1. Suppose $f$ is a positive continuous function on $\mathbb{R}^n$ such that $\lim_{|x| \to \infty} f(x) = 0$ [i.e., for all $\epsilon > 0$, there is an $N$ so that $|f(x)| < \epsilon$ form all $x$ with $|x| > N$]. Show that $f$ is uniformly continuous on $\mathbb{R}^n$.

Solution.

First we see that by Theorem 4.19, $f$ is uniformly continuous on $B_N = \{x \in \mathbb{R}^n : |x| \leq N\}$, for all $N > 0$. So let $\epsilon > 0$ be given and, since $\lim_{|x| \to \infty} f(x) = 0$, there exists $N > 0$ such that $|f(x)| < \epsilon$ if $|x| \geq N$. Since $f$ is uniformly continuous on $B_N$, there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \quad \text{if } |x - y| < \delta \text{ and } x,y \in B_N.$$ 

We see that if $|x| > N$, then either $B_{\delta}(x) \cap B_N = \emptyset$ or there exists $y \in B_{\delta}(x)$ with $|y| < N$, where $B_{\delta}(x) = \{z \in \mathbb{R}^n : |x - z| < \delta\}$. In the first case, if $z \in B_{\delta}(x)$, then $|z| > N$, so

$$|f(x) - f(z)| \leq |f(x)| + |f(z)| < \epsilon + \epsilon = 2\epsilon.$$ 

In the second case, then let $y \in B_{\delta}(x)$ with $|y| < N$, and let $x_0 \in B_{\delta}(x)$ with $|x_0| = N$. We have then

$$|f(x) - f(y)| \leq |f(x) - f(x_0)| + |f(x_0) - f(y)| \leq |f(x)| + |f(x_0)| + |f(x_0) - f(y)| < 3\epsilon,$$

note that $|x_0 - y| < \delta$.

Therefore

$$|f(x) - f(y)| < 3\epsilon,$$

whenever $|x - y| < \delta$. Hence $f$ is uniformly continuous.

2. Let $f$ be a continuous function on $\mathbb{R}$ to $\mathbb{R}$ which does not take on any of its values twice. Is it true that $f$ must either be strictly increasing (in the sense that if $x' < x''$ then $f(x') < f(x'')$) or strictly decreasing?

Solution.

Answer: SIM! (YES!).

Let $x \in \mathbb{R}$. Since $f$ does not take on any of its values twice, then if $y > x$ either $f(x) < f(y)$ or $f(x) > f(y)$. Assume without loss of generality that $f(x) < f(y)$. Then if $z \in \mathbb{R}$, $x < z < y$, we have that $f(z) > f(x)$, because if $f(z) < f(x)$, then since $f(y) > f(x) > f(z)$, by the intermediate value theorem there exists $w$, $z < w < y$ such that $f(w) = f(x)$, a contradiction with the fact that $f$ does not take on any of its values twice. Similarly $f(z) < f(y)$, because of $f(z) > f(y)$, then $f(x) < f(y) < f(z)$ so there would be a $w$, $x < w < z$ such that $f(w) = f(y)$, a contradiction. Hence $f$ is strictly increasing.
Note that if \( f \) is not continuous, then the result is no longer valid. For instance, consider \( f(x) = x \) if \( x \neq 1, 2 \) and \( f(1) = 2, f(2) = 1 \). Then \( f \) is neither strictly increasing, nor strictly decreasing.

3. Suppose that \( \{s_n\}_{n=1}^{\infty} \) and \( \{t_n\}_{n=1}^{\infty} \) are sequences of positive real numbers such that
\[
\liminf_{n \to \infty} \frac{s_n}{t_n} > 0 \text{ and } \limsup_{n \to \infty} \frac{s_n}{t_n} < \infty.
\]
Prove that there is a constant \( M > 0 \) such that
\[
\frac{1}{M}t_n \leq s_n \leq Mt_n \text{ for all } n = 1, 2, 3, ...
\]
Solution.
Since
\[
\limsup_{n \to \infty} \frac{s_n}{t_n} < \infty,
\]
there exists \( M_0 > 0 \) such that
\[
\limsup_{n \to \infty} \frac{s_n}{t_n} < M_0.
\]
By theorem 3.17, there exists \( N_0 \) such that \( n \geq N_0 \) implies \( \frac{s_n}{t_n} < M_0 \). Let
\[
M_1 = \max \left\{ \frac{s_1}{t_1}, ..., \frac{s_{N_0}}{t_{N_0}}, M_0 \right\}.
\]
Then
\[
\frac{s_n}{t_n} < M_0,
\]
for all \( n = 1, 2, 3, .... \)
Similarly, since
\[
\liminf_{n \to \infty} \frac{s_n}{t_n} > 0
\]
there exists \( M_2 > 0 \) such that
\[
\frac{s_n}{t_n} > \frac{1}{M_2},
\]
for all \( n = 1, 2, 3, .... \)
Now let \( M = \max\{M_1, M_2\} \). We have
\[
M > \frac{s_n}{t_n} > \frac{1}{M_2} \geq \frac{1}{M} \Rightarrow \frac{1}{M}t_n \leq s_n \leq Mt_n \text{ for all } n = 1, 2, 3, ...
\]
4. What is the radius of convergence of the power series
\[
\begin{align*}
(a) \sum_{n=0}^{\infty} (2 + (-1)^n)^n z^n & \quad (b) \sum_{n=0}^{\infty} \frac{(3n)!n!}{(4n)!} x^n \\
(c) \sum_{n=1}^{\infty} x^n & \quad (d) \sum_{n=0}^{\infty} n^2 x^n
\end{align*}
\]
Solution.
(a) We have

\[
\limsup \sqrt[n]{(2 + (-1)^n)^n} = \limsup 2 + (-1)^n = 3,
\]
so by theorem 3.39, the radius of convergence is \( R = \frac{1}{3} \).

(b) We have

\[
\frac{(3n+3)![(n+1)!]}{(4n+4)!} = \frac{(3n+3)! (n+1)! (4n)!}{(3n)! n! (4n+4)!} = \frac{(3n+3)(3n+2)(3n+1)(n+1)}{(4n+4)(4n+3)(4n+2)(4n+1)},
\]
so

\[
\lim_{n \to \infty} \frac{(3n+3)(3n+2)(3n+1)(n+1)}{(4n+4)(4n+3)(4n+2)(4n+1)} = \frac{3^3}{4^4},
\]
and the radius of convergence is \( R = \frac{4^4}{3^3} \).

(c) We have

\[
\sqrt[n]{x^n} = x^{n/n} = x^{n-1}
\]
so by the root test, we see that the series converges if \( |x| < 1 \) and diverges if \( |x| \geq 1 \), ie, the radius of convergence is \( R = 1 \).

(d) We have

\[
\lim_{n \to \infty} \frac{(n+1)^2}{n^2} = 1,
\]
hence the radius of convergence is \( R = 1 \).

5. Let \( \sum_{n=1}^{\infty} a_n \) be an absolutely convergent series of numbers and \( \{b_n\}_{n=1}^{\infty} \) a bounded sequence. Show that \( \sum_{n=1}^{\infty} a_n b_n \) is also convergent. Can we still have the same conclusion if \( \sum_{n=1}^{\infty} a_n \) is convergent, but not absolutely convergent?

Solution.
Since \( \sum_{n=1}^{\infty} a_n \) is an absolutely convergent series of numbers,

\[
\sum_{n=1}^{\infty} |a_n| < \infty.
\]

Since \( \{b_n\}_{n=1}^{\infty} \) is a bounded sequence, there exists a \( M > 0 \) such that \( |b_n| < M \) for every \( n \).
Hence

\[
\left| \sum_{n=1}^{\infty} a_n b_n \right| \leq \sum_{n=1}^{\infty} |a_n||b_n| \leq M \sum_{n=1}^{\infty} |a_n| < \infty,
\]
so it converges.
Now consider
\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n}. \]
It is easy to see that the series above converges, but it is not absolutely convergent. Let \( b_n = (-1)^n. \) Then
\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cdot (-1)^n = \sum_{n=1}^{\infty} \frac{1}{n} \text{ does not converges}. \]

6. Assume \( f \) has a finite derivative in \((a, b)\), and is continuous on \([a, b]\) with \( f(a) = f(b) = 0 \). Prove that for every \( \lambda \in \mathbb{R} \) there exists \( c \in (a, b) \) such that \( f'(c) = \lambda f(c) \).

Solution.
By the mean value theorem there exists \( s \in (a, b) \) such that
\[ 0 = f(b) - f(a) = (b - a)f'(s) \Rightarrow f'(s) = 0. \]
So if \( f(s) = 0 \), \( f'(s) = 0 = \lambda f(s) \) for every \( \lambda \in \mathbb{R} \). Otherwise \( f(s) \neq 0 \). Assume without loss of generality that \( f(s) > 0 \), if not, consider \(-f\) instead of \( f \).
Let \( \lambda \in \mathbb{R} \) be given.
Since \( f \) is continuous at \( s \), there exits a neighborhood of \( s \), say \((s - \delta, s + \delta) \subset (a, b), \delta > 0 \), such that \( f \) is positive on it, i.e., \( f(x) > 0 \) for every \( x \in (s - \delta, s + \delta) \).
We see that there exist \( a_1, b_1 \in (s - \delta, s + \delta) \) such that \( a_1 < b_1 \) and \( f(a_1) = f(b_1) \). Indeed, otherwise by the previous problem 2, \( f \) would be strictly increasing or strictly decreasing, but then it is not hard to show that \( f'(x) > 0 \) for all \( x \in (s - \delta, s + \delta) \) or \( f'(x) < 0 \) for all \( x \in (s - \delta, s + \delta) \). This is a contradiction with \( f'(s) = 0 \).
Then we can consider the function \( h \) defined by
\[ h(x) = \log f(x) + \lambda x. \]
By the generalized Mean Value Theorem (Theorem 5.9) with \( g(x) = \lambda x \), there exists \( c \in (a_1, b_1) \subset (a, b) \) such that
\[ [h(b_1) - h(a_1)]g'(c) = [g(b_1) - g(a_1)]h'(c) \Rightarrow \lambda(b_1 - a_1)\lambda = \lambda(b_1 - a_1) \frac{f'(c)}{f(c)} \Rightarrow f'(c) = \lambda f(c). \]

7. Let \( f : \mathbb{R} \to \mathbb{R} \) be a uniformly continuous function. Show that there exist constants \( A \) and \( B \) such that
\[ |f(x)| \leq A + B|x|, \quad \text{for all } x \in \mathbb{R}. \]
Solution.
Since \( f \) is uniformly continuous, there exists \( \delta > 0 \) such that
\[ |f(x) - f(y)| < 1, \]
whenever $|x - y| < \delta$.
Let $x \in \mathbb{R}$. Assume $x \neq 0$. Let $n \in \mathbb{Z}$ be a integer such that $n\delta \leq x < (n + 1)\delta$. We have

$$f(x) - f(0) = f(x) - f(n\delta) + \sum_{j=1}^{n} (f(j\delta) - f((j - 1)\delta))$$

$$\Rightarrow |f(x)| \leq |f(0)| + |f(x) - f(n\delta)| + \sum_{j=1}^{n} |f(j\delta) - f((j - 1)\delta)|$$

$$\Rightarrow |f(x)| \leq |f(0)| + \delta + |n|\delta$$

$$\Rightarrow \frac{|f(x)|}{1 + |x|} \leq \frac{|f(0)|}{1 + |x|} + \frac{\delta(|n| + 1)}{1 + |x|} \leq |f(0)| + \frac{\delta(|n| + 1)}{|n|\delta} \leq |f(0)| + 1 + \frac{1}{|n|}.$$ 

Then $\frac{|f(x)|}{1 + |x|} \leq |f(0)| + 2$, so

$$|f(x)| \leq |f(0)| + 2 + (|f(0)| + 2)|x|.$$

8. Show that a monotone function $f : [a, b] \to \mathbb{R}$ is continuous if and only if its image $f([a, b])$ is an interval.

Solution.
Suppose that $f$ is continuous. Then by theorem 4.22, $f([a, b])$ is connected, since $[a, b]$ is connected. By Theorem 2.47, $f([a, b])$ has the property that if $x \in f([a, b])$, $y \in f([a, b])$, and $x < z < y$, then $z \in f([a, b])$, ie, $f([a, b])$ is a line segment. Since $f([a, b])$ is compact, because $[a, b]$ is compact and $f$ is continuous, $f([a, b])$ is an interval.

Conversely assume that $f([a, b])$ is an interval. Assume that $f$ is not continuous. Then there exists $x \in [a, b]$ such that $f$ is not continuous at $x$. Since $f$ is monotone, the discontinuities are of first type, ie, both limits

$$\lim_{t \to x^-} f(t), \quad \lim_{t \to x^+} f(t)$$

exist, but they are different. If $x = a$, then we only consider the right limit, and if $x = b$ we only consider the left limit.

Assume $x \in (a, b)$ and assume without loss of generality that $f$ is monotone increasing, otherwise consider $-f$ instead of $f$. Then

$$\lim_{t \to x^-} f(t) < \lim_{t \to x^+} f(t).$$

Let $\epsilon > 0$ be such that

$$\lim_{t \to x^-} f(t) < \lim_{t \to x^+} f(t) - \epsilon.$$ 

Since $f$ is monotone increasing

$$f(y) \leq \lim_{t \to x^-} f(t), \quad \text{for } y < x,$$

and

$$f(y) \geq \lim_{t \to x^+} f(t), \quad \text{for } y \geq x.$$
But there exists no $z \in [a, b]$ such that
\[
f(a) \leq \lim_{t \to x^-} f(t) < f(z) = \lim_{t \to x^+} f(t) - \epsilon < \lim_{t \to x^+} f(t) \leq f(b),
\]
a contradiction with $f([a, b])$ being a interval.
One can argue in a similar way if $x = a$, or $x = b$.
Therefore $f$ is continuous.