Chapter 16

The Spin group

We return to the study of rotations. We saw earlier a rotation can be represented by a 3×3 matrix in $SO(3)$. However, as we saw this description is cumbersome for doing calculations. We will give an alternative based on the ring of quaternions which makes this easy. Define the spin group

$$\mathrm{Spin} = \{q \in \mathbb{H} \mid |q| = 1\}$$

Using theorem 15.2, we can see that this is a subgroup of $\mathbb{H}^*$, so it really is a group. The word "spin" comes from physics (as in electron spin); at least I think it does. Usually this group is called Spin(3), but we won't consider any of the other groups in this series.

Lemma 16.1. *If $q \in \mathrm{Spin}$ and $v \in \mathbb{H}$ is imaginary, then $qv\bar{q}$ is imaginary.*

Proof. $Re(v) = 0$ implies that $\bar{v} = -v$. Therefore

$$\overline{qv\bar{q}} = q\bar{v}\bar{q} = -qv\bar{q}$$

This implies $qv\bar{q}$ is imaginary. □

We will identify $\mathbb{R}^3$ with imaginary quaternions by sending $[x, y, z]$ to $xi + yj + zk$. The previous lemma allows us to define a transformation $\mathrm{Rot}(q) : \mathbb{R}^3 \to \mathbb{R}^3$ by $\mathrm{Rot}(q) = qv\bar{q}$ for $q \in \mathrm{Spin}$. This is a linear transformation, therefore it can be represented by a 3×3 matrix.

Lemma 16.2. $\mathrm{Rot} : \mathrm{Spin} \to GL_3(\mathbb{R})$ *is a homomorphism.*

Proof. We have that $\mathrm{Rot}(q_1q_2) = \mathrm{Rot}(q_1)\,\mathrm{Rot}(q_2)$ because $\mathrm{Rot}(q_1q_2)(v) = q_1q_2v\bar{q}_2\bar{q}_1 = \mathrm{Rot}(q_1)\,\mathrm{Rot}(q_2)(v)$. And the lemma follows.

□

Lemma 16.3. $\mathrm{Rot}(q)$ *is an orthogonal matrix.*

Proof. We use the standard characterization of orthogonal matrices that these are exactly the square matrices for which $|Av| = |v|$ for all vectors v. If $v \in \mathbb{R}^3$, $|\mathrm{Rot}(q)(v)|^2 = |qv\bar{q}|^2 = |q|^2|v|^2|\bar{q}|^2 = |v|^2$. □

Lemma 16.4. $\mathrm{Rot}(q) \in SO(3)$.

Proof. There are a number of ways to see this. Geometrically, an orthogonal matrix lies in $SO(3)$ if it takes a right handed orthonormal basis to another right handed basis. In terms of the vector cross products, right handed means that the cross product of the first vector with the second vector is the third. In the exercise 7 of the last chapter, we saw that the imaginary part of the product of two imaginary quaternions is the vector cross product of the corresponding vectors. The right handed basis i, j, k gets transformed to $\mathrm{Rot}(q)i, \mathrm{Rot}(q)j, \mathrm{Rot}(q)k$. Since $qi\bar{q}qj\bar{q} = qij\bar{q} = qk\bar{q}$, we have $\mathrm{Rot}(q)i \times \mathrm{Rot}(q)j = \mathrm{Rot}(q)k$. So this is again right handed. $\square$

Lemma 16.5. *If r is an imaginary quaternion with $|r| = 1$, and $a, b \in \mathbb{R}$ satisfy $a^2 + b^2 = 1$, then $\mathrm{Rot}(a + br)$ is a rotation about r.*

Proof. Let $q = a + br$. It clearly satisfies $|q| = 1$. The lemma follows from

$$\mathrm{Rot}(q)(r) = (a + br)r(a - br) = (ar - b)(a - br) = r$$

$\square$

It remains to determine the angle.

Theorem 16.6. *For any unit vector r viewed as an imaginary quaternion,*

$$\mathrm{Rot}(\cos(\theta) + \sin(\theta)r)$$

is $R(2\theta, r)$.

Proof. Pick a right handed system orthonormal vectors v_1, v_2, v_3 with $v_3 = r$. Then by exercise 7 of the last chapter, $v_1v_2 = v_3$, $v_2v_3 = v_1$, and $v_3v_1 = v_2$. Let $q = \cos(\theta) + \sin(\theta)r$. We have already seen that $\mathrm{Rot}(q)v_3 = v_3$. We also find

$$\begin{aligned} Rot(q)v_1 &= (\cos\theta + \sin\theta v_3)v_1(\cos\theta - \sin\theta v_3) \\ &= (\cos^2\theta - \sin^2\theta)v_1 + (2\sin\theta\cos\theta)v_2 \\ &= \cos(2\theta)v_1 + \sin(2\theta)v_2 \end{aligned}$$

and

$$\begin{aligned} Rot(q)v_2 &= (\cos\theta + \sin\theta v_3)v_2(\cos\theta - \sin\theta v_3) \\ &= -\sin(2\theta)v_1 + \cos(2\theta)v_2 \end{aligned}$$

which means that $\mathrm{Rot}(q)$ behaves like $R(2\theta, r)$. $\square$

Corollary 16.7. *The homomorphism $\mathrm{Rot} : \mathrm{Spin} \to SO(3)$ is onto, and $SO(3)$ is isomorphic to $\mathrm{Spin}/\{\pm 1\}$.*

Proof. Any rotation is given by $R(2\theta, r)$ for some θ and r, so Rot is onto. The kernel of Rot consists of $\{1, -1\}$. Therefore $SO(3) \cong \mathrm{Spin}/\{\pm 1\}$. $\square$

So in other words, a rotation can be represented by an element of Spin uniquely up to a plus or minus sign. This representation of rotations by quaternions is very economical, and, unlike $R(\theta, r)$, multiplication is straigthforward.

16.8 Exercises

1. Suppose we rotate $\mathbb{R}^3$ counterclockwise once around the z axis by 90°, and then around the x axis by 90°. This can expressed as a single rotation. Determine it.

2. Given a matrix $A \in M_{nn}(\mathbb{C})$. Define the adjoint $A^* = \bar{A}^T$. In other words the ijth entry of A^* is $\overline{a}_{ji}$. (This should not be confused with the matrix built out of cofactors which also often called the adjoint.) A matrix A called unitary if $A^*A = I$ and special unitary if in addition $\det A = 1$. Prove that the subset $U(n)$ (or $SU(n)$) of (special) unitary matrices in $GL_n(\mathbb{C})$ forms a subgroup.

3. Let $a + bi + cj + dk \in$ Spin, and let $A \in M_{22}(\mathbb{C})$ be given by (15.1). Prove that $A \in SU(2)$. Prove that this gives an isomorphism Spin $\cong SU(2)$.

4. Consider the quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ studied in a previous exercise. Show this lies in Spin and that its image in $SO(3)$ *is* the subgroup

$$\left\{ \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \mid \text{there are 0 or 2 } -1\text{'s} \right\}$$

   Find the poles (see chapter 14) and calculate the orders of their stabilizers.

5. Let

$$V = \{\frac{1}{\sqrt{3}}[1, 1, 1]^T, \frac{1}{\sqrt{3}}[-1, -1, 1]^T, \frac{1}{\sqrt{3}}[-1, 1, -1]^T, \frac{1}{\sqrt{3}}[1, -1, -1]^T\}$$

   and let

$$\tilde{T} = \{\pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k)\}$$

   be the subgroup of Spin defined in an exercise in the previous chapter. Show that the image T of $\tilde{T}$ in $SO(3)$ has order 12, and that it consists of the union of the set of matrices in exercise 5 and

$$\{R(\theta, r) \mid \theta \in \{\frac{\pi}{6}, \frac{\pi}{3}\}, r \in V\}$$

6. Continuing the last exercise. Show that the T acts as the rotational symmetry group of the regular tetrahedron with vertices in V.