Random walks on barycentric subdivisions and Strichartz hexacarpet

M. Begue\textsuperscript{1}, *D. J. Kelleher\textsuperscript{2}, A. Nelson\textsuperscript{2}, H. Panzo\textsuperscript{2}, R. Pellico\textsuperscript{2}, A. Teplyaev\textsuperscript{2}

\textsuperscript{1}Department of Mathematics
University of Maryland

\textsuperscript{2}Department of Mathematics
University of Connecticut

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We start off with a simplicial complex,

- Take each 1-simplex (line segment) and add a vertex at the midpoint.
- Take each 2-simplex (triangle) add a vertex at the barycenter (average of corners), ad an edge connecting the barycenter to each vertex adjacent to the original 2-simplex.
- ... proceed inductively.
Start off with a simplex
Add midpoints to 1-simplexes
Add edges to 2-simplexes
...proceed inductively
We approximate the hexacarpet with a graph that has as vertices the faces of the \( n \)th barycentric subdivision, which are connect if the faces share and edge. (the second level is pictured above)
This is what the 4th level approximation looks like.
To get a fractal out of this, we look at the symbolic dynamic system
Let $X = \{0, 1, \ldots, 5\}$ where $x$ is any element in $X^*$ and
$v \in \{0, 5\}^\omega$. Suppose $i$ is odd and $j = i + 1 \mod 6$. Then,

$$x_i3v \sim x_j3v \quad \text{and} \quad x_i4v \sim x_j4v.$$  \quad (1)

If $i$ is even ($j$ is still $i + 1 \mod 6$), then

$$x_i1v \sim x_j1v \quad \text{and} \quad x_i2v \sim x_j2v.$$  \quad (2)

This produces a fractal with Cantor boundaries and hexagonal symmetries. We call it the hexacarpet.
Two-dimensional eigenfunction coordinates

(A) ($\varphi_2, \varphi_3$)  
(B) ($\varphi_2, \varphi_4$)  
(C) ($\varphi_2, \varphi_5$)  
(D) ($\varphi_2, \varphi_6$)  
(E) ($\varphi_3, \varphi_4$)  
(F) ($\varphi_3, \varphi_5$)  
(G) ($\varphi_3, \varphi_6$)  
(H) ($\varphi_4, \varphi_5$)  
(I) ($\varphi_4, \varphi_6$)
Three-dimensional eigenfunction coordinates

(A) $(\varphi_2, \varphi_3, \varphi_4)$

(B) $(\varphi_2, \varphi_3, \varphi_5)$

(C) $(\varphi_2, \varphi_3, \varphi_6)$

(D) $(\varphi_2, \varphi_3, \varphi_7)$

(E) $(\varphi_2, \varphi_4, \varphi_5)$

(F) $(\varphi_2, \varphi_4, \varphi_6)$

(G) $(\varphi_2, \varphi_5, \varphi_6)$

(H) $(\varphi_3, \varphi_5, \varphi_6)$

(I) $(\varphi_4, \varphi_5, \varphi_6)$
### Table: Hexacarpet estimates for resistance coefficient $c$ given by $\frac{1}{6} \frac{\lambda_j^n}{\lambda_j^{n+1}}$.
Higher dimensions

We can perform a construction analogous to the hexacarpet on the 3-simplex. This time, our approximating graphs have tetrahedra as vertices, connected if they share a face.

Figure: First and second level graph approximations to the 3-simplectic sponge.
Eigenfunction Pictures - Level 4

D. J. Kelleher

Hexacarpet
<table>
<thead>
<tr>
<th>Original Shape</th>
<th>Triangle</th>
<th>Tetrahedron</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subdivisions ((N))</td>
<td>6</td>
<td>24</td>
</tr>
<tr>
<td>Hausdorff dimension</td>
<td>(\frac{\log(6)}{\log(2)} \approx 2.58)</td>
<td>(\frac{\log(24)}{\log(2)} \approx 4.585)</td>
</tr>
<tr>
<td>Resistance scaling ((\rho))</td>
<td>1.304</td>
<td>2.035</td>
</tr>
<tr>
<td>Spectral dimension</td>
<td>1.74</td>
<td>2.45</td>
</tr>
</tbody>
</table>
For each of the barycentric fractals

1. there exists a unique self-similar local regular conservative Dirichlet form $\mathcal{E}$ with resistance scaling factor $\rho$ and the Laplacian scaling factor $\tau = 6\rho$.

2. the simple random walks on the repeated barycentric subdivisions of a triangle, with the time renormalized by $\tau^n$, converge to the diffusion process, which is the continuous symmetric strong Markov process corresponding to the Dirichlet form $\mathcal{E}$.

3. the spectral zeta function has a meromorphic continuation to $\mathbb{C}$. 
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3. the spectral zeta function has a meromorphic continuation to $\mathbb{C}$. 
Barlow and Bass used a probabilistic interpretation, comparing transition density of a random walk to the diffusion of heat, to prove, in some cases, that the heat kernel can be approximated for time $0 < t \leq 1$

$$p(t, x, y) \approx t^{d_s/2} \exp \left( -c R(x, y)^{d_w/(d_w - 1)} \right)$$

Where $R$ is the effective resistance metric, and

$$d_h = \text{Hausdorff dimension}$$
$$d_s = \text{Spectral dimension}$$
$$d_w = \frac{2d_h}{d_s} = \text{Walk “dimension”} \leq 2$$
\[ p(t, x, y) \approx t^{d_s/2} \exp \left( -c \frac{R(x, y)^{d_w/d_w-1}}{t^{1/(d_w-1)}} \right) \]

Where \( R \) is the effective resistance metric, and

- \( d_h = \) Hausdorff dimension
- \( d_s = \) Spectral dimension
- \( d_w = \frac{2d_h}{d_s} = \) Walk “dimension” \( \leq 2 \)

However, the Hexacarpet and higher dimensional analogues do not quite fit this frame. There may be some logarithmic corrections...
If we look at the graph approximation, the inner “circle” of the graph has length $3 \cdot 2^n$ at the $n$th level. The perimeter of this graph will have $3n2^n$ length.

How do we resize the graph?
Another fractal

What if we consider the triangle, and then perform barycentric subdivision. This time, we keep the simplicial complex, but renormalize each triangle to be equilateral.

In the case of the first subdivision, we get a hexagon.
If we renormalize so that each line segment has length $2^{-n}$ on the $n$th level, the induced geodesic metric converges to a metric on the triangle.

1. The Hausdorff (and self-similarity) dimension of this object is $\log_2(6) \approx 2.58$.

2. Topologically we still have the same object! i.e. the limit is homeomorphic to the Euclidean triangle.

3. The distance from a vertex of an $n$th level simplex to the any point on the opposite side is $2^{-n}$.

4. There are infinitely many geodesics between points.

The same construction can be done when starting with any $n$-simplex.