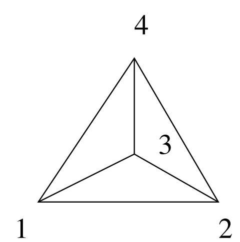
Chapter 6

More counting problems with groups

A polyhedron is a three dimensional version of a polygon. The simplest example is a tetrahedron which is a pyramid with a triangular base.



It is regular if all the triangles, called faces, are equilateral. Let us analyze the rotational symmetries of a regular tetrahedron. Let us call the symmetry group T. We view it as a subgroup of the group S_4 of permutations of the vertices labelled $1, 2, 3, 4$. We can use the orbit-stabilizer theorem to calculate the order of T. Clearly any vertex can be rotated to any other vertex, so the action is transitive. The stabilizer of 4 is the group of rotations keeping it fixed. This consists of the identity I and

$$(123), (132)$$

Therefore $|T| = (4)(3) = 12$. It is easy to list the 9 remaining rotations. There are the rotations keeping 1 fixed:

$$(234), (243)$$

2 fixed:

$$(134), (143)$$

and 3 fixed:

$$(124), (142)$$

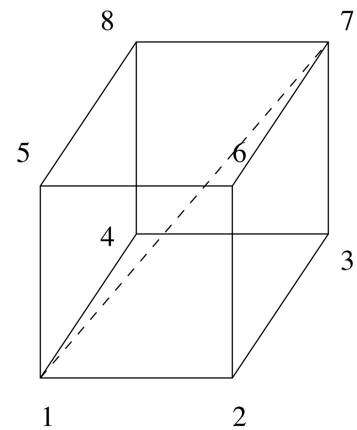


Figure 6.1: cube

We can rotate 180° about the line joining the midpoint of the edges $\overline{13}$ and $\overline{24}$ to get $(13)(24)$. We can do the same thing with other pairs of edges to get

$$(14)(23), (12)(34)$$

In summary

Lemma 6.1. *The symmetry group of a regular tetrahedron is*

$$T = \{I, (123), (132), (134), (143), (124), (142), (13)(24), (14)(23), (12)(34)\}$$

These permutations are exactly the ones than can expressed as an even number of transpositions. This is usually called the alternating group A_4.

Next, we want to analyze the group C of rotational symmetries of the cube

We can view this as a subgroup of S_8. Let us start by writing down all the obvious elements. Of course, we have the identity I. We can rotate the cube 90° about the axis connecting the top and bottom faces to get

$$(1234)(5678)$$

More generally, we have 3 rotations of $90^\circ, 180^\circ, 270^\circ$ fixing each pair of opposite faces. Let us call these type A. There are 2 rotations, other than I, fixing each diagonally opposite pair of vertices such as 1and 7 (the dotted line in picture). Call these type B. For example

$$(254)(368)$$

is type B. We come to the next type, which we call type C. This is the hardest to visualize. To each opposite pair of edges such as $\overline{12}$ and $\overline{78}$, we can connect their midpoints to a get a line L. Now do a 180° rotation about L. Let's count what we have so far:

(I) 1

(A) 3 (rotations) $\times$ 3 (pairs of faces) $= 9$

(B) 2 (rotations) $\times$ 4 (pairs of vertices) $= 8$

(C) 6 (opposite pairs of edges) $= 6$

making 24. To see that this is a complete list, we use the orbit-stabilizer theorem. The action of C is transitive, and Stab(1) consists of I, and two other elements of type B. Therefore $|C| = (8)(3) = 24$. In principle, we have a complete description of C. However, we can do better. There are 4 diagonal lies such as $\overline{17}$. One can see that any non-identity element of C must permute the diagonal lines nontrivially. A bit more formally, we have produced a one to one homomorphism from C to S_4. Since they both have order 24, we can conclude that:

Lemma 6.2. *The symmetry group C of a cube is isomorphic to S_4.*

Let us now turn to counting problems with symmetry.

Question 6.3. *How many dice are there?*

Recall that a die (singular of dice) is gotten by labelling the faces of cube by the numbers 1 through 6. One attempt at a solution goes as follows. Choose some initial labelling, then there as many ways to relabel as there are permutations which is $6! = 720$. This doesn't take into account that there are 24 ways to rotate the cube, and each rotated die should be counted as the same. From this, one may expect that there are $720/24 = 30$ possibilities. This seems more reasonable.

Question 6.4. *How many cubes are there with* 3 *red faces and* 3 *blue?*

Arguing as above, labelling the faces of the cube 1 through 6, there are $\binom{6}{3} = 20$ ways to pick 3 red faces. But this discounts symmetry. On the other hand, dividing by the number of symmetries yields 20/24, which doesn't make sense. Clearly something more sophisticated is required. Let X be a finte set of things such as relabellings of the cube, or colorings of a labelled cube, and suppose that G is a finite set of permutations of X. In fact, we only need to assume that G comes with a homomorphism to S_X. This means that each $g \in G$ determines a permutation of X such that $g_1g_2(x) = g_1(g_2(x))$ for all $g_i \in G, x \in X$. We say that G acts on X. Given $x \in X$, its orbit $\mathrm{Orb}(x) = \{g(x) \mid g \in G\}$, and let X/G be the set of orbits. Since we really want to x and $g(x)$ to be counted as one thing, we should count the number of orbits. Given $g \in G$, let $Fix(g) = \{x \in X \mid g(x) = x\}$ be the set of fixed points.

Theorem 6.5 (Burnside's Formula)**.** *If G is a finite group acting on a finite set X, then*

$$|X/G| = \frac{1}{|G|}\sum_{g\in G} |\operatorname{Fix}(g)|$$

Before starting the proof, we define the stablizer $\mathrm{Stab}(x) = \{g \in G \mid g(x) = x\}$. Theorem 5.15 generalizes, with the same proof, to

$$|G| = |\operatorname{Orb}(x)| \cdot |\operatorname{Stab}(x)|$$

Proof of Burnside. Let

$$S = \{(x, g) \in X \times G \mid g(x) = x\}$$

Consider the map $p : S \to G$ given by $p(x, g) = g$. Then $p^{-1}(g) = \mathrm{Fix}(g)$. Therefore proposition 5.4 applied to p yields

$$|S| = \sum_{g \in G} |p^{-1}(g)| = \sum_{g \in G} |\operatorname{Fix}(g)| \tag{6.1}$$

Next consider the map $q : S \to X$ given by $q(x, g) = x$. Then $q^{-1}(x) = \mathrm{Stab}(x)$. Therefore proposition 5.4 applied to q yields

$$|S| = \sum_{x \in X} |q^{-1}(x)| = \sum_{x \in X} |\operatorname{Stab}(x)|$$

Let us write X as disjoint union of orbits $\mathrm{Orb}(x_1) \cup \mathrm{Orb}(x_2) \cup \ldots$, and group terms of the last sum into these orbits

$$|S| = \sum_{x \in \operatorname{Orb}(x_1)} |\operatorname{Stab}(x)| + \sum_{x \in \operatorname{Orb}(x_2)} |\operatorname{Stab}(x)| + \ldots$$

Each orbit $\mathrm{Orb}(x_i)$ has $|G|/|\mathrm{Stab}(x_i)|$ elements by the orbit-stabilizer theorem. Furthermore, for any $x \in \mathrm{Orb}(x_i)$, we have $|\mathrm{Stab}(x)| = |\mathrm{Stab}(x_i)|$. Therefore

$$\sum_{x \in \operatorname{Orb}(x_i)} |\operatorname{Stab}(x)| = \sum_{x \in \operatorname{Orb}(x_i)} |\operatorname{Stab}(x_i)| = \frac{|G|}{|\operatorname{Stab}(x_i)|} |\operatorname{Stab}(x_i)| = |G|$$

Consequently

$$|S| = \sum_{x \in \operatorname{Orb}(x_1)} |G| + \sum_{x \in \operatorname{Orb}(x_2)} |G| + \ldots = |G| \cdot |X/G|$$

Combining this with equation (6.1) yields

$$|G| \cdot |X/G| = \sum_{g \in G} |\operatorname{Fix}(g)|$$

Dividing by $|G|$ yields the desired formula.

$\square$

Let us say that the action of G on X is fixed point free if $\mathrm{Fix}(g) = \emptyset$ unless g is the identity. In this case the naive formula works.

Corollary 6.6. *If the action is fixed point free,*

$$|X/G| = |X|/|G|$$

Coming back to question 6.3. Let X be the set of relabellings of the cube, and $G = C$ the symmetry group of the cube. Then the action is fixed point free, so that $|X/G| = 720/24 = 30$ gives the correct answer.

The solution to question 6.4 using Burnside's formula is rather messy (the answer is 2). So instead, let us consider the simpler question.

Question 6.7. *How many ways can we color a regular tetrahedron with* 2 *red and* 2 *blue faces?*

Let X be the set of such colorings, and let T be the symmetry group. Then

$$\mathrm{Fix}(I) = X$$

has $\binom{4}{2} = 6$ elements. We can see that

$$\mathrm{Fix}(g) = \emptyset$$

for any 3-cycle such as $g = (123)$ because we would need to have 3 faces the same color for any fixed point. For a fixed point of $g = (13)(24)$, the sides adjacent to $\overline{13}$ and $\overline{24}$ would have to be the same color. Therefore

$$|\,\mathrm{Fix}(g)| = 2$$

The same reasoning applies to $g = (14)(23)$ or $(12)(34)$. Thus

$$|X/T| = \frac{1}{12}(6 + 2 + 2 + 2) = 1$$

Of course, this can be figured out directly.

In general, Burnside's formula can be a bit messy to use. In practice, however, there a some tricks to simplify the sum. Given two elements g_1, g_2 of a group, we say that g_1 is *conjugate* to g_2 if $g_1 = hg_2h^{-1}$ for some $h \in G$. Since we can rewrite this as $g_2 = h^{-1}g_1h$, we can see that the relationship is symmetric. Here are a couple of examples

Example 6.8. *Every element g is conjugate to itself because $g = ege^{-1}$.*

Example 6.9. *In the dihedral group D_n, R is conjugate to R^{-1} because $FRF = FRF^{-1} = R^{-1}$.*

An important example is:

Example 6.10. *In S_n, any cycle is conjugate to any other cycle of the same length.*

The relevance for counting problems is as follows.

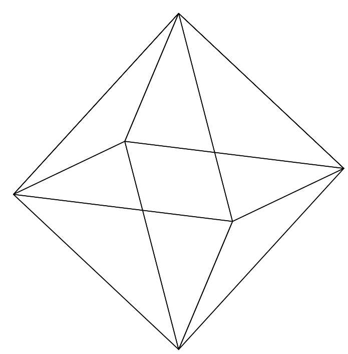
Proposition 6.11. *With the same assumptions as in theorem 6.5, if g_1 is conjugate to g_2, then* $|\operatorname{Fix}(g_1)| = |\operatorname{Fix}(g_2)|$

Proof. Suppose that $g_1 = hg_2h^{-1}$ and $x \in \operatorname{Fix}(g_2)$. Then $g_2 \cdot x = x$. Therefore $g_1(hx) = hg_2h^{-1}hx = hx$. This means that $hx \in \operatorname{Fix}(g_1)$. So we can define a function $f : \operatorname{Fix}(g_2) \to \operatorname{Fix}(g_1)$ by $f(x) = hx$. This has an inverse f^{-1} : $\operatorname{Fix}(g_1) \to \operatorname{Fix}(g_2)$ given by $f(y) = h^{-1}y$. Since f has inverse, it must be one to one and onto. Therefore $|\operatorname{Fix}(g_1)| = |\operatorname{Fix}(g_2)|$. $\square$

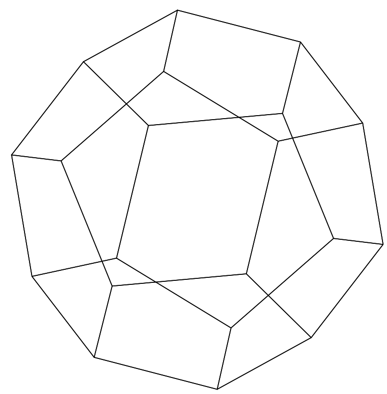
The moral is that only need to calculate $|\operatorname{Fix}(g)|$ once for every element conjugate to g, and weight the factor in Burnside by the number of such elements.

6.12 Exercises

1. Calculate the order of the (rotational) symmetry group for the octrahedron (which most people would call a diamond)



2. Calculate the order of the (rotational) symmetry group for the dodecahedron



(There are 20 vertices, and 12 pentagonal faces.)

3. Let C^+ be the group of symmetries of the cube including both rotations and reflections such as (12)(34)(56)(78) with labeling in figure 6.1. Calculate the order of C^+. Repeat for the tetrahedron, octahedron and dodecahedron.

4. How many ways are the color the sides of a tetrahedron with 2 red faces, 1 and blue and 1 green. (Even if the answer is obvious to you, use Burnside.)

5. Answer question 6.4 using Burnside.

6. How many ways are there to color the sides of a cube with 2 red faces and 4 blue? (Same instructions as above.)

7. Given a group G, and a subgroup H, show that G acts transitively on G/H by $g(\gamma H) = g\gamma H$. Calculate the stabilizer of γH.

8. Prove Cayley's theorem that every group is isomorphic to a group of permutations. (Hint: Use the action G on itself defined as in the previous problem, and use this to construct a one to one homomorphism $G \to S_G$.)

9. Explain why the statement of example 6.10 holds for $n = 3$, and then do the general case.