Gauss is a witch!

Do the problems below. Please write neatly, especially your name! Show all your work and justify all your steps. Write in complete, coherent sentences. I expect and openly encourage you to collaborate on this problem set. However, I will insist that you list your collaborators on the handed in solutions (list them on the top of your first page).

Recall that $\mathrm{GJ1}(i,j)$ is the Gauss–Jordan move on an augmented matrix that interchanges the ith and jth rows. The Gauss–Jordan move $\mathrm{GJ2}(i,\lambda)$ scales the ith row by a non-zero number λ. Finally, the Gauss–Jordan move $\mathrm{GJ3}(i,j)$ replaces the jth row with the sum of the ith and jth row. In all of these moves, all remaining rows are left unchanged.

For a pair of vectors $\mathbf{v}$ and $\mathbf{w}$ of the form

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \qquad \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

we define the dot product to be

$$\mathbf{v}\cdot\mathbf{w} = \sum_{j=1}^{n} v_j w_j = v_1 w_1 + v_2 w_2 + \ldots v_n w_n.$$

We say $\mathbf{v}$ and $\mathbf{w}$ are orthogonal if $\mathbf{v}\cdot\mathbf{w} = 0$.

If A is m by n matrix, we can write

$$A = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_n \end{pmatrix}$$

where the vectors $\mathbf{a}_j \in \mathbf{R}^n$ are called the row vectors. For a vector $\mathbf{x}$ in $\mathbf{R}^n$, we define

$$A\mathbf{x} = \begin{pmatrix} \mathbf{a}_1\cdot\mathbf{x} \\ \mathbf{a}_2\cdot\mathbf{x} \\ \vdots \\ \mathbf{a}_m\cdot\mathbf{x} \end{pmatrix}.$$

Note that $A\mathbf{x}$ is a vector in $\mathbf{R}^m$.

Problem 1. Write each linear system in matrix form $A\mathbf{x} = \mathbf{b}$. Write the augmented matrix $|A:\mathbf{b}|$ for each linear system. Find the reduced form for the augmented matrix using Gauss–Jordan moves. Find all the solutions to each of the linear systems. Describe the space of solutions in terms of intersections of lines/planes in $\mathbf{R}^2$/$\mathbf{R}^3$.

(a)

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_1 - x_2 &= 0 \end{aligned}$$

(b)

$$\begin{aligned} 2x_1 + 3x_2 &= 11 \\ x_1 + 2x_2 &= -5 \end{aligned}$$

(c)

$$\begin{aligned} x_1 - x_2 + x_3 &= 7 \\ 2x_1 + 4x_2 - 3x_3 &= -1 \\ -2x_1 - 3x_2 + 3x_3 &= 1 \end{aligned}$$

Problem 2. Recall the general equation of a circle $S(x_0, y_0, r)$ in $\mathbf{R}^2$ centered at the point (x_0, y_0) of radius r is

$$(x - x_0)^2 + (y - y_0)^2 = r^2.$$

Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be distinct points in $\mathbf{R}^2$. Describe all the circles $S(x_0, y_0, r)$ such that P, Q sit on the circle. Describe the set of centers (x_0, y_0) in $\mathbf{R}^2$. Find a necessary and sufficient condition for when three distinct points P, Q, R sit on a common circle. Find three points P, Q, R that cannot sit on a common circle.

Problem 3. Let $\mathcal{L}$ be an m by n linear system over $\mathbf{R}$ and let $S(\mathcal{L})$ be the space of solutions. Let GJ1, GJ2, and GJ3 be the three Gauss–Jordan operations. Prove that

$$S(\mathcal{L}) = S(\mathrm{GJ1}(\mathcal{L})) = S(\mathrm{GJ2}(\mathcal{L})) = S(\mathrm{GJ3}(\mathcal{L})).$$

[Hint: You need to prove that $(\beta_1, \ldots, \beta_n)$ is a solution to $\mathcal{L}$ if and only if it is a solution to $\mathrm{GJ1}(\mathcal{L})$ (respectively for GJ2 and GJ3).]

Problem 4. Let $\mathcal{L}$ be a 3 by 3 linear system with matrix/vector form $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 0 & 1 & 2t \\ 1 & 2 & 6 \\ t & 0 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

and $t \in \mathbf{R}$ is a parameter.

(a) For what values of t is there a unique solution?

(b) For what values of t are there infinitely many solutions?

(c) For what values of t are there no solutions?

Problem 5. Let

$$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Prove that any vector $\mathbf{u}$ in $\mathbf{R}^2$ can be expressed as a linear combination of $\mathbf{v}$ and $\mathbf{w}$. Prove also that this combination is unique. [We say $\{\mathbf{v}, \mathbf{w}\}$ is a basis for $\mathbf{R}^2$ when this happens.]

Problem 6. Give a geometric description of the all vectors in $\mathbf{R}^2$ of the form $\mathbf{v} + t\mathbf{w}$ where $\mathbf{v}, \mathbf{w}$ are fixed vectors and t is a real parameter. Draw a picture for

$$\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Problem 7. Let $\mathbf{v}$ and $\mathbf{w}$ be orthogonal vectors in $\mathbf{R}^2$ and let

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

where $\theta \in [0, 2\pi]$. Prove that $A\mathbf{v}$ and $A\mathbf{w}$ are orthogonal. Describe geometrically what A does to a vector $\mathbf{v}$.

Problem 8. Let

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}.$$

(a) Prove that for each vector $\mathbf{w} \in \mathbf{R}^2$, there exists a vector $\mathbf{v}$ such that $A\mathbf{v} = \mathbf{w}$. [This shows A is surjective as a linear transformation.]

(b) Prove that if $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{R}^2$ satisfy $A\mathbf{v}_1 = A\mathbf{v}_2$, then $\mathbf{v}_1 = \mathbf{v}_2$. [This shows A is injective.]