Potential Anal (2010) 32:343–362
DOI 10.1007/s11118-009-9154-4

Local Behavior of *p*-harmonic Green's Functions in Metric Spaces

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Received: 17 September 2008 / Accepted: 1 September 2009 / Published online: 23 September 2009

Abstract We describe the behavior of p-harmonic Green's functions near a singularity in metric measure spaces equipped with a doubling measure and supporting a Poincaré inequality.

Keywords Capacity · Doubling measure · Green function · Metric space · Minimizer · Newtonian space · p-Dirichlet integral · p-harmonic · p-Laplace equation and Poincaré inequality · Singular function · Sobolev space

Mathematics Subject Classifications (2000) 31C45 · 35J60

1 Introduction

Holopainen and Shanmugalingam [21] constructed in the metric measure space setting a p-harmonic Green's function, called a singular function there, having most of the characteristics of the Green's function of the Laplace operator. A p-harmonic Green's function lacks, however, one important property: it cannot be used to solve the boundary value problem.

This paper is dedicated to the memory of Professor Juha Heinonen.

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In this paper we study the following question related to the local behavior of a p-harmonic Green's function in a locally doubling metric measure space X supporting a local $(1, p)$-Poincaré inequality: Given a relatively compact domain $\Omega \subset X$, $x \in \Omega$, and a p-harmonic Green's function G with a singularity at x, then can we describe the behavior of G near x?

Capacitary estimates for metric rings play an important role in the study of the asymptotic behavior. Following the ideas in the works of Serrin [37, 38], (see also [32]) such estimates were used in Capogna et al. [7] to establish the local behavior of singular solutions to a large class of nonlinear subelliptic equations which arise in the Carnot–Carathéodory geometry. Sharp capacitary estimates for metric rings with unrelated radii were established in the metric measure space setting in [14].

Here, we confine ourselves to mention that a fundamental example of the spaces included in this paper is obtained by endowing a connected Riemannian manifold M with the Carathéodory metric d associated with a given subbundle of the tangent bundle, see [8, 33]. If such subbundle generates the tangent space at every point, then thanks to the theorem of Chow [11] and Rashevsky [34] (M, d) is a metric space. Such metric spaces are known as sub-Riemannian or Carnot–Carathéodory (CC) spaces. By the fundamental works of Rothschild and Stein [35], Nagel et al. [33], and of Jerison [22], every CC space is locally doubling, and it locally satisfies a (p, p)-Poincaré inequality for any $1 \leq p < \infty$. Another basic example is provided by a Riemannian manifold (M^n, g) with nonnegative Ricci tensor. In such case thanks to the Bishop comparison theorem the doubling condition holds globally, see e.g. [9], whereas a global $(1, 1)$-Poincaré inequality was proved by Buser [6]. An interesting example to which our results apply and that does not fall in any of the two previously mentioned categories is the space of two infinite closed cones $X = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \; x_1^2 + \ldots + x_{n-1}^2 \leq x_n^2\}$ equipped with the Euclidean metric of $\mathbb{R}^n$ and with the Lebesgue measure. This space is Ahlfors regular, and it is shown in Hajłasz–Koskela [17, Example 4.2] that a $(1, p)$-Poincaré inequality holds in X if and only if $p > n$. Another example is obtained by gluing two copies of closed n-balls $\{x \in \mathbb{R}^n : \; |x| \leq 1\}$, $n \geq 3$, along a line segment. In this way one obtains an Ahlfors regular space that supports a $(1, p)$-Poincaré inequality for $p > n - 1$ [31] (see also A. Björn and J. Björn, Nonlinear Potential Theory on Metric Spaces, in preparation). A thorough overview of analysis on metric spaces can be found in Heinonen [18]. One should also consult Semmes [36] and David and Semmes [12].

Our main result in this paper is a quantitative description of the local behavior of a p-harmonic Green's function defined in Holopainen–Shanmugalingam [21]. We shall prove that a Green's function G with a singularity at x_0 in a relatively compact domain satisfies the asymptotic behavior

$$G(x) \approx \left(\frac{d(x, x_0)^p}{\mu(B(x_0, d(x, x_0)))} \right)^{1/(p-1)},$$

where x is uniformly close to x_0. Our approach uses upper gradients à la Heinonen and Koskela [19], and p-harmonic functions that can be characterized in terms of p-energy minimizers among functions with the same boundary values in relatively compact subsets. Following [21] we adopt a definition of the Green's function that uses inequalities for p-capacities of level sets.

We want to stress the fact that even in Carnot groups of homogeneous dimension Q it is not presently known whether such p-harmonic Green's function is unique when $1 < p < Q$. However, in the conformal case, i.e. when $p = Q$, such uniqueness was settled by Balogh et al. in [1].

The paper is organized as follows. In Section 2 we have collected the relevant background, such as the definition of doubling measures, upper gradients, Poincaré inequality, Newton–Sobolev spaces, and capacity. In Section 3 we recall the sharp capacitary estimates for metric rings with unrelated radii recently proved in Garofalo–Marola [14]. In Section 4 we give the definition of the Green's function. We establish its local behavior in Section 5, where we also prove a result on its local integrability. Section 6 closes the paper with a result on the local behavior of Cheeger singular functions. In this section our approach uses Cheeger gradients (see Cheeger [10]) emerging from a differentiable structure that the ambient metric space admits. In particular, p-harmonic functions can thus be characterized in terms of a weak formulation of the p-Laplace equation.

2 Preliminaries

We begin by stating the main assumptions we make on the metric space X and the measure μ.

2.1 General Assumptions

Throughout the paper $X = (X, d, \mu)$ is a locally compact metric space endowed with a metric d and a positive Borel regular measure μ. We assume that for every compact set $K \subset X$ there exist constants $C_K \geq 1$, $R_K > 0$ and $\tau_K \geq 1$, such that for any $x \in K$ and every $0 < r \leq R_K$, $0 < \mu(B) < \infty$, where $B := B(x, r) := \{y \in X : \ d(y, x) < r\}$, and, in particular, one has:

(i) the closed balls $\overline{B}(x, r) = \{y \in X : d(y, x) \leq r\}$ are compact;
(ii) (local doubling condition) $\mu(B(x, 2r)) \leq C_K \mu(B(x, r))$;
(iii) (local weak $(1, p_0)$-Poincaré inequality) there exists $1 < p_0 < \infty$, p_0 does not depend on K, such that for all measurable functions u on X and all upper gradients g_u (see Section 2.3) of u

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq C_K r \Big(\fint_{B(x,\tau_K r)} g_u^{p_0} \, d\mu \Big)^{1/p_0},$$

where $u_{B(x,r)} := \fint_{B(x,r)} u \, d\mu := \int_{B(x,r)} u \, d\mu / \mu(B(x, r))$.

Hereafter, the constants C_K, R_K and τ_K will be referred to as the *local parameters* of K. We also say that a constant C depends on the local doubling constant of K if C depends on C_K.

As we have mentioned in the introduction the above assumptions (i)–(iii) are fulfilled in a wide variety of situations. For instance, all complete Riemannian manifolds with $\mathrm{Ric} \geq 0$ satisfy them globally, in the sense that they hold for all $x \in X$, and every $0 < r < \infty$. Another situation in which the above hypothesis hold globally is that of Carnot groups. More in general, the above hypothesis are fulfilled in all Carnot–Carathéodory spaces. For a detailed discussion of these facts we refer the reader to the papers [15] and [16]. In the case of Carnot–Carathéodory spaces, recall that if the Lie algebra generating vector fields grow at infinity faster than linearly, then the compactness of metric balls of large radii may fail in general. Consider for instance in $\mathbb{R}$ the smooth vector field $X_1 = (1 + x^2)\frac{d}{dx}$. Some direct calculations prove that the distance relative to X_1 is given by $d(x, y) = |\arctan(x) - \arctan(y)|$, and therefore, if $r \geq \pi/2$, we have $B(0, r) = \mathbb{R}$, see [15].

2.2 Local Doubling Property

We note that assumption (ii) implies that for every compact set $K \subset X$ with local parameters C_K and R_K, for any $x \in K$ and every $0 < r \leq R_K$, one has for $1 \leq \lambda \leq R_K/r$,

$$\mu(B(x, \lambda r)) \leq C\lambda^Q \mu(B(x, r)), \tag{2.1}$$

where $Q = \log_2 C_K$, and the constant C depends only on the local doubling constant C_K. The exponent Q serves as a local dimension of the doubling measure μ restricted to the compact set K.

For $x \in X$ we define the *pointwise dimension* $Q(x)$ by

$$\begin{aligned} Q(x) = \sup\{q > 0 :\ \exists C > 0 \ \text{ such that} \\ \lambda^q \mu(B(x, r)) \leq C\mu(B(x, \lambda r)), \\ \text{for all } 1 \leq \lambda < \operatorname{diam} X \text{ and } 0 < r < \infty\}. \end{aligned}$$

The inequality (2.1) readily implies that $Q(x) \leq Q$ for every $x \in K$. Moreover, it follows that

$$\lambda^{Q(x)} \mu(B(x, r)) \leq C\mu(B(x, \lambda r)) \tag{2.2}$$

for any $x \in K$, $0 < r \leq R_K$ and $1 \leq \lambda \leq R_K/r$, and the constant C depends on the local doubling constant C_K. Furthermore, for all $0 < r \leq R_K$ and $x \in K$

$$C_1 r^Q \leq \frac{\mu(B(x, r))}{\mu(B(x, R_K))} \leq C_2 r^{Q(x)}, \tag{2.3}$$

where $C_1 = C(K, C_K)$ and $C_2 = C(x, K, C_K)$.

For more on doubling measures, see, e.g. Heinonen [18] and the references therein.

2.3 Upper Gradients

A nonnegative Borel function g on X is an *upper gradient* of an extended real valued function f on X if for all rectifiable paths γ joining points x and y in X we have

$$|f(x) - f(y)| \leq \int_\gamma g\, ds. \tag{2.4}$$

whenever both $f(x)$ and $f(y)$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise. See Cheeger [10], Shanmugalingam [39], and Heinonen–Koskela [19] for a discussion on upper gradients.

If g is a nonnegative measurable function on X and if Eq. 2.4 holds for p-almost every path, then g is a *weak upper gradient* of f. By saying that Eq. 2.4 holds for p-almost every path we mean that it fails only for a path family with zero p-modulus (see, for example, [39]).

If f has an upper gradient in $L^p(X)$, then it has a *minimal weak upper gradient* $g_f \in L^p(X)$ in the sense that for every p-weak upper gradient $g \in L^p(X)$ of f, $g_f \leq g$ μ-almost everywhere (a.e.), see Corollary 3.7 in Shanmugalingam [40]. The minimal weak upper gradient can be obtained by the formula

$$g_f(x) := \inf_g \limsup_{r\to 0+} \fint_{B(x,r)} g\, d\mu,$$

where the infimum is taken over all upper gradients $g \in L^p(X)$ of f, see Lemma 2.3 in Björn [4].

2.4 Newtonian Spaces

We define Sobolev spaces on the metric space following Shanmugalingam [39]. Let $\Omega \subseteq X$ be nonempty and open. Whenever $u \in L^p(\Omega)$, let

$$\|u\|_{N^{1,p}(\Omega)} = \left(\int_\Omega |u|^p\, d\mu + \inf_g \int_\Omega g^p\, d\mu\right)^{1/p},$$

where the infimum is taken over all weak upper gradients of u. The *Newtonian space* on Ω is the quotient space

$$N^{1,p}(\Omega) = \{u : \|u\|_{N^{1,p}(\Omega)} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(\Omega)} = 0$. The Newtonian space is a Banach space and a lattice, moreover Lipschitz functions are dense, see [39] and Björn et al. [2]. In particular, we have the following lemma [10].

Lemma 2.1 *Let* $u,\ v \in N^{1,p}(X)$*. Then* $g_u = g_v$ μ*-a.e. on* $\{x \in X :\ u(x) = v(x)\}$*, and if* $c \in \mathbb{R}$*, then* $g_u = 0$ μ*-a.e. on* $\{x \in X :\ u(x) = c\}$*.*

To be able to compare the boundary values of Newtonian functions we need a Newtonian space with zero boundary values. Let E be a measurable subset of X. The *Newtonian space with zero boundary values* is the space

$$N_0^{1,p}(E) = \{u|_E : u \in N^{1,p}(X) \text{ and } u = 0 \text{ on } X \setminus E\}.$$

The space $N_0^{1,p}(E)$ equipped with the norm inherited from $N^{1,p}(X)$ is a Banach space, see Theorem 4.4 in Shanmugalingam [40].

We say that u belongs to the *local Newtonian space* $N_{\mathrm{loc}}^{1,p}(\Omega)$ if $u \in N^{1,p}(\Omega')$ for every open $\Omega' \Subset \Omega$ (or equivalently that $u \in N^{1,p}(E)$ for every measurable $E \Subset \Omega$).

We will also need an inequality for Newtonian functions with zero boundary values. If $K \subset X$ is a compact set, and C_K, R_K, τ_K are the local parameters of K (as in (i)–(iii) above), then for $f \in N_0^{1,p}(B(x,r))$, $x \in K$ and $0 < r \leq R_K$, there exists a constant $C > 0$ depending on $p > 1$ and K via the local doubling constant, and the constants in the weak Poincaré inequality, such that

$$\left(\fint_{B(x,r)} |f|^p\, d\mu\right)^{1/p} \leq Cr\left(\fint_{B(x,r)} g_f^p\, d\mu\right)^{1/p} \tag{2.5}$$

for every ball $B(x,r)$ with $r \leq \frac{1}{3}\operatorname{diam} K$. For this result we refer to Kinnunen and Shanmugalingam [29].

2.5 Capacity

Let (K, Ω) be a condenser, i.e., let $\Omega \subset X$ be open and $K \subset \Omega$ compact. The *relative p-capacity* of K with respect to Ω is the number

$$\mathrm{Cap}_p(K, \Omega) = \inf \int_\Omega g_u^p \, d\mu,$$

where the infimum is taken over all functions $u \in N^{1,p}(X)$ such that $u = 1$ on K and $u = 0$ on $X \setminus \Omega$. If such functions do not exist, we set $\mathrm{Cap}_p(K, \Omega) = \infty$.

Observe that the infimum above could be taken over all functions $u \in \mathrm{Lip}_0(\Omega) = \{f \in \mathrm{Lip}(X) : \ f = 0 \text{ on } X \setminus \Omega\}$ such that $u = 1$ on K. In addition, since for us $p > 1$, the relative p-capacity is a Choquet capacity and consequently for all Borel sets E we have

$$\mathrm{Cap}_p(E, \Omega) = \sup\{\mathrm{Cap}_p(K, \Omega) : \ K \subset E, \ K \text{ compact}\}.$$

For other properties as well as equivalent definitions of the capacity we refer to Kilpeläinen et al. [25], Kinnunen–Martio [26, 27], and Kallunki–Shanmugalingam [23].

Finally, we say that a property holds *p-quasieverywhere* (p-q.e.) if the set of points for which the property does not hold is of zero Sobolev p-capacity. The Sobolev p-capacity of a set $E \subset X$ is

$$C_p(E) = \inf \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u = 1$ on E.

2.6 Differentiable Structure

Cheeger [10] demonstrated that metric measure spaces that satisfy assumptions (ii) and (iii) admit a differentiable structure with respect to which a Rademacher–Stepanov theorem holds for Lipschitz functions. This differentiable structure gives rise to an alternative definition of a Sobolev space over the given metric measure space than that of Newtonian space defined above. However, assuming (ii) and (iii) these definitions lead to the same space, see Shanmugalingam [39, Theorem 4.10]. Thanks to a deep theorem by Cheeger the corresponding Sobolev space is reflexive, see [10, Theorem 4.48].

The differentiable structure gives the notion of partial derivatives in the following theorem, see Cheeger [10, Theorem 4.38], and it is compatible with the notion of an upper gradient. We stress here that, while Cheeger's results are stated under the hypothesis that (ii) and (iii) hold globally, their proofs actually only use the local validity of the assumptions themselves, and this is why we can use them in this paper.

Theorem 2.2 (Cheeger) *Let X be a metric measure space equipped with a doubling Borel regular measure μ. Assume that X admits a weak $(1, p_0)$-Poincaré inequality for some $1 < p_0 < \infty$. Then there exist measurable sets U_α with positive measure such that*

$$\mu\left(X \setminus \bigcup_\alpha U_\alpha\right) = 0,$$

and Lipschitz "coordinate charts"

$$\mathcal{X}^\alpha = \left(X_1^\alpha, \ldots, X_{k(\alpha)}^\alpha\right) : X \to \mathbb{R}^{k(\alpha)}$$

such that for each α functions $X_1^\alpha, \ldots, X_{k(\alpha)}^\alpha$ are all linearly independent as functions on U_α and

$$1 \le k(\alpha) \le N,$$

where N is a constant depending only on the doubling constant of μ and the constants in the Poincaré inequality. Moreover, if $f : X \to \mathbb{R}$ is Lipschitz, then there exist unique (up to a set of measure zero) bounded vector-valued functions $d^\alpha f : U_\alpha \to \mathbb{R}^{k(\alpha)}$ such that

$$\lim_{r\to 0+} \sup_{x\in B(x_0,r)} \frac{|f(x) - f(x_0) - d^\alpha f \cdot (\mathcal{X}^\alpha(x) - \mathcal{X}^\alpha(x_0))|}{r} = 0$$

for μ-a.e. $x_0 \in U_\alpha$, where $(\cdot, \cdot)$ denotes the usual inner product in $\mathbb{R}^{k(\alpha)}$.

We can assume that the sets U_α are pairwise disjoint, and extend $d^\alpha f$ by zero outside U_α. Regard $d^\alpha f$ as vectors in $\mathbb{R}^N$ and let $Df := \sum_\alpha d^\alpha f$. By Shanmugalingam [39, Theorem 4.10] and [10, Theorem 4.47], the Newtonian space $N^{1,p_0}(X)$ is equal to the closure in the N^{1,p_0}-norm of the collection of (locally) Lipschitz functions on X, then the derivation operator D can be uniquely extended to all of $N^{1,p_0}(X)$ so that there exists a constant $C > 0$ such that

$$C^{-1}|Df(x)| \le g_f(x) \le C|Df(x)|$$

for all $f \in N^{1,p_0}(X)$ and μ-a.e. $x \in X$. Here the norms $|\cdot|$ can be chosen to be inner product norms. We mention that the possibility of uniquely extending the operator D to all $N^{1,p_0}(X)$ was also independently established by Franchi, Hajłasz and Koskela in [13]. The differential mapping Df satisfies the product and chain rules: if f is a bounded Lipschitz function on X, $u \in N^{1,p_0}(X)$, and $h : \mathbb{R} \to \mathbb{R}$ is continuously differentiable with bounded derivative, then uf and $h \circ u$ both belong to $N^{1,p_0}(X)$ and

$$D(uf) = uDf + fDu;$$
$$D(h \circ u) = (h \circ u)' Du.$$

See the discussion in Cheeger [10] and Keith [24].

2.7 p-Harmonic Functions

Let $\Omega \subset X$ be a domain (an open connected set). A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\Omega)$ is *p-harmonic* in Ω if for all relatively compact sets $\Omega' \subset \Omega$ and for all $\varphi \in N_0^{1,p}(\Omega')$,

$$\int_{\Omega'} g_u^p \, d\mu \le \int_{\Omega'} g_{u+\varphi}^p \, d\mu.$$

It is known that nonnegative p-harmonic functions satisfy Harnack's inequality and the strong maximum principle, there are no non-constant nonnegative p-harmonic functions on all of X, and p-harmonic functions have locally Hölder continuous representatives. See [29].

A nonnegative p-harmonic function on an annulus $B(y, Cr) \setminus B(y, r/C)$ satisfies Harnack's inequality on the sphere $S(y, r) = \{x \in X : \ d(x, y) = r\}$ for sufficiently small r, see Björn et al. [5, Lemma 5.3] and also Korte [30, Theorem 3.3].

We also say that a function $u \in N^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\Omega)$ is *Cheeger p-harmonic* in Ω if in the above definition upper gradients g_u and $g_{u+\varphi}$ are replaced by $|Du|$ and $|D(u+\varphi)|$, respectively. Note that by a result in Cheeger [10], the Cheeger p-harmonic functions are p-quasiminimizers in the sense of, e.g., Kinnunen–Shanmugalingam [29]. Moreover, the Cheeger p-harmonic functions can be characterized in terms of a weak formulation of the p-Laplace equation: u is Cheeger p-harmonic if and only if

$$\int_{\Omega'} |Du|^{p-2} Du \cdot D\varphi \, d\mu = 0$$

for all Ω' and φ as in the above definition.

3 Capacitary Estimates

The aim of this section is to recall sharp capacity estimates for metric rings with unrelated radii proved in [14]. We emphasize an interesting feature of Theorems 3.1 and 3.3 that cannot be observed in the setting of, for example, Carnot groups. That is the dependence of the estimates on the center of the ring. This is a consequence of the fact that in the general setting $Q(x_0) \neq Q$ where $x_0 \in X$, see Section 2. The results in this section will play an important role in the subsequent developments.

Theorem 3.1 (Estimates from below) *Let $\Omega \subset X$ be a bounded open set, $x_0 \in \Omega$, and $Q(x_0)$ be the pointwise dimension at x_0. Then there exists $R_0(\Omega) > 0$ such that for any $0 < r < R \le R_0(\Omega)$ we have*

$$\mathrm{Cap}_{p_0}(\overline{B}(x_0, r), B(x_0, R)) \ge \begin{cases} C_1\left(1 - \dfrac{r}{R}\right)^{p_0(p_0-1)} \dfrac{\mu(B(x_0, r))}{r^{p_0}}, & \text{if } 1 < p_0 < Q(x_0), \\ C_2\left(1 - \dfrac{r}{R}\right)^{Q(x_0)(Q(x_0)-1)} \left(\log \dfrac{R}{r}\right)^{1-Q(x_0)}, & \text{if } p_0 = Q(x_0), \\ C_3\left(1 - \dfrac{r}{R}\right)^{p_0(p_0-1)} \left|(2R)^{\frac{p_0-Q(x_0)}{p_0-1}} - r^{\frac{p_0-Q(x_0)}{p_0-1}}\right|^{1-p_0}, & \text{if } p_0 > Q(x_0), \end{cases}$$

where

$$C_1 = C\left(1 - \frac{1}{2^{\frac{Q(x_0)-p_0}{p_0-1}}}\right)^{p_0-1},$$

$$C_2 = C\frac{\mu(B(x_0, r))}{r^{Q(x_0)}},$$

$$C_3 = C\frac{\mu(B(x_0, r))}{r^{Q(x_0)}}\left(2^{\frac{p_0-Q(x_0)}{p_0-1}} - 1\right)^{p_0-1},$$

with positive constants C depending only on p_0 and the local doubling constant of Ω.

Remark 3.2 Observe that if X supports the weak $(1,1)$-Poincaré inequality, i.e. $p_0 = 1$, these estimates reduce to the capacitary estimates, e.g., in Capogna et al. [7, Theorem 4.1].

Theorem 3.3 (Estimates from above) *Let Ω, x_0, and $Q(x_0)$ be as in Theorem 3.1. Then there exists $R_0(\Omega) > 0$ such that for any $0 < r < R \le R_0(\Omega)$ we have*

$$\mathrm{Cap}_{p_0}(\overline{B}(x_0,r), B(x_0,R)) \le \begin{cases} C_4 \dfrac{\mu(B(x_0,r))}{r^{p_0}}, & \text{if } 1 < p_0 < Q(x_0), \\ C_5 \left(\log \dfrac{R}{r}\right)^{1-Q(x_0)}, & \text{if } p_0 = Q(x_0), \\ C_6 \left| (2R)^{\frac{p_0-Q(x_0)}{p_0-1}} - r^{\frac{p_0-Q(x_0)}{p_0-1}} \right|^{1-p_0}, & \text{if } p_0 > Q(x_0), \end{cases}$$

where C_4 is a positive constant depending only on p_0 and the local doubling constant of Ω, C_5 is a positive constant depending on p_0, x_0, and the local parameters of Ω, and the last constant has the form

$$C_6 = C\left(2^{\frac{p_0-Q(x_0)}{p_0-1}} - 1\right)^{-1},$$

with a positive constant C depending on p_0, x_0, and the local parameters of Ω.

We have the following immediate corollary.

Corollary 3.4 *If $1 < p_0 \le Q(x_0)$, then we have*

$$\mathrm{Cap}_{p_0}(\{x_0\}, \Omega) = 0.$$

4 Green's Functions

We define a Green's function on metric spaces following Holopainen and Shanmugalingam [21]. We note that these authors referred to such functions as singular functions. We consider here a definition that uses inequalities for p-capacities of level sets. Note that Green's function on a Riemannian manifold satisfies an equation for p-capacities of such level sets, see Holopainen [20].

Definition 4.1 Given $1 < p_0 \le Q(x_0)$, let $\Omega \subset X$ be a relatively compact domain, and $x_0 \in \Omega$. An extended real-valued function $G = G(\cdot, x_0)$ is said to be a *Green's function with singularity at* x_0 if the following criteria are satisfied:

1. G is p_0-harmonic and positive in $\Omega \setminus \{x_0\}$,
2. $G|_{X\setminus\Omega} = 0$ p-quasieverywhere and $G \in N^{1,p_0}_{\mathrm{loc}}(X \setminus B(x_0,r))$ for all $r > 0$,
3. x_0 is a singularity, i.e.,

$$\lim_{x\to x_0} G(x) = \infty.$$

4. whenever $0 \le \alpha < \beta$,

$$C_1(\beta-\alpha)^{1-p_0} \le \mathrm{Cap}_{p_0}\left(\Omega^\beta, \Omega_\alpha\right) \le C_2(\beta-\alpha)^{1-p_0},$$

where $\Omega^\beta = \{x \in \Omega : \ G(x) \ge \beta\}$, $\Omega_\alpha = \{x \in \Omega : \ G(x) > \alpha\}$, and $C_1,\ C_2 > 0$ are constants depending only on p_0.

Remark 4.2 (Existence) The existence of Green's functions in the Q-regular metric space setting was first proved by Holopainen and Shanmugalingam in [21]. Being a Q-regular metric measure space means that the measure μ satisfies, for all balls $B(x,r)$ a double inequality

$$C^{-1}r^Q \le \mu(B(x,r)) \le Cr^Q$$

with a fixed constant Q. There are, however, many instances where the Q-regularity condition is not satisfied. For example, systems of vector fields of Hörmander type are, in general, not Q-regular for any $Q > 0$.

In [14] the Q-regularity assumption was removed and the existence of this function class was proved in more general setting. For the proof of the existence, we refer to [21, Theorem 3.4], see also remarks in [14].

Remark 4.3 (Uniqueness) It is not known whether a Green's function is unique in the metric space setting even in the case of Cheeger p-harmonic functions. Indeed, the uniqueness of Green's functions is not even settled in Carnot groups when $1 < p_0 < Q$, where Q is the homogeneous dimension attached to the non-isotropic dilations. However, in this setting the Green's function is known to be unique when $p_0 = Q$, see Balogh et al. [1], and also [20].

5 Local Behavior of p-Harmonic Green's Functions

We begin by recalling that if $K \subset \Omega$ is closed, $u \in N^{1,p_0}(X)$ is a p_0-*potential* of K (with respect to Ω) if

(i) u is p_0-harmonic on $\Omega \setminus K$;
(ii) $u = 1$ on K and $u = 0$ in $X \setminus \Omega$.

By Lemma 3.3 in Holopainen–Shanmugalingam [21] p_0-potentials always exist if $\mathrm{Cap}_{p_0}(K,\Omega) < \infty$. It readily follows that

$$\mathrm{Cap}_{p_0}(K,\Omega) = \int_{\Omega\setminus K} g_u^{p_0}\, d\mu,$$

where u is the p_0-potential of K with respect to Ω.

From now on, we set

$$m(r) = m_G(x_0, r) = \min\{G(x) : \ d(x,x_0) = r\},$$

$$M(r) = M_G(x_0, r) = \max\{G(x) : \ d(x,x_0) = r\},$$

where G is a Green's function with singularity at x_0. We can now state the following growth estimates for a Green's function near a singularity. In what follows, $R_0(\Omega) > 0$ is the constant from Theorems 3.1 and 3.3.

Lemma 5.1 *Let Ω be a relatively compact domain in X, $x_0 \in \Omega$, and $1 < p \leq Q(x_0)$. If G is a Green's function with singularity at x_0 and $0 < R \leq R_0(\Omega)$ is such that $\overline{B}(x_0, R) \subset \Omega$, then for every $0 < r < R$ we have*

$$m_G(x_0, r) \leq C_1 \left(\frac{1}{\mathrm{Cap}_{p_0}\left(\overline{B}(x_0, r), B(x_0, R)\right)} \right)^{1/(p_0-1)} + M_G(x_0, R).$$

If $r_0 \in (0, R)$ is such that $m_G(x_0, r_0) \geq M_G(x_0, R)$, then for every $0 < r < r_0$ we have

$$M_G(x_0, r) \geq C_2 \left(\frac{1}{\mathrm{Cap}_{p_0}\left(\overline{B}(x_0, r), B(x_0, r_0)\right)} \right)^{1/(p_0-1)} + M_G(x_0, R),$$

where the constants C_1 and C_2 both depend only on p_0.

Proof Consider a radius $R > 0$ such that $\overline{B}(x_0, R) \subset \Omega$. Since $G(x) \to \infty$ when x tends to x_0, the maximum principle implies that

$$m(r) \geq m(\rho), \quad 0 < r < \rho < R. \tag{5.1}$$

Define $w = G - M(R)$, and hence $w \leq 0$ on $\partial B(x_0, R)$. Observe that the first inequality in the theorem obviously holds true if $m(r) \leq M(R)$, thus, we might as well assume that

$$m(r) > M(R), \tag{5.2}$$

and consider the function v in the annulus $B(x_0, R) \setminus \overline{B}(x_0, r)$ defined by

$$v = \begin{cases} 0, & \text{if } G \leq M(R), \\ w, & \text{if } M(R) < G < m(r), \\ m_w(r), & \text{if } G \geq m(r). \end{cases}$$

If we extend v by letting $v = m_w(r)$ on $\overline{B}(x_0, r)$, then $v \in N_0^{1,p_0}(B(x_0, R))$. Our assumption (5.2) implies that $m_w(r) = m(r) - M(R) > 0$, so the function

$$\varphi = \frac{v}{m_w(r)},$$

which equals to 1 in $\overline{B}(x_0, r)$, is both an admissible function for the capacity of $\overline{B}(x_0, r)$ with respect to $B(x_0, R)$ and the p_0-potential of the set $\{x \in X : \varphi(x) \geq 1\}$ with respect to the set $\{x \in X : \varphi(x) > 0\}$. Thus one has

$$\begin{aligned} \mathrm{Cap}_{p_0}(\overline{B}(x_0, r), B(x_0, R)) &\leq \int_{B(x_0,R)\setminus\overline{B}(x_0,r)} g_\varphi^{p_0}\, d\mu \\ &= \mathrm{Cap}_{p_0}(\{x \in X : \varphi(x) \geq 1\}, \{x \in X : \varphi(x) > 0\}) \\ &= \mathrm{Cap}_{p_0}(\{x \in X : G(x) \geq m(r)\}, \{x \in X : G(x) > M(R)\}) \\ &\leq C_1 (m(r) - M(R))^{1-p_0}, \end{aligned}$$

where we used criterion 4 from Definition 4.1 and the fact that $\varphi \geq 1$ or $\varphi > 0$ if and only if $G \geq m(r)$ or $G > M(r)$, respectively. This implies the first claim.

To prove the second inequality of the claim, let $r_0 \in (0, R)$ be such that $m(r_0) \geq M(R)$. This implies that $M(r) \geq M(R)$, for all $0 < r < r_0$. Hence, by the maximum principle we have that

$$\{x \in \Omega : \ G(x) \geq M(r)\} \subset \overline{B}(x_0, r)$$

and

$$B(x_0, r_0) \subset \{x \in \Omega : \ G(x) > M(R)\}.$$

It thus follows that

$$\begin{aligned} \mathrm{Cap}_{p_0}(\overline{B}(x_0, r), B(x_0, r_0)) &\geq \mathrm{Cap}_{p_0}(\{x \in \Omega : \ G(x) \geq M(r)\}, B(x_0, r_0)) \\ &\geq \mathrm{Cap}_{p_0}(\{x \in \Omega : \ G(x) \geq M(r)\}, \{x \in \Omega : \ G(x) > M(R)\}) \\ &\geq C_2(M(r) - M(R))^{1-p_0}, \end{aligned}$$

which implies the second claim and the proof is complete. □

We have the following result on the local behavior of a Green's function near a singularity.

Theorem 5.2 *Let Ω be a relatively compact domain in X, and $x_0 \in \Omega$. If G is a Green's function with singularity at x_0, then there exist positive constants C_1, C_2, R_0, and R_1 such that $R_1 \leq \frac{R_0}{2}$ and for any $0 < r < R_1$ and $x \in B(x_0, r)$ we have*

$$C_1\left(\frac{d(x, x_0)^{p_0}}{\mu(B(x_0, d(x, x_0)))}\right)^{1/(p_0-1)} \leq G(x) \leq C_2\left(\frac{d(x, x_0)^{p_0}}{\mu(B(x_0, d(x, x_0)))}\right)^{1/(p_0-1)},$$

when $1 < p_0 < Q(x_0)$, whereas

$$C_1 \log\left(\frac{R_0}{d(x, x_0)}\right) \leq G(x) \leq C_2 \log\left(\frac{R_0}{d(x, x_0)}\right),$$

when $p_0 = Q(x_0)$. Here the constants C_1 and C_2 depend on p_0, x_0, and the local parameters of Ω, whereas constant R_0 depends only on Ω.

Proof Let $R_0 = \min\{r_0, R_0(\Omega)\}$, where $r_0 > 0$ is from the second estimate in Lemma 5.1. We can choose $0 < R_1 < \frac{R_0}{2}$ such that

$$2M(R_0) \leq m(R_1).$$

Then if $0 < r < R_1$ we have from the first estimate in Lemma 5.1, as $m(R_1) \leq m(r)$, that

$$m(r) \leq C_1\left(\frac{1}{\mathrm{Cap}_{p_0}\left(\overline{B}(x_0, r), B(x_0, R_0)\right)}\right)^{1/(p_0-1)} + \frac{m(r)}{2}. \tag{5.3}$$

The Harnack inequality on a sphere (see Björn et al. [5, Lemma 5.3]) implies that there exists a constant $C > 0$ such that

$$M(r) \leq Cm(r).$$

for every $0 < r < R_1 \leq \frac{R_0}{2}$. Let, in particular, $r := d(x, x_0) < R_1$. From Eq. 5.3, the maximum principle, and the Harnack inequality on spheres, we obtain for any $0 < r < R_1$

$$G(x) \leq M(r) \leq Cm(r) \leq C\,\mathrm{Cap}_{p_0}\left(\overline{B}(x_0, r), B(x_0, R_0)\right)^{-1/(p_0-1)}.$$

Thanks to Theorem 3.1 we have

$$G(x) \leq C\left(1 - \frac{r}{R_0}\right)^{-p_0}\left(\frac{r^{p_0}}{\mu(B(x_0, r))}\right)^{1/(p_0-1)} \leq C\left(\frac{r^{p_0}}{\mu(B(x_0, r))}\right)^{1/(p_0-1)},$$

when $1 < p < Q(x_0)$, and

$$G(x) \leq C\log\left(\frac{R_0}{r}\right),$$

when $p = Q(x_0)$. This proves the estimate from above.

To show the estimate from below, observe that the second estimate in Lemma 5.1, the maximum principle, and the Harnack inequality on a sphere imply for $0 < r < R_0$

$$G(x) \geq m(r) \geq C^{-1}M(r) \geq C\,\mathrm{Cap}_{p_0}\left(\overline{B}(x_0, r), B(x_0, R_0)\right)^{-1/(p_0-1)}$$

Applying Theorem 3.3 we conclude for $1 < p_0 < Q(x_0)$

$$G(x) \geq C\left(\frac{r^{p_0}}{\mu(B(x_0, r))}\right)^{1/(p_0-1)},$$

and for $p_0 = Q(x_0)$ that

$$G(x) \geq C\log\left(\frac{R_0}{r}\right).$$

This completes the proof. □

Remark 5.3 Note that if $1 < p_0 < Q(x_0)$ then due to Eq. 2.3, it readily follows that

$$C_1 d(x, x_0)^{(p_0-Q(x_0))/(p_0-1)} \leq G(x) \leq C_2 d(x, x_0)^{(p_0-Q)/(p_0-1)},$$

when $x \in B(x_0, r)$ with $0 < r < \frac{R_0}{2}$. Here the constants C_1 and C_2 depend on p_0, x_0 and the local parameters of Ω.

In general Green's function $G \notin L^{p_0}_{\mathrm{loc}}(\Omega)$, but as a corollary of Theorem 5.2 we have the following integrability result near a singularity. This result follows also from [28, Theorem 5.1 and 5.5] and from the fact that if we set $G(x_0) = \infty$ then G becomes p-superharmonic in Ω.

Corollary 5.4 *Let* $1 < p_0 < Q(x_0)$. *Under the assumptions of Theorem* 6.4*, one has*

(i)

$$G \in \bigcap_{0<q<\frac{Q(x_0)(p_0-1)}{Q-p_0}} L^q_{\mathrm{loc}}(\Omega),$$

(ii)

$$g_G \in \bigcap_{0<q<\frac{Q(x_0)(p_0-1)}{Q-1}} L^q_{\mathrm{loc}}(\Omega),$$

(iii) *If* $p_0 > (Q + Q(x_0) - 1)/Q(x_0)$, *then*

$$G \in \bigcap_{1<q<\frac{Q(x_0)(p_0-1)}{Q-1}} N_0^{1,q}(\Omega).$$

Proof The proof of (i) is an immediate consequence of the estimate from above in Theorem 5.2. To prove (ii), we note that since $1 < p_0 < Q(x_0) \leq Q$,

$$q^* := \frac{Q(x_0)(p_0-1)}{Q-1} < p_0.$$

Applying Hölder's inequality, the Caccioppoli inequality, see Björn–Marola [3, Proposition 7.1], and again Theorem 5.2, we find for $0 < q < p$ and for $\sigma \in (0, r)$

$$\int_{B(x_0,2\sigma)\setminus B(x_0,\sigma)} g_G^q \, d\mu \leq C\sigma^{Q(x_0)-\frac{q(Q-1)}{p_0-1}}.$$

Note that the exponent $Q(x_0) - \frac{q(Q-1)}{p_0-1}$ is strictly positive, when $0 < q < q^*$ and zero when $q = q^*$. This observation gives us that

$$\int_{B(x_0,r)} g_G^q \, d\mu = \sum_{i=0}^{\infty} \int_{B(x_0,2^{-i}r)\setminus B(x_0,2^{-(i+1)}r)} g_G^q \, d\mu$$

$$\leq C\mu(B(x_0,r)) \sum_{i=0}^{\infty} (2^{-i}r)^{Q(x_0)-\frac{q(Q-1)}{p_0-1}} < \infty.$$

This proves (ii). Finally, (iii) follows from (ii) once we observe that the condition $p_0 > (Q + Q(x_0) - 1)/Q(x_0)$ is equivalent to $Q(x_0)(p_0 - 1)/(Q - 1) > 1$. □

6 Cheeger Singular Functions

In this section we study *Cheeger singular functions*, i.e. functions that satisfy *only* conditions 1, 2 and 3 in Definition 4.1 and the notion of a p_0-harmonic function is replaced by that of a Cheeger p_0-harmonic function, see Section 2.7.

Let G' be a function that satisfies conditions 1–3 in Definition 4.1. We begin by defining $K(G')$ by

$$K(G') = \int_\Omega |DG'|^{p_0-2} DG' \cdot D\varphi \, d\mu, \tag{6.1}$$

where $\varphi \in N_0^{1,p_0}(\Omega)$ is such that $\varphi = 1$ in a neighborhood of x_0. If $\varphi_i \in N_0^{1,p_0}(\Omega)$, $i = 1, 2$, and $\varphi_i = 1$ in a neighborhood of x_0 then $\varphi = \varphi_1 - \varphi_2 \in N_0^{1,p_0}(\Omega \setminus \{x_0\})$. This gives us

$$\int_\Omega |DG'|^{p_0-2} DG' \cdot D\varphi_1 \, d\mu = \int_\Omega |DG'|^{p_0-2} DG' \cdot D\varphi_2 \, d\mu.$$

Thus $K(G') = K(G', p_0, \Omega)$, in particular, K does not depend on φ. Another property of $K(G')$ that will play an important role is that

$$K(G') > 0,$$

see Eq. 6.2 below. We obtain the following result on the growth of Cheeger singular functions near a singularity.

Lemma 6.1 *Let Ω be a relatively compact domain in X, $x_0 \in \Omega$, and $1 < p < Q(x_0)$. If G' is a Cheeger singular function, i.e. G' satisfies conditions 1–3 in Definition 4.1, with singularity at x_0 and $0 < R \le R_0(\Omega)$ is such that $\overline{B}(x_0, R) \subset \Omega$, then for every $0 < r < R$ we have*

$$m_{G'}(x_0, r) \le \left(\frac{K(G')}{\mathrm{Cap}_{p_0}\left(\overline{B}(x_0, r), B(x_0, R)\right)} \right)^{1/(p_0-1)} + M_{G'}(x_0, R).$$

If $r_0 \in (0, R)$ is such that $m_{G'}(x_0, r_0) \ge M_{G'}(x_0, R)$, then for every $0 < r < r_0$ we have

$$M_{G'}(x_0, r) \ge C\left(1 - \frac{r}{r_0}\right)^{p_0} \left(\frac{K(G')}{\mathrm{Cap}_{p_0}\left(\overline{B}(x_0, r), B(x_0, r_0)\right)} \right)^{1/(p_0-1)} + M_{G'}(x_0, R),$$

where $C = (C_1/C_4)^{1/(p_0-1)} > 0$, and the constants C_1 and C_4 are as in Theorems 3.1 and 3.3, respectively.

Proof Consider a radius $R > 0$ such that $\overline{B}(x_0, R) \subset \Omega$. Define $w = G' - M(R)$, and hence $w \le 0$ on $\partial B(x_0, R)$. Observe also that the first inequality in the theorem obviously holds true if $m(r) \le M(R)$, thus, we might as well assume that $m(r) > M(R)$. Let functions v and $\varphi = v/m_w(r)$ be defined as in the proof of Lemma 5.1 with G replaced by G'. Then φ can be used in the definition of $K(G')$, see Eq. 6.1. We have

$$\begin{aligned} K(G') &= \int_{B(x_0, R)\setminus\overline{B}(x_0,r)} |DG'|^{p_0-2} DG' \cdot D\varphi \, d\mu \\ &= \frac{1}{m_w(r)} \int_{B(x_0, R)\setminus\overline{B}(x_0,r)} |DG'|^{p_0-2} DG' \cdot Dv \, d\mu. \end{aligned}$$

Observing that $Dv = 0$ whenever $v \ne w$, whereas $Dv = Dw = DG'$ on the set where $v = w$, we conclude

$$K(G') = \frac{1}{m_w(r)} \int_{B(x_0, R)} |Dv|^{p_0} \, d\mu = m_w^{p_0-1} \int_{B(x_0, R)} |D\varphi|^{p_0} \, d\mu. \tag{6.2}$$

Note at this point that Eq. 6.2 proves that $K(G') > 0$. Indeed, if, in fact, $K(G') \le 0$, the Sobolev–Poincaré inequality (2.5) implies that

$$\int_{B(x_0, R)} |v|^{p_0} \, d\mu \le C R^{p_0} \int_{B(x_0, R)} |Dv|^{p_0} \, d\mu \le 0,$$

and, moreover, $v \equiv 0$ in $B(x_0, R)$. This, in turn, would contradict the fact that $G'(x) \to \infty$ when x tends to x_0. This shows that $K(G') > 0$.

Observing that $\varphi = v/m_w(r)$ is an admissible function for the capacity of $B(x_0, r)$ with respect to $B(x_0, R)$, we obtain from Eq. 6.2 that

$$\begin{aligned}\mathrm{Cap}_{p_0}(\overline{B}(x_0, r), B(x_0, R)) &\leq \int_{B(x_0,R)\setminus\overline{B}(x_0,r)} |D(v/m_w(r))|^{p_0}\, d\mu \\ &\leq \frac{1}{m_w(r)^{p_0}} \int_{B(x_0,R)\setminus\overline{B}(x_0,r)} |Dv|^{p_0}\, d\mu \leq m_w(r)^{1-p_0} K(G'). \quad (6.3)\end{aligned}$$

This implies the first claim.

To prove the second inequality of the claim, we observe that $w(x) \to \infty$, when x tends to x_0. As above, $w = G' - M(R)$. Also thanks to Eq. 5.1 one has that

$$m_w(r) \geq m_w(\rho), \quad 0 < r < \rho < R.$$

Let $r_0 \in (0, R)$ be such that $m(r_0) \geq M(R)$. This implies that $w \geq 0$ on $\overline{B}(x_0, r_0)$. For any $0 < r < r_0$ consider the function $\psi : \mathbb{R} \to \mathbb{R}$ defined by

$$\psi(t) = \begin{cases} 1, & \text{in } 0 \leq t \leq r, \\ \dfrac{t^{\frac{p_0-Q(x_0)}{p_0-1}} - r_0^{\frac{p_0-Q(x_0)}{p_0-1}}}{r^{\frac{p_0-Q(x_0)}{p_0-1}} - r_0^{\frac{p_0-Q(x_0)}{p_0-1}}}, & \text{in } r \leq t \leq r_0, \\ 0, & \text{in } r_0 \leq t \leq R. \end{cases}$$

Observe that $\psi \in L^\infty(\mathbb{R})$, $\mathrm{supp}(\psi') \subset [r, r_0]$, and that $\psi' \in L^\infty(\mathbb{R})$, thus ψ is a Lipschitz function. Moreover, $\psi \circ d(x_0, x) \in N^{1,p_0}(B(x_0, R))$. As in the proof of Theorem 4.5 in Garofalo–Marola [14], we obtain

$$\int_{B(x_0,R)} |D\psi|_0^p\, d\mu \leq C_4 \frac{\mu(B(x_0, r))}{r^{p_0}}.$$

On the other hand, if we use Theorem 3.1, for the proof see [14], we have

$$\int_{B(x_0,R)} |D\psi|^{p_0}\, d\mu \leq \frac{C_4}{C_1} \left(1 - \frac{r}{r_0}\right)^{p_0(1-p_0)} \mathrm{Cap}_{p_0}\left(\overline{B}(x_0, r), B(x_0, r_0)\right). \quad (6.4)$$

Since $\psi \circ d(x_0, x)$ is an admissible function for $K(G')$, it follows from Eqs. 6.1, 6.4, and Hölder's inequality that

$$\begin{aligned}K(G')^{p_0/(p_0-1)} &\leq \left(\int_{B(x_0,R)} |D\psi|^{p_0}\, d\mu\right)^{1/(p_0-1)} \int_{B(x_0,r_0)\setminus\overline{B}(x_0,r)} |DG'|^{p_0}\, d\mu \\ &\leq \left(\frac{C_4}{C_1}\right)^{1/(p_0-1)} \left(1 - \frac{r}{r_0}\right)^{-p_0} \mathrm{Cap}_{p_0}\left(\overline{B}(x_0, r), B(x_0, r_0)\right)^{1/(p_0-1)} \\ &\quad \cdot \int_{B(x_0,r_0)\setminus\overline{B}(x_0,r)} |Dw|^{p_0}\, d\mu. \quad (6.5)\end{aligned}$$

Let us introduce the function $\xi \in N^{1,p_0}(B(x_0, R))$ defined by

$$\xi = \begin{cases} 0, & \text{in } \Omega \setminus B(x_0, R), \\ \max\{w, 0\}, & \text{in } B(x_0, R) \setminus B(x_0, r_0), \\ w, & \text{in } B(x_0, r_0) \setminus B(x_0, r) \\ \min\{w, M_w(r)\}, & \text{in } B(x_0, r). \end{cases}$$

Observe that we have $\xi = M_w(r)$ in a neighborhood of x_0. Let

$$I = \{x \in B(x_0, R) : \xi(x) = w(x)\}.$$

Since $|D\xi| = |Dw| = |DG'|$ on I, and $|D\xi| = 0$ on $B(x_0, R) \setminus I$, from Eq. 6.1 we have

$$\begin{aligned}\int_{B(x_0,r_0)\setminus\overline{B}(x_0,r)} |Dw|^{p_0}\, d\mu &\leq \int_I |Dw|^{p_0-2} Dw \cdot Dw\, d\mu \\ &= \int_I |Dw|^{p_0-2} Dw \cdot D\xi\, d\mu = \int_{B(x_0,R)} |Dw|^{p_0-2} Dw \cdot D\xi\, d\mu \\ &= K(G') M_w(r).\end{aligned}$$

By plugging this in Eq. 6.5, we finally conclude that

$$M(r) \geq \left(\frac{C_1}{C_4}\right)^{1/(p_0-1)} \left(1 - \frac{r}{r_0}\right)^{p_0} \cdot \left(\frac{K(G')}{\mathrm{Cap}_{p_0}\left(\overline{B}(x_0, r), B(x_0, r_0)\right)}\right)^{1/(p_0-1)} + M(R).$$

This completes the proof. □

Remark 6.2 By obvious modifications, the preceding argument holds in the case $p_0 = Q(x_0)$ as well.

Remark 6.3 Observe that assuming only conditions 1–3 in Definition 4.1, factor $K(G')$ comes up in the above estimates as opposed to the estimates in Lemma 5.1.

We have the following result on the local behavior of a Cheeger singular function near a singularity. The proof of this result is similar to that of Theorem 5.2, thus, we omit the proof.

Theorem 6.4 *Let Ω be a relatively compact domain in X, and $x_0 \in \Omega$. If G' is a Cheeger singular function, i.e. G' satisfies conditions 1–3 in Definition* 4.1, *with singularity at x_0, then there exist positive constants C_1, C_2, R_0 and R_1 such that $R_1 \leq \frac{R_0}{2}$ and for any $0 < r < R_1$ and $x \in B(x_0, r)$ we have*

$$\begin{aligned}C_1\left(\frac{d(x,x_0)^{p_0}}{\mu(B(x_0, d(x,x_0)))}\right)^{1/(p_0-1)} &\leq G'(x) \\ &\leq C_2\left(\frac{d(x,x_0)^{p_0}}{\mu(B(x_0, d(x,x_0)))}\right)^{1/(p_0-1)},\end{aligned}$$

when $1 < p_0 < Q(x_0)$, whereas

$$C_1 \log\left(\frac{R_0}{d(x,x_0)}\right) \leq G'(x) \leq C_2 \log\left(\frac{R_0}{d(x,x_0)}\right),$$

when $p_0 = Q(x_0)$. Here the constants C_1 and C_2 depend on $K(G')$, p_0, x_0, and the local parameters of Ω, and R_0 depends only on Ω.

The following lemma is well-known and we omit the proof. For instance, see Holopainen [20, Lemma 3.8] or Balogh et al. [1, Lemma 2.9].

Lemma 6.5 *Let K be a closed subset of a relatively compact domain Ω, and let u be the p_0-potential of K with respect to Ω. Then for all $0 \leq \alpha < \beta \leq 1$ one has*

$$\operatorname{Cap}_{p_0}\left(\Omega^\beta, \Omega_\alpha\right) = \frac{\operatorname{Cap}_{p_0}(K, \Omega)}{(\beta - \alpha)^{p_0 - 1}}.$$

We close this paper with the following observation. The proof of Proposition 6.6 is similar to the proof of Lemma 3.16 in Holopainen [20], but we present it here for completeness.

Proposition 6.6 *Let G' be a Cheeger singular function. Then*

$$G = K(G')^{-1/(p_0-1)} G'$$

is a Cheeger Green's function, i.e. in Definition 4.1 p_0-harmonic is replaced by Cheeger p_0-harmonic, so that whenever $0 \leq \alpha < \beta$,

$$\operatorname{Cap}_{p_0}\left(\Omega^\beta, \Omega_\alpha\right) = (\beta - \alpha)^{1-p_0},$$

where Ω^β and Ω_α are as in Definition 4.1.

Proof Observing that the function $\varphi = \min\{G', 1\}$ can be used in Eq. 6.1, and since G' is the p_0-potential of the set $\{x \in \Omega : G' \geq 1\}$ with respect to Ω, we obtain

$$\operatorname{Cap}_{p_0}(\{x \in \Omega : \; G'(x) \geq 1\}, \Omega) = K(G'). \tag{6.6}$$

Let $0 \leq \alpha < \beta$ and suppose first that $\beta \leq K(G')^{-1/(p_0-1)}$. Then one has

$$\begin{aligned}
&\operatorname{Cap}_{p_0}(\{x \in \Omega : G(x) \geq \beta\}, \{x \in \Omega : \; G(x) > \alpha\}) \\
&= \operatorname{Cap}_{p_0}\left(\left\{x \in \Omega : \; G'(x) \geq \beta K(G')^{1/(p_0-1)}\right\}, \left\{x \in \Omega : \; G'(x) > \alpha K(G')^{1/(p_0-1)}\right\}\right) \\
&= (\beta - \alpha)^{1-p_0} K(G')^{-1} \operatorname{Cap}_{p_0}(\{x \in \Omega : G'(x) \geq 1\}, \Omega) \\
&= (\beta - \alpha)^{1-p_0}.
\end{aligned}$$

Let then assume that $K(G')^{-1/(p_0-1)} < \beta$. Equation 6.6 implies that

$$\begin{aligned}
\frac{\operatorname{Cap}_{p_0}(\{x \in \Omega : G(x) \geq \beta\}, \Omega)}{(K(G')^{-1/(p_0-1)}/\beta)^{p_0-1}} &= \operatorname{Cap}_{p_0}\left(\left\{x \in \Omega : \frac{G(x)}{\beta} \geq \frac{K(G')^{-1/(p_0-1)}}{\beta}\right\}, \Omega\right) \\
&= K(G'),
\end{aligned}$$

from which it follows that

$$\operatorname{Cap}_{p_0}(\{x \in \Omega : G(x) \geq \beta\}, \Omega) = \beta^{1-p_0}.$$

Then one has

$$\begin{aligned}
&\operatorname{Cap}_{p_0}(\{x \in \Omega : G(x) \geq \beta\}, \{x \in \Omega : \; G(x) > \alpha\}) \\
&= \operatorname{Cap}_{p_0}(\{x \in \Omega : \; G(x)/\beta \geq 1\}, \{x \in \Omega : \; G(x)/\beta > \alpha/\beta\}) \\
&= (1 - \alpha/\beta)^{1-p_0} \operatorname{Cap}_{p_0}(\{x \in \Omega : G(x) \geq \beta\}, \Omega) \\
&= (\beta - \alpha)^{1-p_0}.
\end{aligned}$$

This completes the proof. □

Acknowledgements The authors would like to thank Anders Björn, Piotr Hajłasz, and Nageswari Shanmugalingam for valuable comments on the manuscript and their interest in the paper.

The paper was completed while the third author was visiting Purdue University in 2007–2008. He thanks the Department of Mathematics for the hospitality and several of its faculty for fruitful discussions.

First author supported in part by NSF CAREER Grant, DMS-0239771. Second author supported in part by NSF Grant DMS-0701001. Third author supported by the Academy of Finland and the Emil Aaltonen foundation.

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