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1. Let $z = e^{x+2y}$, $x = s+t^2$, $y = s^2 - t$

Find $\frac{\partial z}{\partial t}$ at $(s,t) = (1,2)$

Sol. $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$

$= e^{x+2y}\cdot 2t + 2e^{x+2y}\cdot(-1)$

$= 2e^{x+2y}(t-1)$

when $(s,t) = (1,2)$, $x = 5$, $y = -1$

$\frac{\partial z}{\partial t}(1,2) = 2e^{3}$

2. Let $f(x,y) = 3x^2 - 2x + 4y^2$ near $(2,1)$

(a) Draw the xy-plane, and at $(2,1)$, draw an arrow which shows the direction in which f decrease most rapidly. Decribe the angle between the arrow and x-axis.

(b) Indicate in which direction the rate of change is $-\frac{1}{2}$.

Sol. (a) Suppose the angle between the arrow and x-axis is θ

let $u = \langle \cos\theta, \sin\theta \rangle$.

$D_u(f) = f_x\cos\theta + f_y\sin\theta$ ~~$\Rightarrow \tan\theta$~~

$[\text{at } (2,1)] = 10\cos\theta + 8\sin\theta$.

To find the minimum of $D_u(f)$ at $(2,1)$, let $\frac{d}{d\theta}D_u(f) = 0$

$\frac{d}{d\theta}D_u(f) = -f_x\sin\theta + f_y\cos\theta = 0$.

$\therefore \tan\theta = \frac{f_y}{f_x} = \frac{4}{5}$, $\theta_1 = \arctan\frac{4}{5}$ $\theta_2 = \arctan\frac{4}{5} + \pi$

To see which θ is where $D_u(f)$ reaches its minimum, check

$\frac{d^2 D_u(f)}{d\theta^2} = -f_x\cos\theta - f_y\sin\theta$,

$\left.\frac{d^2 D_u(f)}{d\theta^2}\right|_{\theta_1} = -10\frac{5}{\sqrt{41}} + 8\frac{4}{\sqrt{41}} = \frac{-18}{\sqrt{41}} < 0$

$\left.\frac{d^2 D_u(f)}{d\theta^2}\right|_{\theta_2} = -10\frac{-5}{\sqrt{41}} + 8\frac{-4}{\sqrt{41}} = \frac{18}{\sqrt{41}} > 0$

$\Rightarrow$ So, $\theta = \arctan\frac{4}{5} + \pi$ when $D_u(f)$ reaches its minimum

~~(b) let $u = \langle \cos\theta, \sin\theta \rangle$ s.t. $D_u(f) = -\frac{1}{2}$.~~

~~By (a). $D_u(f) = 10\cos\theta + 8\sin\theta$, let it equal $-\frac{1}{2}$~~

(b) let $u = \langle a, b \rangle$. s.t. $D_u(f) = -\frac{1}{2}$

By (a) $D_u(f) = 10a + 8b$. let it equal $-\frac{1}{2}$

Solve $\begin{cases} 10a + 8b = -\frac{1}{2} \\ a^2 + b^2 = 1 \end{cases}$, $\begin{cases} \cancel{a_1 = -20 + 40\sqrt{41}} \\ \cancel{b_1 = -20 - 40\sqrt{41}} \end{cases}$

$\begin{cases} a_1 = -20 + 8\sqrt{2620} \\ b_1 = \dfrac{399 - 160\sqrt{2620}}{16} \end{cases}$ $\begin{cases} a_2 = -20 - 8\sqrt{2620} \\ b_2 = \dfrac{399 + 160\sqrt{2620}}{16} \end{cases}$