Some irreducibility theorems of parabolic induction on the metaplectic group via the Langlands-Shahidi method

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Abstract

Let $\overline{Sp_{2n}(\mathbb{F})}$ be the metaplectic double cover of $Sp_{2n}(\mathbb{F})$ where $\mathbb{F}$ is a local field of characteristic 0. We use the Uniqueness of Whittaker model to define a metaplectic analog to Shahidi local coefficients and we use these coefficients to define gamma factors. We show that these gamma factors are multiplicative and satisfies crude global functional equation. Then, we compute these factors in various cases and obtain explicit formulas for Plancherel measures. These computations are then used to prove some irreducibility theorems for parabolic induction on the metaplectic group over p-adic fields. In particular we show that all principal series representations induced from unitary characters are irreducible. We also prove that parabolic induction from unitary supercuspidal representation of the Siegel parabolic sub group is irreducible if and only if a certain parabolic induction on $SO_{2n+1}(\mathbb{F})$ is irreducible.

Key words: The metaplectic group, The Langlands-Shahidi method, gamma factors.

0 Introduction

Over a period of about thirty years Freydoon Shahidi has developed the theory of local coefficients and its applications. Nowadays this method is known as the Langlands-Shahidi method. The references [35], [36], [37], [38], [39], [40], [41], [42], [44] are among Shahidi's works from the first half of this period. The applications of this theory are numerous; see the surveys [13], [43], [45] and [23] for a partial list. Although this theory addresses quasi-split connected reductive linear algebraic groups, our aim in this paper is to extend this theory to $\overline{Sp_{2n}(\mathbb{F})}$, the metaplectic double cover of the symplectic group over a local field of characteristic 0, which is not a linear algebraic group, and use it to prove a few irreducibility theorems of parabolic induction.

The properties of $\overline{Sp_{2n}(\mathbb{F})}$ enable the extension of the general representation theory of quasi-split connected reductive linear algebraic groups as presented in [48] and [53]. A great part of this extension is already available in the literature; see [22], [10], [1], [2] and [55] for example. An analog to Bruhat decomposition holds in $\overline{Sp_{2n}(\mathbb{F})}$. If $\mathbb{F}$ is a p-adic field, $\overline{Sp_{2n}(\mathbb{F})}$ is an l-group in the sense of Bernstein and Zelevinsky, [7]. Furthermore, in $\overline{Sp_{2n}(\mathbb{F})}$ the analogs of the Cartan and Iwasawa decompositions hold as well. If $\mathbb{F}$ is not 2-adic then $\overline{Sp_{2n}(\mathbb{F})}$ splits over the standard maximal compact subgroup of $Sp_{2n}(\mathbb{F})$. Over any local field (of characteristic different than 2) $\overline{Sp_{2n}(\mathbb{F})}$ splits over the unipotent subgroups of $Sp_{2n}(\mathbb{F})$.

For a subset H of $Sp_{2n}(\mathbb{F})$ we denote by $\overline{H}$ its pre-image in $\overline{Sp_{2n}(\mathbb{F})}$. Let $P = M \ltimes N$ be a parabolic subgroup of $Sp_{2n}(\mathbb{F})$. $\overline{P}$ has a "Levi" decomposition: $\overline{P} = \overline{M} \ltimes \mu(N)$, where μ is an embedding of N in $\overline{Sp_{2n}(\mathbb{F})}$ which commutes with the projection map.

The general theory of Harish-Chandra, see [48] and [53], extends to the metaplectic group. In fact many of the geometric proofs that are given in [7] and [8] apply word for word to the metaplectic group. This includes the general theorems regarding Jacquet modules, L_2-representations, matrix coefficients, intertwining operators, Harish-Chandra's c-functions etc'. Same holds for Harish-Chandra's completeness theorem and the Knapp-Stein dimension theorem which follows from this theorem. We note that although the metaplectic Knapp-Stein dimension theorem may be proven in the same way as its linear analog, such a proof does not exists in the literature. We shall give this proof elsewhere.

Many of the properties mentioned in the last paragraph are common to general n-fold covering groups of classical groups. However, the following property is a special feature of $\overline{Sp_{2n}(\mathbb{F})}$: If g and h commute in $Sp_{2n}(\mathbb{F})$ then the pre-images in $\overline{Sp_{2n}(\mathbb{F})}$ also commute. In particular, the inverse image of a commutative subgroup of $Sp_{2n}(\mathbb{F})$ is commutative. This implies that the irreducible representations of $\overline{T_{Sp_{2n}}(\mathbb{F})}$, the inverse image of the maximal torus of $Sp_{2n}(\mathbb{F})$, are one dimensional. As noted in [6], this is the reason that a Whittaker model for principal series representation of $\overline{Sp_{2n}(\mathbb{F})}$ is unique. In [50] the uniqueness of Whittaker model for any irreducible admissible genuine representations of $\overline{Sp_{2n}(\mathbb{F})}$ is proven. We emphasize that this uniqueness does not hold for general covering groups; see [12] and [1]. It is the uniqueness of Whittaker model that enables a straight forward generalization of the definition of the Shahidi local coefficients to the metaplectic group.

We now give an rough outline of the main results and arguments of this work . Exact definitions and notation are given in the body of the paper. Let $\mathbb{F}$ be a local field of characteristic 0. Let $\overline{P_{\overrightarrow{t}}(\mathbb{F})}$ be the inverse image in $\overline{Sp_{2n}(\mathbb{F})}$ of the parabolic subgroup of $Sp_{2n}(\mathbb{F})$ whose Levi part is isomorphic to $GL_{n_1}(\mathbb{F}) \times GL_{n_2}(\mathbb{F}) \ldots \times GL_{n_r}(\mathbb{F}) \times Sp_{2k}(\mathbb{F})$. Let τ_i be an admissible representation of $GL_{n_i}(\mathbb{F})$ and let $\overline{\sigma}$ be an admissible genuine representation of $\overline{Sp_{2k}(\mathbb{F})}$. Let γ_ψ be the normalized Weil Index. We define $\pi = \big(\otimes_{i=1}^{r}(\gamma_\psi^{-1} \otimes \tau_i)\big) \otimes \overline{\sigma}$ to be a representation $\overline{P_{\overrightarrow{t}}(\mathbb{F})}$ as explained in Section 2.1 and define $I(\pi) = Ind_{\overline{P_{\overrightarrow{t}}(\mathbb{F})}}^{\overline{Sp_{2n}(\mathbb{F})}} \pi$. In Section 3.1 we introduce the metaplectic Shahidi local coefficients

$$C_{\psi}^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{\overrightarrow{t}}(\mathbb{F})}, \overrightarrow{s}, (\otimes_{i=1}^{r}\tau_i) \otimes \overline{\sigma}, w\big).$$

Here w is a Weyl element and $\overrightarrow{s} \in \mathbb{C}^r$. This definition requires of cause that the inducing representations are ψ-generic. Using the local coefficients we also define

$$\gamma(\overline{\sigma} \times \tau, s, \psi) = \frac{C_{\psi}^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{m;k}(\mathbb{F})}, s, \tau \otimes \overline{\sigma}, j_{m,n}(\omega_m'^{-1})\big)}{C_{\psi}^{\overline{Sp_{2m}(\mathbb{F})}}\big(\overline{P_{m;0}(\mathbb{F})}, s, \tau, \omega_m'^{-1}\big)},$$

the γ-factor attached to $\overline{\sigma}$, an irreducible admissible genuine ψ-generic representation of $\overline{Sp_{2k}(\mathbb{F})}$, and τ, an irreducible admissible generic representation of $GL_m(\mathbb{F})$; see (3.8). Here $P_{m;0}$ is the Siegel parabolic subgroup and the ω_m' is the long Weyl element. This definition of the γ-factor is an exact analog to the definition given in Section 6 of [42] for quasi-split connected reductive algebraic groups.

Theorem A. $\gamma(\overline{\sigma} \times \tau, s, \psi)$ is multiplicative in the since of Part 3 of Theorem 3.15 of [42] and satisfies global crude functional equation which is the metaplectic analog to the crude

functional equation proven by Shahidi in Theorem 4.1 of [37], see also Part 4 of Theorem 3.15 of [42].

Theorem B. Let $\mathbb{F}$ be a p-adic field and let τ be an irreducible admissible generic representation of $GL_m(\mathbb{F})$. Then,

$$C_{\psi}^{\overline{Sp_{2m}(\mathbb{F})}}\big(\overline{P_{m;0}(\mathbb{F})}, s, \tau, \omega_m'^{-1}\big) = c(s)\frac{\gamma(\tau, sym^2, 2s, \psi)}{\gamma(\tau, s+\frac{1}{2}, \psi)}. \tag{0.1}$$

The γ factors on the left hand side of (0.1) are the symmetric square and standard γ factors and $c(s)$ is an exponential factor.

Theorem C. Let $\mathbb{F}$ be a p-adic field. Any principal series representation of $\overline{Sp_{2n}(\mathbb{F})}$ induced from unitary characters is irreducible.

Theorem D. Let $\mathbb{F}$ be a p-adic field. If the inducing representations are unitary, supercuspidal and generic then $I(\pi)$ is reducible if and only if there exists $1 \leq i \leq r$ such that τ_i is self dual and

$$\gamma(\overline{\sigma} \times \tau_i, 0, \psi)\gamma(\tau_i, sym^2, 0, \psi) \neq 0.$$

Theorem E. If $\pi = \tau \otimes \gamma_{\psi}^{-1}$, where τ is an irreducible admissible supercuspidal representation of $GL_n(\mathbb{F})$, then $I(\pi)$ is irreducible if and only if $Ind_{P_{SO_{2n+1}}(\mathbb{F})}^{SO_{2n+1}(\mathbb{F})}\tau$ is irreducible. Here $SO_{2n+1}(\mathbb{F})$ is the split odd orthogonal group and $P_{SO_{2n+1}}$ is its maximal parabolic subgroup whose Levi part is $GL_n(\mathbb{F})$.

The main ingredient of the proof of the multiplicativity of $\gamma(\overline{\sigma} \times \tau, s, \psi)$ is a certain decomposition of the intertwining operators. This decomposition resembles the decomposition of the intertwining operators in the linear case. The only small difference is that two Weyl elements may carry cocycle relations. Our choice of Weyl elements is such that these relations are non-trivial only in the field of real numbers and in 2-adic fields. The proof of the crude global functional equation requires some of the local computations of the metaplectic spherical Whittaker functions given in [6].

Theorem B follows essentially from Theorem A and Proposition 5.1 of [42]. The proof of Theorems C and D uses a metaplectic analog to Knapp-Stein dimension Theorem. These irreducibility results are proven via the computation of various Plancherel measures, expressed as a multiplication of two local coefficients. For Theorem C we use the detailed metaplectic rank one computation given in [51]. The proof of Theorem E is achieved by showing that the two corresponding Plancherel measures are essentially equal.

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1 The metaplectic group

1.1 General notations

All The fields in this work are assumed to be of characteristic 0. Let $\mathbb{F}$ be a p-adic field. Let $\mathbb{O}_{\mathbb{F}}$ be the ring of integers of $\mathbb{F}$ and let $\mathbb{P}_{\mathbb{F}}$ be its maximal ideal. Let q be the cardinality

of the residue field $\overline{\mathbb{F}} = \mathbb{O}_{\mathbb{F}}/\mathbb{P}_{\mathbb{F}}$. Let $|\cdot|$ be the absolute value on $\mathbb{F}$ normalized in the usual way: $|\pi| = q^{-1}$. Let ψ be a character of $\mathbb{F}$. ψ is said to be normalized if its restriction to $\mathbb{O}_{\mathbb{F}}$ is trivial while its restriction to $\mathbb{P}_{\mathbb{F}}^{-1}$ is not.

For any field (of characteristic different than 2) we define $(\cdot,\cdot)_{\mathbb{F}}$ to be the quadratic Hilbert symbol of $\mathbb{F}$. The Hilbert symbol defines a non-degenerate bilinear form on $\mathbb{F}^*/\mathbb{F}^{*2}$. For future references we recall some of the properties of the Hilbert symbol:

$$1.\ (a,-a)_{\mathbb{F}} = 1 \qquad 2.\ (aa',b)_{\mathbb{F}} = (a,b)_{\mathbb{F}}(a',b)_{\mathbb{F}} \qquad 3.\ (a,b)_{\mathbb{F}} = (a,-ab)_{\mathbb{F}}. \tag{1.1}$$

Let ψ be a non-trivial character of $\mathbb{F}$. For $a \in \mathbb{F}^*$ let $\gamma_\psi(a)$ be the normalized Weil factor associated with the character of second degree of $\mathbb{F}$ given by $x \mapsto \psi_a(x^2)$ (see Theorem 2 of Section 14 of [54]). We have $\gamma_\psi(\mathbb{F}^{*2}) = 1$ and $\gamma_\psi(\mathbb{F}^*)^4 = 1$. It is known that

$$\gamma_\psi(ab) = \gamma_\psi(a)\gamma_\psi(b)(a,b)_{\mathbb{F}} \tag{1.2}$$

1.2 The symplectic group

Let $\mathbb{F}$ be a field of characteristic different then 2. Let $X = \mathbb{F}^{2n}$ be a vector space of even dimension over $\mathbb{F}$ equipped with $<\cdot,\cdot>: X \times X \to \mathbb{F}$, a non degenerate symplectic form and let $Sp(X) = Sp_{2n}(\mathbb{F})$ be the subgroup of $GL(X)$ of isomorphisms of X onto itself which preserve $<\cdot,\cdot>$. Following Rao, [32], we shall write the action of $GL(X)$ on X from the right. Let

$$E = \{e_1, e_2, \ldots, e_n, e_1^*, e_2^*, \ldots, e_n^*\}$$

be a symplectic basis of X; for $1 \leq i,j \leq n$ we have $< e_i, e_j >=< e_i^*, e_j^* >= 0$ and $< e_i, e_j^* >= \delta_{i,j}$. In this base $Sp(X)$ is realized as the group

$$\{a \in GL_{2n}(\mathbb{F}) \mid aJ_{2n}a^t = J_{2n}\},$$

where $J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0, \end{pmatrix}$. Through this paper we shall identify $Sp_{2n}(\mathbb{F})$ with this realization. For $0 \leq r \leq n$ define $i_{r,n}$ to be an embedding of $Sp_{2r}(\mathbb{F})$ in $Sp_{2n}(\mathbb{F})$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} I_{n-r} & & & \\ & a & & b \\ & & I_{n-r} & \\ & c & & d \end{pmatrix},$$

where $a, b, c, d \in Mat_{r\times r}(\mathbb{F})$, and define $j_{r,n}$ to be an embedding of $Sp_{2r}(\mathbb{F})$ in $Sp_{2n}(\mathbb{F})$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & & b & \\ & I_{n-r} & & \\ c & & d & \\ & & & I_{n-r} \end{pmatrix}.$$

Let $T_{GL_n}(\mathbb{F})$ be the subgroup of diagonal elements of $GL_n(\mathbb{F})$, let $Z_{GL_n}(\mathbb{F})$ be the group of upper triangular unipotent matrices in $GL_n(\mathbb{F})$ and let $B_{GL_n}(\mathbb{F}) = T_{GL_n}(\mathbb{F}) \ltimes Z_{GL_n}(\mathbb{F})$ be

the standard Borel subgroup of $GL_n(\mathbb{F})$. Let $T_{Sp_{2n}}(\mathbb{F})$ be the subgroup of diagonal elements of $Sp_{2n}(\mathbb{F})$ and let $Z_{Sp_{2n}}(\mathbb{F})$ be the following maximal unipotent subgroup of $Sp_{2n}(\mathbb{F})$;

$$\left\{\begin{pmatrix} z & b \\ 0 & \widetilde{z} \end{pmatrix} \mid z \in Z_{GL_n(\mathbb{F})}, b \in Mat_{n\times n}(\mathbb{F}),\, b^t = z^{-1}bz^t\right\},$$

where for $a \in GL_n$ we define $\widetilde{a} = {}^t a^{-1}$. The subgroup $B_{Sp_{2n}}(\mathbb{F}) = T_{Sp_{2n}}(\mathbb{F}) \ltimes Z_{Sp_{2n}}(\mathbb{F})$ of $Sp_{2n}(\mathbb{F})$ is a Borel subgroup. We call it the standard Borel subgroup. A standard parabolic subgroup of $Sp_{2n}(\mathbb{F})$ is defined to be a parabolic subgroup which contains $B_{Sp_{2n}}(\mathbb{F})$. A standard Levi subgroup (unipotent radical) is a Levi part (unipotent radical) of a standard parabolic subgroup. In particular a standard Levi subgroup contains $T_{Sp_{2n}}(\mathbb{F})$ and a standard unipotent radical is contained in $Z_{Sp_{2n}}(\mathbb{F})$.

Let $n_1, n_2, \ldots, n_r, k$ be $r+1$ nonnegative integers whose sum is n. Put $\overrightarrow{t} = (n_1, n_2, \ldots, n_r; k)$. Let $M_{\overrightarrow{t}}$ be the standard Levi subgroup of $Sp_{2n}(\mathbb{F})$ which consists of elements of the form

$$[g_1, g_2, \ldots, g_r, h] = diag(g_1, g_2, \ldots, g_r, I_k, \widetilde{g_1}, \widetilde{g_2}, \ldots, \widetilde{g_r}, I_k)i_{k,n}(h),$$

where $g_i \in GL_{n_i}(\mathbb{F}), h \in Sp_{2k}(\mathbb{F})$. When convenient we shall identify $GL_{n_i}(\mathbb{F})$ with its natural embedding in $M_{\overrightarrow{t}}(\mathbb{F})$. Denote by $P_{\overrightarrow{t}}(\mathbb{F})$ the standard parabolic subgroup of $Sp_{2n}(\mathbb{F})$ that contains $M_{\overrightarrow{t}}(\mathbb{F})$ as its Levi part. Denote by $N_{\overrightarrow{t}}(\mathbb{F})$ the unipotent radical of $P_{\overrightarrow{t}}(\mathbb{F})$. We denote by $P_{Sp_{2n}}(\mathbb{F})$ or simply by $P(\mathbb{F})$, the Siegel parabolic subgroup of $Sp_{2n}(\mathbb{F})$:

$$P_{(n;0)}(\mathbb{F}) = \left\{\begin{pmatrix} a & b \\ 0 & \widetilde{a} \end{pmatrix} \mid a \in GL_n(\mathbb{F}), b \in Mat_{n\times n}(\mathbb{F}), b^t = a^{-1}ba^t\right\}. \tag{1.3}$$

Note that $M_{(n;0)}(\mathbb{F}) \simeq GL_n(\mathbb{F})$. A natural isomorphism is given by

$$g \mapsto \widehat{g} = \begin{pmatrix} g & \\ & \widetilde{g} \end{pmatrix}.$$

Define

$$V = span\{e_1, e_2, \ldots, e_n\}, \quad V^* = span\{e_1^*, e_2^*, \ldots, e_n^*\}.$$

These are two transversal Lagrangian subspaces of X. The Siegel parabolic subgroup is the subgroup of $Sp(X)$ which consists of elements that preserve V^*. Let S be a subset of $\{1, 2, \ldots, n\}$. Define τ_S, a_S to be the following elements of $Sp(X)$:

$$e_i \cdot \tau_S = \begin{cases} -e_i^* & i \in S \\ e_i & \text{otherwise} \end{cases}, \quad e_i^* \cdot \tau_S = \begin{cases} e_i & i \in S \\ e_i^* & \text{otherwise} \end{cases}, \tag{1.4}$$

$$e_i \cdot a_S = \begin{cases} -e_i & i \in S \\ e_i & \text{otherwise} \end{cases}, \quad e_i^* \cdot a_S = \begin{cases} -e_i^* & i \in S \\ e_i^* & \text{otherwise} \end{cases}. \tag{1.5}$$

The elements τ_{S_1}, a_{S_1}, τ_{S_2}, a_{S_2} commute. Note that $a_S \in P(\mathbb{F})$, $a_S^2 = I_{2n}$, and that

$$\tau_{S_1}\tau_{S_2} = \tau_{S_1 \triangle S_2} a_{S_1 \cap S_2}, \tag{1.6}$$

where $S_1 \triangle S_2 = S_1 \cup S_2 \backslash S_1 \cap S_2$. In particular $\tau_S^2 = a_S$. For $S = \{1, 2, \ldots, n\}$ we define $\tau = \tau_S$, in this case $a_S = -I_{2n}$.

Denote by $W'_{Sp_{2n}}(\mathbb{F})$ the subgroup of $Sp_{2n}(\mathbb{Z})$ generated by the elements τ_S, and $\widehat{w_\pi}$, where $S \subseteq \{1, 2, \ldots, n\}$, and $w_\pi \in GL_n(\mathbb{F})$ is defined by $w_{\pi i,j} = \delta_{\pi(i),j}$; π is a permutation in S_n. If $\mathbb{F}$ is a p-adic field then $W'_{Sp_{2n}}(\mathbb{F})$ is a subgroup of $Sp_{2n}(\mathbb{O}_\mathbb{F})$. Note that $W'_{Sp_{2n}}(\mathbb{F})$ modulo its diagonal elements may be identified with the Weyl group of $Sp_{2n}(\mathbb{F})$ denoted by $W_{Sp_{2n}}(\mathbb{F})$. Define $W_{P_{\overrightarrow{t}}}(\mathbb{F})$ to be the subgroup of $W'_{Sp_{2n}}(\mathbb{F})$ which consists of elements w such that

$$M_{\overrightarrow{t}}(\mathbb{F})^w = wM_{\overrightarrow{t}}(\mathbb{F})w^{-1}$$

is a standard Levi subgroup and

$$w\big(Z_{Sp_{2n}}(\mathbb{F}) \cap M_{\overrightarrow{t}}(\mathbb{F})\big)w^{-1} \subset Z_{Sp_{2n}}(\mathbb{F}).$$

This means that up to conjugation by diagonal elements inside the blocks of $M_{\overrightarrow{t}}(\mathbb{F})$, we have

$$w[g_1, g_2, \ldots, g_r, s]w^{-1} = [g^{(\epsilon_1)}_{\pi(1)}, g^{(\epsilon_2)}_{\pi(2)}, \ldots, g^{(\epsilon_r)}_{\pi(r)}, s], \tag{1.7}$$

where π is a permutation of $\{1, 2, \ldots, r\}$, and where for $g \in GL_n(\mathbb{F})$, $\epsilon = \pm 1$ we define

$$g^{(\epsilon)} = \begin{cases} g & \epsilon = 1 \\ \omega_n \widetilde{g} \omega_n & \epsilon = -1 \end{cases},$$

where

$$\omega_n = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}.$$

We may assume, and in fact do, that $W_{P_{\overrightarrow{t}}}(\mathbb{F})$ commutes with $i_{k,n}(Sp_{2k}(\mathbb{F}))$.

For $w \in W_{P_{\overrightarrow{t}}}(\mathbb{F})$, let $P_{\overrightarrow{t}}(\mathbb{F})^w$ be the standard parabolic subgroup whose Levi part is $M_{\overrightarrow{t}}(\mathbb{F})^w$, and let $N_{\overrightarrow{t}}(\mathbb{F})^w$ be its standard unipotent radical.

1.3 Rao's cocycle

In [32], Rao constructs an explicit non-trivial 2-cocycle $c(\cdot, \cdot)$ on $Sp(X)$ which takes values in $\{\pm 1\}$. The set $\overline{Sp(X)} = Sp(X) \times \{\pm 1\}$ is then given a group structure via the formula

$$(g_1, \epsilon_1)(g_2, \epsilon_2) = \big(g_1 g_2, \epsilon_1 \epsilon_2 c(g_1, g_2)\big). \tag{1.8}$$

It is called the metaplectic group. For any subset A of $Sp_{2n}(\mathbb{F})$ we denote by $\overline{A}$ its inverse image in $\overline{Sp_{2n}(\mathbb{F})}$. If $\mathbb{F}$ is either $\mathbb{R}$ or a p-adic field, this group is the unique non-trivial double cover of $Sp_{2n}(\mathbb{F})$. We now describe Rao's cocycle. Detailed proofs can be found in [32]. Define

$$\Omega_j = \{\sigma \in Sp(X) \mid \dim(V^* \cap V^*\sigma) = n - j\}.$$

Note that $P(\mathbb{F}) = \Omega_0$, $\tau_S \in \Omega_{|S|}$ and more generally, if $\alpha, \beta, \gamma, \delta \in Mat_{n\times n}(\mathbb{F})$ and $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in Sp(X)$ then $\sigma \in \Omega_{rank(\gamma)}$. The Bruhat decomposition states that each Ω_j is a single double coset in $P(\mathbb{F}) \backslash Sp(X) / P(\mathbb{F})$, that $\Omega_j^{-1} = \Omega_j$ and that $\bigcup_{j=0}^n \Omega_j = Sp(X)$. In particular every element of $Sp(X)$ has the form $p\tau_S p'$, where $p, p' \in P(\mathbb{F})$, $S \subseteq \{1, 2, \ldots, n\}$.

Let $p_1, p_2 \in P(\mathbb{F})$. Rao defines

$$x(p_1 \tau_S p_2) \equiv \det(p_1 p_2 \mid_{V^*}) \big(mod(\mathbb{F}^*)^2\big), \tag{1.9}$$

and proves that it is a well defined map from $Sp(X)$ to $\mathbb{F}^*/(\mathbb{F}^*)^2$. Note that $x(a_S) \equiv (-1)^{|S|}$. More generally; if $p = \begin{pmatrix} a & b \\ 0 & \widetilde{a} \end{pmatrix} \in P(\mathbb{F})$ then $x(p) \equiv \det(a)$. We shall use the notation $\det(p) = \det(a)$. Also note that $x(\tau_S) \equiv 1$ and that for $g \in \Omega_j$, $p_1, p_2 \in P(\mathbb{F})$,

$$x(g^{-1}) \equiv x(g)(-1)^j, \quad x(p_1 g p_2) \equiv x(p_1)x(g)x(p_2). \tag{1.10}$$

Theorem 5.3 in Rao's paper states that a non-trivial 2-cocycle on $Sp(X)$ can be defined by

$$c(\sigma_1, \sigma_2) = \big(x(\sigma_1), x(\sigma_2)\big)_{\mathbb{F}} \big(-x(\sigma_1)x(\sigma_2), x(\sigma_1\sigma_2)\big)_{\mathbb{F}} \big((-1)^l, d_F(\rho)\big)_{\mathbb{F}} \big(-1,-1\big)_{\mathbb{F}}^{\frac{l(l-1)}{2}} h_{\mathbb{F}}(\rho), \tag{1.11}$$

where ρ is the Leray invariant $-q(V^*, V^*\sigma_1, V^*\sigma_2^{-1})$, $d_{\mathbb{F}}(\rho)$ and $h_{\mathbb{F}}(\rho)$ are its discriminant, and Hasse invariant, and $2l = j_1 + j_2 - j - \dim(\rho)$, where $\sigma_1 \in \Omega_{j_1}$, $\sigma_2 \in \Omega_{j_2}$, $\sigma_1\sigma_2 \in \Omega_j$. We use Rao's normalization of the Hasse invariant. (Note that the cocycle formula just given differs slightly from the one that appears in Rao's paper. There is a small mistake in Theorem 5.3 of [32]. A correction by Adams can be found in [25], Theorem 3.1).

An immediate consequence of Rao's formula is that if g and h commute in $Sp_{2n}(\mathbb{F})$ then their pre-image in $\overline{Sp_{2n}(\mathbb{F})}$ also commute (this may be deduced from more general ideas. See page 39 of [30]). In particular, a pre-image in $\overline{Sp_{2n}(\mathbb{F})}$ of a commutative subgroup of $Sp_{2n}(\mathbb{F})$ is also commutative. This does not hold for general covering groups; see [1] for example.

If $\mathbb{F}$ is a local field then $\overline{Sp_{2n}(\mathbb{F})}$ is a locally compact group. If $\mathbb{F}$ is a p-adic field, $\overline{Sp_{2n}(\mathbb{F})}$ is an l-group in the sense of [7]; since $c(\cdot,\cdot)$ is continuous, it follows that there exists U, an open compact subgroup of $Sp_{2n}(\mathbb{F})$, such that $c(U,U) = 1$. Thus, a system of neighborhoods of $(I_{2n}, 1)$ is given by open compact subgroups of the form $(V, 1)$, where $V \subseteq U$ is an open compact subgroup of $Sp_{2n}(\mathbb{F})$.

From (1.11) and from previous remarks we obtain the following properties of $c(\cdot,\cdot)$; for $\sigma, \sigma' \in \Omega_j$, $p, p' \in P(\mathbb{F})$ we have

$$c(\sigma, \sigma^{-1}) = \big(x(\sigma), (-1)^j x(\sigma)\big)_{\mathbb{F}} (-1,-1)_{\mathbb{F}}^{\frac{j(j-1)}{2}} \tag{1.12}$$

$$c(p\sigma, \sigma' p') = c(\sigma, \sigma')\big(x(p), x(\sigma)\big)_{\mathbb{F}} \big(x(p'), x(\sigma')\big)_{\mathbb{F}} \big(x(p), x(p')\big)_{\mathbb{F}} \big(x(pp'), x(\sigma\sigma')\big)_{\mathbb{F}}. \tag{1.13}$$

As a consequence of (1.13) we obtain

$$c(p, \sigma) = c(\sigma, p) = \big(x(p), x(\sigma)\big)_{\mathbb{F}}. \tag{1.14}$$

Another property of the cocycle noted in [32] is that

$$c(\tau_{S_1}, \tau_{S_2}) = (-1,-1)_{\mathbb{F}}^{\frac{j(j+1)}{2}}, \tag{1.15}$$

where j is the cardinality of $S_1 \cap S_2$. From (1.6), (1.13) and (1.15) we conclude that if S and S' are disjoint then for $p, p' \in P(\mathbb{F})$ we have

$$c(p\tau_S, \tau_{S'}p') = \big(x(p), x(p')\big)_{\mathbb{F}}. \tag{1.16}$$

It follows from (1.14) and (1.1) that

$$(p, \epsilon_1)(\sigma, \epsilon)(p, \epsilon_1)^{-1} = (p\sigma p^{-1}, \epsilon), \tag{1.17}$$

for all $\sigma \in Sp(X)$, $p \in P(\mathbb{F})$, $\epsilon_1, \epsilon \in \{\pm 1\}$. Furthermore, assume that $p \in P(\mathbb{F})$, $\sigma \in Sp(X)$ satisfy $\sigma p \sigma^{-1} \in P(\mathbb{F})$. Then

$$(\sigma, \epsilon_1)(p, \epsilon)(\sigma, \epsilon_1)^{-1} = (\sigma p \sigma^{-1}, \epsilon). \tag{1.18}$$

Indeed, due to (1.17) and the Bruhat decomposition we only need to show that if $\tau_S p {\tau_S}^{-1} \in P(\mathbb{F})$ then

$$c(p, \tau_S)c(\tau_S p, \tau_S^{-1})c(\tau_S, {\tau_S}^{-1}) = 1.$$

Define j to be the cardinality of S. From (1.12) it follows that $c(\tau_S, {\tau_S}^{-1}) = (-1,-1)_{\mathbb{F}}^{\frac{j(j-1)}{2}}$. From (1.14) it follows that $c(p, \tau_S) = 1$. It is left to show that if $V^*\tau_S p \tau_S^{-1} = V^*$ then

$$c(\tau_S p, {\tau_S}^{-1}) = (-1,-1)_{\mathbb{F}}^{\frac{j(j-1)}{2}}. \tag{1.19}$$

Recall that the Leray invariant is stable under the action of $Sp(X)$ on Lagrangian triplets; see Theorem 2.11 of [32]. Therefore,

$$q(V^*, V^*\tau_S p, V^*\tau_S) = q(V^*{\tau_S}^{-1}, V^*\tau_S p {\tau_S}^{-1}, V^*) = q(V^*{\tau_S}^{-1}, V^*, V^*)$$

is an inner product defined on the trivial space. (1.11) implies now (1.19).

We recall Corollary 5.6 in Rao's paper. For $S \subset \{1, 2, \ldots, n\}$ define

$$X_S = span\{e_i, e_i^* \mid i \in S\}.$$

We may now consider x_S and $c_{X_S}(\cdot,\cdot)$ defined by analogy with x and $c(\cdot,\cdot)$. Let S_1 and S_2 be a partition of $\{1, 2, \ldots, n\}$. Suppose that $\sigma_1, \sigma_1' \in Sp(X_{S_1})$ and that $\sigma_2, \sigma_2' \in Sp(X_{S_2})$. Put $\sigma = diag(\sigma_1, \sigma_2)$, $\sigma' = diag(\sigma_1', \sigma_2')$. Rao proves that $c(\sigma, \sigma')$ equals

$$c_{S_1}(\sigma_1, \sigma_1')c_{S_2}(\sigma_2, \sigma_2')\big(x_{S_1}(\sigma_1), x_{S_2}(\sigma_2)\big)_{\mathbb{F}}\big(x_{S_1}(\sigma_1'), x_{S_2}(\sigma_2')\big)_{\mathbb{F}}\big(x_{S_1}(\sigma_1\sigma_1'), x_{S_2}(\sigma_2\sigma_2')\big)_{\mathbb{F}}. \tag{1.20}$$

From (1.20) it follows that $(s, \epsilon) \mapsto \big(i_{r,n}(s), \epsilon\big)$ and $(s, \epsilon) \mapsto \big(j_{r,n}(s), \epsilon\big)$ are two embeddings of $\overline{Sp_{2r}(\mathbb{F})}$ in $\overline{Sp_{2n}(\mathbb{F})}$. We shall continue to denote these embeddings by $i_{r,n}$ and $j_{r,n}$ respectively. Note that the map $g \mapsto (\widehat{g}, 1)$ is not an embedding of $GL_n(\mathbb{F})$ in $\overline{Sp_{2n}(\mathbb{F})}$, although, by (1.14), its restriction to $Z_{GL_n}(\mathbb{F})$ is an embedding.

1.4 Some splittings

For $\mathbb{F}$, a p-adic field of odd residual characteristic it is known (see page 58 of [28]) that $\overline{SL_2(\mathbb{F})}$ splits over $SL_2(\mathbb{O}_{\mathbb{F}})$, the standard maximal compact subgroup of $SL_2(\mathbb{F})$ and that

$$\iota_2 : SL_2(\mathbb{O}_{\mathbb{F}}) \to \{\pm 1\}$$

defined by

$$\iota_2\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} (c,d)_{\mathbb{F}} & 0<|c|<1 \\ 1 & \text{otherwise} \end{cases}$$

is the unique map such that the map

$$k \mapsto \kappa_2(k) = \big(k, \iota_2(k)\big)$$

is an embedding of $SL_2(\mathbb{O}_{\mathbb{F}})$ in $\overline{SL_2(\mathbb{F})}$. More generally, it is known, see [28], that if $\mathbb{F}$ is a p-adic field of odd residual characteristic then $\overline{Sp_{2n}(\mathbb{F})}$ splits over $Sp_{2n}(\mathbb{O}_{\mathbb{F}})$; there exists a map

$$\iota_{2n} : Sp_{2n}(\mathbb{O}_{\mathbb{F}}) \to \{\pm 1\}$$

such that the map

$$k \mapsto \kappa_{2n}(k) = \big(k, \iota_{2n}(k)\big)$$

is an embedding of $Sp_{2n}(\mathbb{O}_{\mathbb{F}})$ in $\overline{Sp_{2n}(\mathbb{F})}$. Since κ_2 is the unique splitting of $SL_2(\mathbb{O}_{\mathbb{F}})$ in $\overline{SL_2(\mathbb{F})}$ and since $Sp_{2n}(\mathbb{O}_{\mathbb{F}})$ is generated by various embeddings of $SL_2(\mathbb{O}_{\mathbb{F}})$ it follows that ι_{2n} is also unique.

Lemma 1.1. *The restrictions of ι_{2n} to $P(\mathbb{F}) \cap Sp_{2n}(\mathbb{O}_{\mathbb{F}})$ and to $W'_{Sp_{2n}}(\mathbb{F})$ are trivial.*

Proof. Since for odd residue characteristic $(\mathbb{O}_{\mathbb{F}}^*, \mathbb{O}_{\mathbb{F}}^*)_{\mathbb{F}} = 1$ we conclude, using (1.14), that ι_{2n} restricted to $P(\mathbb{F}) \cap Sp_{2n}(\mathbb{O}_{\mathbb{F}})$ is a quadratic character and hence has the form $p \mapsto \chi(\det p)$, where χ is a quadratic character of $\mathbb{O}_{\mathbb{F}}^*$. By the inductivity property of Rao's cocycle and by the formula of ι_2 we conclude that

$$\iota_{2n}\big(i_{1,n}(SL_2(\mathbb{O}_{\mathbb{F}})\big) \cap P(\mathbb{F}) = 1.$$

Thus $\chi = 1$. We now move to the second assertion. We note that the group generated by the elements of the form τ_S is the group of elements of the form $\tau_{S_1} a_{S_2}$, where $S, S_1, S_2 \subseteq \{1, 2, \ldots, n)$. The group $\{\widehat{w_\pi} \mid \pi \in S_n\}$ is disjoint from that group and normalizes it. Hence we need only to show that for all $S_1, S_2 \subseteq \{1, 2, \ldots, n\}$, and for all $\pi \in S_n$

$$\iota_{2n}(\widehat{w_\pi} a_{S_1} \tau_{S_2}) = 1.$$

The fact that $\iota_{2n}(\widehat{w_\pi} a_{S_2})$=1 was proved already. We now show that $\iota_{2n}(\tau_{S_1}) = 1$: The fact that for $|S_1| = 1$: $\iota_{2n}(\tau_{S_1}) = 1$ follows from the properties of ι_2 and the inductivity properties of Rao's cocycle. We proceed by induction on the cardinality of S_1; suppose that if $\mid S_1 \mid \leq l$ then $\iota_{2n}(\tau_{S_1}) = 1$. Assume now that $\mid S_1 \mid = l+1$. Write $S_1 = S' \cup S''$, where S' and S'' are two non-empty disjoint sets. By (1.15) we have

$$\iota_{2n}(\tau_{S_1}) = \iota_{2n}(\tau_{S'})\iota_{2n}(\tau_{S''})c(\tau_{S'}, \tau_{S''}) = 1.$$

Finally,

$$\iota_{2n}(\widehat{w_\pi} a_{S_2} \tau_{S_1}) = \iota_{2n}(\widehat{w_\pi})\iota_{2n}(a_{S'})\iota_{2n}(\tau_S)c(\widehat{w_\pi}, a_{S'})c(\widehat{w_\pi} a_{S'}, \tau_S) = 1.$$

□

1.5 Some facts about parabolic subgroups of $\overline{Sp_{2n}(\mathbb{F})}$

Let $\mathbb{F}$ be a local field. By a parabolic subgroup of $\overline{Sp_{2n}(\mathbb{F})}$ we mean an inverse image of a parabolic subgroup of $Sp_{2n}(\mathbb{F})$.

Lemma 1.2. *Let Q a parabolic subgroup of $Sp_{2n}(\mathbb{F})$. Write $Q = M \ltimes N$, a Levi decomposition. Then, there exists a unique function $\mu' : N \to \{\pm 1\}$, such that $n \mapsto \mu(n) = \big(n, \mu'(n)\big)$ is an embedding of N in $\overline{Sp_{2n}(\mathbb{F})}$. Furthermore: $\overline{Q} = \overline{M} \ltimes \mu(N)$. By abuse of language we shall refer to the last equality as the Levi decomposition of $\overline{Q}$.*

Proof. Suppose first that Q is standard. From the fact $\overline{Sp_{2n}(\mathbb{F})}$ splits over N via the trivial section it follows that μ' is a quadratic character of N. Since $N = N^2$ we conclude that μ' is trivial. Using (1.18) we get $\overline{Q} = \overline{M} \ltimes \mu(N)$. Assume now that Q is a general parabolic subgroup. Then, $\overline{Q} = (w,1)\overline{Q'}(w,1)^{-1}$ for some $w \in Sp_{2n}(\mathbb{F})$, and a standard parabolic subgroup Q'. For $n \in Q$ define $n' = w^{-1}nw$. From the proof in the standard case it follows that

$$\mu'(n) = c(w,n')c(wn',w^{-1})c(w^{-1},w) = c(nw,w{-}1)c(w^{-1},w)$$

is the unique function mentioned in the lemma. The fact that $\overline{Q} = \overline{M} \ltimes \mu(N)$ follows also from the standard case. $\square$

Lemma 1.3. *Let $\mathbb{F}$ be a p-adic field. $\overline{Sp_{2k}(\mathbb{O}_\mathbb{F})}$ is a maximal open compact subgroup of $\overline{Sp_{2n}(\mathbb{F})}$. For any parabolic subgroup Q of $Sp_{2n}(\mathbb{F})$ we have*

$$\overline{Sp_{2n}(\mathbb{F})} = (Q,1)\overline{Sp_{2k}(\mathbb{O}_\mathbb{F})} = \overline{Sp_{2k}(\mathbb{O}_\mathbb{F})}(Q,1).$$

If $\mathbb{F}$ is a p-adic field of odd residual characteristic then $\overline{Sp_{2n}(\mathbb{F})} = \overline{Q}\kappa_{2n}\big(Sp_{2n}(\mathbb{O}_\mathbb{F})\big)$.

By an abuse of notation we call the last decomposition an Iwasawa decomposition of $\overline{Sp_{2n}(\mathbb{F})}$.

Proof. This follows immediately from the analogous lemma in the algebraic case. $\square$

1.6 The global metaplectic group

Let $\mathbb{F}$ be a number field, and let $\mathbb{A}$ be its Adele ring. For every place ν of $\mathbb{F}$ we denote by $\mathbb{F}_\nu$ its completion at ν. We denote by $\widehat{Sp_{2n}(\mathbb{A})}$ the restricted product $\prod'_\nu Sp_{2n}(\mathbb{F}_\nu)$ with respect to

$$\Big\{\kappa_{2n}\big(Sp_{2n}(\mathbb{O}_{\mathbb{F}_\nu})\big) \mid \nu \text{ } is\ finite\ and\ odd\Big\}.$$

$\widehat{Sp_{2n}(\mathbb{A})}$ is clearly not a double cover of $Sp_{2n}(\mathbb{A})$. Put

$$C' = \Big\{\prod_\nu (I,\epsilon_\nu) \mid \prod_\nu \epsilon_\nu = 1\Big\}.$$

We define

$$\overline{Sp_{2n}(\mathbb{A})} = C' \setminus \widehat{Sp_{2n}(\mathbb{A})}$$

to be the metaplectic double cover of $Sp_{2n}(\mathbb{A})$. It is shown in page 728 of [21] that

$$k \mapsto C' \prod_\nu (k,1)$$

is an embedding of $Sp_{2n}(\mathbb{F})$ in $\overline{Sp_{2n}(\mathbb{A})}$.

2 Some representation theory of $\overline{Sp_{2n}(\mathbb{F})}$

In this Section $\mathbb{F}$ will denote a p-adic field. Recall that a representation $(V, \overline{\sigma})$ of $\overline{Sp_{2n}(\mathbb{F})}$ is called genuine if

$$\overline{\sigma}(I_{2k}, -1) = -Id_V.$$

This means that $\overline{\sigma}$ does not factor through the projection map $Pr : \overline{Sp_{2n}(\mathbb{F})} \to Sp_{2n}(\mathbb{F})$. Same definition applies to representations of $\overline{M_{\overrightarrow{t}}(\mathbb{F})}$.

2.1 Genuine parabolic induction

For a representation (τ, V) of $GL_n(\mathbb{F})$ and a complex number s we denote by $\tau_{(s)}$ the representation of $GL_n(\mathbb{F})$ in V defined by

$$g \mapsto |\det(g)|^s \tau(g).$$

Put $\overrightarrow{t} = (n_1, n_2, \ldots, n_r; k)$, where $k + \sum_{i=1}^{r} n_i = n$. Let $(\tau_1, V_{\tau_1}), (\tau_2, V_{\tau_2}), \ldots, (\tau_r, V_{\tau_r})$ be r representations of $GL_{n_1}(\mathbb{F}), GL_{n_2}(\mathbb{F}), \ldots, GL_{n_r}(\mathbb{F})$ respectively. Let $(\overline{\sigma}, V_{\overline{\sigma}})$ be a genuine representation of $\overline{Sp_{2k}(\mathbb{F})}$. We shall now describe a representation of $\overline{P_{\overrightarrow{t}}(\mathbb{F})}$ constructed from these representations. We cannot repeat the algebraic construction since generally

$$\overline{M_{\overrightarrow{t}}(\mathbb{F})} \ncong GL_{n_1}(\mathbb{F}) \times GL_{n_2}(\mathbb{F}) \ldots \times GL_{n_r}(\mathbb{F}) \times \overline{Sp_{2k}(\mathbb{F})}$$

(it can be shown that these groups are isomorphic in the case of p-adic fields of odd residual characteristic). Instead we define

$$\big(\otimes_{i=1}^{r}(\gamma_\psi^{-1} \otimes \tau_{i(s_i)})\big) \otimes \overline{\sigma} : \overline{M_{\overrightarrow{t}}(\mathbb{F})} \to GL\big((\otimes_{i=1}^{i=r} V_{\tau_i}) \otimes V_{\overline{\sigma}}\big)$$

by

$$\big(\otimes_{i=1}^{r}(\gamma_\psi^{-1} \otimes \tau_{i(s_i)})\big) \otimes \overline{\sigma}\big(j_{n-k,n}(\widehat{g}), 1)(i_{k,n}(h), \epsilon)\big) = \gamma_\psi^{-1}\big(\det(g)\big)\Big(\otimes_{i=1}^{r} \tau_{i(s_i)}(g_i)\Big) \otimes \overline{\sigma}(h, \epsilon), \tag{2.1}$$

where for $1 \leq i \leq r$, $g_i \in GL_{n_i}(\mathbb{F})$, $g = diag(g_1, g_2, \ldots, g_r) \in GL_{n-k}(\mathbb{F})$, $h \in Sp_{2k}(\mathbb{F})$ and $\epsilon \in \{\pm 1\}$. When convenient we shall use the notation

$$\begin{aligned} \pi(\overrightarrow{s}) &= \big(\otimes_{i=1}^{r}(\gamma_\psi^{-1} \otimes \tau_{i(s_i)})\big) \otimes \overline{\sigma} \qquad (2.2) \\ \pi &= \pi(0). \end{aligned}$$

Lemma 2.1. *$\pi(\overrightarrow{s})$ is a representation of $\overline{M_{\overrightarrow{t}}(\mathbb{F})}$.*

Proof. Let $\alpha = \big(j_{n-k,n}(\widehat{g}), 1\big)\big(i_{k,n}(h), \epsilon\big)$ and $\alpha' = \big(j_{n-k,n}(\widehat{g'}), 1\big)\big(i_{k,n}(h'), \epsilon'\big)$, where for $1 \leq i \leq r$, $g_i, g'_i \in GL_{n_i}(\mathbb{F})$, $g = diag(g_1, g_2, \ldots, g_r)$, $g' = diag(g'_1, g'_2, \ldots, g'_r) \in GL_{n-k}(\mathbb{F})$, $h, h' \in Sp_{2k}(\mathbb{F})$ and $\epsilon, \epsilon' \in \{\pm 1\}$ be two elements in $\overline{M_{\overrightarrow{t}}(\mathbb{F})}$. It is sufficient to show that if $v = (\otimes_{i=1}^{r} v_i) \otimes w$ is a pure tensor in $(\otimes_{i=1}^{i=r} V_{\tau_i}) \otimes V_{\overline{\sigma}}$ then $\pi(\alpha'\alpha)v = \pi(\alpha')\big(\pi(\alpha)v\big)$. Indeed,

$$\pi(\alpha')\big(\pi(\alpha)v\big) = \gamma_\psi^{-1}\big(\det(g)\big)\gamma_\psi^{-1}\big(\det(g')\big)\Big(\otimes_{i=1}^{r} \tau_i(g'_i)\tau_i(g_i)v_i\Big) \otimes \overline{\sigma}(h', \epsilon')\overline{\sigma}(h, \epsilon)w.$$

Due to (1.2) and (1.14), for $p, p' \in P(\mathbb{F})$ we have

$$\gamma_\psi^{-1}\big(\det(p)\big)\gamma_\psi^{-1}\big(\det(p')\big)c(p, p') = \gamma_\psi^{-1}\big(\det(pp')\big). \tag{2.3}$$

Recalling that $\overline{\sigma}$ is genuine we now see that

$$\pi(\alpha')\big(\pi(\alpha)v\big) = \epsilon\epsilon'\gamma_\psi^{-1}\big(\det(gg')\big)c\big(j_{n-k,n}(\widehat{g'}), j_{n-k,n}(\widehat{g})\big)c(h',h)\Big(\otimes_{i=1}^r \tau_i(g_i'g_i)v_i\Big)\otimes\overline{\sigma}(h'h,1)w. \tag{2.4}$$

Next, we note that since $\big(j_{n-k,n}(\widehat{g'}),1\big)$ and $\big(i_{k,n}(h),\epsilon\big)$ commute we have

$$\alpha'\alpha = \Big(j_{n-k,n}(\widehat{gg'}),1\Big)\Big(i_{k,n}(h'h), \epsilon\epsilon' c(h',h)c\big(j_{n-k,n}(\widehat{g'}), j_{n-k,n}(\widehat{g})\big)\Big).$$

(2.4) implies now that $\pi(\alpha'\alpha)v = \pi(\alpha')\big(\pi(\alpha)v\big)$. $\square$

As in the linear case we note that if $\tau_1, \tau_2, \ldots, \tau_r$ and $\overline{\sigma}$ are smooth (admissible) representations then $\pi(\overrightarrow{s})$ is also smooth (admissible). Due to (1.18) it is possible to extend $\pi(\overrightarrow{s})$ to a representation of $P_{\overrightarrow{t}}(\mathbb{F})$ by letting $(N_{\overrightarrow{t}}(\mathbb{F}),1)$ act trivially.

Assuming that $\tau_1, \tau_2, \ldots, \tau_r$ and $\overline{\sigma}$ are smooth we define smooth induction

$$I\big(\pi(\overrightarrow{s})\big) = I(\tau_{1(s_1)}, \tau_{2(s_2)}, \ldots, \tau_{r(s_r)}, \overline{\sigma}) = Ind_{\overline{P_{\overrightarrow{t}}(\mathbb{F})}}^{\overline{Sp_{2n}(\mathbb{F})}}\pi(\overrightarrow{s}) \tag{2.5}$$

and

$$I(\pi) = I(\tau_1, \tau_2, \ldots, \tau_r, \overline{\sigma}) = Ind_{\overline{P_{\overrightarrow{t}}(\mathbb{F})}}^{\overline{Sp_{2n}(\mathbb{F})}}\pi. \tag{2.6}$$

All the induced representations in this paper are assumed to be normalized, i.e., if (π, V) is a smooth representation of H, a closed subgroup of a locally compact group G, then $Ind_H^G\pi$ acts in the space of all right-smooth functions on G that take values in V and satisfy $f(hg) = \sqrt{\frac{\delta_H(h)}{\delta_G(h)}}\pi(h)f(g)$, for all $h \in H, g \in G$. Whenever we induce from a parabolic subgroup (a pre-image of a parabolic subgroup in a metaplectic group) we always mean that the inducing representation is trivial on its unipotent radical (on its embedding in the metaplectic group).

We claim that if the inducing representations are admissible, then $\pi(\overrightarrow{s})$ is also admissible. Indeed, the proof of Proposition 2.3 of [8] which is the p-adic analog of this claim applies to $\overline{Sp_{2n}(\mathbb{F})}$ as well since it mainly uses the properties on an l-group; see Proposition 4.7 of [55] or Proposition 1 of [1].

Next we note that similar to the linear case we can define the Jacquet functor, replacing the role of $Z_{Sp_{2n}}(\mathbb{F})$ with $(Z_{Sp_{2n}}(\mathbb{F}),1)$. The notion of a supercuspidal representation is defined via the vanishing of Jacquet modules along unipotent radicals of parabolic subgroups. The (metaplectic) Jacquet functor has similar properties to those of the Jacquet functor in the linear case; see Section 4.1 of [1] or Proposition 4.7 of [55]. This similarity follows from the fact that $(Z_{Sp_{2n}}(\mathbb{F}),1)$ is a limit of compact groups.

2.2 An application of Bruhat theory

Let $\overrightarrow{t} = (n_1, n_2, \ldots, n_r; k)$ where $n_1, n_2, \ldots, n_r, k$ are $r+1$ non-negative integers whose sum is n. Recall that $W_{P_{\overrightarrow{t}}(\mathbb{F})}$ is the subgroup of Weyl group of $Sp_{2n}(\mathbb{F})$ which consists of elements which maps $M_{\overrightarrow{t}}(\mathbb{F})$ to a standard Levi subgroup and commutes with $i_{k,n}(Sp_{2k}(\mathbb{F}))$; see Section 1.2.

Lemma 2.2. *For $1 \leq i \leq r$ let g_i be an element of $GL_{n_i}(\mathbb{F})$ and let (h, ϵ) be an element of $\overline{Sp_{2k}(\mathbb{F})}$. Denote again $g = diag(g_1, g_2, \ldots, g_r) \in GL_{n-k}(\mathbb{F})$. For $w \in W_{P_{\overrightarrow{t}}}(\mathbb{F})$ we have*

$$(w,1)\big(j_{n-k,n}(\widehat{g}),1\big)\big(i_{k,n}(h),\epsilon\big)(w,1)^{-1} = \big(wj_{n-k,n}(\widehat{g})w^{-1},1\big)\big(i_{k,n}(h),\epsilon\big)$$

Proof. This follows from (1.18) and from the fact that w commutes with h. $\square$

For $1 \leq i \leq r$ let τ_i be an irreducible admissible supercuspidal representation of $GL_{n_i}(\mathbb{F})$. Let $\overline{\sigma}$ be an irreducible admissible supercuspidal genuine representation of $\overline{Sp_{2k}(\mathbb{F})}$. Then,

$$\pi = \big(\otimes_{i=1}^{r}(\gamma_\psi^{-1} \otimes \tau_i)\big) \otimes \overline{\sigma}$$

is an irreducible admissible supercuspidal genuine representation of $\overline{M_{\overrightarrow{t}}(\mathbb{F})}$. For $w \in W_{P_{\overrightarrow{t}}(\mathbb{F})}$ denote by π^w the representation of $\overline{M_{\overrightarrow{t}}(\mathbb{F})^w}$ defined by

$$(m,\epsilon) \mapsto \pi\big((w,1)^{-1}(m,\epsilon)(w,1)\big).$$

Exactly as in the linear case, see Section 2 of [8], $I(\pi)$ and $I(\pi^w)$ have the same Jordan Holder series.

Recalling (1.7), we note that due to Lemma 2.2 and the fact that $\gamma_\psi(a) = \gamma_\psi(a^{-1})$ we have

$$\begin{aligned}\Big((\otimes_{i=1}^{r}\gamma_\psi^{-1} \otimes \tau_{i(s_i)}) \otimes \overline{\sigma}\Big)\Big((w,1)^{-1}\big(j_{n-k,n}(\widehat{g}),1)(i_{k,n}(h),\epsilon\big)(w,1)\Big) \\ = \gamma_\psi^{-1}(\det(g))(\otimes_{i=1}^{r}|\det(g_i)|^{\epsilon_i s_i}\tau_i^{(\epsilon_i)}(g_i)\overline{\sigma}(h,\epsilon),\end{aligned} \tag{2.7}$$

where for $1 \leq i \leq r$, $g_i \in GL_{n_i}(\mathbb{F})$, $g = diag(g_1, g_2, \ldots, g_r) \in GL_{n-k}(\mathbb{F})$, $h \in Sp_{2k}(\mathbb{F})$, $\epsilon \in \{\pm 1\}$ and where $\tau_i^{(\epsilon_i)}(g_i) = \begin{cases}\tau_i(g_i) & \epsilon_i = 1 \\ \tau_i\big((\omega_n\widetilde{g_i}\omega_n\big) & \epsilon_i = -1\end{cases}$, where $t_w^{-1}\big(\omega_n\widetilde{g_i}\omega_n\big)t_w$ is to be understood via the identification of $GL_{n_i}(\mathbb{F})$ with its image in the relevant block of $M_{\overrightarrow{t}}$. Hence, it makes sense to denote by π^w the representation

$$(\otimes_{i=1}^{r}\gamma_\psi^{-1} \otimes (\tau_i^{(\epsilon_i)})_{(\epsilon_i s_i)}) \otimes \overline{\sigma}.$$

Note that $\tau_i^{(-1)} \simeq \widehat{\tau_i}$, where $\widehat{\tau_i}$ is the dual representation of τ_i; see Theorem 4.2.2 of [5] for example.

Define $W(\pi)$ to be the following subgroup of $W_{P_{\overrightarrow{t}}(\mathbb{F})}$:

$$W(\pi) = \{w \in W_{P_{\overrightarrow{t}}(\mathbb{F})} \mid \pi^w \simeq \pi\}. \tag{2.8}$$

π is called *regular* if $W(\pi)$ is trivial and *singular* otherwise. Bruhat theory, [3], implies the following well known result :

Theorem 2.1. *If π is regular then*

$$Hom_{\overline{Sp_{2n}(\mathbb{F})}}\big(I(\pi), I(\pi)\big) \simeq \mathbb{C}. \tag{2.9}$$

See the Corollary in page 177 of [17] for this theorem in the context of connected algebraic reductive p-adic groups, and see Proposition 6 in page 61 of [1] for this theorem in the context of an r-fold cover of $GL_n(\mathbb{F})$. This theorem follows immediately from the description of the Jordan-Holder series of a Jacquet module of a parabolic induction; see Proposition Theorem 5 in page 49 of [1]. This Jordan-Holder series is an exact analog to the the Jordan-Holder series of a Jacquet module of a parabolic induction in the linear case. The proof of Proposition Theorem 5 in page 49 of [1] is done exactly as in the linear case. It uses Bruhat decomposition and a certain filtration (see Theorem 5.2 of [8]) which applies to both linear and metaplectic cases. An immediate corollary of Theorem 2.1 is the following.

Theorem 2.2. *Let $\overrightarrow{t}=(n_1,n_2,\ldots,n_r;k)$ where $n_1,n_2,\ldots,n_r,k$ are $r+1$ non-negative integers whose sum is n. For $1\leq i\leq r$ let τ_i be an irreducible admissible supercuspidal representation of $GL_{n_i}(\mathbb{F})$. Let $\overline{\sigma}$ be an irreducible admissible supercuspidal genuine representation of $\overline{Sp_{2k}(\mathbb{F})}$. Denote $\pi=\big(\otimes_{i=1}^{r}(\gamma_{\psi}^{-1}\otimes\tau_i)\big)\otimes\overline{\sigma}$. If $\tau_i\neq\tau_j$ for all $1\leq i<j\leq n$ and $\tau_i\neq\widehat{\tau_j}$ for all $1\leq i\leq j\leq n$ then (2.9) holds. If we assume in addition that τ_i is unitary for each $1\leq i\leq r$ then $I(\pi)$ is irreducible.*

Proof. From (2.7) it follows that π is regular. Thus, (2.9) follows immediately from Theorem 2.1. Note that since the center of $\overline{Sp_{2k}(\mathbb{F})}$ is finite $\overline{\sigma}$ is unitary. Therefore, the assumption that τ_i is unitary for each $1\leq i\leq r$ implies that π is unitary. Hence $I(\pi)$ is unitary. The irreducibility of $I(\pi)$ follows now from (2.9). □

2.3 The intertwining operator

Let $n_1,n_2,\ldots,n_r,k$ be $r+1$ nonnegative integers whose sum is n. Put $\overrightarrow{t}=(n_1,n_2,\ldots,n_r;k)$. For $1\leq i\leq r$ let (τ_i,V_{τ_i}) be an irreducible admissible representation of $GL_{n_i}(\mathbb{F})$ and let $(\overline{\sigma},V_{\overline{\sigma}})$ be an irreducible admissible genuine representation of $\overline{Sp_{2k}(\mathbb{F})}$. Fix now $w\in W_{P_{\overrightarrow{t}}}(\mathbb{F})$. Define

$$N_{\overrightarrow{t},w}(\mathbb{F})=Z_{Sp_{2n}}(\mathbb{F})\cap(wN_{\overrightarrow{t}}(\mathbb{F})^{-}w^{-1}),$$

where $N_{\overrightarrow{t}}(\mathbb{F})^{-}$ is the unipotent radical opposite to $N_{\overrightarrow{t}}(\mathbb{F})$, explicitly in the $Sp_{2n}(\mathbb{F})$ case:

$$N_{\overrightarrow{t}}(\mathbb{F})^{-}=J_{2n}N_{\overrightarrow{t}}(\mathbb{F})J_{2n}^{-1}.$$

For $g\in\overline{Sp_{2n}(\mathbb{F})}$ and $f\in I\big(\pi(\overrightarrow{s})\big)$ define

$$A_wf(g)=\int_{N_{\overrightarrow{t},w}}f\big(((wt_w),1)^{-1}(n,1)g\big)\,dn,\tag{2.10}$$

where t_w is a particular diagonal element in $M_{\overrightarrow{t}}(\mathbb{F})\cap W'_{Sp_{2n}}(\mathbb{F})$ whose entries are either 1 or -1. The exact definition of t_w will be given in Section 3.1; see (3.5). The appearance of t_w here is technical.

As in the linear case (see Section 2 of [37]) the last integral converges absolutely in some open set of $\mathbb{C}^r$ and has a meromorphic continuation to $\mathbb{C}^r$. In fact, it is a rational function in $q^{s_1},q^{s_2},\ldots,q^{s_r}$. See Chapter 5 of [1] for the context of a p-adic covering group. We define its continuation to be the intertwining operator

$$A_w:I(\tau_{1(s_1)},\tau_{2(s_2)},\ldots,\tau_{r(s_r)},\overline{\sigma})\to I\big((\tau_{\pi(1)}^{(\epsilon_1)})_{(\epsilon_1s_1)},(\tau_{\pi(2)}^{(\epsilon_2)})_{(\epsilon_2s_2)},\ldots,(\tau_{\pi(r)}^{(\epsilon_r)})_{(\epsilon_rs_r)},\sigma\big)$$

Denote $\overrightarrow{s}^{w}=(\epsilon_1s_1,\epsilon_2s_2,\ldots,\epsilon_rs_r)$ and $(\otimes_{i=1}^{r}\tau_i)^{w}=(\otimes_{i=1}^{r}\tau_{\pi(i)}^{(\epsilon_i)})$.

2.4 The Knapp-Stein dimension theorem

We keep the notation and assumptions of the first paragraph of Section 2.2. From Theorem 2.1 it follows that outside a Zariski open set of values of $(q^{s_1},\ldots,q^{s_r})\in(\mathbb{C}^*)^r$,

$$Hom_{\overline{Sp_{2n}(\mathbb{F})}}\Big(I\big(\pi(\overrightarrow{s})\big),I\big(\pi(\overrightarrow{s})\big)\Big)\simeq\mathbb{C}.$$

This implies that for every $w_0\in W_{P_{\overrightarrow{t}}}(\mathbb{F})$ there exists a meromorphic function $\beta(\overrightarrow{s},\tau_1,\ldots,\tau_r,\overline{\sigma},w_0)$ such that

$$A_{w_0^{-1}}A_{w_0}=\beta^{-1}(\overrightarrow{s},\tau_1,\ldots,\tau_n,\overline{\sigma},w_0)Id. \tag{2.11}$$

Remark: In the case of connected reductive quasi-split algebraic group this function differs from the Plancherel measure by a well understood positive function; see Section 3 of [43] for example. For connected reductive quasi-split algebraic groups it is known that if (in addition to all the other assumptions made here) we assume that $\tau_1,\ldots,\tau_r,\overline{\sigma}$ are unitary then $\beta(\overrightarrow{s},\tau_1,\ldots,\tau_r,\overline{\sigma},w_0)$, as a function of $\overrightarrow{s}$ is analytic on the unitary axis. This is (part of) the content of Theorem 5.3.5.2 of [48] or equivalently Lemma V.2.1 of [53]. The proof of this property has a straight forward generalization to the metaplectic group.

Let $\Sigma_{P_{\overrightarrow{t}}(\mathbb{F})}$ be the set of reflections corresponding to the roots of $T_{Sp_{2n}}(\mathbb{F})$ outside $M_{\overrightarrow{t}}(\mathbb{F})$. $W_{P_{\overrightarrow{t}}(\mathbb{F})}$ is generated by $\Sigma_{P_{\overrightarrow{t}}(\mathbb{F})}$. Following [47] we denote by $W''(\pi)$ the subgroup of $W(\pi)$ generated by the elements $w\in\Sigma_{P_{\overrightarrow{t}}(\mathbb{F})}\cap W(\pi)$ which satisfy

$$\beta(\overrightarrow{0},\tau_1,\ldots,\tau_n,\overline{\sigma},w)=0.$$

The Knapp-Stein dimension theorem states the following:

Theorem 2.3. *Let $\overrightarrow{t}=(n_1,n_2,\ldots,n_r;k)$ where $n_1,n_2,\ldots,n_r,k$ are $r+1$ non-negative integers whose sum is n. For $1\leq i\leq r$ let τ_i be an irreducible admissible supercuspidal unitary representation of $GL_{n_i}(\mathbb{F})$. Let $\overline{\sigma}$ be an irreducible admissible supercuspidal genuine representation of $\overline{Sp_{2k}}(\mathbb{F})$. We have:*

$$Dim\big(Hom(I(\pi),I(\pi)\big)=[W(\pi):W''(\pi)].$$

The Knapp-Stein dimension theorem was originally proved for real groups, see [24]. Harish-Chandra and Silberger proved it for algebraic p-adic groups; see [47] and [48]. It is a consequence of Harish-Chandra's completeness theorem; see Theorem 5.5.3.2 of [48]. Although Harish-Chandra's completeness theorem was proved for algebraic p-adic groups its proof holds for the metaplectic case as well; see the remarks on page 99 of [10]. Thus, Theorem 2.3 is a straight forward generalization of the Knapp-Stein dimension theorem to the metaplectic group. The precise details of the proof will appear in a future publication of the author. We note here that the theorem presented in [47] is more general; it deals with square integrable representations. The version given here will be sufficient for our purposes.

3 Local coefficients and gamma factors

3.1 Definitions

Let $\mathbb{F}$ be a local field of characteristic 0 and let ψ be a non-trivial character of $\mathbb{F}$. We shall continue to denote by ψ the non-degenerate character of $Z_{Sp_{2n}}(\mathbb{F})$ given by

$$\psi(z)=\psi\big(z_{(n,2n)}+\Sigma_{i=1}^{n-1}z_{(i,i+1)}\big).$$

From 1.14 it follows that $\overline{Z_{Sp_{2n}}(\mathbb{F})} \simeq Z_{Sp_{2n}}(\mathbb{F}) \times \{\pm 1\}$. We define a character of $\overline{Z_{Sp_{2n}}(\mathbb{F})}$ by $(z, \epsilon) \mapsto \epsilon\psi(z)$, and continue to denote it by ψ. We shall also denote by ψ the characters of $Z_{GL_m(\mathbb{F})}$ identified with $\big(i_{k,n}\big(j_{m,k}(\widehat{Z_{GL_m(\mathbb{F})}})\big), 1\big)$ and of $\overline{Z_{Sp_{2k}}(\mathbb{F})}$ identified with $i_{k,n}\big(\overline{Z_{Sp_{2k}}(\mathbb{F})}\big)$, obtained by restricting ψ as a character of $\overline{Z_{Sp_{2n}}(\mathbb{F})}$.

Let (π, V_π) be a smooth representation of $\overline{Sp_{2n}(\mathbb{F})}$ (of $GL_n(\mathbb{F})$). By a ψ-Whittaker functional on π we mean a linear functional w on V_π satisfying

$$w\big(\pi(z)v\big) = \psi(z)w(v)$$

for all $v \in V_\pi$, $z \in \overline{Z_{Sp_{2n}}(\mathbb{F})}(Z_{GL_n}(\mathbb{F}))$. Define $W_{\pi,\psi}$ to be the space of Whittaker functionals on π with respect to ψ. If $\mathbb{F}$ is archimedean we add smoothness requirements to the definition of a Whittaker functional, see [18] or [36]. π is called ψ-generic or simply generic if it has a non-trivial Whittaker functional with respect to ψ. If w is a non-trivial ψ-Whittaker functional on (π, V) then one may consider $W_w(\pi, \psi)$, the space of complex functions on $\overline{Sp_{2n}(\mathbb{F})}$ (on $GL_n(\mathbb{F})$) of the form

$$g \mapsto w\big(\pi(g)v\big),$$

where $v \in V$. $W_w(\pi, \psi)$ is a representation space of $\overline{Sp_{2n}(\mathbb{F})}$(of $GL_n(\mathbb{F})$). The group acts on this space by right translations. It is clearly a subspace of $Ind_{\overline{Z_{Sp_{2n}}(\mathbb{F})}}^{\overline{Sp_{2n}(\mathbb{F})}}\psi$ (of $Ind_{Z_{GL_n}(\mathbb{F})}^{GL_n(\mathbb{F})}\psi$). From Frobenius reciprocity it follows that if π is irreducible and $\dim(W_{\pi,\psi}) = 1$ then $W_w(\pi, \psi)$ is the unique subspace of $Ind_{\overline{Z_{Sp_{2n}}(\mathbb{F})}}^{\overline{Sp_{2n}(\mathbb{F})}}\psi$ (of $Ind_{Z_{GL_n}(\mathbb{F})}^{GL_n(\mathbb{F})}\psi$) which is isomorphic to π. In this case we drop the index w and we write $W(\pi, \psi)$. One can identify π with $W(\pi, \psi)$ which is called the Whittaker model of π.

For quasi-split algebraic groups, uniqueness of Whittaker functional for irreducible admissible representations is well known, see [46], [14] and [7] for the p-adic case, and see [18] and [46] for the archimedean case. Uniqueness of Whittaker models for irreducible admissible representations of $\overline{Sp_{2n}(\mathbb{F})}$ was proven in [50]:

Theorem 3.1. *Let π be an irreducible admissible representation of $\overline{Sp_{2n}(\mathbb{F})}$, where $\mathbb{F}$ is a p-adic field. Then,*

$$\dim(W_{\pi,\psi}) \leq 1.$$

Uniqueness for $\overline{SL_2(\mathbb{R})}$ was proven in [52]. To prove uniqueness for $\overline{Sp_{2n}(\mathbb{R})}$ for general n, it is sufficient to consider principal series representations. The proof in this case follows from Bruhat Theory. In fact, the proof goes almost word for word as the proof of Theorem 2.2 of [18] for minimal parabolic subgroups. The proof in this case is a heredity proof in the sense of [33]. The reason that this, basically algebraic, proof works for $\overline{Sp_{2n}(\mathbb{R})}$ as well is the fact that if the characteristic of $\mathbb{F}$ is not 2 then the inverse image in $\overline{Sp_{2n}(\mathbb{F})}$ of a maximal torus of $Sp_{2n}(\mathbb{F})$ is commutative. This implies that its irreducible representations are one dimensional. Uniqueness for $\overline{Sp_{2n}(\mathbb{C})}$ follows from the uniqueness for $Sp_{2n}(\mathbb{C})$ since $\overline{Sp_{2n}(\mathbb{C})} = Sp_{2n}(\mathbb{C}) \times \{\pm 1\}$.

We note that uniqueness of Whittaker model does not hold in general for covering groups. See the introduction of [2] for a k fold cover of $GL_n(\mathbb{F})$, see [4] for an application of a theory of local coefficients in a case where Whittaker model is not unique and see [12] for a unique model for genuine representations of a double cover of $GL_2(\mathbb{F})$.

Unless otherwise is mentioned, through the rest of this section $\mathbb{F}$ will denote a p-adic field. We shall define here the metaplectic analog of the local coefficients defined by Shahidi in Theorem 3.1 of [37] for connected reductive quasi split algebraic groups. Let $n_1, n_2, \ldots, n_r, k$ be $r+1$ nonnegative integers whose sum is n. Put $\overrightarrow{t} = (n_1, n_2, \ldots, n_r; k)$. Let $(\tau_1, V_{\tau_1}), (\tau_2, V_{\tau_2}), \ldots, (\tau_r, V_{\tau_r})$ be r irreducible admissible generic representations of $GL_{n_1}(\mathbb{F}), GL_{n_2}(\mathbb{F}), \ldots, GL_{n_r}(\mathbb{F})$ respectively. It is clear that for $s_i \in \mathbb{C}$, $\tau_{i(s_i)}$ is also generic. In fact, if λ is a ψ-Whittaker functional on (τ_i, V_{τ_i}) it is also a ψ-Whittaker functional on $(\tau_{i(s_i)}, V_{\tau_i})$. Let $(\overline{\sigma}, V_{\overline{\sigma}})$ be an irreducible admissible ψ-generic genuine representation of $\overline{Sp_{2k}(\mathbb{F})}$. Let

$$I\big(\pi(\overrightarrow{s})\big) = I(\tau_{1(s_1)}, \tau_{2(s_2)}, \ldots, \tau_{r(s_r)}, \overline{\sigma})$$

be the parabolic induction defined in Section 2.1. Since the inducing representations are generic, then, by a theorem of Rodier, [33], extended to a non linear algebraic setting in [2], $I\big(\pi(\overrightarrow{s})\big)$ has a unique ψ-Whittaker functional. Define $\lambda_{\tau_1,\psi}, \lambda_{\tau_2,\psi}, \ldots, \lambda_{\tau_r,\psi}$ to be non-trivial ψ-Whittaker functionals on $V_{\tau_1}, V_{\tau_2}, \ldots, V_{\tau_r}$ respectively and fix $\lambda_{\overline{\sigma},\psi}$, a non-trivial ψ-Whittaker functional on $V_{\overline{\sigma}}$. Define

$$\epsilon(\overrightarrow{t}) = j_{n-k}(diag(\epsilon_{n_1}, \epsilon_{n_2}, \ldots, \epsilon_{n_r}, \epsilon_{n_1}, \epsilon_{n_2}, \ldots, \epsilon_{n_r}),$$

where $\epsilon_n = diag\big(1, -1, 1 \ldots, (-1)^{n+1}\big) \in GL_n(\mathbb{F})$. We fix J_{2n} as a representative of the long Weyl element of $Sp_{2n}(\mathbb{F})$ and

$$\omega_n = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}$$

as a representative of the long Weyl element of $GL_n(\mathbb{F})$. We now fix

$$w_l(\overrightarrow{t}) = j_{n-k,n}\big(diag(\epsilon_{n_1}\omega_{n_1}, \epsilon_{n_2}\omega_{n_2}, \ldots, \epsilon_{n_r}\omega_{n_r}, \epsilon_{n_1}\omega_{n_1}, \epsilon_{n_2}\omega_{n_2}, \ldots, \epsilon_{n_r}\omega_{n_r})\big)i_{k,n}(-J_{2k})$$

as a representative of the long Weyl element of $P_{\overrightarrow{t}}(\mathbb{F})$. We also define

$$w'_l(\overrightarrow{t}) = w_l(\overrightarrow{t})J_{2n}. \tag{3.1}$$

Note that $w'_l(\overrightarrow{t})$ is a representative of minimal length of the longest Weyl element of $\overline{Sp_{2n}(\mathbb{F})}$ modulo the Weyl group of $M_{\overrightarrow{t}}(\mathbb{F})$. It maps the positive roots outside $M_{\overrightarrow{t}}(\mathbb{F})$ to negative roots and maps the positive roots of $M_{\overrightarrow{t}}(\mathbb{F})$ to positive roots. The presence of $\epsilon(\overrightarrow{t})$ in the definition of $w_l(\overrightarrow{t})$ and $w'_l(\overrightarrow{t})$ is to ensure that

$$\psi\Big(\big(w'_l(\overrightarrow{t}), 1\big)^{-1} n\big(w'_l(\overrightarrow{t}), 1\big)\Big) = \psi(n) \tag{3.2}$$

for all $n \in \overline{M_{\overrightarrow{t}}(\mathbb{F})^{w'_l(\overrightarrow{t})}} \cap \overline{Z_{Sp_{2n}}(\mathbb{F})}$ (the reader may verify this fact by (1.18) and by a matrix multiplication). Consider the integral

$$\lim_{k\to\infty} \int_{N_{\overrightarrow{t}}(\mathbb{P}_{\mathbb{F}}^{-k})^w} \Big((\otimes_{i=1}^r \lambda_{\tau_i,\psi^{-1}}) \otimes \lambda_{\overline{\sigma},\psi}\Big)\Big(f\big(w'_l(\overrightarrow{t}), 1\big)^{-1}(n, 1)\Big)\psi^{-1}(n)\, dn. \tag{3.3}$$

By abuse of notations we shall write

$$\int_{N_{\overrightarrow{t}}(\mathbb{F})^w} \Big((\otimes_{i=1}^r \lambda_{\tau_i,\psi^{-1}}) \otimes \lambda_{\overline{\sigma},\psi}\Big)\Big(f\big(w'_l(\overrightarrow{t}), 1\big)^{-1}(n, 1)\Big)\psi^{-1}(n)\, dn. \tag{3.4}$$

Since $Z_{Sp_{2n}}(\mathbb{F})$ splits in $\overline{Sp_{2n}(\mathbb{F})}$ via the trivial section the integral converges exactly as in the linear case, see Proposition 3.1 of [37] (and see Chapter 4 of [1] for the context of a p-adic covering group). In fact, it converges absolutely in an open subset of $\mathbb{C}^r$. Due to (1.17), it defines, again as in the linear case, a ψ-Whittaker functional on $I(\tau_{1(s_1)}, \tau_{2(s_2)}, \ldots, \tau_{r(s_r)}, \overline{\sigma})$. We denote this functional by

$$\lambda\big(\overrightarrow{s}, (\otimes_{i=1}^r \tau_i) \otimes \overline{\sigma}, \psi\big),$$

where $\overrightarrow{s} = (s_1, s_2, \ldots, s_r)$. Also, since the integral defined in (3.3) is stable for a large enough r it defines a rational function in $q^{s_1}, q^{s_2}, \ldots, q^{s_r}$.

Fix now $w \in W_{P_{\overrightarrow{t}}}(\mathbb{F})$. Let t_w be the unique diagonal element in $M_{\overrightarrow{t}}(\mathbb{F}) \cap W'_{Sp_{2n}}(\mathbb{F})$ whose first entry in each block is 1 such that

$$\psi\big((wt_w, 1)^{-1} n (wt_w, 1)\big) = \psi(n) \tag{3.5}$$

for each $n \in \overline{M_{\overrightarrow{t}}(\mathbb{F})^w} \cap \overline{Z_{Sp_{2n}}(\mathbb{F})}$. (3.5) assures that

$$\lambda\big(\overrightarrow{s}^w, (\otimes_{i=1}^r \tau_i)^w \otimes \overline{\sigma}, \psi\big) \circ A_w$$

is another ψ-Whittaker functional on $I\big(\pi(\overrightarrow{s})\big)$. It now follows from the uniqueness of Whittaker functional that there exists a complex number

$$C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{\overrightarrow{t}}(\mathbb{F})}, \overrightarrow{s}, (\otimes_{i=1}^r \tau_i) \otimes \overline{\sigma}, w\big)$$

defined by the property

$$\lambda\big(\overrightarrow{s}, (\otimes_{i=1}^r \tau_i) \otimes \overline{\sigma}, \psi\big) = C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{\overrightarrow{t}}(\mathbb{F})}, \overrightarrow{s}, (\otimes_{i=1}^r \tau_i) \otimes \overline{\sigma}, w\big) \lambda\big(\overrightarrow{s}^w, (\otimes_{i=1}^r \tau_i)^w \otimes \overline{\sigma}, \psi\big) \circ A_w. \tag{3.6}$$

It is called the **local coefficient** and it clearly depends only on $\overrightarrow{s}, w$ and the isomorphism classes of $\tau_1, \tau_2, \ldots, \tau_r$ (not on a realization of the inducing representations nor on $\lambda_{\tau_1,\psi}, \lambda_{\tau_2,\psi}, \ldots, \lambda_{\tau_r,\psi}\ \psi^{-1}, \lambda_{\overline{\sigma},\psi}$). Also it is clear by the above remarks that

$$\overrightarrow{s} \mapsto C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{\overrightarrow{t}}(\mathbb{F})}, \overrightarrow{s}, (\otimes_{i=1}^r \tau_i) \otimes \overline{\sigma}, w\big)$$

defines a rational function in function in $q^{s_1}, q^{s_2}, \ldots, q^{s_r}$. Note that (3.6) implies that the zeros of the local coefficient are among the poles of the intertwining operator.

Remark: Assume that the residue characteristic of $\mathbb{F}$ is odd. Then, by Lemma 1.1 $\kappa_{2n}(w) = (wt_w, 1)$ for all $w \in W_{P_{\overrightarrow{t}}}(\mathbb{F})$ and $\kappa_{2n}\big(w'_l(\overrightarrow{t})\big) = \big(w'_l(\overrightarrow{t}), 1\big)$. Keeping the Adelic context in mind, whenever one introduces local integrals that contain a pre-image of $w \in W'_{Sp_{2n}}(\mathbb{F}) \subset Sp_{2n}(\mathbb{O}_\mathbb{F})$ inside $\overline{Sp_{2n}(\mathbb{F})}$ one should use elements of the form $\kappa_{2n}(w)$.

Let $\overrightarrow{t} = (n_1, n_2, \ldots, n_r; k)$ where $n_1, n_2, \ldots, n_r, k$ are $r+1$ non-negative integers whose sum is n. For each $1 \leq i \leq r$ let τ_i be an irreducible admissible generic representation of $GL_{n_i}(\mathbb{F})$ and let $\overline{\sigma}$ be an irreducible admissible generic genuine representation of $\overline{Sp_{2k}(\mathbb{F})}$. From the definition of the local coefficients it follows that

$$\begin{aligned} &\beta(\overrightarrow{s}, \tau_1, \ldots, \tau_n, \overline{\sigma}, w_0) \\ &= C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{\overrightarrow{t}}(\mathbb{F})}, \overrightarrow{s}, (\otimes_{i=1}^r \tau_i) \otimes \overline{\sigma}, w_0\big) C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{\overrightarrow{t}}(\mathbb{F})}, \overrightarrow{s^{w_0}}, ((\otimes_{i=1}^r \tau_i) \otimes \overline{\sigma})^{w_0}, w_0^{-1}\big), \end{aligned} \tag{3.7}$$

Recalling Theorem 2.3, the significance of the local coefficients for questions of irreducibility of a parabolic induction is clear.

Let $\overline{\sigma}$ be an irreducible admissible generic irreducible admissible generic genuine representation of $\overline{Sp_{2k}(\mathbb{F})}$. Let τ be an irreducible admissible ψ-generic representation of $GL_m(\mathbb{F})$. Put $n = m + k$. We define:

$$\gamma(\overline{\sigma} \times \tau, s, \psi) = \frac{C_{\psi}^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{m;k}(\mathbb{F})}, s, \tau \otimes \overline{\sigma}, j_{m,n}(\omega_m'^{-1})\big)}{C_{\psi}^{\overline{Sp_{2m}(\mathbb{F})}}\big(\overline{P_{m;0}(\mathbb{F})}, s, \tau, \omega_m'^{-1}\big)}, \tag{3.8}$$

where $\omega_m' = \begin{pmatrix} & -\omega_m \\ \omega_m & \end{pmatrix}$. It is clearly a rational function in q^s. Note that if $k = 0$ then $\overline{\sigma}$ is the non-trivial character of the group of 2 elements and $\gamma(\overline{\sigma} \times \tau, s, \psi) = 1$.

This definition of the γ-factor is an exact analog to the definition given in Section 6 of [42] for quasi-split connected algebraic groups over a non-archimedean field. We note that similar definitions hold for the case $\mathbb{F} = \mathbb{R}$. In this case the local coefficients are meromorphic functions.

3.2 Multiplicativity of the local coefficients and gamma factor

Let $\mathbb{F}$ be a local field of characteristic 0. Let $\overline{\sigma}$ be a genuine irreducible admissible ψ-generic representation of $\overline{Sp_{2k}(\mathbb{F})}$. Let τ be an irreducible admissible ψ-generic representation of $GL_m(\mathbb{F})$. For two nonnegative integers l, r such that $l + r = m$ denote by $P^0_{l,r}(\mathbb{F})$ the standard parabolic subgroup of $GL_m(\mathbb{F})$ whose Levi part is

$$M^0_{l,r}(\mathbb{F}) = \begin{pmatrix} GL_l(\mathbb{F}) & \\ & GL_r(\mathbb{F}) \end{pmatrix}.$$

Denote its unipotent radical by $N^0_{l,r}(\mathbb{F})$. Let τ_l, τ_r be two irreducible admissible ψ-generic representations of $GL_l(\mathbb{F})$ and $GL_r(\mathbb{F})$ respectively. In the p-adic case; see [39], Shahidi defines

$$\gamma(\tau_l \times \tau_r\, s, \psi) = \chi_{\tau_r}(-I_r)^l C_{\psi}^{GL_m(\mathbb{F})}\big(P^0_{l,r}(\mathbb{F}), (\frac{s}{2}, \frac{-s}{2}), \tau_l \otimes \widehat{\tau_r}, \varpi_{r,l}\big), \tag{3.9}$$

where $\varpi_{r,l} = \begin{pmatrix} & I_r \\ I_l & \end{pmatrix}$, χ_{τ_r} is the central character of τ_r and $C_{\psi}^{GL_m(\mathbb{F})}(\cdot, \cdot, \cdot, \cdot)$, the $GL_m(\mathbb{F})$ local coefficient in the right-hand side defined via a similar construction to the one presented above. In the same paper the author proves that the γ-factor defined that way is the same arithmetical factor defined by Jacquet, Piatetskii-Shapiro and Shalika via the Rankin-Selberg method, see [20]. Due to the remark given in the introduction of [40], see page 974 after Theorem 1, we take (3.9) as a definition in archimedean fields as well. The archimedean γ-factor defined in this way agrees also with the definition given via the Rankin-Selberg method, see [19].

For future use we note the following:

$$\varpi_{r,l}^{-1} = \varpi_{r,l}^t = \varpi_{l,r}, \quad \varpi_{r,l} M^0_{l,r}(\mathbb{F}) \varpi_{l,r} = M^0_{r,l}(\mathbb{F}), \tag{3.10}$$

and

$$\psi(\varpi_{l,r} n \varpi_{r,l}) = \psi(n), \tag{3.11}$$

for all $n \in Z_{GL_m}(\mathbb{F}) \cap M^0_{l,r}(\mathbb{F})$. $\varpi_{r,l}$ is a representative of the long Weyl element of $GL_n(\mathbb{F})$ modulo the long Weyl element of $M^0_{l,r}(\mathbb{F})$.

Theorem 3.2. *Assume that $\tau = Ind^{GL_m(\mathbb{F})}_{P^0_{l,r}(\mathbb{F})} \tau_l \otimes \tau_r$, where τ_l, τ_r are two irreducible admissible generic representations of $GL_l(\mathbb{F})$ and $GL_r(\mathbb{F})$ respectively, where $l + r = m$, then*

$$\gamma(\overline{\sigma} \times \tau, s, \psi) = \gamma(\overline{\sigma} \times \tau_l, s, \psi)\gamma(\overline{\sigma} \times \tau_r, s, \psi). \tag{3.12}$$

Theorem 3.3. *Assume that $\overline{\sigma} = Ind^{\overline{Sp_{2k}(\mathbb{F})}}_{P_{l;r}(\mathbb{F})} (\gamma_\psi^{-1} \otimes \tau_l) \otimes \overline{\sigma_r}$, where τ_l is an irreducible admissible generic representation of $GL_l(\mathbb{F})$ and $\overline{\sigma_r}$ is an irreducible admissible genuine ψ-generic representation of $\overline{Sp_{2r}(\mathbb{F})}$, where $l + r = k$. Let τ be an irreducible admissible generic representation of $GL_m(\mathbb{F})$. Then,*

$$\gamma(\overline{\sigma}\times\tau, s, \psi) = \chi^l_\tau(-I_m)\chi^m_{\tau_l}(-I_l)(-1, -1)^{ml}_{\mathbb{F}}\gamma(\overline{\sigma_r}\times\tau, s, \psi)\gamma(\widehat{\tau_l}\times\tau, s, \psi)\gamma(\tau_l\times\tau, s, \psi). \tag{3.13}$$

We start with proving Theorem 3.2. We proceed through the following lemmas:

Lemma 3.1. *With notations in Theorem 3.2 we have:*

$$C^{\overline{Sp_{2n}(\mathbb{F})}}_\psi\big(\overline{P_{m;k}(\mathbb{F})}, s, \tau \otimes \overline{\sigma}, j_{m,n}(\omega'^{-1}_m)\big) = C^{\overline{Sp_{2n}(\mathbb{F})}}_\psi\big(\overline{P_{l,r;k}(\mathbb{F})}, (s, s), \tau_l \otimes \tau_r \otimes \overline{\sigma}, j_{m,n}(\omega'^{-1}_m)\big). \tag{3.14}$$

In particular, for $m = n$, that is when $k = 0$ and $\overline{\sigma}$ is the non trivial representation of the group of two elements, we have

$$C^{\overline{Sp_{2m}(\mathbb{F})}}_\psi(\overline{P_{m;0}(\mathbb{F})}, s, \tau, \omega'^{-1}_m) = C^{\overline{Sp_{2m}(\mathbb{F})}}_\psi(\overline{P_{l,r;0}(\mathbb{F})}, (s, s), \tau_l \otimes \tau_r, \omega'^{-1}_m). \tag{3.15}$$

Proof. We find it convenient to assume that the inducing representations τ_l, τ_r and $\overline{\sigma}$ are given in their ψ-Whittaker model. In this realization $f \mapsto f(I_l)$, $f \mapsto f(I_r)$ and $f \mapsto f(I_{2n}, 1)$ are ψ-Whittaker functionals on τ_l, τ_r and $\overline{\sigma}$ respectively. We realize the space on which τ acts as a space of functions

$$f : GL_m(\mathbb{F}) \times GL_l(\mathbb{F}) \times GL_r(\mathbb{F}) \to \mathbb{C}$$

which are smooth from the right in each variable and which satisfies

$$f(\begin{pmatrix} a & * \\ & b \end{pmatrix} g, nl_o, n'r_0) = |\det(a)|^{\frac{r}{2}}|\det(b)|^{\frac{-l}{2}}\psi(n)\psi(n')f(g, l_0a, r_0b),$$

for all $g \in GL_m(\mathbb{F})$, $l_0, a \in GL_l(\mathbb{F})$, $r_0, b \in GL_r(\mathbb{F})$, $n \in Z_{GL_l}(\mathbb{F})$, $n' \in Z_{GL_r}(\mathbb{F})$. In this realization τ acts by right translations of the first argument (see pages 11 and 65 of [49] for similar realizations). According to the $GL_m(\mathbb{F})$ analog to the construction given in Section 3.1, i.e., Proposition 3.1 of [37] for $GL_m(\mathbb{F})$, and due to (3.10) and (3.11), a ψ-Whittaker functional on τ is given by

$$\lambda_{\tau,\psi}(f) = \int_{N^0_{r,l}(\mathbb{F})} f(\varpi_{l,r}n, I_l, I_r)\psi^{-1}(n)\, dn. \tag{3.16}$$

We realize the representation space of $I(\tau_{(s)}, \overline{\sigma})$ as a space of functions

$$f : \overline{Sp_{2n}(\mathbb{F})} \times \overline{Sp_{2k}(\mathbb{F})} \times GL_m(\mathbb{F}) \times GL_l(\mathbb{F}) \times GL_r(\mathbb{F}) \to \mathbb{C}$$

which are smooth from the right in each variable and which satisfy

$$f\big((j_{m,n}(\widehat{m}),1)i_{k,n}(h)us, ny, \begin{pmatrix} a & * \\ & b \end{pmatrix} g, n'l_o, n''r_0\big) = f(s, yh, gm, l_0a, r_0b) \quad (3.17)$$
$$\gamma_\psi^{-1}\big(\det(m)\big)|\det(m)|^{\frac{2k+m+1}{2}+s}|\det(a)|^{\frac{r}{2}}|\det(b)|^{\frac{-l}{2}}\psi(n)\psi(n')\psi(n''),$$

for all $s \in \overline{Sp_{2n}(\mathbb{F})}$, $h, y \in \overline{Sp_{2k}(\mathbb{F})}$, $\begin{pmatrix} a & * \\ & b \end{pmatrix} \in P^0_{l,r}(\mathbb{F})$, $m, g \in GL_m(\mathbb{F})$, $l_0 \in GL_l(\mathbb{F})$, $r_0 \in GL_r(\mathbb{F})$, $u \in (N_{m;k}(\mathbb{F}),1)$, $n \in \overline{Z_{Sp_{2k}}(\mathbb{F})}$, $n' \in Z_{GL_l}(\mathbb{F})$, $n'' \in Z_{GL_r}(\mathbb{F})$. In this realization $\overline{Sp_{2n}(\mathbb{F})}$ acts by right translations of the first argument. Due to (3.16), we have

$$\lambda(s,\tau\otimes\overline{\sigma},\psi)(f) = \int_{N_{m;k}(\mathbb{F})} \lambda_{\tau,\psi}\Big(f\big(w'_l(m;k)n,1),(I_{2k},1),I_m,I_l,I_r\big)\Big)\psi^{-1}(n)dn =$$

$$\int_{n\in N_{m;k}(\mathbb{F})}\int_{n'\in N^0_{r,l}(\mathbb{F})} f\big((j_{m,n}(-\widehat{\epsilon_m}\omega'_m)n,1),(I_{2k},1),\varpi_{l,r}n',I_l,I_r\big)\psi^{-1}(n')\psi^{-1}(n)\,dn'\,dn. \tag{3.18}$$

Note that $\omega'_m = \begin{pmatrix} & -I_m \\ I_m & \end{pmatrix}\begin{pmatrix} \omega_m & \\ & \omega_m \end{pmatrix}$. Thus, $x(-\widehat{\epsilon_m}\omega'_m) = (-1)^m$. Due to (3.17) and (1.14), we observe that for $n \in N_{m;k}(\mathbb{F}), n' \in M^0_{l,r}(\mathbb{F})$

$$f\Big(\big(j_{m,n}(-\widehat{\epsilon_m}\omega'_m)n,1),(I_{2k},1\big),\varpi_{l,r}n',I_l,I_r\Big) =$$
$$(-1,-1)^{mrl}_{\mathbb{F}}\gamma_\psi^{-1}(-1^{rl})f\Big(\big(j_{m,n}(-\widehat{\varpi_{l,r}}\widehat{n'}\widehat{\epsilon_m}\omega'_m)n,1\big),(I_{2k},1),I_m,I_l,I_r\Big).$$

We shall write

$$-\widehat{\varpi_{l,r}}\widehat{n'}\widehat{\epsilon_m}\omega'_m = -\widehat{\varpi_{l,r}}\widehat{\epsilon_m}\omega'_m\widehat{\breve{n}'},$$

where for $g \in GL_m(\mathbb{F})$ we define

$$\breve{g} = (\epsilon_m\omega_m)^{-1}\widetilde{g}(\epsilon_m\omega_m).$$

Since $n \mapsto \breve{n}$ maps $N^0_{r,l}(\mathbb{F})$ to $N^0_{l,r}(\mathbb{F})$, we get by (3.2) and (3.17):

$$\lambda(s,\tau\otimes\overline{\sigma},\psi)(f) = \tag{3.19}$$
$$(-1,-1)^{mrl}_{\mathbb{F}}\gamma_\psi^{-1}(-1^{lr})\int_{N_{l,r;k}(\mathbb{F})} f\big(j_{m,n}(-\widehat{\varpi_{l,r}}\widehat{\epsilon_m}\omega'_m)n,1),(I_{2k},1),I_m,I_l,I_r\big)\psi^{-1}(n)\,dn.$$

(Clearly the change of integration variable does not require a correction of the measure). We turn now to $I(\tau_{l(s)},\tau_{r(s)},\overline{\sigma})$. We realize the space of this representation as a space of functions

$$f : \overline{Sp_{2n}(\mathbb{F})} \times \overline{Sp_{2k}(\mathbb{F})} \times GL_l(\mathbb{F}) \times GL_r(\mathbb{F}) \to \mathbb{C}$$

which are smooth from the right in each variable and which satisfy

$$f((j_{m,n}\widehat{\begin{pmatrix} a & \\ & b \end{pmatrix}},1)i_{k,n}(h)us, ny, n'l_o, n''r_0) = f(s, yh, l_0a, r_0b) \tag{3.20}$$
$$\gamma_\psi^{-1}(\det\begin{pmatrix} a & \\ & b \end{pmatrix})|\det(\begin{pmatrix} a & \\ & b \end{pmatrix})|^{\frac{2k+m+1}{2}+s}|\det(a)|^{\frac{r}{2}}|\det(b)|^{\frac{-l}{2}}\psi(n)\psi(n')\psi(n''),$$

for all $s \in \overline{Sp_{2n}(\mathbb{F})}$, $h, y \in \overline{Sp_{2k}(\mathbb{F})}$, $\begin{pmatrix} & a \\ & & b \end{pmatrix} \in M^0_{l,r}(\mathbb{F})$, $l_0 \in GL_l(\mathbb{F})$, $r_0 \in GL_r(\mathbb{F})$, $u \in \big(N_{l,r;k}(\mathbb{F}), 1\big)$, $n \in \overline{Z_{Sp_{2k}}(\mathbb{F})}$, $n' \in Z_{GL_l}(\mathbb{F})$, $n'' \in Z_{GL_r}(\mathbb{F})$. In this realization $\overline{Sp_{2n}(\mathbb{F})}$ acts by right translations of the first argument. Recall that in (3.1) we have defined $w'_l(\overrightarrow{t})$ to be a particular representative of minimal length of the longest Weyl element of $\overline{Sp_{2n}(\mathbb{F})}$ modulo the Weyl group of $M_{\overrightarrow{t}}(\mathbb{F})$. Since

$$w'_l(l, r; k) = j_{m,n}(-\widehat{g_0}\widehat{\varpi_{l,r}}\widehat{\epsilon_m}\omega'_m),$$

where

$$g_0 = \begin{pmatrix} & (-I_l)^r \\ & & I_r \end{pmatrix},$$

We have:

$$\lambda\big((s,s), \tau_l \otimes \tau_r \otimes \overline{\sigma}, \psi\big)(f) = \int_{N_{l,r;k}(\mathbb{F})} f\Big(\big(j_{m,n}(-\widehat{g_0}\widehat{\varpi_{l,r}}\widehat{\epsilon_m}\omega'_m)n, 1\big), (I_{2k}, 1), I_l, I_r\Big)\psi^{-1}(n)\, dn. \tag{3.21}$$

Note that

$$\begin{aligned}
& f\Big(\big(j_{m,n}(-\widehat{g_0}\widehat{\varpi_{l,r}}\widehat{\epsilon_m}\omega'_m)n, 1\big), (I_{2k}, 1), I_l, I_r\Big) \\
&= (-1,-1)^{mlr}_{\mathbb{F}}\gamma_\psi(-1^{lr}) f\Big(\big(j_{m,n}(-\widehat{g_0}\widehat{\varpi_{l,r}}\widehat{\epsilon_m}\omega'_m)n, 1\big), (I_{2k}, 1), (-I_l)^r, I_r\Big) \\
&= (-1,-1)^{mlr}_{\mathbb{F}}\gamma_\psi(-1^{lr})\chi^r_{\tau_l}(-I_l)(f\Big(\big(j_{m,n}(-\widehat{\varpi_{l,r}}\widehat{\epsilon_m}\omega'_m)n, 1\big), (I_{2k}, 1), I_l, I_r\Big).
\end{aligned}$$

Thus,

$$\begin{gathered}
\lambda\big((s,s), \tau_l \otimes \tau_r \otimes \overline{\sigma}, \psi\big)(f) = (-1,-1)^{mlr}_{\mathbb{F}}\gamma_\psi(-1^{lr})\chi^r_{\tau_l}(-I_l) \\
\int_{N_{l,r;k}(\mathbb{F})} f\Big(\big(j_{m,n}(-\widehat{\varpi_{l,r}}\widehat{\epsilon_m}\omega'_m)n, 1\big), (I_{2k}, 1), I_l, I_r\Big)\psi^{-1}(n)
\end{gathered} \tag{3.22}$$

For $f \in I(\tau_{(s)}, \overline{\sigma})$ define

$$\widetilde{f} : \overline{Sp_{2m}(\mathbb{F})} \times \overline{Sp_{2k}(\mathbb{F})} \times GL_l(\mathbb{F}) \times GL_r(\mathbb{F}) \to \mathbb{C}$$

by $\widetilde{f}(s, y, r_0, l_0) = f(s, y, I_m, r_0, l_0)$. The map $f \mapsto \widetilde{f}$ is an $\overline{Sp_{2m}(\mathbb{F})}$ isomorphism from $I(\tau_{(s)}, \overline{\sigma})$ to $I(\tau_{l(s)}, \tau_{r(s)}, \overline{\sigma})$. Comparing (3.19) and (3.22)we see that

$$\lambda(s, \tau \otimes \overline{\sigma}, \psi)(f) = \chi^r_{\tau_l}(-I_l)\lambda\big((s,s), \tau_l \otimes \tau_r \otimes \overline{\sigma}, \psi\big)(\widetilde{f}). \tag{3.23}$$

We now introduce an intertwining operator

$$A_{j_{m,n}(\omega'^{-1}_m)} : I(\tau_{(s)}, \overline{\sigma}) \to I(\widehat{\tau}_{(-s)}, \overline{\sigma}),$$

defined by

$$A_{j_{m,n}(\omega'^{-1}_m)}(s, y, m, l_0, r_0) = \int_{j_{m,n}(N_{(m;0)}(\mathbb{F}))} f\big((j_{m,n}(\widehat{\epsilon_m}\omega'_m)n, 1)s, y, m, l_0, r_0\big) dn.$$

Note that for $f \in I(\tau_{(s)}, \overline{\sigma})$ we have

$$A_{j_{m,n}(\omega_m'^{-1})} f : \overline{Sp_{2m}(\mathbb{F})} \times \overline{Sp_{2k}(\mathbb{F})} \times GL_m(\mathbb{F}) \times GL_l(\mathbb{F}) \times GL_r(\mathbb{F}) \to \mathbb{C}$$

is smooth from the right in each variable and satisfies

$$A_{j_{m,n}(\omega_m'^{-1})}(f)((j_{m,n}(\widehat{m}),1)i_{k,n}(h)us, ny, \begin{pmatrix} a & * \\ & b \end{pmatrix} g, n'l_o, n''r_0) =$$

$$\gamma_\psi^{-1}(\det(m))|\det(m)|^{\frac{2k+m+1}{2}-s}|\det(a)|^{\frac{r}{2}}|\det(b)|^{\frac{-l}{2}}\psi(n)\psi(n')\psi(n'')\psi f(s, yh, g\breve{m}, l_0a, r_0b), \tag{3.24}$$

for all $s \in \overline{Sp_{2n}(\mathbb{F})}$, $h, y \in \overline{Sp_{2k}(\mathbb{F})}$, $\begin{pmatrix} a & * \\ & b \end{pmatrix} \in P^0_{l,r}(\mathbb{F})$, $m, g \in GL_m(\mathbb{F})$, $l_0 \in GL_l(\mathbb{F})$, $r_0 \in GL_r(\mathbb{F})$, $u \in \big(N_{m;k}(\mathbb{F}), 1\big)$, $n \in \overline{Z_{Sp_{2k}}(\mathbb{F})}$, $n' \in Z_{GL_l}(\mathbb{F})$, $n'' \in Z_{GL_r}(\mathbb{F})$. Since $\breve{\breve{g}} = g$ and since $\breve{\varpi}_{l,r} = h_0 \varpi_{r,l}$, where

$$h_0 = \begin{pmatrix} (-I_r)^l & \\ & (-I_l)^r \end{pmatrix},$$

we have,

$$\begin{aligned}
&\lambda\big(-s, \breve{\tau} \otimes \overline{\sigma}, \psi\big)(f) \\
&= \int_{n \in N_{m;k}(\mathbb{F})} \int_{n' \in N^0_{r,l}(\mathbb{F})} f\big((j_{m,n}(-\widehat{\epsilon_m}\omega_m')n, 1), (I_{2k}, 1), h_0 h_0 \varpi_{l,r} n', I_l, I_r\big)\psi^{-1}(n')\psi^{-1}(n)\, dn'\, dn \\
&= \chi^r_{\tau_l}(-I_l)\chi^l_{\tau_r}(-I_r)(-1,-1)^{mrl}_{\mathbb{F}}\gamma_\psi^{-1}(-1^{lr}) \\
&\int_{n \in N_{m;k}(\mathbb{F})} \int_{n' \in N^0_{r,l}(\mathbb{F})} f\big((j_{m,n}(-\varpi_{r,l}\breve{n'}\widehat{\epsilon_m}\omega_m')n, 1), (I_{2k}, 1), I_m, I_l, I_r\big)\psi^{-1}(n')\psi^{-1}(n)\, dn'\, dn \\
&= \chi^r_{\tau_l}(-I_l)\chi^l_{\tau_r}(-I_r)(-1,-1)^{mrl}_{\mathbb{F}}\gamma_\psi^{-1}(-1^{lr}) \\
&\int_{n \in N_{m;k}(\mathbb{F})} \int_{n' \in N^0_{r,l}(\mathbb{F})} f\big((j_{m,n}(-\varpi_{r,l}\widehat{\epsilon_m}\omega_m')n'n, 1), (I_{2k}, 1), I_m, I_l, I_r\big)\psi^{-1}(n')\psi^{-1}(n)\, dn'\, dn.
\end{aligned}$$

We have shown:

$$\begin{aligned}
\lambda\big(-s, \breve{\tau} \otimes \overline{\sigma}, \psi\big)(f) &= \chi^r_{\tau_l}(-I_l)\chi^l_{\tau_r}(-I_r)(-1,-1)^{mrl}_{\mathbb{F}}\gamma_\psi^{-1}(-1^{lr}) \\
&\int_{n \in N_{r,l,k}(\mathbb{F})} f\big((j_{m,n}(-\varpi_{r,l}\widehat{\epsilon_m}\omega_m')n'n, 1), (I_{2k}, 1), I_m, I_l, I_r\big)\psi^{-1}(n)\, dn.
\end{aligned} \tag{3.25}$$

Consider now

$$\widetilde{A}_{j_{m,n}(\omega_m'^{-1})} : I(\tau_{l(s)}, \tau_{r(s)}, \overline{\sigma}) \to I(\widehat{\tau}_{r(-s)}, \widehat{\tau}_{l(-s)}, \overline{\sigma}),$$

defined by

$$\widetilde{A}_{j_{m,n}(\omega_m'^{-1})}(s, y, l_0, r_0) = \int_{j_{m,n}(N_{m;0})} f\big((j_{m,n}(\widehat{\epsilon_m}\omega_m'^{-1})n, 1)s, y, l, r\big) dn.$$

Note that for $f \in I(\tau_{l(s)}, \tau_{r(s)}, \overline{\sigma})$, we have

$$\widetilde{A}_{j_{m,n}(\omega_m'^{-1})} f : \overline{Sp_{2n}(\mathbb{F})} \times \overline{Sp_{2k}(\mathbb{F})} \times GL_l(\mathbb{F}) \times GL_r(\mathbb{F}) \to \mathbb{C}$$

is smooth from the right in each variable and satisfies

$$\widetilde{A}_{j_{m,n}(\omega_m'^{-1})}(f)\big((j_{m,n}\widehat{\begin{pmatrix} b & \\ & a \end{pmatrix}},1)i_{k,n}(h)us,ny,n'l_o,n''r_0\big)=f(s,yh,l_0\breve{a},r_0\breve{b}) \tag{3.26}$$
$$\gamma_\psi^{-1}(\det\begin{pmatrix} b & \\ & a \end{pmatrix})|\det(\begin{pmatrix} b & \\ & a \end{pmatrix})|^{\frac{2k+m+1}{2}-s}|\det(a)|^{\frac{r}{2}}|\det(b)|^{\frac{-l}{2}}\psi(n)\psi(n')\psi(n''),$$

for all $s\in\overline{Sp_{2n}(\mathbb{F})}$, $h,y\in\overline{Sp_{2k}(\mathbb{F})}$, $\begin{pmatrix} b & \\ & a \end{pmatrix}\in M^0_{r,l}$, $l_0\in GL_l(\mathbb{F})$, $r_0\in GL_r(\mathbb{F})$, $u\in\big(N_{r,l;k}(\mathbb{F}),1\big)$, $n\in\overline{Z_{Sp_{2k}}(\mathbb{F})}$, $n'\in Z_{GL_l}(\mathbb{F})$, $n''\in Z_{GL_r}(\mathbb{F})$. Similar to (3.22) we have:

$$\lambda\big((-s,-s),\breve{\tau}_r\otimes\breve{\tau}_l,\psi\big)(f)=(-1,-1)_{\mathbb{F}}^{mlr}\gamma_\psi(-1^{lr})\chi^l_{\tau_r}(-r_l)$$
$$\int_{N_{r,l;k}(\mathbb{F})}f\Big(\big(j_{m,n}(-\widehat{g_0}\widehat{\varpi_{r,l}}\widehat{\epsilon_m}\omega_m')n,1\big),(I_{2k},1),I_l,I_r\Big)\psi^{-1}(n) \tag{3.27}$$

For $f\in I(\breve{\tau}_{(-s)}\otimes\overline{\sigma})$ define

$$\widehat{f}:\overline{Sp_{2m}(\mathbb{F})}\times\overline{Sp_{2k}(\mathbb{F})}\times GL_l(\mathbb{F})\times GL_r(\mathbb{F})\to\mathbb{C}$$

by

$$\widehat{f}(s,y,r_0,l_0)=f(s,y,I_m,r_0,l_0).$$

The map $f\mapsto\widehat{f}$ is an $\overline{Sp_{2n}(\mathbb{F})}$ isomorphism from $I(\breve{\tau}_{(-s)},\overline{\sigma})$ to $I(\breve{\tau}_{r(-s)},\breve{\tau}_{l(-s)},\overline{\sigma})$. By (3.25) and (3.27) we have,

$$\lambda(-s,\breve{\tau}\otimes\overline{\sigma},\psi)(f)=\chi^r_{\tau_l}(-I_l)\lambda\big((-s,-s),\breve{\tau}_r\otimes\breve{\tau}_l\otimes\overline{\sigma},\psi\big)(\widehat{f}). \tag{3.28}$$

We use (3.23), (3.28) and the fact that for all $f\in I(\tau^s\otimes\overline{\sigma})$, we have

$$\widetilde{A}_{j_{m,n}(\omega_m'^{-1})}(\widetilde{f})=\widehat{A_{j_{m,n}(\omega_m'^{-1})}(f)},$$

to complete the lemma:

$$C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{m,k;0}(\mathbb{F})},s,\tau\otimes\overline{\sigma},j_{m,n}(\omega_m'^{-1})\big)=\frac{\lambda(s,\tau\otimes\overline{\sigma},\psi)(f)}{\lambda(-s,\breve{\tau}\otimes\overline{\sigma},\psi)\big(A_{j_{m,n}(\omega_m'^{-1})}(f)\big)}=$$

$$\frac{\lambda\big((s,s),\tau_l\otimes\tau_r\otimes\overline{\sigma},\psi\big)(\widetilde{f})}{\lambda\big((-s,-s),\breve{\tau}_r\otimes\breve{\tau}_l\otimes\overline{\sigma},\psi\big)\big(\widetilde{A}_{j_{m,n}(\omega_m'^{-1})}(\widetilde{f})\big)}=C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{l,r;k}(\mathbb{F})},(s,s),\tau_l\otimes\tau_r\otimes\overline{\sigma},j_{m,n}(\omega_m')\big).$$

□

The heart of the proof of Theorem 3.2 is the following lemma which is a slight modification of the following argument, originally proved for algebraic groups (see [37] for example): If w_1,w_2 are two Weyl elements such that $l(w_1w_2)=l(w_1)+l(w_2)$, where $l(\cdot)$ is the length function in the Weyl group, then $A_{w_1}\circ A_{w_2}=A_{w_1w_2}$. See Lemma 1 of Chapter VII of [1] for a proof of this factorization in the case of an r-fold cover of $GL_n(\mathbb{F})$.

Lemma 3.2.

$$C_{\psi}^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{l,r;k}(\mathbb{F})},(s,s),\tau_l\otimes\tau_r\otimes\overline{\sigma},j_{m,n}(\omega_m'^{-1})\big)=\phi_{\psi}^{-1}(r,l,\tau_r)c_1(s)c_2(s)c_3(s),$$

where

$$c_1(s)=C_{\psi}^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{l,r;k}(\mathbb{F})},(s,s),\tau_l\otimes\tau_r\otimes\overline{\sigma},j_{m,n}(w_1^{-1})\big),$$
$$c_2(s)=C_{\psi}^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{l,r;k}(\mathbb{F})},(s,-s),\tau_l\otimes\widehat{\tau_r}\otimes\overline{\sigma},j_{m,n}(w_2^{-1})\big),$$
$$c_3(s)=C_{\psi}^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{l,r;k}(\mathbb{F})},(-s,s),\widehat{\tau_r}\otimes\tau_l\otimes\overline{\sigma},j_{m,n}(w_3^{-1})\big)$$

and where

$$w_1=\begin{pmatrix} I_l & & & \\ & & & -\omega_r \\ & & I_l & \\ & \omega_r & & \end{pmatrix},\quad w_2=\widehat{\varpi_{l,r}},\ w_3=\begin{pmatrix} I_r & & & \\ & & & -\omega_l \\ & & I_r & \\ & \omega_l & & \end{pmatrix},$$

and

$$\phi_{\psi}(r,l,\tau_r)=(-1,-1)_{\mathbb{F}}^{\frac{l^2(l-1)}{2}}\chi_{\tau_r}(-I_r)^l\gamma_{\psi}^{-1}\ \big((-1)^{rl}\big).$$

In particular :

$$C_{\psi}^{\overline{Sp_{2m}(\mathbb{F})}}\big(\overline{P_{l,r;0}(\mathbb{F})},(s,s),\tau_l\otimes\tau_r,\omega_m'^{-1}\big)=\phi_{\psi}^{-1}(r,l,\tau_r)c_1'(s)c_2'(s)c_3'(s),$$

where

$$c_1'(s)=C_{\psi}^{\overline{Sp_{2m}(\mathbb{F})}}\big(\overline{P_{l,r;0}(\mathbb{F})},(s,s),\tau_l\otimes\tau_r,w_1^{-1}\big)$$
$$c_2'(s)=C_{\psi}^{\overline{Sp_{2m}(\mathbb{F})}}\big(\overline{P_{l,r;0}(\mathbb{F})},(s,-s),\tau_l\otimes\widehat{\tau_r},w_2^{-1}\big)$$
$$c_3'(s)=C_{\psi}^{\overline{Sp_{2m}(\mathbb{F})}}\big(\overline{P_{l,r;0}(\mathbb{F})},(-s,s),\widehat{\tau_r}\otimes\tau_l,w_3^{-1}\big).$$

Proof. We keep the realizations used in Lemma 3.1 and most of its notations. Consider the following three intertwining operators:

$$\begin{aligned} A_{j_{m,n}(w_1^{-1})}&:I(\tau_{l(s)},\tau_{r(-s)},\overline{\sigma})\to I(\tau_{l(s)},\breve{\tau}_{r(-s)},\overline{\sigma}) \\ A_{j_{m,n}(w_2^{-1})}&:I(\tau_{l(s)},\breve{\tau}_{r(-s)},\overline{\sigma})\to I(\breve{\tau}_{r(-s)},\tau_{l(s)},\overline{\sigma}) \\ A_{j_{m,n}(w_3^{-1})}&:I(\breve{\tau}_{r(-s)},\tau_{l(s)},\overline{\sigma})\to I(\breve{\tau}_{r(-s)},\breve{\tau}_{l(-s)},\overline{\sigma}) \end{aligned} \tag{3.29}$$

Suppose that we show that

$$A_{j_{m,n}(w_3^{-1})}\circ A_{j_{m,n}(w_2^{-1})}\circ A_{j_{m,n}(w_1^{-1})}=\phi_{\psi}(r,l,\tau_r)A_{j_{m,n}(\omega_m'^{-1})}, \tag{3.30}$$

This will finish the lemma at once since

$$C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{l,r;k}(\mathbb{F})},(s,s),\tau_l\otimes\tau_r\otimes\overline{\sigma},j_{m,n}(\omega'_m)\big)=\frac{\lambda\big((s,s),\tau_l\otimes\tau_r\otimes\overline{\sigma},\psi\big)(f)}{\lambda\big((-s,-s),\breve{\tau}_r\otimes\breve{\tau}_l\otimes\overline{\sigma},\psi\big)\big(A_{j_{m,n}(\omega_m'^{-1})}(f)\big)}=$$

$$\frac{\lambda\big((s,s),\tau_l\otimes\tau_r\otimes\overline{\sigma},\psi\big)(f)}{\lambda\big((s,-s),\tau_l\otimes\breve{\tau}_r\otimes\overline{\sigma},\psi\big)\big(A_{j_{m,n}(w_1^{-1})}(f)\big)}\frac{\lambda\big((s,-s),\tau_l\otimes\breve{\tau}_r\otimes\overline{\sigma},\psi\big)\big(A_{j_{m,n}(w_1^{-1})}(f)\big)}{\lambda\big((-s,s),\breve{\tau}_r\otimes\tau_l\otimes\overline{\sigma},\psi\big)\big(A_{j_{m,n}(w_2^{-1})}\circ A_{j_{m,n}(w_1^{-1})}(f)\big)}$$

$$\frac{\lambda\big((-s,s),\breve{\tau}_r\otimes\tau_l\otimes\overline{\sigma},\psi\big)\big(A_{j_{m,n}(w_2^{-1})}\circ A_{j_{m,n}(w_1^{-1})}(f)\big)}{\lambda\big((-s,-s),\breve{\tau}_r\otimes\breve{\tau}_l\otimes\overline{\sigma},\psi\big)\big(A_{j_{m,n}(w_3^{-1})}\circ A_{j_{m,n}(w_2^{-1})}\circ A_{j_{m,n}(w_1^{-1})}(f)\big)}.$$

By definition the right-hand side equals

$$\phi_\psi^{-1}(r,l,\tau_r)c_1(s)c_2(s)c_3(s).$$

Thus, we prove (3.30). It is sufficient to prove it for $Re(s)>>0$ where $A_{j_{m,n}(\omega_m'^{-1})}$ is given by an absolutely convergent integral. Our argument will use Fubini's theorem, whose use will be justified by (3.33).

$$A_{j_{m,n}(w_3^{-1})}\circ A_{j_{m,n}(w_2^{-1})}\circ A_{j_{m,n}(w_1^{-1})}(f)(s,y,l,r)= \tag{3.31}$$
$$\int_{N_{w_3}(\mathbb{F})}A_{j_{m,n}(w_2^{-1})}\circ A_{j_{m,n}(w_1^{-1})}(f)\big(\big(j_{m,n}(t_{w_3}w_3)n_3,1\big)s,y,l,r\big)dn_3=$$
$$\int_{N_{w_3}(\mathbb{F})}\int_{N_{w_2}(\mathbb{F})}A_{j_{m,n}(w_1^{-1})}(f)\big(\big(j_{m,n}(t_{w_2}w_2)n_2,1\big)\big(j_{m,n}(t_{w_3}w_3)n_3,1\big)s,y,l,r\big)dn_2dn_3=$$
$$\int_{N_{w_3}(\mathbb{F})}\int_{N_{w_2}(\mathbb{F})}\int_{N_{w_1}(\mathbb{F})}f\big(\big(j_{m,n}(t_{w_1}w_1)n_1,1\big)\big(j_{m,n}(t_{w_2}w_2)n_2,1\big)\big(j_{m,n}(t_{w_3}w_3)n_3,1\big)s,y,l,r\big)dn_1dn_2dn_3,$$

where, as a straight forward computation will show, $N_{w_1}(\mathbb{F})$ is the group of elements of the form

$$\begin{pmatrix} I_l & & & 0_l & & \\ & I_r & & & z & \\ & & I_k & & & 0_k \\ & & & I_l & & \\ & & & & I_r & \\ & & & & & I_k \end{pmatrix},\quad z\in Mat^{sym}_{r\times r}(\mathbb{F}),$$

$N_{w_2}(\mathbb{F})$ is the group of elements of the form

$$\begin{pmatrix} I_l & z & & & & \\ & I_r & & & & \\ & & I_k & & & \\ & & & I_l & & \\ & & & -z^t & I_r & \\ & & & & & I_k \end{pmatrix},\quad z\in Mat_{l\times r}(\mathbb{F}),$$

$N_{w_3}(\mathbb{F})$ is the group of elements of the form

$$\begin{pmatrix} I_r & & & 0_r & & \\ & I_l & & & z & \\ & & I_k & & & 0_k \\ & & & I_r & & \\ & & & & I_l & \\ & & & & & I_k \end{pmatrix},\quad z\in Mat^{sym}_{l\times l}(\mathbb{F}),$$

and where $t_{w_1} = \widehat{\begin{pmatrix} I_l & \\ & \epsilon_r \end{pmatrix}}$, $\qquad t_{w_2} = I_m, \qquad t_{w_3} = \widehat{\begin{pmatrix} I_r & \\ & \epsilon_l \end{pmatrix}}$. We consider the first argument of f in (3.31): By (1.14) and (1.18) we have:

$$\begin{aligned}
&\big(j_{m,n}(t_{w_1}w_1)n_1, 1\big)\big(j_{m,n}(t_{w_2}w_2)n_2, 1\big)\big(j_{m,n}(t_{w_3}w_3)n_3, 1\big) \\
&= \big(j_{m,n}(t_{w_1}w_1), 1\big)(n_1, 1)\big(j_{m,n}(t_{w_2}w_2), 1\big)(n_2, 1)\big(j_{m,n}(t_{w_3}w_3), 1\big)(n_3, 1) \\
&= \big(j_{m,n}(t_{w_1}w_1), 1\big)\big(j_{m,n}(t_{w_2}w_2), 1\big)\big(j_{m,n}(t_{w_3}w_3), 1\big)(n_1'n_2'n_3, 1) \\
&= (I_{2n}, \epsilon)\big(j_{m,n}(t_{w_1}w_1t_{w_2}w_2t_{w_3}w_3), 1\big)(n_1'n_2'n_3, 1) \\
&= (I_{2n}, \epsilon)\big(j_{m,n}(\widehat{\begin{pmatrix} I_l & \\ & I_r(-1)^l \end{pmatrix}}\widehat{\epsilon_m}\omega_m'), 1\big)(n_1'n_2'n_3, 1),
\end{aligned} \tag{3.32}$$

where

$$n_1' = j_{m,n}(w_3t_{w_3}w_2t_{w_2})^{-1}(n_1)j_{m,n}(w_2t_{w_2}w_3t_{w_3})^{-1}, \quad n_2' = j_{m,n}(w_3t_{w_3})^{-1}(n_2)j_{m,n}(w_3t_{w_3}),$$

and where

$$\epsilon = c(t_{w_1}w_1, t_{w_2}w_2)c(t_{w_1}w_1t_{w_2}w_2, t_{w_3}w_3).$$

We compute:

$$j_{m,n}(w_3t_{w_3}w_2t_{w_2})^{-1}\begin{pmatrix} I_l & & & 0_l & & \\ & I_r & & & z & \\ & & I_k & & & 0_k \\ & & & I_l & & \\ & & & & I_r & \\ & & & & & I_k \end{pmatrix} j_{m,n}(w_2t_{w_2}w_3t_{w_3})^{-1} = \begin{pmatrix} I_r & & & z & & \\ & I_l & & & & \\ & & I_k & & & \\ & & & I_r & & \\ & & & & I_l & \\ & & & & & I_k \end{pmatrix},$$

$$j_{m,n}(w_3t_{w_3})^{-1}\begin{pmatrix} I_l & z & & & & \\ & I_r & & & & \\ & & I_k & & & \\ & & & I_l & & \\ & & & -z^t & I_r & \\ & & & & & I_k \end{pmatrix} j_{m,n}(w_3t_{w_3}) = \begin{pmatrix} I_l & & & & z' & \\ & I_r & & z'^t & & \\ & & I_k & & & \\ & & & I_l & & \\ & & & & I_r & \\ & & & & & I_k \end{pmatrix},$$

where $z' = z\omega_r t_{w_3}$. Hence we can change the three integrals in (3.31) to a single integration on $j_{m,n}\big(N_{m;0}(\mathbb{F})\big)$ without changing the measure and obtain

$$\begin{aligned}
&A_{j_{m,n}(w_3^{-1})} \circ A_{j_{m,n}(w_2^{-1})} \circ A_{j_{m,n}(w_1^{-1})}(f)(s, y, l, r) \\
&= \epsilon \int_{j_{m,n}(N_{m;0})} f\Bigg(\big(j_{m,n}(\widehat{\begin{pmatrix} I_l & \\ & I_r(-1)^l \end{pmatrix}}\widehat{\epsilon_m}\omega_m')n, 1\big)s, y, l, r\Bigg)dn \\
&= \epsilon\chi_{\tau_r}(-I_r)^l\gamma_\psi^{-1}\ \big((-1)^{rl}\big)A_{j_{m,n}(\omega_m'^{-1})}(f)(s, y, l, r).
\end{aligned} \tag{3.33}$$

It is left to show that $\epsilon = (-1, -1)_{\mathbb{F}}^{\frac{l^2(l-1)}{2}}$. Indeed, we have

$$t_{w_1}w_1 = \begin{pmatrix} I_l & & & \\ & \epsilon_r\omega_r & & \\ & & I_l & \\ & & & \epsilon_r\omega_r \end{pmatrix}\begin{pmatrix} I_l & & & \\ & & & -I_r \\ & & I_l & \\ & I_r & & \end{pmatrix}, \quad t_{w_2}w_2 = \begin{pmatrix} & I_l & & \\ I_r & & & \\ & & & I_l \\ & & I_r & \end{pmatrix}.$$

Thus, by (1.14) we have $c(t_{w_1}w_1, t_{w_2}w_2) = (-1,-1)_{\mathbb{F}}^{\frac{r^3 l(r-1)}{2}}$. Since

$$t_{w_1}w_1 t_{w_2}w_2 = \begin{pmatrix} I_l & & & \\ & \epsilon_r\omega_r & & \\ & & I_l & \\ & & & \epsilon_r\omega_r \end{pmatrix}\begin{pmatrix} & I_l & & \\ I_r & & & \\ & & & I_l \\ & & I_r & \end{pmatrix}\begin{pmatrix} & & & -I_r \\ & & I_l & \\ & -I_r & & \\ & & & I_l \end{pmatrix},$$

$$t_{w_3}w_3 = \begin{pmatrix} I_r & & & \\ & & & -I_l \\ & & I_r & \\ & I_l & & \end{pmatrix}\begin{pmatrix} I_r & & & \\ & \epsilon_l\omega_l & & \\ & & I_r & \\ & & & \epsilon_r\omega_r \end{pmatrix},$$

we conclude, using (1.16), that $c(t_{w_1}w_1t_{w_2}w_2, t_{w_3}w_3) = (-1,-1)_{\mathbb{F}}^{\frac{r^3 l(r-1)}{2}+\frac{l^2(l-1)}{2}}$. $\square$

Lemma 3.3. *Keeping the notations of the previous lemmas we have:*

$$C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{l,r;k}(\mathbb{F})}, (s,s), \tau_l\otimes\tau_r\otimes\overline{\sigma}, j_{m,n}(w_1^{-1})\big) = C_\psi^{\overline{Sp_{2(r+k)}(\mathbb{F})}}\big(\overline{P_{r;k}(\mathbb{F})}, s, \tau_r\otimes\overline{\sigma}, j_{r,r+k}(\omega_r'^{-1})\big). \tag{3.34}$$

$$C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{l,r;k}(\mathbb{F})}, (s,-s), \tau_l\otimes\widehat{\tau}_r\otimes\overline{\sigma}, j_{m,n}(w_2^{-1})\big) = C_\psi^{GL_m(\mathbb{F})}\big(P^0_{l,r}(\mathbb{F}), (s,-s), \tau_l\otimes\widehat{\tau}_r, \varpi_{l,r}^{-1}\big). \tag{3.35}$$

$$C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{r,l;k}(\mathbb{F})}, (-ss,s), \widehat{\tau}_r\otimes\tau_l\otimes\overline{\sigma}, j_{m,n}(w_3^{-1})\big) = C_\psi^{\overline{Sp_{2(l+k)}(\mathbb{F})}}\big(\overline{P_{l;k}(\mathbb{F})}, s, \tau_l\otimes\overline{\sigma}, j_{l,l+k}(\omega_r'^{-1})\big). \tag{3.36}$$

In particular:

$$C_\psi^{\overline{Sp_{2m}(\mathbb{F})}}\big(\overline{P_{l,r;0}(\mathbb{F})}, (s,s), \tau_l\otimes\tau_r, w_1^{-1}\big) = C_\psi^{\overline{Sp_{2r}(\mathbb{F})}}\big(\overline{P_{r;0}(\mathbb{F})}, s, \tau_r, \omega_r'^{-1}\big).$$
$$C_\psi^{\overline{Sp_{2m}(\mathbb{F})}}\big(\overline{P_{l,r;0}(\mathbb{F})}, (s,-s), \tau_l\otimes\widehat{\tau}_r, w_2^{-1}\big) = C_\psi^{GL_m(\mathbb{F})}\big(P^0_{l,r}(\mathbb{F}), (s,-s), \tau_l\otimes\widehat{\tau}_r, \varpi_{l,r}^{-1}\big).$$
$$C_\psi^{\overline{Sp_{2m}(\mathbb{F})}}\big(\overline{P_{r,l;0}(\mathbb{F})}, (-s,s), \widehat{\tau}_r\otimes\tau_l\otimes\overline{\sigma}, w_3^{-1}\big) = C_\psi^{\overline{Sp_{2l}(\mathbb{F})}}\big(\overline{P_{l;0}(\mathbb{F})}, s, \tau_l, \omega_r'^{-1}\big).$$

Proof. We prove (3.34) and (3.35) only. (3.36) is proven exactly as (3.34). We start with (3.34): As in Lemma 3.1 we realize $I(\tau_{r(s)}, \overline{\sigma})$ as a space of functions

$$f : \overline{Sp_{2(r+k)}(\mathbb{F})} \times \overline{Sp_{2k}(\mathbb{F})} \times GL_r(\mathbb{F}) \to \mathbb{C}$$

which are smooth from the right in each variable and which satisfies

$$f(j_{r,r+k}(\widehat{b})i_{k,n}(h)us, ny, n'r_0) = \gamma_\psi^{-1}(\det(b))|\det(b)|^{k+r+\frac{1}{2}}\psi(n)\psi(n')\psi f(s, yh, r_0b), \tag{3.37}$$

for all $s\in\overline{Sp_{2(r+k)}(\mathbb{F})}$, $h,y\in\overline{Sp_{2k}(\mathbb{F})}$, $b,r_0\in GL_r(\mathbb{F})$, $u\in(N_{l;k}(\mathbb{F}),1)$, $n\in\overline{Z_{Sp_{2k}}(\mathbb{F})}$ $n'\in Z_{GL_m}(\mathbb{F})$. For $f\in I(\tau_{l(s)}, \tau_{r(s)}, \overline{\sigma})$, $g\in\overline{Sp_{2n}(\mathbb{F})}$ we define

$$f_g : \overline{Sp_{2(r+k)}(\mathbb{F})} \times \overline{Sp_{2k}(\mathbb{F})} \times GL_r(\mathbb{F}) \to \mathbb{C}$$

by

$$f_g(s,y,r) = f(i_{r+k,n}(g)s, y, I_l, r).$$

Recalling (3.20) we note that $f_g \in I(\tau_{r(s)}, \overline{\sigma})$. We want to write the exact relation between $\lambda\big((s,s), \tau_l \otimes \tau_r \otimes \overline{\sigma}, \psi\big)$ and $\lambda\big(s, \tau_r \otimes \overline{\sigma}, \psi\big)$. To do so we consider the left argument of f in (3.21): We decompose $n \in N_{l,r;k}(\mathbb{F})$ as $n = n'n''$, where

$$n' \in i_{r+k,n}(N_{r,k}(\mathbb{F})),\ n'' \in U_0(\mathbb{F}) = \left\{ u = \begin{pmatrix} I_l & * & * & * & * & * \\ & I_r & 0r\times k & * & 0r\times r & 0r\times k \\ & & I_k & * & 0k\times r & 0k\times k \\ & & & I_l & & \\ & & & * & I_r & \\ & & & * & & I_k \end{pmatrix} \mid u \in Sp_{2n}(\mathbb{F}) \right\}.$$

We have

$$\begin{aligned}
&(w_l'(l,r,k)n, 1) \\
&= \left(\begin{pmatrix} I_l & & & & & \\ & & & & \epsilon_r\omega_r & \\ & & I_k & & & \\ & & & I_l & & \\ & -\epsilon_r\omega_r & & & & \\ & & & & & I_k \end{pmatrix} \begin{pmatrix} & & & \epsilon_l\omega_l & & \\ & I_r & & & & \\ & & I_k & & & \\ -\epsilon_l\omega_l & & & & & \\ & & & & I_r & \\ & & & & & I_k \end{pmatrix} n'n'', 1 \right) \\
&= \left(i_{r+k,n}\big(j_{r,r+k}(-\widehat{\epsilon_r}\omega_r')\big) n' \begin{pmatrix} & & & \epsilon_l\omega_l & & \\ & I_r & & & & \\ & & I_k & & & \\ -\epsilon_l\omega_l & & & & & \\ & & & & I_r & \\ & & & & & I_k \end{pmatrix} n'', 1 \right)
\end{aligned}$$

Thus, for an appropriate ϵ independent of n' and n'', we have:

$$\begin{aligned}
&(w_l'(l,r,k)n, 1) = \\
&\left(i_{r+k,n}\big(j_{r,r+k}(-\widehat{\epsilon_r}\omega_r')\big) n', 1 \right) \left(\begin{pmatrix} & & & \epsilon_l\omega_l & & \\ & I_r & & & & \\ & & I_k & & & \\ -\epsilon_l\omega_l & & & & & \\ & & & & I_r & \\ & & & & & I_k \end{pmatrix} n'', \varepsilon \right).
\end{aligned}$$

We denote the right element of the last line by $g(n'')$, and we see that

$$\begin{aligned}
&\lambda\big((s,s), \tau_l \otimes \tau_r \otimes \overline{\sigma}, \psi\big)(f) \\
&= \int_{U_0(\mathbb{F})} \int_{N_{r;k}(\mathbb{F})} f_{g(n'')}\big(j_{r,r+k}(-\widehat{\epsilon_r}\omega_r')n', (I_{2k}, 1), I_r\big)\psi^{-1}(n')\psi^{-1}(n'')\, dn'\, dn'' \\
&= \int_{U_0(\mathbb{F})} \lambda\big(s, \tau_r \otimes \overline{\sigma}, \psi\big)(f_{g(n'')})\psi^{-1}(n'')\, dn''.
\end{aligned} \tag{3.38}$$

For $f \in I(\tau_{l(s)}, \breve{\tau}_{r(-s)}, \overline{\sigma})$, $s \in \overline{Sp_{2n}(\mathbb{F})}$ we define f_g as we did for $I(\tau_{l(s)}, \tau_{r(s)}, \overline{\sigma})$. In this case $f_g \in I(\breve{\tau}_{r(-s)}, \overline{\sigma})$. Exactly as (3.38) we have

$$\lambda\big((s,-s), \tau_l \otimes \breve{\tau}_r \otimes \overline{\sigma}, \psi\big)(f) = \int_{U_0(\mathbb{F})} \lambda\big(-s, \breve{\tau}_r \otimes \overline{\sigma}, \psi\big)(f_{g(n'')})\psi^{-1}(n'')dn''. \tag{3.39}$$

Let

$$A_{j_{m,n}(w_1^{-1})} : I(\tau_{l(s)}, \tau_{r(s)}, \overline{\sigma}) \to I(\tau_{l(s)}, \breve{\tau}_{r(-s)}, \overline{\sigma})$$

be as in lemma 3.2 and let

$$A_{j_{r,r+k}(\omega_r'^{-1})} : I(\tau_{r(s)}, \overline{\sigma}) \to I(\breve{\tau}_{r(-s)}, \overline{\sigma})$$

be the intertwining operator defined by

$$A_{j_{r,r+k}(\omega_r'^{-1})}(f) = \int_{j_{r,r+k}(N_{r;0}(\mathbb{F}))} f(j_{r,r+k}(\epsilon_r \omega_r' n), (I_{2k}, 1), I_r)\psi^{-1}(n)dn$$

Using the fact that $A_{j_{r,r+k}(\epsilon_r \omega_r')^{-1}}(f_g) = \big(A_{j_{m,n}(w_1^{-1})}(f)\big)_g$ we prove (3.34):

$$\begin{aligned}
&C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{l,r;k}(\mathbb{F})}, (s,s), \tau_l \otimes \tau_r \otimes \overline{\sigma}, j_{m,n}(w_1^{-1})\big)\\
&= \frac{\lambda\big((s,s), \tau_l \otimes \tau_r \otimes \overline{\sigma}, \psi\big)\big(f\big)}{\lambda\big((s,-s), \tau_l \otimes \breve{\tau}_r \otimes \overline{\sigma}, \psi\big)\big(A_{j_{m,n}(w_1^{-1})}(f)\big)}\\
&= \frac{\int_{U_0(\mathbb{F})} \lambda\big(s, \tau_r \otimes \overline{\sigma}, \psi\big)\big(f_{g(n'')}\big)\psi^{-1}(n'')dn''}{\int_{U_0(\mathbb{F})} \lambda\big(-s, \breve{\tau}_r \otimes \overline{\sigma}, \psi\big)\big(A_{j_{r,r+k}(\epsilon_r \omega_r')^{-1}}(f_{g(n'')})\big)\psi^{-1}(n'')dn''}\\
&= \frac{\int_{U_0(\mathbb{F})} C_\psi^{\overline{Sp_{2(r+k)}(\mathbb{F})}}\big(\overline{P_{r;k}(\mathbb{F})}, s, \tau_r \otimes \overline{\sigma}, j_{r,r+k}(\omega_r'^{-1})\big)\lambda\big(-s, \breve{\tau}_r \otimes \overline{\sigma}, \psi\big)\big(A_{j_{r,r+k}(\epsilon_r \omega_r')^{-1}}(f_{g(n'')})\big)\psi^{-1}(n'')dn''}{\int_{U_0(\mathbb{F})} \lambda\big(-s, \breve{\tau}_r \otimes \overline{\sigma}, \psi\big)\big(A_{j_{r,r+k}(\epsilon_r \omega_r')^{-1}}(f_{g(n'')})\big)\psi^{-1}(n'')dn''}\\
&= C_\psi^{\overline{Sp_{2(r+k)}(\mathbb{F})}}\big(\overline{P_{r;k}(\mathbb{F})}, s, \tau_r \otimes \overline{\sigma}, j_{r,r+k}(\omega_r'^{-1})\big).
\end{aligned}$$

To prove (3.35) one uses similar arguments. The key point is that for $f \in I(\tau_{l(s)}, \breve{\tau}_{r(-s)}, \overline{\sigma})$, $g \in \overline{Sp_{2n}(\mathbb{F})}$, the function

$$f_g : GL_m(\mathbb{F}) \times GL_l(\mathbb{F}) \times GL_r(\mathbb{F}) \to \mathbb{C}$$

defined by

$$f_g(a, l, r) = |\det a|^{-\frac{2k+m+1}{2}} \gamma_\psi(a) f\bigg(\big(j_{m,n}(a), 1\big)g, (I_{2k}, 1), l, r\bigg),$$

lies in $I(\tau_{l(s)}, \breve{\tau}_{r(-s)})$. □

These three lemmas provide the proof of Theorem 3.2

$$
\begin{aligned}
&\gamma(\overline{\sigma}\times\tau,s,\psi)\\
&=\frac{C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}(\overline{P_{m;k}(\mathbb{F})},s,\tau\otimes\overline{\sigma},j_{m,n}(\omega_m'^{-1}))}{C_\psi^{\overline{Sp_{2m}(\mathbb{F})}}(\overline{P_{m;0}(\mathbb{F})},s,\tau,\omega_m'^{-1})}\\
&=\frac{C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}(\overline{P_{l,r;k}(\mathbb{F})},(s,s),\tau_l\otimes\tau_r\otimes\overline{\sigma},j_{m,n}(\omega_m'^{-1}))}{C_\psi^{\overline{Sp_{2m}(\mathbb{F})}}(\overline{P_{l,r;0}(\mathbb{F})},(s,s),\tau_l\otimes\tau_r,\omega_m'^{-1})}\\
&=\frac{C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{l,r;k}(\mathbb{F})},(s,s),\tau_l\otimes\tau_r\otimes\overline{\sigma},j_{m,n}(w_1^{-1})\big)}{C_\psi^{\overline{Sp_{2m}(\mathbb{F})}}\big(\overline{P_{l,r;0}(\mathbb{F})},(s,s),\tau_l\otimes\tau_r,w_1^{-1}\big)}\frac{C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{l,r;k}(\mathbb{F})},(s,-s),\tau_l\otimes\widehat{\tau}_r\otimes\overline{\sigma},j_{m,n}(w_2^{-1})\big)}{C_\psi^{\overline{Sp_{2m}(\mathbb{F})}}\big(\overline{P_{l,r;0}(\mathbb{F})},(s,-s),\tau_l\otimes\widehat{\tau}_r,w_2^{-1}\big)}\\
&\quad\frac{C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{l,r;k}(\mathbb{F})},(-s,s),\widehat{\tau}_r\otimes\tau_l\otimes\overline{\sigma},j_{m,n}(w_3^{-1})\big)}{C_\psi^{\overline{Sp_{2m}(\mathbb{F})}}\big(\overline{P_{l,r;0}(\mathbb{F})},(-s,s),\widehat{\tau}_r\otimes\tau_l,w_3^{-1}\big)}\\
&=\frac{C_\psi^{\overline{Sp_{2(r+k)}(\mathbb{F})}}\big(\overline{P_{r;k}(\mathbb{F})},s,\tau_r\otimes\overline{\sigma},j_{r,r+k}(\omega_r'^{-1})\big)}{C_\psi^{\overline{Sp_{2r}(\mathbb{F})}}\big(\overline{P_{r;0}(\mathbb{F})},s,\tau_r,\omega_r'^{-1}\big)}\frac{C_\psi^{GL_m(\mathbb{F})}\big(P^0_{l,r}(\mathbb{F}),(s,-s),\tau_l\otimes\widehat{\tau}_r,\varpi_{l,r}^{-1}\big)}{C_\psi^{GL_m(\mathbb{F})}\big(P^0_{l,r}(\mathbb{F}),(s,-s),\tau_l\otimes\widehat{\tau}_r,\varpi_{l,r}^{-1}\big)}\\
&\quad\frac{C_\psi^{\overline{Sp_{2(l+k)}(\mathbb{F})}}\big(\overline{P_{l;k}(\mathbb{F})},s,\tau_l\otimes\overline{\sigma},j_{l,l+k}(\omega_r'^{-1})\big)}{C_\psi^{\overline{Sp_{2l}(\mathbb{F})}}\big(\overline{P_{l;0}(\mathbb{F})},s,\tau_l,\omega_r'^{-1}\big)}=\gamma(\overline{\sigma}\times\tau_l,s,\psi)\gamma(\overline{\sigma}\times\tau_r,s,\psi).
\end{aligned}
$$

The proof of Theorem 3.13 is achieved through similar steps to those used in the proof of Theorem 3.2. We outline them: First one proves an analog to Lemma 3.1 and shows that

$$C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{m;k}(\mathbb{F})},s,\tau\otimes\overline{\sigma},j_{m,n}(\omega_m'^{-1})\big)=C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{m,l;r}(\mathbb{F})},(s,0),\tau\otimes\tau_l\otimes\overline{\sigma_r},j_{m,n}(\omega_m'^{-1})\big). \tag{3.40}$$

Then one gives an analog to Lemma 3.2: Using a decomposition of $A_{j_{m,n}(\omega_m'^{-1})}$, one shows that

$$C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{m,l;r}(\mathbb{F})},(s,0),\tau\otimes\tau_l\otimes\overline{\sigma_r},j_{m+l,n}(\omega_m'^{-1})\big)=(-1,-1)_{\mathbb{F}}^{ml}k_1(s)k_2(s)k_3(s), \tag{3.41}$$

where

$$
\begin{aligned}
k_1(s)&=C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{m,l;r}(\mathbb{F})},(s,0),\tau\otimes\tau_l\otimes\overline{\sigma_r},j_{m+l,n}(w_4^{-1})\big),\\
k_2(s)&=C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{l,m;r}(\mathbb{F})},(0,s),\tau_l\otimes\tau\otimes\overline{\sigma_r},j_{m+l,n}(w_5^{-1})\big),\\
k_3(s)&=C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{m,l;r}(\mathbb{F})},(0,-s),\tau_l\otimes\widehat{\tau}\otimes\overline{\sigma_r},j_{m+l,n}(w_6^{-1})\big)
\end{aligned}
$$

and where

$$w_4=\widehat{\varpi_{m,l}},\quad w_5=\begin{pmatrix} I_l &&& \\ &&& -\omega_m \\ && I_l & \\ & \omega_m && \end{pmatrix},\quad w_6=\widehat{\varpi_{l,m}}.$$

The third step is, an analog to Lemma 3.3:

$$C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{m,l;r}(\mathbb{F})},(s,0),\tau\otimes\tau_l\otimes\overline{\sigma_r},j_{m+l,n}(w_4^{-1})\big)=C_{\psi^{-1}}^{GL_{m+l}(\mathbb{F})}\big(P^0_{m,l}(\mathbb{F})(\frac{s}{2},\frac{-s}{2}),\tau\otimes\tau_l,\varpi_{m,l}^{-1}\big),$$

$$C_{\psi}^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{l,m;r}(\mathbb{F})},(0,s),\tau_l\otimes\tau\otimes\overline{\sigma_r},j_{m+l,n}(w_5^{-1})\big)=C_{\psi}^{\overline{Sp_{2(m+r)}(\mathbb{F})}}\big(\overline{P_{l;r}(\mathbb{F})},s,\tau\otimes\overline{\sigma_r},j_{m,m+k}(\omega_m'^{-1})\big),$$
$$C_{\psi}^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{l,m;r}(\mathbb{F})},(0,-s),\tau_l\otimes\widehat{\tau}\otimes\overline{\sigma_r},j_{m+l,n}(w_6^{-1})\big)=C_{\psi}^{GL_{m+l}(\mathbb{F})}\big(P^0_{m,l}(\mathbb{F})(\frac{s}{2},\frac{-s}{2}),\tau_l\otimes\widehat{\tau},\varpi_{l,m}^{-1}\big). \tag{3.42}$$

Combining (3.40), (3.41), (3.42) we have:

$$\gamma(\overline{\sigma}\times\tau,s,\psi)=(-1,-1)_{\mathbb{F}}^{ml}\gamma(\overline{\sigma_r}\times\tau,s,\psi)$$
$$C_{\psi^{-1}}^{GL_{m+l}(\mathbb{F})}\big(P^0_{m,l}(\mathbb{F})(\frac{s}{2},\frac{-s}{2}),\tau\otimes\tau_l,\varpi_{m,l}^{-1}\big)C_{\psi^{-1}}^{GL_{m+l}(\mathbb{F})}\big(P^0_{m,l}(\mathbb{F})(\frac{s}{2},\frac{-s}{2}),\tau_l\otimes\widehat{\tau},\varpi_{l,m}^{-1}\big).$$

With (3.9) we finish.

3.3 Some computations

Lemma 3.4. *Let $\eta_1,\eta_2,\ldots,\eta_k$ be k characters of $\mathbb{F}^*$ and let $\gamma_\psi^{-1}\otimes\chi=(\gamma_\psi^{-1}\circ\det)\otimes\chi$ be the character of $\overline{T_{Sp_{2k}}(\mathbb{F})}$ defined by*

$$\big(diag(t_1,t_2,\ldots,t_k,t_1^{-1},t_2^{-1},\ldots,t_k^{-1}),\epsilon\big)\mapsto\epsilon\gamma_\psi^{-1}(t_1t_2\ldots,t_k)\prod_{i=1}^{k}\eta_i(t_i).$$

Let $\alpha_1,\alpha_2,\ldots,\alpha_m$ be m characters of $\mathbb{F}^*$ and let μ be the character of $T_{GL_m}(\mathbb{F})$ defined by

$$Diag(t_1,t_2,\ldots,t_m)\mapsto\prod_{i=1}^{m}\alpha_i(t_i).$$

Define $\overline{\sigma}$ and τ to be the corresponding principal series representations:

$$\overline{\sigma}=I(\chi)=Ind_{\overline{B_{SP_{2k}}(\mathbb{F})}}^{\overline{Sp_{2k}(\mathbb{F})}}\gamma_\psi^{-1}\otimes\chi,\quad\tau=I(\mu)=Ind_{B_{GL_m(\mathbb{F})}}^{GL_m(\mathbb{F})}\mu.$$

There exists $c\in\{\pm1\}$ such that

$$\gamma(\overline{\sigma}\times\tau,s,\psi)=c\prod_{i=1}^{k}\prod_{j=1}^{m}\gamma(\alpha_j\times\eta_i^{-1},s,\psi)\gamma(\eta_i\times\alpha_j,s,\psi). \tag{3.43}$$

Proof. Note that $\tau\simeq Ind_{P^0_{1,m-1}(\mathbb{F})}^{GL_m(\mathbb{F})}\alpha_1\otimes\tau'$, where $\tau'=Ind_{B_{GL_{m-1}}(\mathbb{F})}^{GL_{m-1}(\mathbb{F})}\otimes_{i=2}^{m}\alpha_i$. Theorem 3.2 implies that

$$\gamma(\overline{\sigma}\times\tau,s,\psi)=\gamma(\overline{\sigma}\times\alpha_1,s,\psi)\gamma(\overline{\sigma}\times\tau',s,\psi).$$

Repeating this argument $m-1$ more times we observe that

$$\gamma(\overline{\sigma}\times\tau,s,\psi)=\prod_{j=1}^{m}\gamma(\overline{\sigma}\times\alpha_j,s,\psi). \tag{3.44}$$

Next we note that $\overline{\sigma}=Ind_{\overline{P_{1;k-1}(\mathbb{F})}}^{\overline{Sp_{2k}(\mathbb{F})}}(\gamma_\psi^{-1}\otimes\eta_1)\otimes\overline{\sigma'}$, where $\overline{\sigma'}=Ind_{\overline{B_{Sp_{2(k-1)}}(\mathbb{F})}}^{\overline{Sp_{2(k-1)}(\mathbb{F})}}\gamma_\psi^{-1}\otimes\big(\otimes_{j=1}^{k-1}\eta_j\big)$. By using Theorem 3.3 we observe that for all $1\leq j\leq m$. There exists $c'\in\{\pm1\}$ such that

$$\gamma(\overline{\sigma}\times\alpha_j,s,\psi)=c'\gamma(\overline{\sigma'}\times\alpha_j,s,\psi)\gamma(\alpha_j\times\eta_1^{-1},s,\psi)\gamma(\eta_1\times\alpha_j,s,\psi). \tag{3.45}$$

By Repeating this argument $k-1$ more times for each $1\leq j\leq m$ and by using (3.44) we finish. □

Lemma 3.5. *Let χ be a character of $\mathbb{F}^*$ viewed as a character of $\overline{B_{SL_2(\mathbb{F})}}$. There exists an exponential factor, $\epsilon'(\chi, s, \psi)$, such that*

$$C_{\psi}^{\overline{SL_2(\mathbb{F})}}\Big(\overline{B_{SL_2(\mathbb{F})}}, s, \chi, \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\Big) = \epsilon'(\chi, s, \psi)\frac{\gamma(\chi^2, 2s, \psi)}{\gamma(\chi, s+\frac{1}{2}, \psi)}, \tag{3.46}$$

If $\mathbb{F}$ is p-adic field of odd residual characteristic, χ is unramified and ψ is normalized then $\epsilon'(\chi, s, \psi) = 1$.

Proof. This lemma was proven in [51] for p-adic fields and for the field of real numbers. We remak that the computations are extremely technical in the case of 2-adic fields. We now give a short proof for $\mathbb{F} = \mathbb{C}$.

Since $\overline{SL_2(\mathbb{C})} = SL_2(\mathbb{C}) \times \{\pm 1\}$ and since $\gamma_\psi(\mathbb{C}^*) = 1$ it follows that

$$C_{\psi}^{SL_2(\mathbb{C})}\big(B_{SL_2}(\mathbb{C}), s, \chi, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\big) = C_{\psi}^{\overline{SL_2(\mathbb{C})}}\big(\overline{B_{SL_2}(\mathbb{C})}, s, \chi, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\big). \tag{3.47}$$

Recall that any character of $\mathbb{C}^*$ has the form

$$\chi(re^{i\theta}) = \chi_{n,s_0}(re^{i\theta}) = r^{s_0}e^{in\theta},$$

for some $s_0 \in \mathbb{C}$, $n \in \mathbb{Z}$. We may assume that $s_0 = 0$ or equivalently that $\chi = \chi_{n,0}$. Theorem 3.13 of [40] states that

$$C_{\psi}^{SL_2(\mathbb{C})}\big(B_{SL_2}(\mathbb{C}), s, \chi, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\big) = c'(s)\frac{L(\chi^{-1}, 1-s)}{L(\chi, s)}, \tag{3.48}$$

where $c'(s)$ is an exponential factor. The corresponding local L-function is defined by

$$L_{\mathbb{C}}(\chi_{n,0}, s) = (2\pi)^{-(s+\frac{|n|}{2})}\Gamma(s+\frac{|n|}{2}).$$

Due to (3.47) and (3.48) we only have to show that

$$\frac{\Gamma(1+\frac{|n|}{2}-s)}{\Gamma(\frac{|n|}{2}+s)} = 2\frac{\Gamma(1+n-2s)\Gamma(\frac{1}{2}+\frac{|n|}{2}+s)}{\Gamma(n+2s)\Gamma(\frac{1}{2}+\frac{|n|}{2}-s)}$$

This fact follows from the classical duplication formula

$$\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\Gamma(2z).$$

□

Lemma 3.6. *Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be m characters of $\mathbb{F}^*$ and let μ be the character of $T_{GL_m}(\mathbb{F})$ defined by*

$$Diag(t_1, t_2, \ldots, t_m) \mapsto \prod_{i=1}^{m} \alpha_i(t_i).$$

We also regard μ as a character of $B_{GL_m}(\mathbb{F})$. Define τ to be the corresponding principal series representation:

$$\tau = I(\mu) = Ind_{B_{GL_m(\mathbb{F})}}^{GL_m(\mathbb{F})}\mu.$$

There exists an exponential function $c(s)$ such that

$$C_{\psi}^{\overline{Sp_{2m}(\mathbb{F})}}\big(\overline{P_{m;0}(\mathbb{F})}, s, \tau, \omega_m'^{-1}\big) = c(s)\frac{\gamma(\tau, sym^2, 2s, \psi)}{\gamma(\tau, s+\frac{1}{2}, \psi)} \tag{3.49}$$

If $\mathbb{F}$ is a p-adic field of odd residual characteristic, ψ is normalized and τ is unramified then $c(s) = 1$

Remark: In Section 4.3 we shall show that (3.49) holds for every irreducible admissible generic representation τ of $GL_m(\mathbb{F})$; see Theorem 4.3.

Proof. We first prove by induction that there exists $d \in \{\pm 1\}$ such that

$$\begin{aligned} &C_{\psi}^{\overline{Sp_{2m}(\mathbb{F})}}(\overline{P_{m;0}(\mathbb{F})}, s, \tau, \omega_m'^{-1}) \\ &= d\prod_{i=1}^{m} C_{\psi}^{\overline{SL_2(\mathbb{F})}}\Big(\overline{B_{SL_2}(\mathbb{F})}, s, \alpha_i, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\Big) \prod_{i=1}^{m-1} C_{\psi}^{GL_{m+1-i}(\mathbb{F})}\big(P^0_{1,m-i}(\mathbb{F}), (s,-s), \alpha_i \otimes \widehat{\tau}_i, \varpi^{-1}_{1,m-i}\big) \end{aligned} \tag{3.50}$$

where for $1 \leq i \leq m-2$, $\tau_i = Ind_{B_{GL_{m-i}}(\mathbb{F})}^{GL_{m-i}(\mathbb{F})} \otimes_{j=i+1}^{m} \alpha_j$ and $\tau_{m-1} = \alpha_m$. Furthermore, $d = 1$ if $\mathbb{F}$ is a p-adic field of odd residual characteristic and τ is unramified. For $m = 1$ there is nothing to prove. Suppose now that the theorem is true for $m-1$. With our enumeration this means that there exists $d' \in \{\pm 1\}$ such that

$$\begin{aligned} &C_{\psi}^{\overline{Sp_{2(m-1)}(\mathbb{F})}}(\overline{P_{m-1;0}(\mathbb{F})}, s, \tau_1, \omega_{m-1}'^{-1}) \\ &= d'\prod_{i=2}^{m} C_{\psi}^{\overline{SL_2(\mathbb{F})}}(\overline{B_{SL_2(\mathbb{F})}}, s, \alpha_i, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)) \prod_{i=2}^{m-1} C_{\psi}^{GL_{m-i}(\mathbb{F})}\big(P_{1,m-i}(\mathbb{F})^0, (s,-s), \alpha_i \otimes \widehat{\tau}_i, \varpi^{-1}_{1,m-1-i}\big) \end{aligned}$$

and that $d' = 1$ if $\mathbb{F}$ is a p-adic field of odd residual characteristic and τ is unramified. (3.50) follows now once we observe that since $\tau \simeq Ind_{P^0_{1,m-1}}^{GL_m(\mathbb{F})} \alpha_1 \otimes \tau_1$ it follows from Lemmas 3.1, 3.2 and 3.3 that

$$\begin{aligned} &C_{\psi}^{\overline{Sp_{2m}(\mathbb{F})}}(\overline{P_{m;0}(\mathbb{F})}, s, \tau, \omega_m'^{-1}) = d'' C_{\psi}^{\overline{SL_2(\mathbb{F})}}\big(\overline{B_{SL_2}(\mathbb{F})}, s, \alpha_1, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\big) \\ &C_{\psi}^{\overline{Sp_{2(m-1)}(\mathbb{F})}}\big(\overline{P_{m-1;0}(\mathbb{F})}, s, \tau_1, \omega_{m-1}'^{-1}\big) C_{\psi}^{GL_m(\mathbb{F})}\big(P^0_{1,m-1}(\mathbb{F}), (s,-s), \alpha_1 \otimes \widehat{\tau}_1, \varpi^{-1}_{1,m-1}\big). \end{aligned}$$

for some $d'' \in \{\pm 1\}$. If $\mathbb{F}$ is a p-adic field of odd residual characteristic and τ is unramified then $d'' = 1$.

From (3.9) and from the known properties of $\gamma(\tau, s, \psi)$ (see [39] or [20]) it follows that for every $1 \leq i \leq m-1$ there exists $d_i \in \{\pm 1\}$ such that

$$C_{\psi}^{GL_{m+1-i}(\mathbb{F})}\big(P^0_{1,m-i}(\mathbb{F}), (s,-s), \alpha_i \otimes \widehat{\tau}_i, \varpi^{-1}_{1,m-i}\big) = d_i \prod_{j=i+1}^{m} \gamma(\alpha_i\alpha_j, 2s, \psi). \tag{3.51}$$

and that $d_i = 1$ provided that $\mathbb{F}$ is a p-adic field and τ is unramified. From (3.46) it follows that

$$\prod_{i=1}^{m} C_{\psi}^{\overline{SL_2(\mathbb{F})}}\Big(\overline{B_{SL_2}(\mathbb{F})}, s, \alpha_i, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\Big) = c'(s)\prod_{i=1}^{m} \frac{\gamma(\alpha_i^2, 2s, \psi)}{\gamma(\alpha_i, s+\frac{1}{2}, \psi)}, \tag{3.52}$$

where $c'_{\mathbb{F}}(s)$ is an exponential function that equals 1 if $\mathbb{F}$ is a p-adic field of odd residual characteristic, ψ is normalized and τ is unramified. Plugging (3.51) and (3.52) into (3.50) we get

$$C_{\psi}^{\overline{Sp_{2m}(\mathbb{F})}}\big(\overline{P_{m;0}(\mathbb{F})}, s, \tau, \omega_m'^{-1}\big) = c(s)\prod_{i=1}^{m}\Big(\frac{\gamma(\alpha_i^2, 2s, \psi)}{\gamma(\alpha_i, s+\frac{1}{2}, \psi)}\prod_{j=i+1}^{m}\gamma(\alpha_i\alpha_j, 2s, \psi)\Big).$$

By definition, (3.49) now follows.

□

4 An analysis of Whittaker coefficients of an Eisenstein series

4.1 Unramified computations

We keep all the notations we used in Lemma 3.4 but we add the following restrictions; we assume that $\mathbb{F}$ is a p-adic field of odd residual characteristic, that ψ is normalized and that χ and μ are unramified. We define the local unramified L-function of $\overline{\sigma} \times \tau$ with respect to ψ:

$$L_{\psi}(\overline{\sigma} \times \tau, s) = \prod_{1 \leq i \leq k}\prod_{1 \leq j \leq m} L(\eta_i\alpha_j, s)L(\eta_i^{-1}\alpha_j, s). \tag{4.1}$$

The subscript ψ is in place due to the dependence on γ_{ψ}^{-1} in the definition of $\overline{\sigma}$.

Similar to the linear case, $I(\chi_{(s)})$ has a one dimensional $\kappa_{2k}\big(Sp_{2k}(\mathbb{O}_{\mathbb{F}})\big)$ invariant subspace. Let $f^0_{\chi_{(s)}}$ be the normalized spherical vector of $I(\chi_{(s)})$, i.e., the unique $\kappa_{2k}\big(Sp_{2k}(\mathbb{O}_{\mathbb{F}})\big)$ invariant vector with the property $f^0_{\chi_{(s)}}(I_{2k}, 1) = 1$. For $f \in I(\chi_{(s)})$ the corresponding Whittaker function is defined by

$$W_f(g) = \frac{1}{C_{\chi_{(s)}}}\int_{Z_{Sp_{2k}}(\mathbb{F})} f\big((J_{2k}u, 1)g\big)\psi^{-1}(u)\,du,$$

where

$$C_{\chi_{(s)}} = \prod_{i=1}^{k}\big(1+\eta_i(\pi)q^{-(s+\frac{1}{2})}\big)\prod_{1 \leq i < j \leq k}\big((1-q^{-1}\eta_i(\pi)\eta_j(\pi)^{-1})(1-q^{-1}\eta_i(\pi)\eta_j(\pi)q^{-2s})\big).$$

With the normalization above, Theorem 1.2 of [6] states that $W_{f^0_{\chi_{(s)}}} = W^0_{\chi_{(s)}}$, where $W^0_{\chi_{(s)}}$ is the normalized spherical function in $W(I(\chi_{(s)}), \psi)$. To be exact we note that in [6], the ψ^{-1}-Whittaker functional is computed. This difference manifests itself only in the $SL_2(\mathbb{F})$ computation presented in page 387 of [6]. Consequently the left product defining $C_{\chi_{(s)}}$ presented in [6] differs slightly from the one given here. Let f^0_{μ} be the normalized spherical vector of $I(\mu)$. Define

$$W_f(g) = \frac{1}{D_{\mu}}\int_{Z_{GL_m}(\mathbb{F})} f(\omega_m ug)\psi^{-1}(u)\,du,$$

where

$$D_{\mu} = \prod_{1 \leq i < j \leq m}(1-q^{-1}\alpha_i\alpha_j^{-1}).$$

Denote by W_{μ}^{0} the normalized spherical function of $W(I(\mu),\psi)$. Theorem 5.4 of [9] states that $W_{f_{\mu}^{0}}=W_{\mu}^{0}$. Let $\lambda_{\tau,\psi}$ and $\lambda_{\overline{\sigma},\psi}$ be Whittaker functionals on $I(\mu)$ and $I(\chi)$ respectively. Note that

$$\lambda_{\tau,\psi}\big(\tau(g)f\big)=W_f(g), \qquad \lambda_{\overline{\sigma},\psi}\big(\overline{\sigma}(s)f\big)=W_f(s). \tag{4.2}$$

Similar to Section 3.2, we realize

$$I_1=Ind_{\overline{P_{m;k}(\mathbb{F})}}^{\overline{Sp_{2n}(\mathbb{F})}}(\gamma_{\psi}^{-1}\otimes\tau_{(s)})\otimes\overline{\sigma}$$

as a space of complex functions on $\overline{Sp_{2n}(\mathbb{F})}\times GL_m(\mathbb{F})\times\overline{Sp_{2k}(\mathbb{F})}$ which are smooth from the right in each argument and which satisfy

$$\begin{aligned}&f\big((j_{m,n}(\widehat{g}),1)i_{k,n}(y)nh,bg_0,(b',\epsilon)y_0\big)\\&\quad=\epsilon\gamma_{\psi}^{-1}\big(\det(g)\det(b')\big)|det(g)|^{s+\frac{n+k+1}{2}}\delta_{B_{GL_m(\mathbb{F})}}(b)\delta_{B_{Sp_{2n}}(\mathbb{F})}(b')\mu(b)\chi(b)f(h,g_0g,y_0y),\end{aligned}$$

For all $g,g_0\in GL_m(\mathbb{F})$, $y,y_0\in\overline{Sp_{2k}(\mathbb{F})}$, $n\in(N_{m;k},1)$, $h\in\overline{Sp_{2n}(\mathbb{F})}$, $b\in B_{GL_m(\mathbb{F})}$, $(b',\epsilon)\in\overline{B_{Sp_{2k}(\mathbb{F})}}$. We realize

$$I_1'=Ind_{\overline{P_{m;k}(\mathbb{F})}}^{\overline{Sp_{2n}(\mathbb{F})}}\big(\gamma_{\psi}^{-1}\otimes W_{(s)}(\tau,\psi)\big)\otimes W(\overline{\sigma},\psi)$$

as we did in Lemma 3.1. An isomorphism $T_1:I_1\to I_1'$ is given by

$$(T_1f)(h,g,y)=\frac{1}{C_{\chi}D_{\mu}}\int_{n_1\in Z_{GL_m(\mathbb{F})}}\int_{n_2\in Z_{Sp_{2k}(\mathbb{F})}}f\big(s,\omega_m n_1 g,(J_{2k}n_2,1)y\big)\psi^{-1}(n_1)\psi^{-1}(n_2).$$

Let $f_{I_1}^{0}\in I_1$ be the unique function such that

$$f_{I_1}^{0}\big((I_{2n},1),I_m,(I_{2k},1)\big)=1$$

and such that for all $o\in\kappa_{2n}\big(Sp_{2n}(\mathbb{O}_{\mathbb{F}})\big)$, $g\in GL_m(\mathbb{F})$, $y\in\overline{Sp_{2k}(\mathbb{F})}$ we have

$$f_{I_1}^{0}(o,g,y)=f_{\mu}^{0}(g)\cdot f_{\chi}^{0}(y).$$

Let $f_{I_1'}^{0}\in I_1'$ be the unique function such that

$$f_{I_1'}^{0}\big((I_{2n},1),I_m,(I_{2k},1)\big)=1$$

and such that for all $o\in\kappa_{2n}\big(Sp_{2n}(\mathbb{O}_{\mathbb{F}})\big)$, $g\in GL_m(\mathbb{F})$, $y\in\overline{Sp_{2k}(\mathbb{F})}$ we have

$$f_{I_1'}^{0}(o,g,y)=W_{\mu}^{0}(g)\cdot W_{\chi}^{0}(y).$$

According (4.2): $T_1(f_{I_1}^{0})=f_{I_1'}^{0}$. We denote by $\lambda'(s,\tau\otimes\overline{\sigma},\psi)$ the Whittaker functional on I_1' constructed in the usual way.

Lemma 4.1.

$$\lambda'(s,\tau\otimes\overline{\sigma},\psi)f_{I_1'}^{0}=\frac{L(\tau,s+\frac{1}{2})}{L(\tau,sym^2,2s+1)L_{\psi}(\overline{\sigma}\otimes\tau,s+1)}. \tag{4.3}$$

Proof. For $f \in I_1$ we have:

$$\begin{aligned} &\lambda'(s,\tau\otimes\overline{\sigma},\psi)\big(T_1(f)\big) \qquad (4.4)\\ &= \int_{N_{m;k}(\mathbb{F})} (T_1 f)\big((j_{m,n}(\omega'_m)u,1), I_m, (I_{2k},1)\big)\psi^{-1}(u)du \\ &= \frac{1}{C_\chi D_\mu}\int_{N_{Sp_{2n}(\mathbb{F})}} f\big((J_{2n}u,1), I_m, (I_{2k},1)\big)\psi^{-1}(u)du. \end{aligned}$$

In particular

$$\lambda'(s,\tau\otimes\overline{\sigma},\psi)f^0_{I'_1} = \frac{1}{C_\chi D_\mu}\int_{N_{Sp_{2n}(\mathbb{F})}} f^0_{I_1}\big((J_{2n}u,1), I_m, (I_{2k},1)\big)\psi^{-1}(u)du.$$

Let $\mu_{(s)}\otimes\chi$ be the character of $T_{Sp_{2n}}(\mathbb{F})$ defined by

$$\big(j_{m,n}(\widehat{t})i_{k,n}(t')\big) \mapsto |\det(t)|^s\mu(t)\chi(t'),$$

where $t \in T_{GL_m}(\mathbb{F})$, $t' \in T_{Sp_{2k}}(\mathbb{F})$. We realize $I(\mu_{(s)}\otimes\chi) = Ind^{\overline{Sp_{2n}(\mathbb{F})}}_{\overline{B_{Sp_{2n}}(\mathbb{F})}}\gamma_\psi^{-1}\otimes\mu_{(s)}\otimes\chi$, in the obvious way. For the Whittaker functional defined on this representation space,

$$\lambda(s,\chi\otimes\mu)(f) = \int_{N_{Sp_{2n}(\mathbb{F})}} f(J_{2n}u,1)\psi^{-1}(u)du,$$

we have

$$\lambda(s,\mu\otimes\chi)(f^0_{I(\mu_s\otimes\chi)}) = C_{\mu_{(s)}\otimes\chi}. \qquad (4.5)$$

The isomorphism $T_2 : I_1 \to I(\mu_{(s)}\otimes\chi)$ defined by

$$(T_2 f)(h) = f\big(h, I_m, (I_{2k},1)\big),$$

whose inverse is given by

$$(T_2^{-1}f)(h,g,y) = \gamma_\psi\big(\det(g)\big)|det(g)|^{\frac{n+k+1}{-2}} f\big((j_{m,n}(\widehat{g}),1)i_{k,n}(y)h\big),$$

has the property:

$$T_2(f^0_{I_1}) = f^0_{I(\mu_s\otimes\chi)}. \qquad (4.6)$$

Using (4.4), (4.5) and (4.6) we observe that

$$C_{\mu_{(s)}\otimes\chi} = \lambda(s,\chi\otimes\mu)(f^0_{I(\chi_s\otimes\mu)}) = \int_{N_{Sp_{2n}(\mathbb{F})}} \big(T_2(f^0_{I_1})\big)\big(J_{2n}u,1\big)\psi^{-1}(u)du =$$

$$\int_{N_{Sp_{2n}(\mathbb{F})}} f^0_{I_1}\big((J_{2n}u,1), I_m, (I_{2k},1)\big)\psi^{-1}(u)du = C_\chi D_\mu\lambda'(s,\tau\otimes\overline{\sigma},\psi)f^0_{I'_1}. \qquad (4.7)$$

Since

$$\frac{C_{\mu_{(s)}\otimes\chi}}{C_\chi D_\mu} = \frac{L(\tau,s+\frac{1}{2})}{L(\tau,sym^2,2s+1)L_\psi(\overline{\sigma}\otimes\tau,s+1)},$$

the lemma is proved. □

Remark: In the case $k=0$ (4.3) reduces to

$$\lambda'(s,\tau\otimes\overline{\sigma},\psi)f^0_{I'_1}=\frac{L(\tau,s+\frac{1}{2})}{L(\tau,sym^2,2s+1)}.$$

This case appears in the introduction of [6].

Let $A^{\mu_{(s)}\otimes\chi}_{j_{m,n}(\omega'^{-1}_m)}$, $A_{j_{m,n}(\omega'^{-1}_m)}$, $A'_{j_{m,n}(\omega'^{-1}_m)}$ be the intertwining operators defined on $I(\mu_{(s)}\otimes\chi)$, I_1, I'_1 respectively.

Lemma 4.2.

$$\lambda'(-s,\widehat{\tau}\otimes\overline{\sigma},\psi)\big(A'_{j_{m,n}(\omega'^{-1}_m)}(f^0_{I'_1})\big) \tag{4.8}$$
$$=\frac{L(\widehat{\tau},-s+\frac{1}{2})L_\psi(\overline{\sigma}\otimes\tau,s)L(\tau,sym^2,2s)}{L(\widehat{\tau},sym^2,-2s+1)L_\psi(\overline{\sigma}\otimes\widehat{\tau},-s+1)L_\psi(\overline{\sigma}\otimes\tau,s+1)L(\tau,sym^2,2s+1)}.$$

Proof. Application of Lemma 3.4 of [6] to the relevant Weyl element proves that

$$A^{\mu_{(s)}\otimes\chi}_{j_{m,n}(\omega'^{-1}_m)}(f^0_{\mu_{(s)}\otimes\chi})=K_{\mu_{(s)}\otimes\chi}f^0_{I(\mu^{-1}_{(-s)}\otimes\chi)},$$

where

$$K_{\mu_{(s)}\otimes\chi}=\frac{L(\overline{\sigma}\otimes\tau,s)}{L(\overline{\sigma}\otimes\tau,s+1)}\frac{L(\tau,sym^2,2s)}{L(\tau,sym^2,2s+1)}.$$

We define

$$\widetilde{T_1}:Ind^{\overline{Sp_{2n}(\mathbb{F})}}_{\overline{P_{m;k}(\mathbb{F})}}(\gamma_\psi^{-1}\otimes\tau^{(-1)}_{(-s)})\otimes\overline{\sigma}\to Ind^{\overline{Sp_{2n}(\mathbb{F})}}_{\overline{P_{m;k}(\mathbb{F})}}\big(\gamma_\psi^{-1}\otimes W_{(-s)}(\tau^{(-1)},\psi)\big)\otimes W(\overline{\sigma},\psi)$$

and

$$\widetilde{T_2}:Ind^{\overline{Sp_{2n}(\mathbb{F})}}_{\overline{P_{m;k}(\mathbb{F})}}(\gamma_\psi^{-1}\otimes\tau^{(-1)}_{(-s)})\otimes\overline{\sigma}\to Ind^{\overline{Sp_{2n}(\mathbb{F})}}_{\overline{B_{Sp_{2n}}(\mathbb{F})}}\gamma_\psi^{-1}\otimes\mu^{-1}_{(-s)}\otimes\chi$$

by analogy with T_1 and T_2. Note that $\widetilde{T_1}$ commutes with $A_{j_{m,n}}$ and that $\widetilde{T_2}$ commutes with $A_{j_{m,n}(\omega'^{-1}_m)}$. Therefore,

$$\widetilde{T_1}\big(A_{j_{m,n}(\omega'^{-1}_m)}(I_1)\big)\subseteq A'_{j_{m,n}(\omega'^{-1}_m)}(I'_1)$$

and

$$\widetilde{T_2}:\big(A_{j_{m,n}(\omega'^{-1}_m)}(I_1)\big)\subseteq A^{\mu_{(s)}\otimes\chi}_{j_{m,n}(\omega'^{-1}_m)}\big(I(\mu_{(s)}\otimes\chi)\big).$$

We denote by $\widetilde{f^0_{I_1}}$ and $\widetilde{f^0_{I'_1}}$ the spherical functions of $A_{j_{m,n}(\omega'^{-1}_m)}(I_1)$ and $A'_{j_{m,n}(\omega'^{-1}_m)}(I'_1)$ respectively. Since T_1, T_2, ,$\widetilde{T_1}$, $\widetilde{T_2}$ map a normalized spherical function to a normalized spherical function and since a straightforward computation shows that

$$\widetilde{T_1}\widetilde{T_2}^{-1}A^{\mu_{(s)}\otimes\chi}_{j_{m,n}(\omega'^{-1}_m)}=A'_{j_{m,n}(\omega'^{-1}_m)}T_1T_2^{-1},$$

we have

$$A'_{j_{m,n}(\omega'^{-1}_m)}(f^0_{I'_1})=\widetilde{T_1}\widetilde{T_2}^{-1}A^{\mu_{(s)}\otimes\chi}_{j_{m,n}(\omega'^{-1}_m)}T_2T_1^{-1}(\widetilde{f^0_{I'_1}})=K_{\mu_{(s)}\otimes\chi}\widetilde{f^0_{I'_1}}.$$

From this and from (4.7) we conclude that

$$\lambda'(-s,\widehat{\tau}\otimes\overline{\sigma},\psi)\big(A'_{j_{m,n}(\omega'^{-1}_m)}(f^0_{I'_1})\big)=K_{\mu_{(s)}\otimes\chi}\lambda'(-s,\widehat{\tau}\otimes\overline{\sigma},\psi)(f^0_{I'_1})=K_{\mu_{(s)}\otimes\chi}\frac{C_{\widehat{\mu}_{-s}\otimes\chi}}{C_\chi D_{\breve{\mu}}}.$$

This finishes the proof of this lemma. ∎

4.2 Crude functional equation.

The argument presented here depends largely on the theory of Eisenstein series developed by Langlands, [27], for reductive groups. Moeglin and Wladspurger extended this theory to coverings groups; see [29]. Throughout this section, $\mathbb{F}$ will denote a number field. For every place ν of $\mathbb{F}$, denote by $\mathbb{F}_\nu$ the completion of $\mathbb{F}$ at ν. Let $\mathbb{A}$ be the adele ring of $\mathbb{F}$. We fix a non-trivial character ψ of $\mathbb{F}\backslash\mathbb{A}$. We write $\psi(x)=\prod_\nu \psi_\nu(x_\nu)$, where for almost all finite ν, ψ_ν is normalized. As in the local case, ψ will also denote a character of $Z_{GL_m}(\mathbb{A})$, $\overline{Z_{Sp_{2n}}(\mathbb{A})}$ and of their subgroups.

Let τ and $\overline{\sigma}$ be a pair of irreducible automorphic cuspidal representations of $GL_m(\mathbb{A})$ and $\overline{Sp_{2k}(\mathbb{A})}$ respectively. Let τ and $\overline{\sigma}$ act in the spaces V_τ and $V_{\overline{\sigma}}$ respectively. We assume that $\overline{\sigma}$ is genuine and globally ψ-generic, i.e., that

$$\int_{n\in Z_{Sp_{2k}}(\mathbb{F})\backslash Z_{Sp_{2k}}(\mathbb{A})} \phi_{\overline{\sigma}}(n,1)\psi^{-1}(n)\,dn \neq 0 \tag{4.9}$$

for some $\phi_{\overline{\sigma}} \in V_{\overline{\sigma}}$. Fix isomorphisms $T_1 : \otimes'_\nu \tau_\nu \to \tau$ and $T_2 : \otimes'_\nu \overline{\sigma}_\nu \to \overline{\sigma}$. Here, for each place ν of $\mathbb{F}$, τ_ν and $\overline{\sigma}_\nu$ are the local components. Outside a finite set of places S, containing the even places and those at infinity, τ_ν and $\overline{\sigma}_\nu$ come together with a chosen spherical vectors $\alpha^0_{\tau_\nu}$ and $\beta^0_{\overline{\sigma}_\nu}$ respectively. We may assume, and in fact do, that ψ_ν is normalized for all $\nu \notin S$.

Let $T = T_1 \otimes T_2$. We identify $(\otimes'_\nu \tau_\nu)\otimes(\otimes'_\nu \overline{\sigma}_\nu)$ with $\otimes'_\nu(\tau_\nu \otimes \overline{\sigma}_\nu)$ in the obvious way. We also identify the image of T with the space of cusp forms on $GL_m(\mathbb{A}) \times \overline{Sp_{2k}(\mathbb{A})}$ generated by the functions $(g,h) \mapsto \phi_\tau(g)\phi_{\overline{\sigma}}(h)$, here $g \in GL_m(\mathbb{A})$, $h \in \overline{Sp_{2k}(\mathbb{A})}$, $\phi_\tau \in V_\tau$ and $\phi_{\overline{\sigma}} \in V_{\overline{\sigma}}$. T then is an isomorphism $T : \otimes'_\nu(\tau_\nu \otimes \overline{\sigma}_\nu) \to \tau \otimes \overline{\sigma}$. Denote for $\phi \in V_{\tau\otimes\overline{\sigma}}$.

$$W_\phi(g,h) = \int_{n_1\in Z_{GL_m}(\mathbb{F})\backslash Z_{GL_m}(\mathbb{A})} \int_{n_2\in Z_{Sp_{2k}}(\mathbb{F})\backslash Z_{Sp_{2k}}(\mathbb{A})} \phi\big(n_1 g,(n_2,1)h\big)\psi^{-1}(n_1)\psi^{-1}(n_2)dn_2\,dn_1.$$

By our assumption (4.9), there exists $\phi \in V_{\tau\otimes\overline{\sigma}}$ such that $W_\phi \neq 0$ is not the zero function. Note that the linear functional

$$\lambda_{\tau\otimes\overline{\sigma},\psi}(\phi) = W_\phi\big(I_m,(I_{2n},1)\big)$$

is a non-trivial (global) ψ-Whittaker functional on $V_{\tau\otimes\overline{\sigma}}$, i.e,

$$\lambda_{\tau\otimes\overline{\sigma},\psi}\big(\tau\otimes\overline{\sigma}(n_1,n_2)\phi\big) = \psi(n_1)\psi(n_2)\lambda_{\tau\otimes\overline{\sigma},\psi}(\phi),$$

for all $(n_1,n_2) \in GL_m(\mathbb{A}) \times \overline{Sp_{2k}(\mathbb{A})}$. The last fact and the local uniqueness of Whittaker functional imply that

Lemma 4.3. *There exists a unique, up to scalar, global ψ-Whittaker functional on $\tau\otimes\overline{\sigma}$:*

$$\phi \mapsto \lambda_{\tau\otimes\overline{\sigma},\psi}(\phi) = \int_{Z_{GL_m}(\mathbb{F})\backslash Z_{GL_m}(\mathbb{A})} \int_{Z_{Sp_{2k}}(\mathbb{F})\backslash Z_{Sp_{2k}}(\mathbb{A})} \phi\big(n_1,(n_2,1)\big)\psi^{-1}(n_1)\psi^{-1}(n_2)dn_2\,dn_1.$$

For each ν let us fix a non-trivial ψ_ν Whittaker functional $\lambda_{\tau_\nu\otimes\overline{\sigma}_\nu,\psi_\nu}$ on $V_{\tau_\nu\otimes\overline{\sigma}_\nu}$ at each place ν, such that if $\tau_\nu\otimes\overline{\sigma}_\nu$ is unramified then

$$\lambda_{\tau_\nu\otimes\overline{\sigma}_\nu,\psi_\nu}(\alpha^0_{\tau_\nu}\otimes\beta^0_{\overline{\sigma}_\nu}) = 1.$$

Then, by normalizing $\lambda_{\tau_\nu\otimes\overline{\sigma}_\nu,\psi_\nu}$ at one ramified place, we have

$$\lambda_{\tau\otimes\overline{\sigma},\psi}(\phi)=\prod_\nu \lambda_{\tau_\nu\otimes\overline{\sigma}_\nu,\psi_\nu}(v_{\tau_\nu}\otimes v_{\overline{\sigma}_\nu}),$$

where $\phi=T\big(\bigotimes_\nu(v_{\tau_\nu}\otimes v_{\overline{\sigma}_\nu})\big)$, i.e., ϕ corresponds to a pure tensor.

We shall realize each local representation

$$I_\nu(\tau_{\nu(s)},\overline{\sigma}_\nu)=Ind_{P_{m;k}(\mathbb{F}_\nu)}^{\overline{Sp_{2n}(\mathbb{F}_\nu)}}(\gamma_{\psi_\nu}^{-1}\otimes\tau_{\nu(s)})\otimes\overline{\sigma}_\nu$$

as the space of smooth from the right functions

$$f:\overline{Sp_{2n}(\mathbb{F}_\nu)}\to V_{\tau_\nu}\otimes V_{\overline{\sigma}_\nu}$$

satisfying

$$f\big((j_{m,n}(\widehat{g}),1)i_{k,n}(y)nh\big)=\gamma_{\psi_\nu}^{-1}(g)|\det(g)|_\nu^{s+\frac{n+k+1}{2}}\tau_\nu(g)\otimes\overline{\sigma}_\nu(y)f(h)$$

for all $g\in GL_m(\mathbb{F}_\nu)$, $y\in\overline{Sp_{2k}(\mathbb{F}_\nu)}$, $n\in\big(N_{m,k}(\mathbb{F}_\nu),1\big)$, $h\in\overline{Sp_{2n}(\mathbb{F}_\nu)}$. For each place where τ_ν and $\overline{\sigma}_\nu$ are unramified we define $f_\nu^{0,s}\in I_\nu(\tau_{\nu(s)},\overline{\sigma}_\nu)$ to be the normalized spherical function, namely, $f_\nu^{0,0}(I_{2n},1)=\alpha_{\tau_\nu}^0\otimes\beta_{\overline{\sigma}_\nu}^0$. We shall realize the global representation

$$I(\tau_{(s)},\overline{\sigma})=Ind_{P_{m;k}(\mathbb{A})}^{\overline{Sp_{2n}(\mathbb{A})}}(\gamma_\psi^{-1}\otimes\tau_{(s)})\otimes\overline{\sigma}$$

as a space of functions

$$f:\overline{Sp_{2n}(\mathbb{A})}\times GL_m(\mathbb{A})\times\overline{Sp_{2k}(\mathbb{A})}\to V_{\tau\otimes\overline{\sigma}}$$

smooth from the right in the first variable such that

$$f\big((j_{m,n}(\widehat{g}),1)i_{k,n}(y)nh,g_0,y_0\big)=\gamma_\psi^{-1}(g)|\det(g)|^{s+\frac{n+k+1}{2}}\tau(g)\otimes\overline{\sigma}(y)f(h,g_0g,y_0y),$$

for all $g,g_0\in GL_m(\mathbb{A})$, $y,y_0\in\overline{Sp_{2k}(\mathbb{A})}$, $n\in\big(N_{m,k}(\mathbb{A}),1\big)$, $h\in\overline{Sp_{2n}(\mathbb{A})}$, and such that for all $h\in\overline{Sp_{2n}(\mathbb{A})}$ the map $(g,y)\mapsto f(h,g,y)$ lies in $V_{\tau\otimes\overline{\sigma}}$.

$I(\tau_{(s)},\overline{\sigma})$ is spanned by functions of the form $f(g)=T\big((\otimes_\nu f_\nu(g_\nu)\big)$, where $f_\nu\in I_\nu(\tau_{\nu(s)},\overline{\sigma}_\nu)$ and for almost all ν: $f_\nu=f_\nu^0$ (for a fixed g, $f(g)$ is a cuspidal automorphic form corresponding to a pure tensor).

We note that for $(p,1)\in\big(P_{m;k}(\mathbb{F}),1\big)$ we have $f\big((p,1)g\big)=f(g)$. This follows from the fact that $\prod_\nu\gamma_{\psi_\nu}^{-1}(a)=1$ for all $a\in\mathbb{F}^*$. Hence, it makes sense to consider Eisenstein series: For a holomorphic smooth section $f_s\in I(\tau_{(s)},\overline{\sigma})$ define

$$E(f_s,g)=\sum_{\gamma\in P_{m;k}(\mathbb{F})\backslash Sp_{2n}(\mathbb{F})}f_s\big((\gamma,1)g,I_m,(I_{2k},1)\big)$$

It is known that the series in the right-hand side converges absolutely for $Re(s)>>0$, see Section II.1.5 of [29] and that it has a meromorphic continuation to the whole complex plane, see Section IV.1.8 of [29]. We continue to denote this continuation by $E(f_s,g)$.

We introduce the ψ- Whittaker coefficient

$$E_\psi(f_s, g) = \int_{u \in Z_{Sp_{2n}(\mathbb{F})} \backslash Z_{Sp_{2n}}(\mathbb{A})} E\big(f_s, (u,1)g\big)\psi^{-1}(u)\, du.$$

Note that no question of convergence arises here since $Z_{Sp_{2n}}(\mathbb{F})\backslash Z_{Sp_{2n}}(\mathbb{A})$ is compact. It is also clear that $E_\psi(f_s, g)$ is meromorphic in the whole complex plane.

Lemma 4.4. *$\prod_{\nu \notin S} L_{\psi_\nu}(\overline{\sigma}_\nu \otimes \tau_\nu, s)$ converges absolutely for $Re(s) >> 0$. This product has a meromorphic continuation on $\mathbb{C}$. We shall denote this continuation by $L^S_\psi(\overline{\sigma} \otimes \tau, s)$. We have:*

$$E_\psi\big(f_s, (I_{2n}, 1)\big) = \frac{L^S(\tau, s + \frac{1}{2})}{L^S(\tau, sym^2, 2s+1) L^S_\psi(\overline{\sigma} \otimes \tau, s+1)} \prod_{\nu \in S} \lambda\big(s, \tau_\nu \otimes \overline{\sigma}_\nu, \psi\big)(f_\nu). \qquad (4.10)$$

Recall that $\prod_{\nu \notin S} L(\tau_\nu, s)$ and $\prod_{\nu \notin S} L(\tau_\nu, sym^2, s)$ converge absolutely for $Re(s) >> 0$ and that these products have a meromorphic continuation on $\mathbb{C}$; see [26]. These continuations are denoted by $L^S(\tau, s)$ and $L^S(\tau, sym^2, s)$ respectively.

Proof. Recall that in Section 1.2 we have denoted by $W_{Sp_{2n}}$ the Weyl group of $Sp_{2n}(\mathbb{F})$. We now and denote by $W_{M_{m;k}}$ the Weyl group of $M_{m;k}$. We fix Ω, a complete set of representatives of $W_{M_{m;k}} \backslash W_{Sp_{2n}}$. Recall the Bruhat decomposition

$$Sp_{2n}(\mathbb{F}) = \dot{\bigcup_{w \in \Omega}} P_{m,k}(\mathbb{F}) w B_{Sp_{2n}}(\mathbb{F}).$$

Clearly for $w \in \Omega$:

$$P_{m,k}(\mathbb{F}) w B_{Sp_{2n}}(\mathbb{F}) = P_{m,k}(\mathbb{F}) w Z_{Sp_{2n}}(\mathbb{F}).$$

Also, for $w \in \Omega$, $p_1, p_2 \in P_{m;k}(\mathbb{F})$, $u_1, u_2 \in Z_{Sp_{2n}}(\mathbb{F})$ we have: If $p_1 w u_1 = p_2 w u_2$ then

$$u_2 u_1^{-1} \in Z_w(\mathbb{F}) = Z_{Sp_{2n}}(F) \cap w^{-1} Z_{Sp_{2n}}(\mathbb{F}) w.$$

Thus, every element γ of $Sp_{2n}(\mathbb{F})$ can be expressed as $g = pwu$, where $p \in P_{m,k}(\mathbb{F})$ and $w \in \Omega$ are determined uniquely and $u \in Z_{Sp_{2n}}(\mathbb{F})$ is determined uniquely modulo $Z_w(\mathbb{F})$ from the left (note that if $W_{M_{m;k}} w_1 = W_{M_{m;k}} w_2$ it does not follow that $Z_{w_1}(\mathbb{F}) = Z_{w_1}(\mathbb{F})$. This is why we started from fixing Ω). Thus, for $f_s \in I(\tau_{(s)}, \overline{\sigma})$ and $Re(s) >> 0$ we have:

$$\begin{aligned}
&E_\psi\big(f_s, (I_{2n}, 1)\big) &(4.11)\\
&= \int_{u \in Z_{Sp_{2n}}(\mathbb{F}) \backslash Z_{Sp_{2n}}(\mathbb{A})} \sum_{\gamma \in P_{m;k}(\mathbb{F}) \backslash Sp_{2n}(\mathbb{F})} f_s\big((\gamma u, 1), I_m, (I_{2k}, 1)\big)\psi^{-1}(u) du \\
&= \int_{u \in Z_{Sp_{2n}}(\mathbb{F}) \backslash Z_{Sp_{2n}}(\mathbb{A})} \psi^{-1}(u) \sum_{w \in \Omega} \sum_{n \in Z_w(\mathbb{F}) \backslash Z_{Sp_{2n}}(\mathbb{F})} f_s\big((wnu, 1), I_m, (I_{2k}, 1)\big) du \\
&= \sum_{w \in \Omega} \int_{u \in Z_{Sp_{2n}}(\mathbb{F}) \backslash Z_{Sp_{2n}}(\mathbb{A})} \psi^{-1}(u) \sum_{n \in Z_w(\mathbb{F}) \backslash Z_{Sp_{2n}}(\mathbb{F})} f_s\big((wnu, 1), I_m, (I_{2k}, 1)\big) du \\
&= \sum_{w \in \Omega} \int_{u \in Z_w(\mathbb{F}) \backslash Z_{Sp_{2n}}(\mathbb{A})} \psi^{-1}(u) f_s\big(wu, I_m, (I_{2k}, 1)\big) du.
\end{aligned}$$

We now choose $w_0 = w_l'(m;k)$ (see (3.1)) as the representative of J_{2n} in Ω. We note that $Z_{w_0} = Z_{Sp_{2n}} \cap M_{n,k}$. By the same argument used in page 182 of [35] one finds that

$$\int_{u \in Z_w(\mathbb{F}) \backslash Z_{Sp_{2n}}(\mathbb{A})} \psi^{-1}(u) f_s\big(wu, I_m, (I_{2k}, 1)\big) = 0, \tag{4.12}$$

for all $w \in \Omega$, $w \neq w_0$. Thus, from (4.11) we have:

$$\begin{aligned} & E_\psi\big(f_s, (I_{2n}, 1)\big) \\ &= \int_{u \in Z_{Sp_{2n}}(\mathbb{F}) \cap M_{n,k}(\mathbb{F}) \backslash Z_{Sp_{2n}}(\mathbb{A})} \psi^{-1}(u) f_s\big(w_0 u, I_m, (I_{2k}, 1)\big) du \\ &= \int_{u \in Z_{Sp_{2n}}(\mathbb{F}) \cap M_{n,k}(\mathbb{F}) \backslash \big(Z_{Sp_{2n}}(\mathbb{A}) \cap M_{n,k}(\mathbb{A})\big) N_{m;k}(\mathbb{A})} \psi^{-1}(u) f_s\big(w_0 u, I_m, (I_{2k}, 1)\big) du \\ &= \int_{n \in N_{m;k}(\mathbb{A})} \psi^{-1}(n) \int_{u \in Z_{Sp_{2n}}(\mathbb{F}) \cap M_{n,k}(\mathbb{F}) \backslash \big(Z_{Sp_{2n}}(\mathbb{A}) \cap M_{n,k}(\mathbb{A})\big)} \psi^{-1}(u) f_s\big(w_0 u n, I_m, (I_{2k}, 1)\big) du\, dn \\ &= \int_{n \in N_{m;k}(\mathbb{A})} \psi^{-1}(n) \int_{n_1 \in Z_{GL_m}(\mathbb{F}) \backslash Z_{GL_m}(\mathbb{A})} \int_{n_2 \in Z_{Sp_{2k}}(\mathbb{F}) \backslash Z_{Sp_{2k}}(\mathbb{A})} \psi^{-1}(n_2) \psi^{-1}(n_1) f_s\big(w_0 n, n_1, (n_2, 1)\big) du\, dn. \end{aligned} \tag{4.13}$$

Recall that S is a finite set of places of $\mathbb{F}$, such that for all $\nu \notin S$, ν is finite and odd, $\tau_\nu \otimes \overline{\sigma}_\nu$ is unramified and ψ_ν is normalized. Assume now that f_s corresponds to the following pure tensor of holomorphic smooth sections, $f_s(g) = T\big(\otimes_\nu f_{s,\nu}(g_\nu)\big)$, where $f_{s,\nu} \in V_{I_\nu(\tau_{\nu(s)}, \overline{\sigma})}$ and for $\nu \notin S$: $f_{s,\nu} = f_\nu^{0,s}$. By Lemma 4.3, we have

$$E_\psi\big(f_s, (I_{2n}, 1)\big) = \prod_\nu \int_{n \in N_{m;k}(\mathbb{F}_\nu)} \lambda_{\tau_\nu \otimes \overline{\sigma}_\nu, \psi_\nu}\Big(\big(\rho(w_0 n) f_{s\nu}\big)\Big) \psi^{-1}(n)\, dn = \prod_\nu \lambda\big(s, \tau_\nu \otimes \overline{\sigma}_\nu, \psi\big)(f_{s\nu}) \tag{4.14}$$

(see Section 3.3 of [34] for the general arguments about Eulerian integrals). The last equation should be understood as a global metaplectic analog to Rodier's local algebraic heredity. (4.14) and (4.3) imply that for $Re(s) >> 0$

$$E_\psi\big(f_s, (I_{2n}, 1)\big) = \frac{L^S(\tau, s + \frac{1}{2})}{L^S(\tau, sym^2, 2s+1) \prod_{\nu \notin S} L_\psi(\overline{\sigma}_\nu \otimes \tau_\nu, s+1)} \prod_{\nu \in S} \lambda\big(s, \tau_\nu \otimes \overline{\sigma}_\nu, \psi\big)(f_{s,\nu}). \tag{4.15}$$

We claim that we may choose f_s as above such that for all $\nu \in S$

$$\lambda\big(s, \tau_\nu \otimes \overline{\sigma}_\nu, \psi\big)(f_{s,\nu}) = 1 \tag{4.16}$$

for all $s \in \mathbb{C}$. Indeed, we choose $f_{s,\nu}$ which is supported on the open Bruhat cell

$$\overline{P_{m;k}(\mathbb{F}_\nu)}\big(w_l'(m;k) Z_{m;k}(\mathbb{F}_\nu), 1\big)$$

which satisfies

$$f_{s,\nu}(w_l'(m;k)z, 1)(g, y) = \phi(z) W_{\tau_\nu}(g) W_{\overline{\sigma}_\nu}(y).$$

Here $z \in Z_{m;k}(\mathbb{F}_\nu)$, $g \in GL_m(\mathbb{F}_\nu)$, $y \in \overline{Sp_{2k}(\mathbb{F}_\nu)}$, ϕ is a properly chosen smooth compactly supported function on $Z_{m;k}(\mathbb{F}_\nu)$ and $W_{\tau_\nu}(I_m) = W\overline{\sigma}_\nu(I_k, 1) = 1$. We have

$$\lambda\big(s, \tau_\nu \otimes \overline{\sigma}_\nu, \psi\big)(f_\nu) = \int_{Z_{m;k}(\mathbb{F}_\nu)} \phi(z) \psi^{-1}(z_{n,n})\, dn.$$

We now choose ϕ such that (4.16) holds. For such a choice we have

$$E_\psi\big(f_s,(I_{2n},1)\big)=\frac{L^S(\tau,s+\frac{1}{2})}{L^S(\tau,sym^2,2s+1)\prod_{\nu\notin S}L_\psi(\overline{\sigma}_\nu\otimes\tau_\nu,s+1)}.$$

The absolute convergence of $\prod_{\nu\notin S}L_\psi(\overline{\sigma}_\nu\otimes\tau_\nu,s)$ for $Re(s)>>0$ is clear now. Furthermore, the fact that this product has a meromorphic continuation to $\mathbb{C}$ follows from the meromorphic continuations of $E_\psi\big(f_s,(I_{2n},1)\big)$, $L^S(\tau,s)$ and $L^S(\tau,sym^2,s)$. Finally, the validity of (4.10) for all s follows from (4.15).

□

Theorem 4.1.

$$\prod_{\nu\in S}\gamma(\overline{\sigma}_\nu\times\tau_\nu,s,\psi_\nu)=\frac{L^S_\psi(\overline{\sigma}\times\tau,s)}{L^S_\psi(\overline{\sigma}\times\widehat{\tau},1-s)}. \tag{4.17}$$

Proof. The global functional equation for the Eisenstein series states that

$$E(f_s,g)=E\big(A(f_s,g)\big),$$

where A is the global intertwining operator; see Section IV.1.10 of [29]. We compute the ψ-Whittaker coefficient of both sides of the last equation. By (4.8) we have

$$\begin{aligned}&\frac{L^S(\tau,s+\frac{1}{2})}{L^S(\tau,sym^2,2s+1)L^S_\psi(\overline{\sigma}\otimes\tau,s+1)}\prod_{\nu\in S}\lambda\big(s,\tau_\nu\otimes\overline{\sigma}_\nu,\psi\big)(f_{s_\nu})\\&=\frac{L^S(\widehat{\tau},-s+\frac{1}{2})L^S_\psi(\overline{\sigma}\otimes\tau,s)L^S(\tau,sym^2,2s)}{L^S(\widehat{\tau},sym^2,-2s+1)L^S_\psi(\overline{\sigma}\otimes\widehat{\tau},-s+1)L^S_\psi(\overline{\sigma}\otimes\tau,s+1)L^S(\tau,sym^2,2s+1)}\\&\prod_{\nu\in S}\lambda\big(-s,\widehat{\tau_\nu}\otimes\overline{\sigma}_\nu,\psi\big)\big(A_{j_{m,n}(\omega'^{-1}_m)}(f^0_{I'_1})\big),\end{aligned}$$

Or equivalently, by the definition of the local coefficients

$$\prod_{\nu\in S}C_{\psi_\nu}^{\overline{Sp_{2n}(\mathbb{F}_\nu)}}\big(\overline{P_{m;k}(\mathbb{F}_\nu)},s,\tau_\nu\otimes\overline{\sigma}_\nu,j_{m,n}(\omega'^{-1}_m)\big)=\frac{L^S(\widehat{\tau},-s+\frac{1}{2})L^S(\tau,sym^2,2s)L^S_\psi(\overline{\sigma}\otimes\tau,s)}{L^S(\tau,s+\frac{1}{2})L^S(\widehat{\tau},sym^2,-2s+1)L^S_\psi(\overline{\sigma}\otimes\widehat{\tau},1-s)}. \tag{4.18}$$

In particular, for $k=0$

$$\prod_{\nu\in S}C_{\psi_\nu}^{\overline{Sp_{2m}(\mathbb{F}_\nu)}}\big(\overline{P_{m;0}(\mathbb{F}_\nu)},s,\tau_\nu,j_{m,n}(\omega'^{-1}_m)\big)=\frac{L^S(\widehat{\tau},-s+\frac{1}{2})L^S(\tau,sym^2,2s)}{L^S(\tau,s+\frac{1}{2})L^S(\widehat{\tau},sym^2,-2s+1)}. \tag{4.19}$$

Dividing (4.18) and (4.19) we get (4.17). □

4.3 Computation of $C_\psi^{\overline{Sp_{2m}(\mathbb{F})}}\big(\overline{P_{m;0}(\mathbb{F})},s,\tau,\omega'^{-1}_m\big)$ for generic representations

Theorem 4.2. *Let $\mathbb{F}$ be a p-adic field and let τ be an irreducible admissible supercuspidal representation of $GL_m(\mathbb{F})$. There exists an exponential function $c_\mathbb{F}(s)$ such that*

$$C_\psi^{\overline{Sp_{2m}(\mathbb{F})}}\big(\overline{P_{m;0}(\mathbb{F})},s,\tau,\omega'^{-1}_m\big)=c_\mathbb{F}(s)\frac{\gamma(\tau,sym^2,2s,\psi)}{\gamma(\tau,s+\frac{1}{2},\psi)}.$$

Proof. Since τ is supercuspidal it is also generic. Proposition 5.1 of [42] implies now that there exists a number field $\mathbb{K}$, a non-degenerate character $\widetilde{\psi}$ of $Z_{GL_n(\mathbb{K})} \backslash Z_{GL_n(\mathbb{A})}$ and an irreducible cuspidal representation $\pi \simeq \otimes_\nu \pi_\nu$ of $GL_n(\mathbb{A})$ such that
1. $\mathbb{K}_{\nu_0} = \mathbb{F}$ for some place ν_0 of $\mathbb{K}$.
2. $\widetilde{\psi}_{\nu_0} = \psi$.
3. $\pi_{\nu_0} = \tau$.
4. For any finite place $\nu \neq \nu_0$ of $\mathbb{K}$, π_ν is unramified.
Define S to be the finite set of places of $\mathbb{K}$ which consists of ν_0, of the infinite and even places and of the finite places where $\widetilde{\psi}_\nu$ is not normalized. From the fourth part of Theorem 3.5 of [41] it follows that

$$\prod_{\nu \in S} \gamma_{\mathbb{F}_\nu}(\pi_\nu, s, \psi_\nu) = \frac{L^S(\pi, s)}{L^S(\widehat{\pi}, 1-s)}$$

and that

$$\prod_{\nu \in S} \gamma_{\mathbb{F}_\nu}(\pi_\nu, s, sym^2, \psi_\nu) = \frac{L^S(\pi, sym^2, s)}{L^S(\widehat{\pi}, sym^2, 1-s)}.$$

Therefore, (4.19) can be written as

$$\prod_{\nu \in S} C_{\psi_\nu}^{\overline{Sp_{2m}(\mathbb{F}_\nu)}}\big(\overline{P_{m;0}(\mathbb{F}_\nu)}, s, \pi_\nu, j_{m,n}(\omega_m'^{-1})\big) = \prod_{\nu \in S} \frac{\gamma(\pi_\nu, sym^2, 2s, \psi)}{\gamma(\pi_\nu, s+\frac{1}{2}, \psi)}.$$

This implies that this theorem will be proven once we show that for all $\nu \in S$, $\nu \neq \nu_0$, there exists an exponential function $c_\nu(s)$ such that

$$C_{\psi_\nu}^{\overline{Sp_{2m}(\mathbb{F}_\nu)}}\big(\overline{P_{m;0}(\mathbb{F}_\nu)}, s, \pi_\nu, j_{m,n}(\omega_m'^{-1})\big) = c_\nu(s) \frac{\gamma_{\mathbb{F}_\nu}(\pi_\nu, sym^2, 2s, \psi)}{\gamma_{\mathbb{F}_\nu}(\pi_\nu, s+\frac{1}{2}, \psi)}.$$

Since for all for all $\nu \in S$, $\nu \neq \nu_0$, π_ν is the generic constituent of a principal series representation series, this follows from Lemma 3.6. $\square$

Theorem 4.3. *Let $\mathbb{F}$ be a p-adic field and let τ be an irreducible admissible generic representation of $GL_m(\mathbb{F})$. There exists an exponential function $c_{\mathbb{F}}(s)$ such that*

$$C_{\psi}^{\overline{Sp_{2m}(\mathbb{F})}}\big(\overline{P_{m;0}(\mathbb{F})}, s, \tau, \omega_m'^{-1}\big) = c_{\mathbb{F}}(s) \frac{\gamma(\tau, sym^2, 2s, \psi)}{\gamma(\tau, s+\frac{1}{2}, \psi)}. \tag{4.20}$$

Proof. By Chapter II of [7], τ may be realized as a sub-representation of

$$\tau' = Ind_{Q(\mathbb{F})}^{GL_m(\mathbb{F})}(\otimes_{i=1}^r \tau_i),$$

where $Q(\mathbb{F})$ is a standard parabolic subgroup of $GL_m(\mathbb{F})$ whose Levi part, $M(\mathbb{F})$, is isomorphic to

$$GL_{n_1}(\mathbb{F}) \times GL_{n_2}(\mathbb{F}) \ldots \times GL_{n_r}(\mathbb{F})$$

and where for all $1 \leq i \leq r$, τ_i is an irreducible admissible supercuspidal representation of $GL_{n_i}(\mathbb{F})$. Since for all $1 \leq i \leq r$, τ_i has a unique Whittaker model it follows from the

heredity property of the Whittaker model that τ' has a unique Whittaker model; see [33]. This implies that τ is the generic constituent of τ'. Hence,

$$C_{\psi}^{\overline{Sp_{2m}(\mathbb{F})}}\big(\overline{P_{m;0}(\mathbb{F})}, s, \tau, \omega_m'^{-1}\big) = C_{\psi}^{\overline{Sp_{2m}(\mathbb{F})}}\big(\overline{P_{m;0}(\mathbb{F})}, s, \tau', \omega_m'^{-1}\big).$$

Thus, it is sufficient to prove (4.20) replacing τ with τ'. By similar arguments to those used in Lemmas 3.4 and 3.6 one shows that there exists $d \in \{\pm 1\}$ such that

$$C_{\psi}^{\overline{Sp_{2m}(\mathbb{F})}}\big(\overline{P_{m;0}(\mathbb{F})}, s, \tau', \omega_m'^{-1}\big) = d\prod_{i=1}^{r}\Big(C_{\psi}^{\overline{Sp_{n_i}(\mathbb{F})}}\big(\overline{P_{n_i;0}(\mathbb{F})}, s, \tau_i, \omega_{n_i}'^{-1}\big)\prod_{j=i+1}^{r}\gamma(\tau_i \times \tau_j, 2s, \psi)\Big).$$

Since for $1 \leq i \leq r$, τ_i are irreducible admissible supercuspidal representations it follows from Theorem 4.2 that there exits an exponential factor, $c_{\mathbb{F}}(s)$, such that

$$C_{\psi}^{\overline{Sp_{2m}(\mathbb{F})}}\big(\overline{P_{m;0}(\mathbb{F})}, s, \tau', \omega_m'^{-1}\big) = c_{\mathbb{F}}(s)\prod_{i=1}^{r}\Big(\frac{\gamma(\tau_i, sym^2, 2s, \psi)}{\gamma(\tau_i, s+\frac{1}{2}, \psi)}\prod_{j=i+1}^{r}\gamma(\tau_i \times \tau_j, 2s, \psi)\Big).$$

Using the known multiplicativity of the symmetric square γ-factor (see Part 3 of Theorem 3.5 of [41]) we finish.

□

5 Irreducibility theorems

In this section we shall assume that $\mathbb{F}$ is a p-adic field. We shall use various definitions and notation given in the previous sections. Among them are π^w and the notion of a regular and of a singular representation (see Section 2.2), $W_{P_{\vec{t}}}(\mathbb{F})$ (see Section 1.2), $W(\pi)$ (see (2.8)) and $\Sigma_{P_{\vec{t}}(\mathbb{F})}$ (see Section 2.4).

Lemma 5.1. *Let β_1 and β_2 be two characters of $\mathbb{F}^*$. Denote $\beta = \beta_1\beta_2^{-1}$. If $\mathbb{F}$ is a p-adic field then*

$$C_{\psi}^{GL_2(\mathbb{F})}\big(B_{GL_2}(\mathbb{F}), (s_1, s_2), \beta_1 \otimes \beta_2, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\big) = \frac{\beta(\pi^{m(\beta)-n})q^{(n-m(\beta)n)(s_1-s_2)}}{G(\beta, \psi^{-1})}\frac{L\big(\beta^{-1}, 1-(s_1-s_2)\big)}{L\big(\beta, (s_1-s_2)\big)}, \tag{5.1}$$

where n is the conductor of ψ. If $\mathbb{F} = \mathbb{R}$ then

$$C_{\psi}^{GL_2(\mathbb{R})}\big(B_{GL_2}(\mathbb{R}), (s_1, s_2), \beta_1 \otimes \beta_2, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\big) = \beta_2(-1)e^{\frac{-i\pi\beta(-1)}{2}}(2\pi)^{-(s_1-s_2)}\frac{L\big(\beta^{-1}, 1-(s_1-s_2)\big)}{L\big(\beta, (s_1-s_2)\big)}. \tag{5.2}$$

Proof. See Lemma 2.1 of [38] for the p-adic case and see Theorem 3.1 of [40] for the real case.

□

Theorem 5.1. *Let $\alpha_1, \ldots, \alpha_n$ be n unitary characters of $\mathbb{F}^*$. Let α be the character of $\overline{T_{Sp_{2n}}(\mathbb{F})}$ defined by*

$$\big(diag(a_1, \ldots, a_n, a^{-1}, \ldots, a_n^{-1}), \epsilon\big) \mapsto \epsilon\gamma_\psi^{-1}(\prod_{i=1}^{n} a_i) \prod_{i=1}^{n} \alpha_i(a_i).$$

Then $I(\alpha)$ is irreducible.

Proof. Since α is unitary, $I(\alpha)$ is also unitary. Therefore, the irreducibility of $I(\alpha)$ will follow once we show that

$$Hom_{\overline{Sp_{2n}(\mathbb{F})}}\big(I(\alpha), I(\alpha)\big) \simeq \mathbb{C}.$$

For $1 \leq i < j \leq n$ define

$$w_{(i,j)} = \begin{pmatrix} I_{i-1} & & & & \\ & & & 1 & \\ & & I_{j-i-2} & & \\ & 1 & & & \\ & & & & I_{n-j+1} \end{pmatrix}$$

and

$$w'_{(i,j)} = \tau_{\{i,j\}} w_{(i,j)}.$$

A routine exercise shows that

$$\Sigma_{B_{Sp_{2n}}(\mathbb{F})} = \{w_{(i,j)} \mid 1 \leq i < j \leq n\} \cup \{\tau_{\{r\}} \mid 1 \leq r \leq n\} \cup \{w'_{(i,j)} \mid 1 \leq < i < j \leq n\}. \quad (5.3)$$

Note that,

$$w_{(i,j)} diag(\widehat{a_1, \ldots}, a_n) w_{(i,j)}^{-1} = diag(a_1, \ldots, a_{i-1}, a_j, a_{i+1}, \widehat{\ldots}, a_{j-1}, a_i, a + j + 1, \ldots, a_n),$$

$$w_{\{r\}} diag(\widehat{a_1, \ldots}, a_n) w_{\{r\}}^{-1} = diag(a_1, \ldots, a_{r-1}, \widehat{a_r^{-1}}, a_{r+1}, \ldots, a_n)$$

and that

$$w'_{(i,j)} diag(\widehat{a_1, \ldots}, a_n) w'^{-1}_{(i,j)} = diag(a_1, \ldots, a_{i-1}, a_j^{-1}, a_{i+1}, \widehat{\ldots}, a_{j-1}, a_i^{-1}, a + j + 1, \ldots, a_n).$$

Therefore $w_{(i,j)} \in W(\alpha) \Leftrightarrow \alpha_i = \alpha_j$, $\tau_{\{r\}} \in W(\alpha) \Leftrightarrow \alpha_r$ is quadratic and $w'_{(i,j)} \in W(\alpha) \Leftrightarrow \alpha_i = \alpha_j^{-1}$. Furthermore, $W(\alpha)$ is generated by $\Sigma_{B_{Sp_{2n}(\mathbb{F})}} \cap W(\alpha)$. Thus, using Theorem 2.3 and (3.7), the proof of this theorem amounts to showing that

$$C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}(\overline{P_{\vec{t}}(\mathbb{F})}, \vec{s}, (\otimes_{i=1}^{n}\alpha_i)\otimes, w) C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}(\overline{P_{\vec{t}}(\mathbb{F})}, \vec{s^w}, (\otimes_{i=1}^{n}\alpha_i)^w, w^{-1}) = 0 \quad (5.4)$$

for all $w \in \Sigma_{B_{Sp_{2n}}(\mathbb{F})} \cap W(\alpha)$. We prove it for each of the three types in the right-hand side of (5.3).

Suppose that $w_{(i,j)} \in W(\alpha)$. We write:

$$w_{(i,j)} = w_{(i,i+1)} w_{(i+1,i+2)} \ldots w_{(j-1,j)} w_{(j-2,j-1)} \ldots w_{(i+1,i)}. \quad (5.5)$$

We claim that the expression in the right-hand side of (5.5) is reduced. Indeed,

$$\Sigma'_{B_{Sp_{2n}}(\mathbb{F})}=\{w_{(i,+1)}\mid 1\leq< i<n\}\cup\{\tau_{\{1\}}\}\subset\Sigma_{B_{Sp_{2n}(\mathbb{F})}}$$

is the subset of reflections corresponding to simple roots and the length of $w_{(i,j)}$ is $2(j-i)-1$ (any claim about the length of a given Weyl element w may be verified by counting the number of positive root subgroups mapped by w to negative root subgroups). Thus, we may use the same argument as in Lemma 3.2 and conclude that there exists $c\in\{\pm 1\}$ such that

$$C_{\psi}^{\overline{Sp_{2n}(\mathbb{F})}}\big(B_{Sp_{2n}}(\mathbb{F}),\overrightarrow{s},(\otimes_{i=1}^{n}\alpha_i)),w_{(i,j)}\big)= \\ c\Big(\prod_{k=i}^{j-2}f_k(s)f'_k(s)\Big)C_{\psi}^{\overline{Sp_{2n}(\mathbb{F})}}\big(B_{Sp_{2n}}(\mathbb{F}),\overrightarrow{s}^{\widetilde{w}},(\otimes_{i=1}^{n}\alpha_i)^{\widetilde{w}},w_{(j-1,j)}\big), \tag{5.6}$$

where

$$f_k(\overrightarrow{s})=C_{\psi}^{\overline{Sp_{2n}(\mathbb{F})}}\big(B_{Sp_{2n}}(\mathbb{F}),\overrightarrow{s}^{w_{(k)}},(\otimes_{i=1}^{n}\alpha_i)^{w_{(k)}},w_{(k,k+1)}\big)$$
$$f'_k(\overrightarrow{s})=C_{\psi}^{\overline{Sp_{2n}(\mathbb{F})}}\big(B_{Sp_{2n}}(\mathbb{F}),\overrightarrow{s}^{w'_{(k)}},(\otimes_{i=1}^{n}\alpha_i)^{w'_{(k)}},w_{(k,k+1)}\big),$$

where

$$\begin{aligned} w_{(k)} &= w_{(i,i+1)}w_{(i+1,i+2)}\cdots w_{(k-1,k)},\\ \widetilde{w} &= w_{(i,i+1)}w_{(i+1,i+2)}\cdots w_{(j-1,j)}w_{(j-2,j-1)},\\ w'_{(k)} &= w_{(i,i+1)}w_{(i+1,i+2)}\cdots w_{(j-1,j)}w_{(j-2,j-1)}w_{(j-1,j)}w_{(j-2,j-1)}\cdots w_{(k+1,k+2)}.\end{aligned}$$

Since all the local coefficients in the right-hand side of (5.6) correspond to simple reflections we may use the same argument as in Lemma 3.3 and conclude that

$$\begin{aligned} &C_{\psi}^{\overline{Sp_{2n}(\mathbb{F})}}\big(B_{Sp_{2n}}(\mathbb{F}),\overrightarrow{s}^{w_{(k)}},(\otimes_{i=1}^{n}\alpha_i)^{w_{(k)}},w_{(k,k+1)}\big)=C_{\psi}^{GL_2(\mathbb{F})}\big(B_{GL_2}(\mathbb{F}),(s_i,s_{i+k}),\alpha_i\otimes\alpha_{i+k},\omega_2,\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\big),\\ &C_{\psi}^{\overline{Sp_{2n}(\mathbb{F})}}\big(B_{Sp_{2n}}(\mathbb{F}),\overrightarrow{s}^{\widetilde{w}},(\otimes_{i=1}^{n}\alpha_i)^{\widetilde{w}},w_{(j-1,j)}\big)=C_{\psi}^{GL_2(\mathbb{F})}\big(B_{GL_2}(\mathbb{F}),(s_i,s_j),\alpha_i\otimes\alpha_j,\omega_2,\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\big),\\ &C_{\psi}^{\overline{Sp_{2n}(\mathbb{F})}}\big(B_{Sp_{2n}}(\mathbb{F}),\overrightarrow{s}^{w'_{(k)}},(\otimes_{i=1}^{n}\alpha_i)^{w'_{(k)}},w_{(k,k+1)}\big)=C_{\psi}^{GL_2(\mathbb{F})}\big(B_{GL_2}(\mathbb{F}),(s_{i+k},s_j),\alpha_{i+k}\otimes\alpha_j,\omega_2,\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\big).\end{aligned} \tag{5.7}$$

Since $\alpha_1,\ldots,\alpha_k$ are unitary, (5.7) and (5.1) implies that for $i\leq k\leq j-2$,

$$C_{\psi}^{\overline{Sp_{2n}(\mathbb{F})}}\big(B_{Sp_{2n}}(\mathbb{F}),\overrightarrow{s}^{w_{(k)}},(\otimes_{i=1}^{n}\alpha_i)^{w_{(k)}},w_{(k,k+1)}\big)$$

and

$$C_{\psi}^{\overline{Sp_{2n}(\mathbb{F})}}\big(B_{Sp_{2n}}(\mathbb{F}),\overrightarrow{s}^{w'_{(k)}},(\otimes_{i=1}^{n}\alpha_i)^{w'_{(k)}},w_{(k,k+1)}\big)$$

are holomorphic at $\overrightarrow{s}=0$. Also, since $w_{(i,j)}\in W(\alpha)$ implies that $\alpha_i=\alpha_j$, (5.7) and (5.1) imply that

$$C_{\psi}^{\overline{Sp_{2n}(\mathbb{F})}}\big(B_{Sp_{2n}}(\mathbb{F}),\overrightarrow{s}^{\widetilde{w}},(\otimes_{i=1}^{n}\alpha_i)^{\widetilde{w}},w_{(j-1,j)}\big)$$

vanishes for $\overrightarrow{s}=0$. Recalling (5.6) we now conclude that if $w=w_{(i,j)}\in W(\alpha)$ then (5.4) holds.

Suppose now that $\tau_{\{r\}} \in W(\alpha)$. We write

$$\tau_{\{r\}} = w_{(r,r+1)} w_{(r+1,r+2)} \cdots w_{(n-1,n)} \tau_{\{n\}} w_{(n-1,n)} w_{(n-2,n-1)} \cdots w_{(r+1,r)}. \tag{5.8}$$

The reader may check that the expression in the right-hand side of (5.8) is reduced. We now use the same arguments we used for $w = w_{(i,j)}$: We decompose

$$C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}\big(B_{Sp_{2n}}(\mathbb{F}), \overrightarrow{s}, (\otimes_{i=1}^n \alpha_i), \tau_{\{r\}}\big)$$

into $1 + 2(n-i)$ local coefficients. $2(n-i)$ of them are of the form (5.1). These factors are holomorphic at $\overrightarrow{s} = 0$. The additional local coefficient, the one corresponding to $\tau_{\{n\}}$ is

$$C_\psi^{\overline{SL_2(\mathbb{F})}}\big(\overline{B_{SL_2}(\mathbb{F})}, s_r, \alpha_r, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\big).$$

Lemma 3.5 implies that there exists $c \in \mathbb{C}^*$ such that

$$\begin{aligned} & C_\psi^{\overline{SL_2(\mathbb{F})}}\big(\overline{B_{SL_2}(\mathbb{F})}, s, \chi, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\big) C_\psi^{\overline{SL_2(\mathbb{F})}}\big(\overline{B_{SL_2}(\mathbb{F})}, -s, \chi^{-1}, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\big) \\ & = c \frac{L_{\mathbb{F}}(\chi^2, -2s+1)}{L_{\mathbb{F}}(\chi^2, 2s)} \frac{L_{\mathbb{F}}(\chi^2, 2s+1)}{L_{\mathbb{F}}(\chi^2, -2s)}. \end{aligned}$$

Since $\tau_{\{r\}} \in W(\alpha)$ implies that α_r is quadratic, we now conclude that (5.4) holds for $w = \tau_{\{r\}}$.

Finally, assume that $w'_{(i,j)} \in W(\alpha)$. We write it as a reduced product of simple reflections:

$$\begin{aligned} w'_{(i,j)} = \quad & w_{(j,j+1)} w_{(j+1,j+2)} \cdots w_{(n-1,n)} \tau_{\{n\}} w_{(n-1,n)} w_{(n-2,n-1)} \cdots w_{(j+1,i)} \\ & w_{(i,i+1)} w_{(i+1,i+2)} \cdots w_{(j-1,j)} w_{(j-2,j-1)} \cdots w_{(i+1,i)} \\ & w_{(j,j+1)} w_{(j+1,j+2)} \cdots w_{(n-1,n)} \tau_{\{n\}} w_{(n-1,n)} w_{(n-2,n-1)} \cdots w_{(j+1,i)}. \end{aligned}$$

We then decompose

$$C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}\big(B_{Sp_{2n}}(\mathbb{F}), \overrightarrow{s}, (\otimes_{i=1}^n \alpha_i), w'_{(i,j)}\big)$$

into $1 + 2(n-i)$ local coefficients coming either from $GL_2(\mathbb{F})$ or from $\overline{SL_2(\mathbb{F})}$. All these local coefficients are holomorphic at $\overrightarrow{s} = 0$. The factor corresponding to $w_{(j,j-1)}$ equals

$$C_\psi^{GL_2(\mathbb{F})}\big(B_{GL_2}(\mathbb{F}), (s_i, -s_j), \alpha_i \otimes \alpha_j^{-1}, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\big).$$

Since $w'_{(i,j)} \in W(\alpha)$ implies that $\alpha_i = \alpha_j^{-1}$ we conclude, using (5.1), that (5.4) holds for $w = w'_{(i,j)}$, provided that $w'_{(i,j)} \in W(\alpha)$. $\square$

Remarks:
1. Assume that $\mathbb{F}$ is a p-adic field of odd residual characteristic. For the irreducibility of principal series representations of $\overline{SL_2(\mathbb{F})}$, induced from unitary characters see [31]. For the irreducibility of principal series representations induced from unitary characters to the $\mathbb{C}^1$ cover of $Sp_4(\mathbb{F})$ see [55]. For a proof of Theorem 5.1 which uses the theta correspondence; see [16].
2. One can show that Theorem 2.3 applies also to the field of real numbers in the case of a parabolic induction from unitary characters of $\overline{B_{Sp_{2n}}(\mathbb{R})}$. Thus, repeating the same argument used in this section, replacing (5.1) with (5.2), one concludes that Theorem 5.1 applies for the real case as well.

Theorem 5.2. *Let $\overrightarrow{t}=(n_1,n_2,\ldots,n_r;k)$ where $n_1,n_2,\ldots,n_r,k$ are $r+1$ non-negative integers whose sum is n. For $1\leq i\leq r$ let τ_i be an irreducible admissible supercuspidal unitary representation of $GL_{n_i}(\mathbb{F})$ and let $\overline{\sigma}$ be an an irreducible admissible supercuspidal ψ-generic genuine representation of $\overline{Sp_{2k}(\mathbb{F})}$. Denote $\pi=\big(\otimes_{i=1}^r(\gamma_\psi^{-1}\otimes\tau_i)\big)\otimes\overline{\sigma}$. $I(\pi)$ is reducible if and only if there exists $1\leq i\leq r$ such that τ_i is self dual and*

$$C_\psi^{\overline{Sp_{2(k+n_i)}(\mathbb{F})}}\big(\overline{P_{n_i;k}(\mathbb{F})},0,\tau_i\otimes\overline{\sigma},j_{n_i,k+n_i}(\omega_{n_i}'^{-1})\big)\neq 0 \tag{5.9}$$

Proof. Since $j_{n_i,n}(\omega_{n_i}'^{-1})$ is of order two as a Weyl element it follows that if τ_i is self dual then

$$C_\psi^{\overline{Sp_{2(k+n_i)}(\mathbb{F})}}\big(\overline{P_{n_i;k}(\mathbb{F})},0,\tau_i\otimes\overline{\sigma},j_{n_i,k+n_i}(\omega_{n_i}'^{-1})\big)=0$$

if and only if

$$C_\psi^{\overline{Sp_{2(k+n_i)}(\mathbb{F})}}\big(\overline{P_{n_i;k}(\mathbb{F})},s,\tau_i\otimes\overline{\sigma},j_{n_i,k+n_i}(\omega_{n_i}'^{-1})\big)C_\psi^{\overline{SP_{k+2n_i}(\mathbb{F})}}\big(\overline{P_{n_i;k}(\mathbb{F})},-s,\widehat{\tau_i}\otimes\overline{\sigma},j_{n_i,k+n_i}(\omega_{n_i}'^{-1})\big) \tag{5.10}$$

vanishes at $s=0$. Thus, since $I(\pi)$ is unitary, we only have to show that

$$dim\big(Hom_{\overline{Sp_{2n}(\mathbb{F})}}\big(I(\pi),I(\pi)\big)\big)>1 \tag{5.11}$$

if and only if there exits $1\leq i\leq r$ such that τ_i is self dual and (5.10) does not vanish at $s=0$.

Suppose first that there exits $1\leq i\leq r$ such that τ_i is self dual and (5.10) does not vanish at $s=0$. Since for any $w\in W_{P_{\overrightarrow{t}}}(\mathbb{F})$, $I(\pi)$ and $I(\pi^w)$ have the same Jordan Holder series we may assume that $i=r$. It follows from (2.7) that $w_0=j_{n_r,n}(\omega_{n_r}'^{-1})\in\sigma_{P_{\overrightarrow{t}}}(\mathbb{F})\cap W(\pi)$. Since w_0 is a simple reflection we may use a similar argument to the one used in Lemma 3.3 and conclude that

$$C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{\overrightarrow{t}}(\mathbb{F})},\overrightarrow{s},\big(\otimes_{i=1}^r\tau_i\big)\otimes\overline{\sigma},w_0\big)=C_\psi^{\overline{Sp_{2(k+n_i)}(\mathbb{F})}}\big(\overline{P_{n_i;k}(\mathbb{F})},s_r,\tau_i\otimes\overline{\sigma},w_0\big),$$

where $\overrightarrow{s}=(s_1,s_2,\ldots,s_r)$. Thus, our assumption implies that

$$C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}(\overline{P_{\overrightarrow{t}}(\mathbb{F})},\overrightarrow{s},(\otimes_{i=1}^r\tau_i)\otimes\overline{\sigma},w_0)C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}(\overline{P_{\overrightarrow{t}}(\mathbb{F})},\overrightarrow{s^{w_0}},((\otimes_{i=1}^r\tau_i)\otimes\overline{\sigma})^{w_0},w_0\big) \tag{5.12}$$

does not vanish at $s=0$. Theorem 2.3 and (3.7) imply now that (5.11) holds.

We now assume that for any $1\leq i\leq r$, if τ_i is self dual then (5.10) vanishes at $s=0$. Again, by theorem 2.3 and (3.7) we only have to show that (5.12) vanishes at $\overrightarrow{s}=0$ for any $w_0\in\Sigma_{P_{\overrightarrow{t}}(\mathbb{F})}\cap W(\pi)$. Similar to the proof of Theorem 5.1, there are three possible types of $w_0\in\Sigma_{P_{\overrightarrow{t}}(\mathbb{F})}\cap W(\pi)$:
Type 1. $\tau_i\simeq\tau_j$ for some $1\leq i<j\leq r$ then the if and only if the Weyl element that inter change the $GL_{n_i}(\mathbb{F})$ and the $GL_{n_j}(\mathbb{F})$ blocks lies in $\Sigma_{P_{\overrightarrow{t}}(\mathbb{F})}\cap W(\pi)$.
Type 2. $\tau_i\simeq\widehat{\tau_j}$ for some $1\leq i<j\leq r$ if and only if the Weyl element that inter change the $GL_{n_i}(\mathbb{F})$ with the "dual" $GL_{n_j}(\mathbb{F})$ blocks lies in $\Sigma_{P_{\overrightarrow{t}}(\mathbb{F})}\cap W(\pi)$.
Type 3. τ_i is self dual for some $1\leq i\leq r$ if and only if the Weyl element that inter change the $GL_{n_i}(\mathbb{F})$ with its "dual" block lies in $\Sigma_{P_{\overrightarrow{t}}(\mathbb{F})}\cap W(\pi)$.

In fact, by switching from π to π^w for some $w \in W_{P_{\overrightarrow{t}}}(\mathbb{F})$, we may assume that there are no elements in $\Sigma_{P_{\overrightarrow{t}}(\mathbb{F})} \cap W(\pi)$ of type 2. Indeed, Let $I \subseteq \{1, 2, \ldots, r\}$ such that

$$\{1, 2, \ldots, r\} = \dot{\bigcup_{i \in I}} A_i,$$

where A_i are the equivalence classes

$$A_i = \{1 \leq j \leq r \mid \tau_i \simeq \tau_j \text{ or } \tau_i \simeq \widehat{\tau_j}\}.$$

By choosing $w \in W_{P_{\overrightarrow{t}}}(\mathbb{F})$ properly we may assume that $\tau_i \simeq \tau_j$ for all $j \in A_i$. Thus, we only prove that (5.12) vanishes at $\overrightarrow{s} = 0$ for any $w_0 \in \Sigma_{P_{\overrightarrow{t}}(\mathbb{F})} \cap W(\pi)$ of type 1 or 3.

Assume that $\tau_i \simeq \tau_j$. Let $w_0 \in \Sigma_{P_{\overrightarrow{t}}(\mathbb{F})} \cap W(\pi)$ be the corresponding Weyl element. We decompose (5.12) into a product of local coefficients corresponding to simple reflections which may be shown to be equal to local coefficients of the form

$$C_\psi^{GL_{n_p+n_q}(\mathbb{F})}\big(P^0_{n_p,n_q}(\mathbb{F}), (s_p, s_q), \tau_p \otimes \tau_q, \varpi_{q,p}\big) C_\psi^{GL_{n_q+n_p}(\mathbb{F})}\big(P^0_{n_q,n_p}(\mathbb{F}), (s_q, s_p), \tau_q \otimes \tau_p, \varpi_{p,q}\big). \tag{5.13}$$

All these factors are analytic at $(0, 0)$; see Theorem 5.3.5.2 of [48]. One of these factors corresponds to $(p, q) = (i, j)$. Since by assumption $\tau_i \simeq \tau_j$, the well-known reducibility theorems for parabolically induced representation of $GL_n(\mathbb{F})$ (see the first remark on page 1119 of [15], for example) implies that the factor that corresponds to $p = i$, $q = j$ vanishes at $(0, 0)$. This shows that (5.12) vanishes at $\overrightarrow{s} = 0$ for any $w_0 \in \Sigma_{P_{\overrightarrow{t}}(\mathbb{F})} \cap W(\pi)$ of type 1.

Assume that τ_i is self dual. Let $w_0 \in \Sigma_{P_{\overrightarrow{t}}(\mathbb{F})} \cap W(\pi)$ be the corresponding Weyl element. We decompose (5.12) into a product which consist of elements of the form (5.13) and of factor of the form (5.10). All the factors of the form (5.13) are analytic and $(0, 0)$. Since τ_i is self dual, by our assumption the other factor vanishes at $s = 0$. This shows that (5.12) vanishes at $\overrightarrow{s} = 0$ for any $w_0 \in \Sigma_{P_{\overrightarrow{t}}(\mathbb{F})} \cap W(\pi)$ of type 3. □

Corollary 5.1. *We keep the notations and assumptions of Theorem 5.2. $I(\pi)$ is reducible if and only if there exists $1 \leq i \leq r$ such that τ_i is self dual and*

$$\gamma(\overline{\sigma} \times \tau_i, 0, \psi)\gamma(\tau_i, sym^2, 0, \psi) \neq 0 \tag{5.14}$$

Proof. Let τ be an irreducible admissible generic representation of $GL_m(\mathbb{F})$. From the definition of $\gamma(\overline{\sigma} \times \tau, s, \psi)$, (3.8), and from Theorem 4.3 it follows that

$$C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}\big(\overline{P_{m;k}(\mathbb{F})}, s, \tau \otimes \overline{\sigma}, j_{m,n}(\omega'^{-1}_m)\big) = c(s)\gamma(\overline{\sigma} \times \tau, s, \psi)\frac{\gamma(\tau, sym^2, 2s, \psi)}{\gamma(\tau, s + \frac{1}{2}, \psi)}$$

for some exponential factor $c(s)$. By (6.1.4) in page 108 of [39] we have

$$\gamma(\widehat{\tau}, 1 - s, \psi)\gamma(\tau, s, \psi) = \tau(-I_m) \in \{\pm 1\}. \tag{5.15}$$

Therefore, if we assume in addition that τ is self dual we know that $\gamma(\tau, \frac{1}{2}, \psi) \in \{\pm 1\}$. This implies that (5.9) may be replaced with (5.14) □

The following two corollaries follow immediately from Theorem 5.2.

Corollary 5.2. *With the notations and assumptions of Theorem 5.2, $I(\pi)$ is irreducible if and only if $I(\tau_i,\overline{\sigma})$ is irreducible for every $1 \le i \le r$.*

Corollary 5.3. *Let $\overrightarrow{t} = (n_1, n_2, \ldots, n_r; 0)$ where $n_1, n_2, \ldots, n_r$ are r non-negative integers whose sum is n. For $1 \le i \le r$ let τ_i be an irreducible admissible supercuspidal unitary representation of $GL_{n_i}(\mathbb{F})$. Denote $\pi = \otimes_{i=1}^{r}(\gamma_{\psi}^{-1} \otimes \tau_i)$. $I(\pi)$ is reducible if and only if there exits $1 \le i \le r$ such that τ_i is self dual and $\gamma(\tau_i, sym^2, 0) \neq 0$.*

6 A comparison with $SO_{2n+1}(\mathbb{F})$

In this section $\mathbb{F}$ is again a p-adic field. Let $SO_{2n+1}(\mathbb{F})$ be the special orthogonal group:

$$SO_{2n+1}(\mathbb{F}) = \{g \in GL_{2n+1}(\mathbb{F}) \mid gJ'_{2n+1}g^t = J'_{2n+1},\ \det(g) = 1\},$$

where $J'_n = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix}$. Denote by $B_{SO_{2n+1}}(\mathbb{F})$, $N_{SO_{2n+1}}(\mathbb{F})$ the standard Borel subgroup and its unipotent radical respectively (see page 2 of [49] for example). Let ψ be a non-trivial character of $\mathbb{F}$. We continue to denote by ψ the character of $N_{SO_{2n+1}}(\mathbb{F})$ defined by

$$\psi(u) = \psi(\sum_{k=1}^{n} u_{k,k+1}).$$

We also view ψ a character of any subgroup of $N_{SO_{2n+1}}(\mathbb{F})$. Let $P_{SO_{2n+1}}(\mathbb{F})$ be the standard parabolic subgroup of $SO_{2n+1}(\mathbb{F})$ whose Levi part and unipotent radical are

$$M_{SO_{2n+1}}(\mathbb{F}) = \{\begin{pmatrix} g & & \\ & 1 & \\ & & g* \end{pmatrix} \mid g \in GL_n(\mathbb{F})\} \simeq GL_n(\mathbb{F}),$$

$$U_{SO_{2n+1}}(\mathbb{F}) = \{\begin{pmatrix} I_n & x & z \\ & 1 & x' \\ & & I_n \end{pmatrix} \in SO_{2n+1}(\mathbb{F})\},$$

where $g^* = J'_{2n+1}{}^t g^{-1}, x' = -{}^t x J'_n$. Define

$$\omega''_n = \begin{pmatrix} & & \omega_n \\ & (-1)^n & \\ \omega_n & & \end{pmatrix} \in SO_{2n+1},$$

and let τ be a generic representation of $GL_n(\mathbb{F})$ identified with $M_{SO_{2n+1}}(\mathbb{F})$. The local coefficient

$$C_{\psi}^{SO_{2n+1}(\mathbb{F})}(P_{SO_{2n+1}}(\mathbb{F}), s, \tau, \omega_n''^{-1})$$

is defined in the same way as in Section 3.1 via Shahidi's general construction; see Theorem 3.1 of [37]. From the second part of Theorem 3.5 of [41] it follows there exists $c \in \mathbb{C}^*$ such that

$$C_{\psi}^{SO_{2n+1}(\mathbb{F})}(P_{SO_{2n+1}}(\mathbb{F}), s, \tau, \omega_n''^{-1}) = c\gamma(\tau, sym^2, 2s, \psi).$$

Furthermore, if τ is unramified then $c = 1$. In Theorem 4.3 we have proven that

$$C_\psi^{\overline{Sp_{2m}(\mathbb{F})}}\big(\overline{P_{m;0}(\mathbb{F})}, s, \tau, \omega_m'^{-1}\big) = c_{\mathbb{F}}(s)\frac{\gamma(\tau, sym^2, 2s, \psi)}{\gamma(\tau, s+\frac{1}{2}, \psi)},$$

where $c_{\mathbb{F}}(s)$ is an exponential factor which equals 1 if $\mathbb{F}$ is a p-adic field of odd residual characteristic, ψ is normalized and τ is unramified. Recalling (5.15) we have proved the following.

Lemma 6.1. *Let τ be an irreducible admissible generic representation of $GL_n(\mathbb{F})$. There exits an exponential function $c(s)$ such that*

$$\begin{aligned} &C_\psi^{SO_{2n+1}(\mathbb{F})}(P_{SO_{2n+1}}(\mathbb{F}), s, \tau, \omega_n''^{-1})C_\psi^{SO_{2n+1}(\mathbb{F})}(P_{SO_{2n+1}}(\mathbb{F}), -s, \widehat{\tau}, \omega_n''^{-1}) \\ &= c(s)C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}(\overline{P_{n;0}(\mathbb{F})}, s, \tau, \omega_n'^{-1})C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}(\overline{P_{n;0}(\mathbb{F})}, -s, \widehat{\tau}, \omega_n'^{-1}). \end{aligned} \tag{6.1}$$

$c(s) = 1$ provided that $\mathbb{F}$ is a p-adic field of odd residual characteristic, ψ is normalized and τ is unramified.

Theorem 6.1. *Let τ be an irreducible admissible self dual supercuspidal representation of $GL_n(\mathbb{F})$. Then,*

$$I(\tau) = Ind_{\overline{P_{n;0}(\mathbb{F})}}^{\overline{Sp_{2n}(\mathbb{F})}}\big((\gamma_\psi^{-1}\circ\det)\otimes\tau\big)$$

is irreducible if and only if

$$I'(\tau) = Ind_{P_{SO_{2n+1}}(\mathbb{F})}^{SO_{2n+1}(\mathbb{F})}\tau$$

is irreducible.

Proof. In both cases we are dealing with a representation induced from a singular representation of a maximal parabolic subgroup. Therefore, applying Theorem 2.3 and (3.7) to these representations, the theorem follows from Lemma 6.1. □

Remarks:

1. One can replace the assumption that τ is self dual and replace it with the assumption that τ is unitary, since by Theorem 2.9 the commuting algebras of these representations are one dimensional if τ is not self dual.

2. Theorem 6.1 may be proved without a direct use of Lemma 6.1. One just has to recall Corollary 5.3 and the well known fact that $I'(\tau)$ is irreducible if and only if $\gamma(\tau, sym^2, 0) \neq 0$; see [44]. However, the last fact follows also from the Knapp-Stein dimension theory and from the theory of local coefficients. In fact, Lemma 6.1 gives more information than Theorem 6.1. This Lemma implies that $\beta(s, \tau, \omega_n'^{-1})$ has the same analytic properties as the Plancherel measure attached to $SO_{2n+1}(\mathbb{F})$, $P_{SO_{2n+1}}(\mathbb{F})$ and τ.

3. Recently, using a different method, Gan and Savin proved that similar connection between the the parabolic inductions $Ind_{\overline{Q(\mathbb{F})}}^{\overline{Sp_{2n}(\mathbb{F})}}\big((\tau\otimes\gamma_\psi^{-1}\circ\det)\otimes\overline{\sigma}\big)$ and $Ind_{Q'(\mathbb{F})}^{SO_{2n+1}(\mathbb{F})}\big(\tau\otimes\theta_\psi(\overline{\sigma})\big)$ holds. Here $\mathbb{F}$ is a p-adic field, $Q(\mathbb{F})$ is the standard parabolic subgroup of $Sp_{2n}(\mathbb{F})$ which has $GL_m(\mathbb{F}) \times Sp_{2k}(\mathbb{F})$ as its Levi part, $Q'(\mathbb{F})$ is the standard parabolic subgroup of $SO_{2n+1}(\mathbb{F})$ which has $GL_m(\mathbb{F}) \times SO_{2k+1}(\mathbb{F})$ as its Levi part ($r + m = n$), τ is an irreducible supercuspidal generic representation of $GL_m(\mathbb{F})$ and $\overline{\sigma}$ is an irreducible genuine supercuspidal generic

representation of $\overline{Sp_{2k}(\mathbb{F})}$. Here $\theta_\psi(\overline{\sigma})$ is the generic representation of $SO_{2k+1}(\mathbb{F})$ obtained from $\overline{\sigma}$ by the local theta correspondence, see [13].

Corollary 6.1. *Let τ be an irreducible admissible self dual supercuspidal representation of $GL_m(\mathbb{F})$. Let $\overline{\sigma}$ be a generic genuine irreducible admissible supercuspidal representation of $\overline{Sp_{2k}(\mathbb{F})}$. If $I(\tau)$ is irreducible then $I(\tau, \overline{\sigma})$ is irreducible if and only if $\gamma(\overline{\sigma} \times \tau, 0, \psi) = 0$*

Proof. Recalling Theorem 5.2, we only have to show that $\gamma(\overline{\sigma} \times \tau, 0, \psi) = 0$ if and only if

$$C_\psi^{\overline{Sp_{2n}(\mathbb{F})}}(\overline{P_{m;k}(\mathbb{F})}, 0, \tau \otimes \overline{\sigma}, j_{m,n}(\omega_m'^{-1})) = 0.$$

Therefore, from (3.8), the definition of $\gamma(\overline{\sigma} \times \tau, 0, \psi)$, the proof is done once we show that

$$C_\psi^{\overline{Sp_{2m}(\mathbb{F})}}(\overline{P_{m;0}(\mathbb{F})}, s, \tau, \omega_m'^{-1})$$

is analytic and non-zero in $s = 0$. The analyticity of this local coefficient at $s = 0$ follows since by (6.1)

$$C_\psi^{\overline{Sp_{2m}(\mathbb{F})}}(\overline{P_{m;0}(\mathbb{F})}, s, \tau, \omega_m'^{-1}) C_\psi^{\overline{Sp_{2m}(\mathbb{F})}}(\overline{P_{m;0}(\mathbb{F})}, -s, \tau, \omega_m'^{-1})$$

has the same analytic properties as

$$C_\psi^{SO_{2m+1}(\mathbb{F})}(P_{SO_{2m+1}}(\mathbb{F}), s, \tau, \omega_m''^{-1}) C_\psi^{SO_{2m+1}(\mathbb{F})}(P_{SO_{2m+1}}(\mathbb{F}), -s, \widehat{\tau}, \omega_m''^{-1})$$

which is known to be analytic in $s = 0$; see Theorem 5.3.5.2 of [48] (note that that last assertion does not relay on the fact that τ is self dual). The fact that

$$C_\psi^{\overline{Sp_{2m}(\mathbb{F})}}(\overline{P_{m;0}(\mathbb{F})}, s, \tau, \omega_m'^{-1}) \neq 0$$

follows from Theorem 2.3 and the assumption that $I(\tau)$ is irreducible. □

The corollaries below follow from [44]:

Corollary 6.2. *Let τ be as in Theorem 6.1. Assume that $n \geq 2$. Then $I(\tau)$ is irreducible if and only if*

$$I''(\tau) = Ind_{P_{n;0}(\mathbb{F})}^{Sp_{2n}(\mathbb{F})} \tau$$

is reducible.

Proof. Theorem 1.2 of [44] states that $I''(\tau)$ is irreducible if and only if $I'(\tau)$ is reducible. □

Corollary 6.3. *Let τ be as in Theorem 6.1. If n is odd then $I(\tau)$ is irreducible.*

Proof. Corollary 9.2 of [44] states that under the conditions in discussion $I''(\tau)$ is reducible. □

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