now, if $x \in B(x_0,r)$ and $0 < t < r$ we have

$$F(x,t) \geqslant C$$

so that (2.5) gives the following necessary condition for (2.4)

$$\int_{B(x_0,r)} \sup_{0<t<r} \varphi(x,t)^p \, \mathrm{d}\mu(x) \leqslant C|B(x_0,r)|. \tag{2.6}$$

Using the atomic decomposition in Section 2.6 of [S2] and proceeding as in the proof of the Proposition on page 70 there, we see that (2.6) is also sufficient for (2.4). One only needs to observe that the proof of the existence of an atomic decomposition for the tent spaces is based on the Whitney decomposition, and the latter is available in any space of homogeneous type. We refer the reader to [S2].

The above being said, we next observe that

$$M_\alpha f(x) = \sup_{0<t<\infty} \varphi(x,t)F(x,t),$$

where $\varphi(x,t) = t^\alpha$. Therefore, the conclusion of Lemma 2.3 can be formulated as (2.4) with such choice of $\varphi(x,t)$. But then, the necessary and sufficient condition (2.6) would read

$$\int_{B(x_0,r)} \sup_{0<t<r} t^{\alpha p} \, \mathrm{d}\mu(x) \leqslant C|B(x_0,r)|,$$

or, equivalently, $M_{\alpha p}\mu(x_0) \leqslant C$. This completes the proof of Lemma 2.3. □

We are now ready to give the

Proof of Theorem 2.1. With p and s given as in the statement of the theorem we take $\alpha = s$, $\theta = 1/s$, $\mathrm{d}\mu = V^s \, \mathrm{d}x$. Then

$$\int_B I_1 f^p V \, \mathrm{d}x = \int_B I_{\alpha\theta} f^p V \, \mathrm{d}x,$$

$$\text{(Lemma 2.2)} \leqslant C \int_B M_\alpha f^{\theta p} M f^{(1-\theta)p} V \,\mathrm{d}x,$$

$$\begin{aligned}\text{(Hölder inequality)} \;\leqslant\; & C\left(\int_B M_\alpha f^p V^{1/\theta}\,\mathrm{d}x\right)^\theta \left(\int_B M f^p\,\mathrm{d}x\right)^{1-\theta} \\ =\; & C\left(\int_B M_\alpha f^p\,\mathrm{d}\mu\right)^\theta \left(\int_B M f^p\,\mathrm{d}x\right)^{1-\theta}. \end{aligned} \tag{2.7}$$

The first factor in the right-hand side of (2.7) is estimated using Lemma 2.3, since $V \in M_X^{s,ps}(B)$ implies

$$M_{ps}\mu = M_{ps}(V^s) \leqslant \|V\|^s_{M_X^{s,ps}(B)} < \infty.$$

For the second factor we use the L^p-continuity of the Hardy–Littlewood maximal function in a space of homogeneous type, see [C]. We conclude from (2.7)

$$\int_B I_1 f^p V\,\mathrm{d}x \leqslant C\|V\|_{M_X^{s,ps}(B)} \int_B |f|^p\,\mathrm{d}x.$$

The proof of Theorem 2.1 is completed. □

Although we will not use the next result in the proof of Theorem 1.6, we include it because of its large independent interest.

THEOREM 2.8. *Let* $0 < \alpha < Q$, $1 < p < Q/\alpha$, $\alpha p < \lambda \leqslant Q$. *Then there exists a positive constant* $C = C(C_1, \lambda, p, \alpha)$ *such that*

$$\|I_\alpha f\|_{M_X^{q,\lambda}(B)} \leqslant C\|f\|_{M_X^{p,\lambda}(B)}, \tag{2.9}$$

where $1/q = 1/p - \alpha/\lambda$.

If $p = 1$, *and* $B_r = B(y,r)$, $y \in B$, $0 < r < R_0$, *one has for* $q = \lambda/(\lambda - \alpha)$

$$\sup_{\mu>0} \mu^q\,|\{x \in B \cap B_r \mid |I_\alpha f(x)| > \mu\}| \leqslant C\frac{|B_r|}{r^\lambda}\|f\|^q_{M_X^{1,\lambda}(B)}.$$

Proof. First we consider the case $1 < p < Q/a$. Let $f \in M_X^{p,\lambda}(B)$, $x \in B$. Then

$$\begin{aligned} I_\alpha f(x) \;=\; & \int_{B(x,\varepsilon)} f(y)\frac{d(x,y)^\alpha}{|B(x,d(x,y))|}\,\mathrm{d}y \\ & + \int_{B\cap B(x,\varepsilon)^c} f(y)\frac{d(x,y)^\alpha}{|B(x,d(x,y))|}\,\mathrm{d}y = I_\alpha^1 + I_\alpha^2. \end{aligned}$$

Proceeding as in the proof of Lemma 2.2 we obtain

$$
\begin{aligned}
|I_\alpha^1| &\leqslant \sum_{k=0}^{\infty} \int_{2^{-(k+1)}\varepsilon < d(x,y) < 2^{-k}\varepsilon} |f(y)| \frac{d(x,y)^\alpha}{|B(x,d(x,y))|}\,\mathrm{d}y \\
&\leqslant \sum_{k=0}^{\infty} \frac{(2^{-k}\varepsilon)^\alpha}{|B(x,2^{-(k+1)}\varepsilon)|} \int_{B(x,2^{-k}\varepsilon)} |f(y)|\,\mathrm{d}y, \\
&\qquad \text{(by the doubling condition (1.3))} \\
&\leqslant C'\varepsilon^\alpha Mf(x).
\end{aligned} \tag{2.10}
$$

We now estimate I_α^2. Let k_0 be an integer such that $2^{k_0}\varepsilon \leqslant 2R < 2^{k_0+1}\varepsilon$, and let R_k denote the 'ring' $(B(x,2^{k+1}\varepsilon) \setminus B(x,2^k\varepsilon)) \cap B$. Moreover, set $f \equiv 0$ in B^c. We then have, for some $0 < \sigma < 1$,

$$
\begin{aligned}
|I_\alpha^2| &\leqslant \int_{(B(x,2R)\setminus B(x,\varepsilon))\cap B} |f(y)| \frac{d(x,y)^\alpha}{|B(x,d(x,y))|}\,\mathrm{d}y \\
&\leqslant \sum_{k=0}^{k_0} \left(\int_{R_k} |f(y)|^p \frac{d(x,y)^{\alpha\sigma p}}{|B(x,d(x,y))|^{\sigma p}}\,\mathrm{d}y \right)^{1/p} \\
&\quad \cdot \left(\int_{R_k} \frac{d(x,y)^{\alpha(1-\sigma)(p/(p-1))}}{|B(x,d(x,y))|^{(1-\sigma)(p/(p-1))}} \right)^{(p-1)/p} \\
&\qquad \text{(again by the doubling condition (1.3))} \\
&\leqslant \sum_{k=0}^{k_0} \frac{(2^{k+1}\varepsilon)^{\alpha\sigma-\lambda/p}}{|B(x,2^k\varepsilon)|^{\sigma-1/p}} \|f\|_{M_X^{p,\lambda}(B)} \frac{(2^{k+1}\varepsilon)^{\alpha(1-\sigma)}}{|B(x,2^k\varepsilon)|^{1-\sigma-(1-1/p)}} \\
&\leqslant C''\|f\|_{M_X^{p,\lambda}(B)}\varepsilon^{\alpha-\lambda/p}.
\end{aligned} \tag{2.11}
$$

Combining estimates (2.10) and (2.11) we obtain

$$
|I_\alpha f(x)| \leqslant C\{\varepsilon^\alpha Mf(x) + \varepsilon^{\alpha-\lambda/p}\|f\|_{M_X^{p,\lambda}(B)}\}.
$$

The choice

$$
\varepsilon = \left(\frac{Mf(x)}{\|f\|_{M_X^{p,\lambda}(B)}} \right)^{-p/\lambda}
$$

gives

$$
|I_\alpha f(x)| \leqslant C(Mf(x))^{1-\alpha p/\lambda} \|f\|_{M_X^{p,\lambda}(B)}^{\alpha p/\lambda}. \tag{2.12}
$$

The conclusion now follows from the L^p-continuity properties of the Hardy–Littlewood maximal operator in spaces of homogeneous type, see [C].

We turn our attention to the case $p = 1$. Let I_α^1 and I_α^2 be as above. First of all we notice that the estimate (2.12) holds also when $p = 1$. Secondly, we note that, thanks to the doubling condition (1.3), a covering lemma of Vitali type can be established, see [S1]. As a consequence, one can easily prove the weak L^1 continuity of Mf. We thus deduce from (2.12) for $\mu > 0$

$$\begin{aligned}
&|\{x \in B \cap B_r \mid |I_\alpha f(x)| > \mu\}| \\
&\leqslant \left| \left\{ x \in B \cap B_r \mid Mf(x) > \left(\frac{\mu}{C\|f\|^{\alpha/\lambda}_{M_X^{1,\lambda}(B)}} \right)^{\lambda/(\lambda-\alpha)} \right\} \right| \\
&\leqslant \frac{C\|f\|^{\alpha/(\lambda-\alpha)}_{M_X^{1,\lambda}(B)}}{\mu^{\lambda/(\lambda-\alpha)}} \int_{B\cap B_r} |f|\,\mathrm{d}x \\
&\leqslant \frac{C}{\mu^q} \frac{|B \cap B_r|}{r^\lambda} \|f\|^{\lambda/(\lambda-\alpha)}_{M_X^{1,\lambda}(B)} \\
&\leqslant \frac{C}{\mu^q} \frac{|B_r|}{r^\lambda} \|f\|^q_{M_X^{1,\lambda}(B)}.
\end{aligned}$$

This concludes the proof. □

REMARK. One of the main applications of Theorem 2.8 is the following improvement of the Sobolev inequality: Let $0 < \lambda \leqslant Q$, $1 < p < \lambda$, $q = \lambda p/(\lambda - p)$, $\beta \geqslant 1$. Then there exists $C > 0$ such that if $u \in \mathcal{L}^{1,1}(\beta B)$ and $Xu \in M_X^{p,\lambda}(\beta B)$ one has

$$\|u - u_B\|_{M_X^{q,\lambda}(B)} \leqslant C\|Xu\|_{M_X^{p,\lambda}(\beta B)}.$$

Moreover, if $u \in \overset{\circ}{S}{}^{1,1}(B)$ with $Xu \in M_X^{p,\lambda}(B)$, then

$$\|u\|_{M_X^{q,\lambda}(B)} \leqslant C\|Xu\|_{M_X^{p,\lambda}(B)}.$$

These embedding results follow immediately from an application of (2.9) and Theorem 3.1 below. In the case of Hörmander type vector fields, they were proved in [L3], [L4] with an approach based on the Rothschild–Stein lifting theorem [RS].

Our next result is an approximation property of Morrey spaces. For the standard spaces $M_n^{p,\lambda}$, it was first proved in the cited paper [CF]. In order to generalize it to the context of a space of homogeneous type, we need to establish some preliminary results. The first one is related to Lemma 2.3.

LEMMA 2.13. *For* $1 < p < \infty$, *there exists a positive constant* $C = C(p, C_1)$ *such that for any measurable functions in* B $\phi \geqslant 0$ *and* f, *one has*

$$\int_B (Mf(x))^p \phi(x)\,\mathrm{d}x \leqslant C \int_B |f(x)|^p M\phi(x)\,\mathrm{d}x.$$

Lemma 2.13 is due to C. Fefferman and E. Stein [FS] in the Euclidean setting. A slight variant of their arguments, see also [GR, Thm II.2.12], generalizes the conclusion to spaces of homogeneous type. For the reader's convenience we include the proof.

Proof of Lemma 2.13. Without loss of generality, we may assume $M\phi(x) < \infty$ for a.e. $x \in B$. Let μ and ν be two positive measures on B such that $d\mu(x) = M\phi(x)\,\mathrm{d}x$ and $d\nu(x) = \phi(x)\,\mathrm{d}x$. We need to prove that M is a bounded operator from $L^p(B, d\mu)$ to $L^p(B, d\nu)$, and this will be done using Marcinkiewicz interpolation theorem. First we consider the case $p = \infty$. If $M\phi(x) = 0$ for some $x \in B$, then $\phi(x) = 0$ for a.e. $x \in B$ and the conclusion is immediate. Suppose $M\phi(x) > 0$ for every $x \in B$ and let $\alpha > \|f\|_{L^\infty(B,d\mu)}$. We have

$$\int_{\{|f|>\alpha\}} M\phi(x)\,\mathrm{d}x = 0$$

and therefore $|\{x \in B \mid |f(x)| > \alpha\}| = 0$, which implies $|f(x)| \leqslant \alpha$ for a.e. $x \in B$. Hence also $Mf(x) \leqslant \alpha$ for a.e. $x \in B$, so that $\|Mf\|_{L^\infty(B,d\nu)} \leqslant \alpha$. This gives the (∞, ∞) result

$$\|Mf\|_{L^\infty(B,d\nu)} \leqslant \|f\|_{L^\infty(B,d\mu)}.$$

It will now suffice to show that M is of weak type (1,1). Without loss of generality, we may assume $f \in L^1(\mathbb{R}^n)$, $f \geqslant 0$. Let $t > 0$ be given, and let $E_t = \{x \in B \mid Mf(x) > t\}$. By Theorem III.1.3 and the proof of Theorem III.2.1 in [CW], there exists a collection of balls $\{B(x_i, kr_i)\}$ with $k > 1$ such that:

(i) the balls $B(x_i, r_i)$ are disjoint;
(ii) $E_t = \cup_i B(x_i, kr_i)$;
(iii) $(1/|B(x_i, r_i)|) \int_{B(x_i,r_i)} f(x)\,\mathrm{d}x > t$ for any $i \in N$.

Then

$$\int_{E_t} \phi(x)\,\mathrm{d}x \leqslant \sum_i \int_{B(x_i,kr_i)} \phi(x)\,\mathrm{d}x$$

(by the doubling condition (1.3))

$$\leqslant C \sum_i \frac{|B(x_i, r_i)|}{|B(x_i, kr_i)|} \int_{B(x_i,kr_i)} \phi(x)\,\mathrm{d}x$$

$$\leqslant \frac{C}{t}\sum_i \frac{1}{|B(x_i,kr_i)|}\int_{B(x_i,kr_i)}\phi(x)\,\mathrm{d}x\int_{B(x_i,r_i)}f(x)\,\mathrm{d}x$$

$$\leqslant \frac{C}{t}\sum_i \int_{B(x_i,r_i)}f(x)M\phi(x)\,\mathrm{d}x$$

$$\leqslant \frac{C}{t}\int_B f(x)M\phi(x)\,\mathrm{d}x.$$

This concludes the proof of the Lemma. □

Next, we recall an extension of a result due to Coifman and Rochberg [CR]. Since the proof is a step by step repetition of its Euclidean counterpart, we will omit it and refer the reader to [GR, Thm III.3.4].

LEMMA 2.14. *Let μ be a positive Borel measure such that $M\mu(x) < \infty$ for* a.e. $x \in B$. *Then for every $\gamma \in (0,1)$ the function $w = (M\mu)^\gamma$ is an A_1-weight in B. In particular, there exists a constant $C > 0$ such that*

$$Mw(x) \leqslant Cw(x) \quad \text{for a.e. } x \in B.$$

In the sequel, we will need the following estimate for the maximal function of the characteristic function of a ball.

LEMMA 2.15. *Let $y \in B$ and $0 < r < R_0$. If χ denotes the characteristic function of $B(y,r)$, then*

$$M\chi(x) \leqslant \frac{|B(y,r)|}{|B(x,d(x,y)-r)|},$$

for every $x \in B$ with $d(x,y) > r$.

Proof. It suffices to observe that, if $d(x,y) > r$, then

$$B(x,\rho)\cap B(y,r) \neq \emptyset \Leftrightarrow \rho \geqslant d(x,y) - r.$$

Hence

$$M\chi(x) = \sup_{\rho>0}\frac{|B(x,\rho)\cap B(y,r)|}{|B(x,\rho)|} \leqslant \frac{|B(y,r)|}{|B(x,d(x,y)-r)|}.$$ □

Our next result shows that for any given $V \in M_X^{s,ps}(B)$ there exists an A_1-weight in the same class majorizing V. Precisely, we have the following (cfr. [CF]).

LEMMA 2.16. *Let $V \in M^{s,ps}(B)$, $1 < p < Q$, $1 < s \leqslant Q/p$. Then $(MV^{s_1})^{1/s_1} \in A_1 \cap M_X^{s,ps}(B)$ for any $1 < s_1 < s$.*

Proof. We know that $(MV^{s_1})^{1/s_1} \in A_1$ by virtue of Lemma 2.14. Now let $y \in B$, $0 < r < \min(2R, R_0)$ and let χ denote the characteristic function of $B_r = B(y, r)$. We infer from Lemma 2.13 that there exists a positive constant C such that

$$\int_{B_r \cap B} (MV^{s_1})^{s/s_1}\,\mathrm{d}x \leqslant C \int_B |V|^s M\chi\,\mathrm{d}x.$$

Let k_0 be an integer such that $2^{k_0} r \leqslant 2R < 2^{k_0+1} r$. Let also R_k denote the set $(B(y, 2^{k+1}r) \setminus B(y, 2^k r)) \cap B$. Applying Lemma 2.15 we obtain

$$\int_{B_r \cap B} (MV^{s_1})^{s/s_1}\,\mathrm{d}x \leqslant C \left\{ \int_{B_r} |V|^s\,\mathrm{d}x + \sum_{k=0}^{k_0} \int_{R_k} |V|^s M\chi\,\mathrm{d}x \right\}$$

$$\leqslant C \left\{ \frac{|B_r \cap B|}{r^{ps}} \|V\|^s_{M_X^{s,ps}(B)} + \sum_{k=0}^{k_0} \int_{R_k} |V|^s \frac{|B_r|}{|B(x, d(x,y) - r)|}\,\mathrm{d}x \right\}$$

$$\leqslant C \left\{ \frac{|B_r \cap B|}{r^{ps}} \|V\|^s_{M_X^{s,ps}(B)} + \sum_{k=0}^{k_0} |B_r| \int_{R_k} |V|^s \frac{1}{|B(x, 2^{k-1}r)|}\,\mathrm{d}x \right\}$$

(by the doubling condition (1.3))

$$\leqslant C \left\{ \frac{|B_r \cap B|}{r^{ps}} \|V\|^s_{M_X^{s,ps}(B)} + \sum_{k=0}^{k_0} |B_r| \int_{R_k} |V|^s \frac{1}{|B(x, 2^{k+2}r)|}\,\mathrm{d}x \right\}$$

$$\leqslant C \left\{ \frac{|B_r \cap B|}{r^{ps}} \|V\|^s_{M_X^{s,ps}(B)} + \sum_{k=0}^{k_0} \frac{|B_r|}{|B(y, 2^{k+1}r)|} \int_{B(y,2^{k+1}r)\cap B} |V|^s\,\mathrm{d}x \right\}$$

$$\leqslant C \frac{|B_r \cap B|}{r^{ps}} \|V\|^s_{M_X^{s,ps}(B)} \left\{ 1 + \frac{|B_r|}{|B_r \cap B|} \right\}.$$

At this point we need only to recall that Carnot–Carathéodory balls satisfy the interior corkscrew condition (see [GN2]) and thus, in particular, the quotient $|B_r|/|B \cap B_r|$ is bounded above by a positive constant. The proof is now complete. □

3. Proof of Theorem 1.6

In this section we prove Theorem 1.6 along with some generalizations.

The following basic result is crucial to the proof of Theorem 1.6.

THEOREM 3.1. *Let U be as in Theorem* 1.6 *and suppose that* (1.3) *and* (1.4) *hold.*

Then for any $B = B(x_0, R)$, *with* $x_0 \in U$ *and* $0 < R < R_0$, *there exist constants* $C > 0$ *and* $\beta \geqslant 1$ *depending on* C_1 *and* C_2 *such that for* $u \in \mathcal{L}^{1,1}(\beta B))$ *and* a.e. $x \in B$

$$|u(x) - u_B| \leqslant C \int_{\beta B} |Xu(y)| \frac{d(x, y)}{|B(x, d(x, y))|} \, \mathrm{d}y. \tag{3.2}$$

Furthermore, if $u \in \overset{\circ}{S}{}^{1,1}(B)$, *then one has for* a.e. $x \in B$

$$|u(x)| \leqslant C \int_{B} |Xu(y)| \frac{d(x, y)}{|B(x, d(x, y))|} \, \mathrm{d}y. \tag{3.3}$$

Theorem 3.1 was proved in [FLW1] for a C^∞ system X of vector fields of Hörmander type. In the same context an elementary proof based only on (1.3), (1.4) and on the size estimates of the fundamental solution was first discovered in [CDG4]. Subsequently, Franchi, Lu and Wheeden [FLW2] made the interesting observation that to obtain (3.2) one needs only (1.3), (1.4) and the following size estimate for the metric balls: for any $0 < t < 1$

$$|B(x, tR)| \leqslant C_1^{-1} t^{1+\gamma} |B(x, R)| \quad \text{for some } \gamma > 0. \tag{3.4}$$

Recently, Franchi and Wheeden [FW] have proved the important fact that, even for a system of locally Lipshitz vector fields, 3.4) holds with $\gamma = 0$, and that moreover such linear growth is enough to deduce (3.2) from (1.3) and (1.4).

Proof of Theorem 1.6. For $u \in \mathcal{L}^{1,p}(\beta B)$ and $I_1(f)(x) = \int_{\beta B} |f(y)|(d(x, y)/|B(x, d(x, y))|) \, \mathrm{d}y$, Theorem 3.1 gives

$$|u(x) - u_B| \leqslant C \, I_1(|Xu|)(x) \quad \text{for a.e. } x \in B.$$

Invoking Theorem 2.1 we conclude

$$\begin{aligned} &\int_B |u(x) - u_B|^p |V(x)| \, \mathrm{d}x \\ &\quad \leqslant C \int_{\beta B} I_1(|Xu|)(x)^p |V(x)| \, \mathrm{d}x \\ &\quad \leqslant C \|V\|_{M_X^{s,ps}(\beta B)} \int_{\beta B} |Xu(x)|^p \, \mathrm{d}x. \end{aligned}$$

This proves Theorem 1.6 when $u \in \mathcal{L}^{1,p}(\beta B)$. If, instead, $u \in \overset{\circ}{S}{}^{1,p}(B)$, then we read the conclusion from (3.3) and Theorem 2.1. □

In the case in which the measure $d\mu = V\,\mathrm{d}x$, with $V \geqslant 0$, is also doubling, i.e., for any given bounded set $U \subset \mathbb{R}^n$, $x_0 \in U$, $0 < R < R_0$ one has

$$\int_{B(x_0,2R)} V\,\mathrm{d}y \leqslant C_3 \int_{B(x_0,R)} V\,\mathrm{d}y, \tag{3.5}$$

for some $C_3 > 0$, then we can prove a more general (and global) version of Theorem 1.6. To state it we recall the definition of the Poincaré–Sobolev domains in [GN1], abbreviated in (PS)-domains hereafter.

DEFINITION 3.6. *An open set* $\Omega \subset \mathbb{R}^n$ *is called a* (PS)-domain *if there exist a covering* $\{B\}_{B\in\mathcal{F}}$ *of* Ω *by metric balls, and numbers* $N > 0$, $\alpha \geqslant 1$, $\nu \geqslant 1$ *such that*

(i) $\sum_{B\in\mathcal{F}} \chi_{(\alpha+1)B} \leqslant N\chi_\Omega$;
(ii) *There exists a* (*central*) *ball* $B_0 \in \mathcal{F}$ *such that for any* $B \in \mathcal{F}$ *one can find a chain* $B_0, B_1, \ldots, B_{s(B)} = B$, *with* $B_i \cap B_{i+1} \supset \tilde{B}_i$ *for some ball* $\tilde{B}_i$, *for which* $N\tilde{B}_i \supset B_i \cup B_{i+1}$;
(iii) *For any* $i = 0, \ldots, s(B)$ *one has* $B \subset \nu B_i$.

The above definition is clearly purely metrical. In [GN1] it was proved that a complete theory of Sobolev and isoperimetric inequalities can be developed for (PS)-domains from the two elementary constituents (1.3), (1.4). The importance of (PS)-domains rests in the following chain of inclusions, which holds true in the Euclidean setting and, except for the first one, also in our very general context (see [GN1] for a complete discussion)

$$\text{Lip} \subset \text{NTA} \subset (\varepsilon,\delta) \subset \text{John} \subset (\text{PS}).$$

We mention that it was proved in [GN1], [BKL] that, in fact

$$\text{John} = (\text{PS}).$$

We have the following global version of Theorem 1.6.

THEOREM 3.7. *Suppose that* (1.3) *and* (1.4) *hold. For a bounded set* $U \subset \mathbb{R}^n$, *let* $\Omega \subset \overline{\Omega} \subset U$ *be a* (PS)-*domain such that* $\operatorname{diam}(\Omega) < R_0/2$. *For* p *and* s *as in Theorem* 1.6 *let* $V \in M_X^{s,ps}(\Omega)$, $V \geqslant 0$, *satisfy* (3.5). *Then, there exists a positive constant* C *such that for any* $u \in \mathcal{L}^{1,p}(\Omega)$

$$\int_\Omega |u - u_\Omega|^p V\,\mathrm{d}x \leqslant C\|V\|_{M_X^{s,ps}(\Omega)} \int_\Omega |Xu|^p\,\mathrm{d}x.$$

We do not present here the rather technical proof of Theorem 3.7, but refer the reader to [GN1] for complete details of the chaining argument that leads from a local inequality, such as that in Theorem 1.6, to a global one, where both integrals in the left-and right-hand side are performed on the same (PS)-domain Ω. The arguments in [GN1] need to be suitably modified due to the fact that we are using in the left-hand side the measure $d\mu = V\,\mathrm{d}x$ instead of Lebesgue measure. Such modifications, however, are easily accomplished by using (3.5).

Several examples of large families of (PS)-domains in Carnot–Carathéodory spaces were given in [CG], [GN1], [GN2]. We only recall that each Carnot–Carathéodory ball of sufficiently small radius is a (PS)-domain (see [FGW] and [GN1]). As a consequence of Theorem 3.7 we can then strengthen the first conclusion of Theorem 1.6 as follows

$$\int_B |u - u_B|^p |V|\,\mathrm{d}x \leqslant C\|V\|_{M_X^{s,ps}(B)} \int_B |Xu|^p\,\mathrm{d}x,$$

for every $u \in \mathcal{L}^{1,p}(B)$.

4. Regularity of Solutions to Quasilinear Sub-elliptic Equations

In this section we are concerned with equations of the type (1.9). Throughout the section U will denote a bounded open set, with doubling constants C_1 and R_0 as in (1.3) and homogeneous dimension $Q = Q(U) > 0$. We also let $B = B(x_0, R)$ be a fixed metric ball, with $x_0 \in U$ and $0 < R < R_0$, such that $4B \subset U$. We consider measurable functions $A: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$, $f: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ and suppose $A = (A_1, \ldots, A_m)$. We assume that A_i, $i = 1, \ldots, m$, and f satisfy the following structural conditions: *There exist* $p \in (1, Q)$, $g_1 \geqslant 0$ *and measurable functions* f_1, f_2, f_3, g_2, g_3, h_3 *on* $\mathbb{R}^n$, *such that for* a.e. $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ *and* $\xi \in \mathbb{R}^m$

$$(S)\ \begin{cases} |A(x,u,\xi)| \leqslant g_1|\xi|^{p-1} + g_2(x)|u|^{p-1} + g_3(x), \\ |f(x,u,\xi)| \leqslant f_1(x)|\xi|^{p-1} + f_2(x)|u|^{p-1} + f_3(x), \\ A(x,u,\xi)\cdot\xi \geqslant |\xi|^p - f_2(x)|u|^p - h_3(x). \end{cases}$$

The relevant integrability requirements on the functions f_i, g_i, h_i in the structural assumptions (S) are as follows: for some $\varepsilon \in (0,1)$

(i) $g_2, g_3 \in M_X^{q,q(p-1)}(U)$, with $p/(p-1) < q \leqslant Q/(p-1)$;
(ii) $f_2, f_3, h_3 \in M_X^{q,q(p-\varepsilon)}(U)$, with $1 < q < Q/(p-\varepsilon)$;
(iii) $f_1 \in M_X^{q,(1-\varepsilon)q}(U)$, with $p < q < Q/(1-\varepsilon)$.

A function $u \in L^p_{\mathrm{loc}}(U)$ is said to belong to $\mathcal{L}^{1,p}_{\mathrm{loc}}(U)$ if $\varphi u \in \mathcal{L}^{1,p}(U)$ for every $\varphi \in C_0^1(U)$. We say that $u \in \mathcal{L}^{1,p}_{\mathrm{loc}}(U)$ is a (weak) solution to (1.9) if for every $\varphi \in \overset{\circ}{S}{}^{1,p}(U)$

$$\sum_{j=1}^m \int_U A_j(x,u,Xu)X_j\varphi\,\mathrm{d}x = \int_U f(x,u,Xu)\varphi\,\mathrm{d}x.$$

It is worth noting that with the choice

$$A_j(x,u,\xi) = A_j(\xi) = |\xi|^{p-2}\xi_j, \quad j = 1,\dots,m, \quad f \equiv 0,$$

one can recover (1.10) from (1.9).

Our aim is to prove three lemmas, which allow to bound the 'lower order terms' f_i, g_i, h_i appearing in (S) in the proof of the local boundedness and of the Harnack inequality for solutions to (1.9).

LEMMA 4.1. *Let* $q \in (p/(p-1), Q/(p-1)]$ *and* $g \in M_X^{q,q(p-1)}(B)$. *Then, there exists* $C = C(U,X) > 0$ *such that for any* $\varphi, \psi \in \overset{\circ}{S}{}^{1,p}(B)$

$$\int_B g\psi|\varphi|^{p-1}\,\mathrm{d}x \leqslant C\|g\|_{M_X^{q,q(p-1)}(B)} \cdot \left(\int_B |X\varphi|^p\,\mathrm{d}x\right)^{(p-1)/p}\left(\int_B |\psi|^p\,\mathrm{d}x\right)^{1/p}.$$

Proof. Observe first that $g^{p/(p-1)} \in M_X^{((p-1)/p)q,(p-1)q}(B)$, with

$$\|g^{p/(p-1)}\|_{M_X^{((p-1)/p)q,(p-1)q}(B)} = \|g\|^{p/(p-1)}_{M_X^{q,q(p-1)}(B)}.$$

We then apply Hölder's inequality and Theorem 1.6 to obtain, for $\varphi, \psi \in \overset{\circ}{S}{}^{1,p}(B)$,

$$\begin{aligned}\int_B g\psi|\varphi|^{p-1}\,\mathrm{d}x &\leqslant \left(\int_B |\psi|^p\right)^{1/p}\left(\int_B |g|^{p/p-1}|\varphi|^p\,\mathrm{d}x\right)^{(p-1)/p}\\ &\leqslant C\|g\|_{M_X^{q,q(p-1)}(B)}\cdot\left(\int_B |X\varphi|^p\right)^{(p-1)/p}\cdot\left(\int_B|\psi|^p\right)^{1/p}. \qquad \square\end{aligned}$$

LEMMA 4.2. *Let* $\varepsilon \in (0,1)$, $1 < q < Q/(p-\varepsilon)$, $\max(1, q^{(p-\varepsilon)/p}) < s < \min(q, Q/p)$ *and* $h \in M_X^{q,q(p-\varepsilon)}(B)$. *Set* $\delta = p(1-s/q)$. *Then there exists* $C = C(U,X) > 0$ *such that for any* $\varphi \in \overset{\circ}{S}{}^{1,p}(B)$

$$\int_B h\varphi^p\,\mathrm{d}x \leqslant CR^{\varepsilon-\delta}\|h\|_{M_X^{q,q(p-\varepsilon)}(B)} \cdot \left(\int_B |\varphi|^p\,\mathrm{d}x\right)^{\delta/p}\left(\int_B |X\varphi|^p\,\mathrm{d}x\right)^{(p-\delta)/p}.$$

Proof. It is easy to check that $h^{p/p-\delta} \in M_X^{s,ps}(B)$, and

$$\|h^{p/p-\delta}\|^{(p-\delta)/p}_{M_X^{s,ps}(B)} \leqslant R^{\varepsilon-\delta}\|h\|_{M_X^{q,q(p-\varepsilon)}(B)}. \tag{4.3}$$

Proceeding as in the proof of Lemma 4.1 we obtain, for any $\varphi \in \overset{\circ}{S}{}^{1,p}(B)$,

$$\int_B h\varphi^p \,\mathrm{d}x \leqslant \left(\int_B |\varphi|^p \,\mathrm{d}x\right)^{\delta/p} \left(\int_B |h|^{p/(p-\delta)}|\varphi|^p \,\mathrm{d}x\right)^{(p-\delta)/p}$$

$$\leqslant C\|h^{p/(p-\delta)}\|^{(p-\delta)/p}_{M_X^{s,ps}(B)} \cdot \left(\int_B |\varphi|^p \,\mathrm{d}x\right)^{\delta/p}$$

$$\cdot \left(\int_B |X\varphi|^p \,\mathrm{d}x\right)^{(p-\delta)/p}. \tag{4.4}$$

Combining (4.3) and (4.4) we infer the conclusion of the Lemma. □

LEMMA 4.5. *Let* $\varepsilon \in (0,1)$, $f \in M_X^{q,(1-\varepsilon)q}(B)$ *with* $p < q < Q/(1-\varepsilon)$, *and set* $\sigma = 1 - ps/q$ *for* $\max(1, (q/p)(1-\varepsilon)) < s < \min(q/p, Q/p)$. *Then there exists* $C = C(U, X) > 0$ *such that for any* $\varphi, \psi \in \overset{\circ}{S}{}^{1,p}(B)$

$$\int_B f\varphi|\psi|^{p-1} \,\mathrm{d}x$$

$$\leqslant CR^{\varepsilon-\sigma}\|f\|_{M_X^{q,(1-\varepsilon)q}(B)} \left(\int_B |\varphi|^p \,\mathrm{d}x\right)^{\sigma/p}$$

$$\cdot \left(\int_B |X\varphi|^p \,\mathrm{d}x\right)^{(1-\sigma)/p} \left(\int_B |\psi|^p \,\mathrm{d}x\right)^{(p-1)/p}.$$

Proof. Since it is completely analogous to the one of Lemma 4.2, we omit it. □

Using Lemmas 4.1, 4.2, and 4.5 one can prove the following.

THEOREM 4.6 (Harnack inequality). *Let* $u \in \mathcal{L}^{1,p}_{\mathrm{loc}}(U)$ *be a nonnegative solution to* (1.9). *Then*

$$\operatorname*{ess\,sup}_B u \leqslant C(\operatorname*{ess\,inf}_B u + K).$$

Here, K denotes a positive constant which depends solely on the norms of the functions f_i, g_i, h_i, appearing in the structural assumptions (S), in the appropriate Morrey spaces.

Since the proof of Theorem 4.6 follows very closely the one of Theorem 3.1 in [CDG1] we will not present it and leave the easy modifications to the interested reader. We only mention that one has to rely on Lemmas 4.1, 4.2, and 4.5, instead of Hölder's inequality and Sobolev embedding theorem, in order to bound the relevant integrals involving the 'lower order terms' f_i, g_i, h_i.

In a classical fashion, it is possible to infer from Theorem 4.6 the Hölder continuity of solutions to (1.9) with respect to the control distance d.

THEOREM 4.7. *Let* $u \in \mathcal{L}^{1,p}_{\mathrm{loc}}(U)$ *be a* (*weak*) *solution to* (1.9), *and suppose*

$$\operatorname*{ess\,sup}_U |u| = M < \infty.$$

Then, there exist $C > 0$ *and* $0 < \alpha < 1$, *depending on* U *and* M, *such that*

$$\operatorname*{ess\,sup}_{x,y\in U} \frac{|u(x) - u(y)|}{d(x, y)^\alpha} \leqslant C.$$

REMARK. We explicitly observe that our results here properly include the ones in [CDG1], thanks to the inclusions

$$L^{Q/(p-1)}(U) \overset{\subset}{\neq} M_X^{q,q(p-1)}(U), \qquad \frac{p}{p-1} < q \leqslant \frac{Q}{p-1},$$

$$L^{Q/(p-\varepsilon)}(U) \overset{\subset}{\neq} M_X^{q,q(p-\varepsilon)}(U), \qquad 1 < q < \frac{Q}{p-\varepsilon},$$

$$L^{Q/(1-\varepsilon)}(U) \overset{\subset}{\neq} M_X^{q,q(1-\varepsilon)}(U), \qquad p < q < \frac{Q}{1-\varepsilon}.$$

Moreover, Theorem 4.6 generalizes to the subelliptic context the Harnack inequality obtained in [Za] for quasilinear elliptic equations. Finally, we would like to compare our results with the ones obtained by G. Lu ([L5]). In [L5], the author considers a wider class of equations. Namely, the second inequality in (S) is there replaced by

$$|f(x, u, \xi)| \leqslant b_0|\xi|^p + f_1(x)|\xi|^{p-1} + f_2(x)|u|^{p-1} + f_3(x).$$

When $b_0 = 0$, however, our integrability assumptions on the functions f_i, g_i, h_i are weaker than the ones requested by Lu and therefore, in this respect, our results are more general than the ones in [5].

5. An Extension of a Result by V. G. Maz'ya

In this section we prove that the embedding

$$\int_B |u(x)|^p |V(x)|\,\mathrm{d}x \leqslant C \int_B |Xu|^p\,\mathrm{d}x, \qquad u \in \overset{\circ}{S}{}^{1,p}(B), \tag{5.1}$$

holds under suitable generalizations of the criteria (1.11) and (1.12), established by Maz'ya [M1].

We need to recall the notion of *contact p-capacity* associated to a system $X = \{X_1, \dots, X_m\}$ (see [D3], [CDG3]).

DEFINITION 5.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set, and $K \subset \Omega$ be compact. For $1 \leqslant p < Q$ we define the (X, p)-capacity of the condenser (K, Ω) as*

$$\mathrm{cap}_p(K, \Omega) = \inf \left\{ \int_\Omega |Xu|^p \, \mathrm{d}x \mid u \in C_0^\infty(\Omega), u = 1 \text{ in } K \right\}.$$

Our first result is a sufficient condition for the inequality (5.1).

THEOREM 5.3. *Let $U \subset \mathbb{R}^n$ be a bounded set with homogeneous dimension Q, and let R_0 be as in the Hypothesis. Assume $1 \leqslant p < Q$, $0 < R < R_0$, $x_0 \in U$ and $V \in L^1_{\mathrm{loc}}(B)$, where $B = B(x_0, R)$. If*

$$\sup_{\substack{K \subset \mathbb{R}^n \\ K \text{ compact}}} \frac{\int_K |V(x)| \, \mathrm{d}x}{\mathrm{cap}_p(K, B)} \leqslant \beta,$$

then there exists a positive constant C_p such that (5.1) holds with $C = C_p \beta$.

This result is a straightforward consequence of the following.

THEOREM 5.4. *Assume p and B are as in Theorem 5.3. Let $u \in \overset{\circ}{S}{}^{1,p}(B)$ and, for $t > 0$, define $\mathcal{N}_t = \{x \in B \mid |u(x)| \geqslant t\}$. Then*

$$\int_0^\infty \mathrm{cap}_p(\mathcal{N}_t, B) d(t^p) \leqslant C_p \int_B |Xu|^p \, \mathrm{d}x,$$

with $C_p = 2^{2p-1}$.

Since the proof of this theorem is a step by step repetition of its classical counterpart, we omit it and refer the reader to, e.g., [M2, Rem. 4.1] or [M3, Rem. 2.3.1].

We explicitly observe that in the Euclidean context the choice of the constant C_p in Theorem 5.4 is not optimal. In fact, Maz'ya [M2, Thm 4.3], see also [M3, Thm 2.3.1], proved that C_p can be chosen as $p^p/((p-1)^{p-1})$ ($C_p = 1$ when $p = 1$). A generalization of this sharper result to the present context would require a nontrivial extension of the subelliptic coarea formula presently available (see [GN1]), which goes beyond the scope of this paper. For the applications to partial differential equations, however, a non-optimal constant suffices.

Since it is clear that the inequality

$$\sup_{\substack{K \subset \mathbb{R}^n \\ K \text{ compact}}} \frac{\int_K |V(x)| \, \mathrm{d}x}{\mathrm{cap}_p(K, B)} < \infty \tag{5.5}$$

is necessary for (5.1), we immediately obtain the following characterization.

COROLLARY 5.6. *Under the assumptions of Theorem* 5.3, *the inequality* (5.1) *holds if, and only if, condition* (5.5) *is satisfied*.

Next, we recall the subelliptic Sobolev embedding theorem (see [D2], [CDG1], [L1], [L2] and also [CDG2], [FGaW] for the case $p = 1$)).

THEOREM 5.7. *Let* p, Q, B, R *be as in Theorem* 5.3. *Then there exists* $C > 0$ *such that for any* $u \in \overset{\circ}{S}{}^{1,p}(B)$ *one has*

$$\left(\frac{1}{|B|}\int_B |u|^{kp}\,\mathrm{d}x\right)^{1/kp} \leqslant CR\left(\frac{1}{|B|}\int_B |Xu|^p\,\mathrm{d}x\right)^{1/p},$$

where $1 \leqslant k \leqslant Q/(Q-p)$.

We can now prove the following 'isoperimetric' inequality.

PROPOSITION 5.8. *Let* p, Q, B, R *be as in Theorem* 5.3. *Then, there exists* $C > 0$ *such that*

$$\mathrm{cap}_p(K,B) \geqslant C|K|^{(Q-p)/Q}\left(\frac{|B|^{1/Q}}{R}\right)^p,$$

for any compact set $K \subset B$.

Proof. Let $\phi \in C_0^\infty(B)$, $\phi = 1$ in K. By Theorem 5.7,

$$\left(\frac{1}{|B|}\int_B |\phi|^{kp}\,\mathrm{d}x\right)^{1/kp} \leqslant CR\left(\frac{1}{|B|}\int_B |X\phi|^p\,\mathrm{d}x\right)^{1/p}.$$

On the other hand, clearly

$$\left(\frac{1}{|B|}\int |\phi|^{pQ/(Q-p)}\,\mathrm{d}x\right)^{(Q-p)/pq} \geqslant \left(\frac{|K|}{|B|}\right)^{(Q-p)/pq}.$$

Hence

$$\left(\frac{|K|}{|B|}\right)^{(Q-p)/Q} \leqslant C\frac{R^p}{|B|}\int_B |X\phi|^p\,\mathrm{d}x.$$

The conclusion follows from the definition of (X,p)-capacity. □

At this point we observe that, as a consequence of the doubling condition (1.3), the quotient $|B|^{1/Q}/R$ is bounded from below. Keeping this fact in mind and combining Theorem 5.3 and Proposition 5.8, we obtain a non-capacitary criterion for (5.1).

COROLLARY 5.9. *Under the assumptions of Theorem* 5.3, *suppose that*

$$\sup_{\substack{K\subset\mathbb{R}^n\\ K\ \text{compact}}} \frac{\int_K |V|\,\mathrm{d}x}{|K|^{(Q-p)/Q}} \leqslant \beta,$$

for some $\beta > 0$. *Then* (5.1) *holds.*

Acknowledgements

We would like to thank B. Franchi and R. L. Wheeden for making their yet unpublished manuscript [FW] available to us. We also thank D. M. Nhieu for helpful conversations and the referee for several valuable comments, which have improved the exposition of the paper.

References

[B] Baouendi, M. S.: Sur une classe d'opérateurs elliptiques dégénérés, *Bull. Soc. Math. France* **195** (1967), 45–87.

[BKL] Buckley, S., Koskela, P., and Lu, G.: *Boman Equals John*, Proc. of the 16th Nevanlinna Coll., (Joensuu, 1995), de Gruyter, Berlin (1996).

[C] Calderón, A. P.: Inequalities for the maximal function relative to a metric, *Studia Math*. **57** (1976), 297–306.

[CDG1] Capogna, L., Danielli, D., and Garofalo, N.: An embedding theorem and the Harnack inequality for nonlinear subelliptic equations, *Comm. P.D.E*. **18** (1993), 1765–1794.

[CDG2] Capogna, L., Danielli, D., and Garofalo, N.: The geometric Sobolev embedding for vector fields and the isoperimetrc inequality, *Comm. Anal. Geom*. **2**(2) (1994), 203–215.

[CDG3] Capogna, L., Danielli, D., and Garofalo, N.: Capacitary estimates and the local behavior of solutions of nonlinear subelliptic equations, *Amer. J. Math*. **118** (1996), 1153–1196.

[CDG4] Capogna, L., Danielli, D., and Garofalo, N.: Subelliptic mollifiers and a basic pointwise estimate of Poincaré type, *Math. Zeit*. **226** (1997), 147–154.

[CG] Capogna, L., and Garofalo, N.: *NTA Domains for Carnot–Carathéodory Metrics and a Fatou Type Theorem*, preprint (1995).

[CWW] Chang, S. Y. A., Wilson, J. M. and Wolff, T. H.: Some weighted norm inequalities concerning the Schrödinger operators, *Comment. Math. Helvetici* **60** (1985), 217–246.

[ChW] Chanillo, S., and Wheeden, R. L.: L^p estimates for fractional integrals and Sobolev inequalities with applications to Schrödinger operators, *Comm. P.D.E*. **10** (1985), 1077–1166.

[CF] Chiarenza, F. and Frasca, M.: A remark on a paper by C. Fefferman, *Proc. Amer. Math. Soc*. **108** (1990), 407–409.

[CR] Coifman, R. and Rochberg, R.: Another characterization of BMO, *Proc. Amer. Math. Soc*. **79** (1980), 249–254.

[CW] Coifman, R. and Weiss, G.: *Analyse Harmonique Non-commutative sur Certains Espaces Homogenes*, Springer-Verlag (1971).

[D1] Danielli, D.: A compact embedding theorem for a class of degenerate Sobolev spaces, *Rend. Sem. Mat. Torino* **49** (1991), 399–420.

[D2] Danielli, D.: Formules de représentation et théoreèmes d'inclusion pour des opérateurs sous-elliptiques, *C. R. Acad. Sc., Paris, Serie I* **314** (1992), 987–990.

[D3] Danielli, D.: Regularity at the boundary for solutions of nonlinear subelliptic equations, *Indiana Univ. Math. Jour*. **44**(1) (1995), 269–286.

[F] Fefferman, C.: The uncertainty principle, *Bull. Amer. Math. Soc*. **9** (1983), 129–206.

[FP] Fefferman, C. and Phong, D. H.: Pseudo-differential operators with positive symbols, *Seminaire Goulaouic-Meyer-Schwartz* **23** (1981).

[FSC] Fefferman, C. and Sanchez-Calle, A.: Fundamental solutions for second order subelliptic operators, *Ann. Math*. **124** (1986), 247–272.

[FS] Fefferman, C. and Stein, E. M.: Some maximal inequalities, *Amer. J. Math*. **93** (1971), 107–115.

[FoS] Folland, G. B. and Stein, E. M.: *Hardy Spaces on Homogeneous Groups*, Princeton Univ. Press (1982).

[Fr] Franchi, B.: Weighted Sobolev–Poincaré inequalities and pointwise estimates for a class of degenerate elliptic equations, *Trans. Amer. Math. Soc*. **327** (1991), 125–158.

[FGaW] Franchi, B., Gallot, S. and Wheeden, R.: Sobolev and isoperimetric inequalities for degenerate metrics, *Math. Ann*. **300** (1994), 557–571.

[FGW] Franchi, B., Gutierrez, C. E. and Wheeden, R. L.: Weighted Sobolev–Poincaré inequalities for Grushin type operators, *Comm. P.D.E*. **19** (1994), 523–604.

[FL1] Franchi, B. and Lanconelli, E.: *Une Metrique Associeé à une Classe D'operateurs Elliptiques Degénérés*, Proceedings of the meeting 'Linear Partial and Pseudo Differential Operators', Rend. Sem. Mat. Torino (1984), 105–114.

[FL2] Franchi, B. and Lanconelli, E.: Hölder regularity theorem for a class of linear non uniform elliptic operators with measurable coefficients, *Ann. Scuola Norm. Sup. Pisa (IV)* **10** (1983), 523–451.

[FL3] Franchi, B. and Lanconelli, E.: Une condition géométrique pour l'inégalité de Harnack, *J. Math. Pures Appl*. **64** (1985), 237–256.

[FLW1] Franchi, B., Lu, G. and Wheeden, R. L.: Representation formulas and weighted Poincaré inequalities for Hörmander vector fields, *Ann. Inst. Fourier Grenoble* **45** (1995), 577–604.

[FLW2] Franchi, B., Lu, G. and Wheeden, R. L.: A relationship between Poincaré type inequalities and representation formulas in spaces of homogeneous type, *Inter. Math. Res. Not*. **1** (1996), 1–14.

[FSe] Franchi, B. and Serapioni, R.: Pointwise estimates for a class of strongly degenerate elliptic operators, *Ann. Scuola Norm. Sup. Pisa (IV)* **14** (1987), 527–568.

[FW] Franchi, B. and Wheeden, R. L.: *Some Remarks about Poincaré Type Inequalities and Representation Formulas in Metric Spaces of Homogeneous Type*, preprint.

[GR] Garcia Cuerva, J. and Rubio de Francia, J. L.: Weighted norm inequalities and related topics, *North-Holland Mat. Stud*. **116** (1985).

[GN1] Garofalo, N. and Nhieu, D. M.: Isoperimetric and Sobolev inequalities for Carnot–Carathéodory spaces and the existence of minimal surfaces, *Comm. Pure Appl. Math*. **10** (1996), 1081–1144.

[GN2] Garofalo, N. and Nhieu, D. M.: Lipschitz continuity, global smooth approximation and extension theorems for Sobolev functions in Carnot-Carathéodory spaces, *Jour. d'Anal. Math*. to appear.

[G] Grushin, V. V.: On a class of hypoelliptic operators, *Math USSR Sbornick* **12**(3) (1970), 458–476.

[Ha] Hajlasz, P.: Sobolev spaces on an arbitrary metric space, *Pot. Anal*. **5** (1996), 403–415.

[He] Hedberg, L.: On certain convolution inequalities, *Proc. Amer. Math. Soc*. **36** (1972), 505–510.

[HW] Hedberg, L. and Wolff, T. H.: Thin sets in nonlinear potential theory, *Ann. Inst. Fourier Grenoble* **23** (1983), 161–187.

[H] Hörmander, L.: Hypoelliptic second-order differential equations, *Acta Math*. **119** (1967), 147–171.

[J] Jerison, D. S.: The Poincaré inequality for vector fields satisfying Hörmander condition, *Duke Math. Jour*. **53** (1986), 503–523.

[KS] Kerman, R. and Sawyer, E. T.: The trace inequality and eigenvalue estimates for Schrödinger operators, *Ann. Inst. Fourier Grenoble* **36** (1986), 207–228.

[L1] Lu, G.: Weighted Poincaré and Sobolev inequalities for vector fields satisfying Hörmander's condition and applications, *Rev. Mat. Iberoamericana* **8** (1992), 367–439.

[L2] Lu, G.: The sharp Poincaré inequality for free vector fields: an endpoint result, *Rev. Mat. Iberoamericana* **18** (1994), 453–466.

[L3] Lu, G.: Embedding theorems on Campanato–Morrey spaces for vector fields and applications, *C. R. Acad. Sci., Paris, Serie I* **320** (1995), 429–434.

[L4] Lu, G.: Embedding theorems on Campanato–Morrey spaces for vector fields of Hörmander type, *Approx. Theory Appl*., to appear.

[L5] Lu, G.: Embedding theorems into Lipschitz and BMO spaces and applications to quasilinear subelliptic differential equations, *Publ. Matem*. **40** (1996), 301–329.

[M1] Maz'ya, V. G.: The negative spectrum of the n-dimensional Schrödinger operator, *Dokl. Akad. Nauk SSSR* **144** (1962), 721–722 (Russian). English translation: *Soviet Math. Dokl*. **3** (1962), 808–810.

[M2] Maz'ya, V. G.: On some integral inequalities for functions of many variables. In the book: *Problems in Mathematical Analysis*, Leningrad: LGU, **3** (1972), p. 33–68 (Russian). English translation: *Jour. Soviet Math*. **1** (1973), 205–234.

[M3] Maz'ya, V. G.: *Sobolev spaces*, Springer-Verlag (1985).

[MV] Maz'ya, V. G. and Verbitsky, I. E.: Capacitary inequalities for fractional integrals, with applications to partial differential equations and Sobolev multiplier, *Ark. Mat*. **33** (1995), 81–115.

[NSW] Nagel, A., Stein, E. M. and Wainger, S.: Balls and metrics defined by vector fields I: Basic properties, *Acta Math*. **155** (1985), 103–147.

[OR] Oleinik, O. A. and Radkevich, E. V.: *Second Order Equations with Non-negative Characteristic Form* (Mathematical Analysis 1969), Moscow: Itogi Nauki (1971) [Russian], English translation: Providence, R.I., Amer. Math. Soc. (1973).

[PS] Phillips, R. S. and Sarason, L.: Elliptic-parabolic equations of the second order, *J. Math. Mech.* **17** (1967/8), 891–917.

[RS] Rothschild, L. and Stein, E. M.: Hypoelliptic differential operators and nilpotent groups, *Acta Math.*, **137** (1976), 247–320.

[SC] Sanchez-Calle, A.: Fundamental solutions and geometry of sum of squares of vector fields, *Inv. Math*. **78** (1984), 145–160.

[Sa] Sawyer, E. T.: *Weighted Norm Inequalities for Fractional Maximal Operators*, 1980 Seminar on Harmonic Analysis (Montréal, Que., 1980) Herz, C. and Rigelhot, R., (eds), 283–309, CMS Conf. Proc. 1, Amer. Math. Soc., Providence, R.I., 1981.

[Sc] Schecther, M.: *Spectra of Partial Differential Operators*, North–Holland (1971).

[Se1] Serrin, J.: *Local behavior of solutions of quasilinear equations*, Acta Math. **111** (1964), 247–302.

[Se2] Serrin, J.: Isolated singularities of solutions of quasilinear equations, *Acta Math*. **113** (1965), 219–240.

[S1] Stein, E. M.: *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press (1970).

[S2] Stein, E. M.: *Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton Univ. Press (1993).

[Xu] Xu, C. J.: Subelliptic variational problems, *Bull. Soc. Math. France* **118** (1990), 147–169.

[Za] Zamboni, P.: Harnack's inequality for quasilinear elliptic equations with coefficients in Morrey spaces, *Rend. Sem. Mat. Univ. Padova* **89** (1993), 87–95.

[Z] Ziemer, W. P.: *Weakly Differentiable Functions*, Springer-Verlag (1989).