Sub-Riemannian geometry in models of the visual cortex

Scott Pauls
Department of Mathematics
Dartmouth College

4th Symposium on Analysis and PDE
Purdue, May 27, 2009
Mathematical structure in the visual cortex

“This paper is devoted to a mathematical model of the brain utilizing the structures of modern differential geometry and topology. To some, this may seem a bit much.”

VI - the primary visual cortex

- How do we see?
  - Light hits the retina, stimulates neurons (rods and cones)
  - Signal transmitted to the LGN, which aggregates and “pre-processes” signal.
  - LGN projects to the primary visual cortex VI, transmitting signal.
  - VI is populated by a large number of cells of many different types.
  - We will focus on the most numerous, the “simple cells”
Simple cells

- Direct recordings in monkeys (Hubel, Weisel, 50s-60s) showed that simple cells respond to oriented line segments in space.
- In other words, simple cells are tuned to spatial position and orientation.
Moreover, simple cells have a hypercolumn structure – a stack of orientations is associated to each spatial position.
Contact structure

V1 \sim \mathbb{R}^2 \times S^1

- What is the contact structure?
- Neurons are almost always connected to nearby neurons, but there are also “long range” connections as well.
- Hypothesis: one role of long range connections is to facilitate image recognition. In particular, it should recognize lines and curves – they recognize consistent orientations over different spatial points.
Contact structure

\[ \theta(x, y) = \tan^{-1} \left( -\frac{I_x(x, y)}{I_y(x, y)} \right) \]
Contact structure

- Favor connections between
  \[ (x_1, y_1, \theta_1) \sim (x_2, y_2, \theta_2) \]
  if \( \theta_1 = \theta_2 \) and \( (x_2 - x_1, y_2 - y_1) \parallel \theta \)

- Assuming a hypercolumn is spatially localized on the cortex, a contact form would be
  \[ \omega = -\sin(\theta) \, dx + \cos(\theta) \, dy \]

- Basis for the horizontal bundle:
  \[ X_1 = \cos(\theta) \, \partial_x + \sin(\theta) \, \partial_y \]
  \[ X_2 = \partial_\theta \]
Experimental evidence

Metric structure

- Hoffman (1989): contact structure but no explicit metric structure
- Petitot, Petit-Tondut (1999): explore metric structures – Riemannian and sub-Riemannian
Roto-translation model

• Horizontal directions:

\[ X_1 = \cos(\theta) \partial_x + \sin(\theta) \partial_y \]
\[ X_2 = \partial_\theta \]

• Vertical direction:

\[ X_3 = [X_1, X_2] = -\sin(\theta) \partial_x + \cos(\theta) \partial_y \]

• Metric:

\[ d_R((x_1, y_1, \theta_1), (x_2, y_2, \theta_2)) = \inf \left\{ \int_\gamma \langle \dot{\gamma}, \dot{\gamma} \rangle^{\frac{1}{2}} \right\} \]
Applicability of Citti-Sarti model

- Arrangement of neurons is optimal for a map from a higher dimensional space to $R^2$.
- Does the roto-translation model reflect this?
Consider a map $F : \mathbb{R}^2 \times S^1 \rightarrow \mathbb{R}^2$ and attempt to minimize metric distortion.

We use two different distances to test hypothesis, the Riemannian distance on $\mathbb{R}$ and the sub-Riemannian distance.

We attempt to minimize

\[
\left( \int (d_R(x, y) - |F(x) - F(y)|^2)^\frac{1}{2} \right)
\]

where the integral is taken over the domain of $F$.

We implement this numerically using the Metropolis-Monte Carlo method to deform random initial maps to minima.
Results

- Small scale: pinwheels
- Large scale:
  - Riemannian
  - Sub-Riemannian #1
  - Sub-Riemannian #2
Occlusion and disocclusion
Image representations in the rototranslation group.

- An grayscale image is a function: $I : \mathbb{R}^2 \rightarrow \mathbb{R}$
- It’s representation in the rototranslation group is a graph over the $xy$-plane given by

$$\theta(x, y) = \tan^{-1} \left( -\frac{I_x(x, y)}{I_y(x, y)} \right)$$

- If a piece of the image is missing, i.e. an open region $\Omega$ missing from the domain of $I$, the disocclusion problem is then the problem of filling in the function over $\Omega$ while matching the boundary values on $\partial \Omega$. 
Citti-Sarti model and occlusions

- Citti and Sarti provide a solution:
  - Use a geometric diffusion (essentially sub-Riemannian heat flow) to move the representation of the image in the roto-translation group.
  - After some time, perform “non-maximal suppression” to concentrate the function on a surface.
  - They prove that upon iterating this procedure infinitely often, the limiting surface is a sub-Riemannian minimal surface satisfying the Dirichlet boundary conditions.
Minimal Surface Equation

- Write a surface as $\theta - \theta(x, y) = 0$ where
  
  $$\theta(x, y) = \tan^{-1} \left( -\frac{I_x(x, y)}{I_y(x, y)} \right)$$

- The sub-Riemannian unit normal is
  
  $$\nu_H = \frac{N_H}{|N_H|}$$

where

$$N_H = X_1(\theta - \theta(x, y)) X_1 + X_2(\theta - \theta(x, y)) X_2$$

$$= -X_1(\theta(x, y)) X_1 + X_2$$

$$= (\cos(\theta)\theta_x + \sin(\theta)\theta_y)X_1 + X_2$$
Smooth minimal surfaces in the roto-translation group

- In joint work with R. Hladky, we explicitly solve the smooth minimal surface problem in this setting.

Summary of results:

- Smooth minimal surfaces in $\mathbb{R}$ are ruled surfaces (Cheng, Hwang, Malchiodi and Yang, 2005)

- The rules are either circular arcs given by
  $$(x_c + R \sin(\theta_0 + \alpha t), y_c + R \cos(\theta_0 + \alpha t), \theta_0 + \alpha t)$$
  for real numbers $x_c, y_c, \theta_0, R, \alpha$, or lines given by
  $$(x_0 + R \cos(\theta_0) t, y_0 + R \cos(\theta_0) t, \theta_0)$$
  for real numbers $x_0, y_0, \theta_0, R$
Constructing disocclusions

- Fix a boundary curve \((\beta(t), \theta(t)), t \in [0, 2\pi]\) where
  \[
  \theta(t) = \tan^{-1} \left( -\frac{I_x \circ \beta(t)}{I_y \circ \beta(t)} \right)
  \]

- Using the model for allowable connections, construct a (multi-valued) function \(u\) that associates to \(t\) all values \(u(t)\) so that \(\beta(t)\) can be connected to \(\beta(u(t))\).

- Using the explicit form of the rules, for each \((t, u(t))\) pair, construct an arc \(A(t, u(t))\) joining the two.

- Any algorithm based on this method now depends on picking coherent choices of \(A\) for all \(t\), forming a minimal surface which solves the disocclusion problem.
Results

- There exist obstructions to the construction of smooth minimal spanning surfaces.
- Even when there are no obstructions, the spanning surfaces, when translated into image data, may introduce ambiguities.
- Under some conditions, we can prove that smooth spanning surfaces exist.
- We have an algorithm which will produce an image disocclusion (if it exists) with complexity $O(n^2)$ where $n$ is the number of pixels.
Examples

- For simplicity, we will consider circular occlusions where the behavior of the disocclusion mechanism described above is guided by the transversality function:

  \[ Q(t) = \theta(t) - \phi_\beta(t) \]

  where \( \phi_\beta(t) \) is defined by

  \[ \frac{\beta'(t)}{|\beta'(t)|} = (-\sin(\phi_\beta(t)), \cos(\phi_\beta(t))) \]

- Geometrically, \( Q \) measure the angle between \( \beta' \) and \( \nabla I \)
Deg $Q = -1$, $Q' \neq 0$
Conflicting data: \( \text{deg } Q = -1, Q' \) has zeros
Obstruction: $\deg Q = 0$
$|\deg Q| > 1$, multiple completions

(k) $u(t)$  (l) $Q$  (m) Completion I  (n) Completion II
Digital image disocclusion
Digital image disocclusion
Remaining questions

- Non-smooth minimal surfaces
  - Gluing theorems
  - Numeric results (G. Petrics)

- Absolute minimizers

- Extensions: higher order models, V2-V4, binocular vision, etc.
Example