(25) 1. Let $A$ be a positive definite matrix and $x_1, \ldots, x_n$ an orthogonal basis with respect to which $A$ is diagonal. Let $C$ be nonsingular and define

$$B = C^T A C.$$

Note: this does NOT say $B$ is similar to $A$. That would mean $B = C^{-1} A C$.

(a) Using any appropriate criterion, show that $B$ is positive definite.

Easiest way: Show if $y^T B y > 0$. But

$$y^T B y = y^T C^T A C y = y^T A y > 0$$

and $y^T A y > 0$ since $A$ is nonsingular!

People write about $X^T B X$, but things like that are not in our discussion of positive definite.

Note: if you show that $B$ is symmetric and $\det B > 0$,

you have not shown $B$ is positive definite.

(b) Suppose we have $x_j = C y_j$ where $C$ is from (a). Must the $\{y_j\}$ also be orthogonal? Linearly independent? Justify.

Here people gave lots of hand-waving. The $y_j$ are linearly independent, but no reason to be orthogonal.

Orthogonal? All we are saying is that $C$ is nonsingular.

So $C$ could take $i$ to $i$ and $j$ to $j$.

Then $C_i$ and $C_j$ are NOT orthogonal!

Linear? Suppose $x_1 y_1 + x_2 y_2 + \ldots + x_n y_n = 0$

Then $Y = (Y_i)$ has rank $< n$. But

$Y = X C^{-1}$ (here $Y^T$ and $X$ are right) and

$\det X C^{-1} \neq 0$ since $X$ and $C$ are nonsingular.

So $\det Y$ can't be $0$!

Note: several students said the $y_j$ were orthogonal but not linearly independent. Get clear at first please.
(c) Suppose that $A$ is the matrix

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Some people said 11, but there is no eigenvalue.

What can you say about the eigenvalues of $B$, with $B$ defined in (a)?

All we can say (p 324) is that $B$ has one eigenvalue $m > 0$.

The book (and lecturer) should have made this clearer. I made this example $C = \begin{pmatrix} 7 & 3 \\ 4 & 2 \end{pmatrix}$, $A = \begin{pmatrix} 15 & -7 \\ 0 & 5 \end{pmatrix}$. I then set $B = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$, so char eqn $\lambda^2 - 130\lambda + 20$. So 1, 5 are not eigenvalues.

(25) (a) Show that the matrix

$$A = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

is positive definite (use any criterion that works for you).

You should mention that $A$ is symmetric.

Then principal minors are 5, 9.

(b) Sketch the ellipse $5x^2 + 8xy + 5y^2 = 1$ in the $xy$-plane.

$\lambda = 9$ eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\lambda = 1$ eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Limit vectors $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

There exists $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
(c) Write $A = U \Lambda U^T$ with $U$ unitary and $\Lambda$ diagonal (so show the matrices $U$ and $\Lambda$).

\[
U = \begin{bmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & 0 \\
0 & 9 \\
\end{bmatrix}
\]

(d) Can we choose $U$ to be triangular in this situation? Explain.

(Bad question. Answer was meant to be "no" since the eigenvectors won't be triangular.)

(e) Find the max and min of the Rayleigh quotients $x^T A x / x^T x$.

$8$ is max (largest eigenvalue)

$1$ is smallest (least eigenvalue)
3. Let $A$ the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(a) Find the eigenvalues of $A$: $A$ is skew so values are $0, 0, 1$.

(b) From the information in (a) write down the possible Jordan form $J$ that $A$ might have (arrange this so that the diagonal elements of $J$ are nonincreasing).

- $0$ block can be $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
- $1$ block has to be $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(c) Which of the possibilities in (b) is the correct Jordan form, and explain why.

We have to work a bit. The eigenspace for $\lambda = 1$ has dimension 1, but for $\lambda = 0$ it is not clear. But the matrix $A$ has rank 1, so its nullspace has dimension 2, in fact, basis vectors for $0$-eigenspace are $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. For $1$-space it is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

(d) Let $x_1, x_2, x_3$ be the basis for which $J$ has the form in (c). Is this guaranteed to be orthogonal? Explain (you are not expected to produce this basis!)

In this example, $x_1 = x_2 = x_3$, but that is all. But we should think about it.

For the $0$-space, we know the nullspace has dimension 2. But there is no reason for any two vectors in the nullspace to be $\perp$ (they are, through, and by Gram-Schmidt we could arrange that). Thus the vectors in the $1$-subspace are only linearly independent from those in the $0$-subspace (check the book!). So no reason to expect orthogonal.