

Course Notes for MA 460. Version 5.

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1 Definitions and Basic Facts.

The goal of this course is to help you become expert in geometry, so that you can teach it with confidence and pleasure. We begin more or less where you left off in high school, and then advance rather quickly to a higher level.

Here's the big picture. The goal of geometry is to know the truth about geometrical figures, just as the goal of astronomy is to know the truth about planets and stars, and the goal of chemistry is to know the truth about atoms, molecules and chemical reactions. To achieve this goal we must have a way of knowing which statements are true and which statements are false. In sciences like astronomy and chemistry the way to do this is to test each statement against careful observations. But this method is not foolproof, because no observation can be perfectly accurate. For example, Newton's laws in physics were tested by thousands of observations over almost 200 years, but the small uncertainties in these observations concealed the fact that there is a more accurate and fundamental theory than Newton's, namely Einstein's theory of relativity.

Mathematics is the only science in which there is a way to achieve complete certainty, with no possibility of errors due to observation. The way mathematicians do this is to begin with a few very simple statements (called Basic Facts or Axioms) that can be accepted as true on the basis of intuition. These are then used to prove everything else. This means that as long as the Basic Facts are correct, everything else is *guaranteed* to be completely correct.

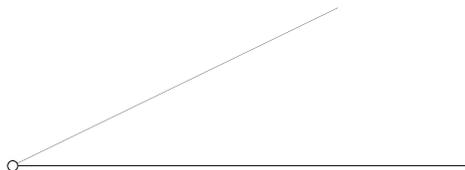
Before we can state the Basic Facts and begin to prove things, it's helpful to have some special words (such as congruent or parallel) that refer to geometric figures. Thus we begin with a section of Definitions.

1.1 Definitions.

In order to achieve complete certainty in our proofs we must avoid any uncertainty or ambiguity in the way we use geometric terms. We need to be sure that we all mean

the same thing by these terms and that we're always using them in the same way. The way to accomplish this is to begin with precise definitions. I assume that you already know the meanings of some simple terms like point, line, line segment, and ray.

Angle An angle is the figure formed by two rays which begin at the same point; this point is called the vertex of the angle.



Straight angle A straight angle is an angle in which the two rays point in opposite directions on the same line. Here is a picture:



You know that we measure angles in degrees. But what is a degree?

Degree A degree is the $\frac{1}{180}$ th part of a straight angle.

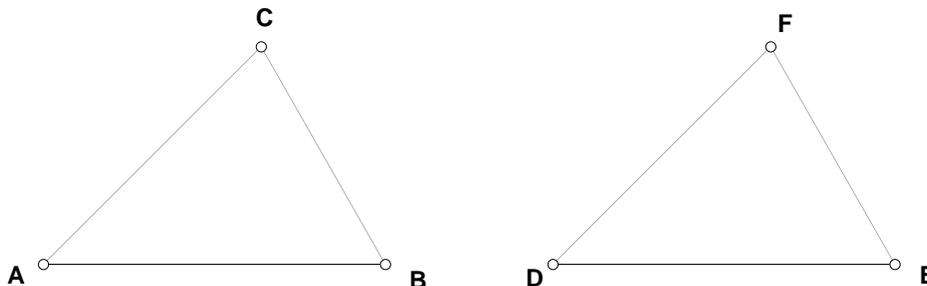
(Note: we will not consider angles bigger than 180°).

Right Angle A 90° angle is called a right angle.

Perpendicular Two lines are called perpendicular if they form a right angle.

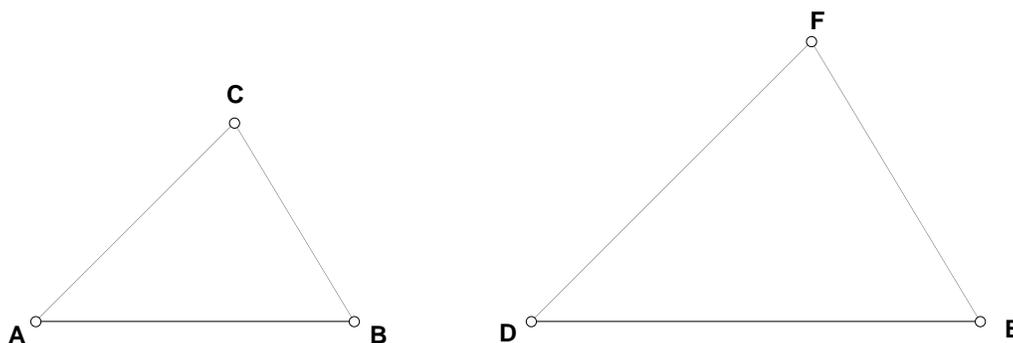
Triangle A triangle consists of three points (called vertices) and the three line segments which connect them (called sides). The vertices are not allowed to be collinear (that is, they are not allowed to all lie on the same line).

Congruent Triangles Two triangles $\triangle ABC$ and $\triangle DEF$ are congruent (written $\triangle ABC \cong \triangle DEF$) if all three corresponding angles and all three corresponding sides are equal. Here is a picture:



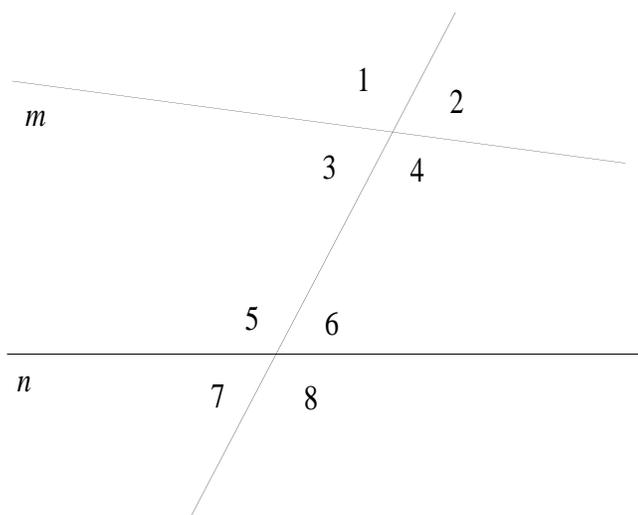
(Note: when we say that $\triangle ABC$ is congruent to $\triangle DEF$, we mean that the vertices match up in that order, that is, A matches with D , B matches with E , and C matches with F .)

Similar Triangles Two triangles $\triangle ABC$ and $\triangle DEF$ are similar (written $\triangle ABC \sim \triangle DEF$) if all three corresponding angles are equal. (Note: a common mistake is to say that the definition of similarity tells you that the corresponding sides are proportional. This is not part of the definition, it is Basic Fact BF 4 in Section 1.2). Here is a picture:



Parallel Lines Two lines are parallel if they do not intersect. (See Section 2.8 for a discussion of this definition.)

Corresponding angles If two lines m and n are crossed by a third line (the third line is called a transversal), then the following pairs of angles are called “corresponding angles:” $\angle 1$ and $\angle 5$, $\angle 2$ and $\angle 6$, $\angle 3$ and $\angle 7$, $\angle 4$ and $\angle 8$.



Midpoint of a line segment The midpoint of a segment AB is the point M on the segment for which $MA = MB$.

Angle bisector The bisector of an angle is the line that goes through the vertex of the angle and splits the angle into two equal parts.

Quadrilateral A quadrilateral consists of four points A , B , C , and D (called vertices) and the line segments AB , BC , CD and AD (called sides).

Parallelogram A quadrilateral is a parallelogram if the opposite sides are parallel. (Note: a common mistake is to say that the definition of parallelogram tells you that the opposite sides are equal; this is not part of the definition, it is Theorem 11.)

Rectangle A quadrilateral is a rectangle if it has four right angles.

Square A quadrilateral is a square if it has four equal sides and four right angles.

Comment on the definitions of similar triangles and parallelogram. As I've mentioned, the definition of similarity doesn't include the fact that corresponding sides of similar triangles are proportional, and the definition of parallelogram doesn't include the statement that opposite sides are equal. This means that when you want to say that similar triangles have equal angles, you use the definition of similarity, but when you want to say that similar triangles have proportional sides, you have to use Basic Fact 4, which is stated in Section 1.2. And when you want to say that opposite sides of a parallelogram are parallel, you use the definition of parallelogram, but when you want to say that opposite sides of a parallelogram are equal, you have to use Theorem 11.

Since this makes your life more complicated, I should explain why we do it this way. As I've said, our method for achieving complete certainty in geometry is to begin with Basic Facts that we can accept on the basis of intuition and then to prove everything else using only those. But for this method to work, we have to be careful to know exactly what we're assuming. In particular, we must be careful to distinguish between Basic Facts (which are statements about reality that we're assuming) and Definitions (which are agreements about the meanings of words). The fact that when two triangles have equal angles they also have proportional sides is a (rather surprising) statement about reality, it isn't a statement about the meanings of the words. If we included this fact in the definition of the word similar, we would be making a hidden assumption about reality.

Here's an analogy that can make this clearer. If we want to prove the statement that there is water on Mars, we can't do it by including this statement in the definition of the word Mars or of the word water. We can only prove it by making observations, for example by sending a spacecraft to Mars and finding water there.

This way of using definitions is also what we do in ordinary life. For example, the dictionary definition of “president” is “the highest executive officer in a modern republic;” the definition does not include the statement that Washington was the first president of the United States, although this is a true fact.

1.2 Basic Facts.

Before we can begin to prove things, we have to have as a starting point a list of intuitively clear facts that we accept without proof. Here is a list of basic facts that you know from high school which will be the starting points for our proofs. (When we get to Euclid, we will see that most of these basic facts can themselves be proved by starting from even simpler facts: Euclid gives proofs of BF 1, BF 2, BF 3, BF 4 and the second part of BF 5).

An important point. In geometry it’s more satisfying to prove things than to base them on intuition. Because of this, we want to keep the list of Basic Facts as short as possible. We shouldn’t add a statement to the list of Basic Facts if there’s a way to prove it from the other Basic Facts.

BF 1 SSS: if two triangles have three pairs of corresponding sides equal, then the triangles are congruent.

BF 2 SAS: if two triangles have two pairs of corresponding sides and the included angles equal, then the triangles are congruent. (Note: it is possible for two triangles to have two pairs of corresponding sides and a pair of *nonincluded* corresponding angles equal, and still not be congruent—can you draw an example of this?).

BF 3 ASA: if two triangles have two pairs of corresponding angles and the included side equal, then the triangles are congruent. (Note: as you know from high school, the AAS criterion for congruence is also valid, but we don’t include it in the list of Basic Facts because we can prove it—see the end of Section 2.2).

BF 4 If two triangles are similar then their corresponding sides are proportional: that is, if $\triangle ABC$ is similar to $\triangle DEF$ then

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$

(Note: BF 4 is a “one-way street”—it does not say that if the corresponding sides are proportional then the triangles are similar. Eventually we will *prove* that if the corresponding sides are proportional then the triangles are similar.)

BF 5 If two parallel lines ℓ and m are crossed by a transversal, then all corresponding angles are equal. If two lines ℓ and m are crossed by a transversal, and at least one pair of corresponding angles are equal, then the lines are parallel.

(Note: BF 5 is a two-way street: you can use it in either direction.)

Here are some simpler Basic Facts which are included for completeness: we will need these in order to give reasons for every step in our proofs. (Euclid gives proofs of BF 10, BF 11, BF 12, BF 13, BF 14 and part of BF 8.)

BF 6 The whole is the sum of its parts; this applies to lengths, angles, areas and arcs.

BF 7 Through two given points there is one and only one line. (This means two things. First, it is possible to draw a line through two points. Second, if two lines have two or more points in common they must really be the same line).

BF 8 On a ray there is exactly one point at a given distance from the endpoint. (This means two things. First, it is possible to find a point on the ray at a given distance from the endpoint. Second, if two points on the ray have the same distance from the endpoint they must really be the same point.)

BF 9 It is possible to extend a line segment to an infinite line.

BF 10 It is possible to find the midpoint of a line segment.

BF 11 It is possible to draw the bisector of an angle.

BF 12 Given a line ℓ and a point P (which may be either on ℓ or not on ℓ) it is possible to draw a line through P which is perpendicular to ℓ .

BF 13 Given a line ℓ and a point P not on ℓ , it is possible to draw a line through P which is parallel to ℓ .

BF 14 If two lines are each parallel to a third line then they are parallel to each other.

BF 15 The area of a rectangle is the base times the height.

2 Some familiar theorems from high-school geometry.

Now we can start proving things. The facts we state from now on will be called Theorems, not Basic Facts. Both Basic Facts and Theorems are true statements about

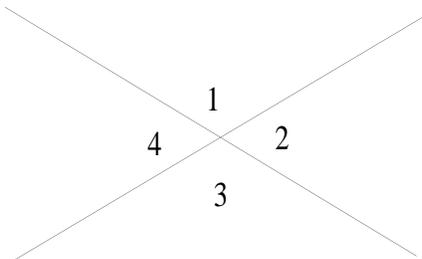
reality—the difference between them is that a Basic Fact is something we accept without proof (based on intuition) while a Theorem is something we can prove by using the Basic Facts. (Basic Facts are also called Axioms).

A very important point. In proving theorems, we are allowed to use only three ingredients: definitions, Basic Facts, and theorems that have *already* been proved. We have to be careful not to use Theorems that have not yet been proved because we have to avoid circular arguments: we aren't allowed to use one theorem to prove a second and then use the second to prove the first.

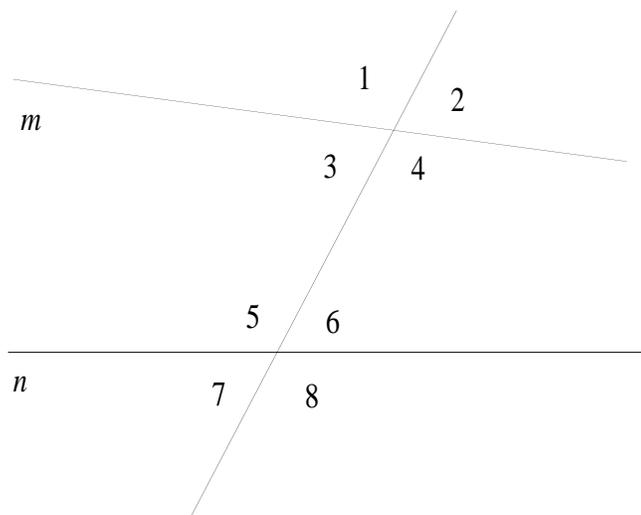
2.1 Angles formed by intersecting and parallel lines.

Before continuing we need a few more definitions:

Adjacent and Vertical Angles When two lines cross, four angles are formed. The pairs of angles that share a side are called adjacent and the pairs of angles that do not share a side are called vertical. In the picture, the adjacent pairs are $\angle 1$ and $\angle 2$, $\angle 1$ and $\angle 4$, $\angle 2$ and $\angle 3$, $\angle 3$ and $\angle 4$. The vertical pairs are $\angle 1$ and $\angle 3$, $\angle 2$ and $\angle 4$.



Interior and exterior angles If two lines m and n are crossed by a transversal, then angles 1, 2, 7 and 8 are called exterior angles and angles 3, 4, 5, and 6 are called interior angles.



Alternate interior angles In the picture above, the following pairs of angles are called “alternate interior pairs:” $\angle 3$ and $\angle 6$, $\angle 4$ and $\angle 5$.

Theorem 1. *When two lines cross,*

(a) *adjacent angles add up to 180° , and*

(b) *vertical angles are equal.*

Proof Every possible picture which illustrates the theorem will look like Figure 1, so it is enough if we show in Figure 1 that $\angle 1$ and $\angle 2$ add up to 180° and that $\angle 1$ and $\angle 3$ are equal. Now

$$(*) \quad \angle 1 + \angle 2 = \angle ABC$$

by BF 6, and

$$(**) \quad \angle ABC = 180^\circ$$

by the definition of degree. Combining (*) and (**) gives

$$(***) \quad \angle 1 + \angle 2 = 180^\circ$$

which is what we were to prove for part (a). An exactly similar argument shows that

$$(***) \quad \angle 2 + \angle 3 = 180^\circ$$

and we conclude that $\angle 1 = \angle 3$ by (***), (***) and algebra. This is what we were to prove for part (b). QED

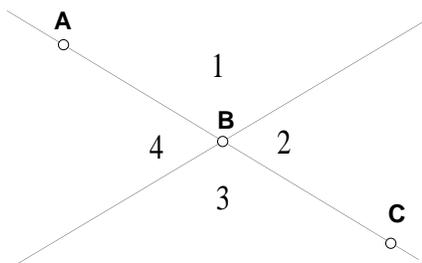


Figure 1

Note: we always end a proof by writing QED. This is an abbreviation for the Latin phrase “Quod erat demonstrandum” which means “this is what was to be proved.”

Comment on the use of pictures in proofs. We are allowed to use a picture in a proof as long as we use a *typical* picture, that is, one which has only the features guaranteed by the hypothesis, and no special features. An argument which is valid for such a picture will automatically be valid for every other picture which satisfies the hypothesis. Examples of the sort of thing you should be careful about: if the theorem is to be valid for every rectangle, your proof shouldn’t be based on a picture of a square; if the theorem is to be valid for every triangle, your proof shouldn’t be based on a picture of an isosceles triangle.

Our next theorem collects some useful variations on BF 5.

Theorem 2. *Suppose that ℓ and m are two lines crossed by a transversal.*

- (a) *If ℓ and m are parallel, then both pairs of alternate interior angles are equal. If at least one pair of alternate interior angles are equal, then ℓ and m are parallel.*
- (b) *If ℓ and m are parallel, then each pair of interior angles on the same side of the transversal adds up to 180° . If at least one pair of interior angles on the same side of the transversal adds up to 180° , then ℓ and m are parallel.*

Proof Every possible picture that illustrates the Theorem will look like Figure 2.

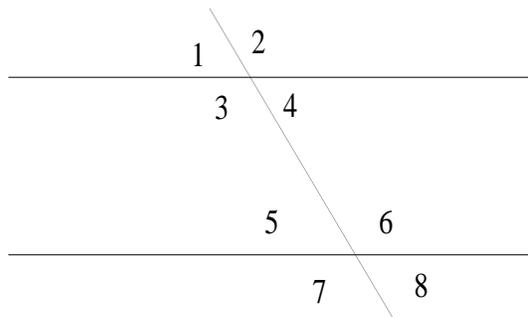


Figure 2

Part (a) contains two statements, and we have to prove both. For the first statement, we are given that ℓ and m are parallel, and we have to prove that the two pairs of alternate interior angles are equal. We know that

$$(*) \quad \angle 1 = \angle 5$$

by BF 5. We also know that

$$(**) \quad \angle 1 = \angle 4$$

by Theorem 1(b). Combining $(*)$ and $(**)$ tells us that $\angle 4 = \angle 5$. An exactly similar argument gives the equality of the other alternate interior pair.

For the second statement in part (a), we are given that a pair of alternate interior angles are equal. It's enough to give the proof when this pair is $\angle 4$ and $\angle 5$, because the proof for the other pair is exactly similar. So we are given

$$(*) \quad \angle 4 = \angle 5$$

Theorem 1(b) tells us that

$$(**) \quad \angle 4 = \angle 1$$

Combining $(*)$ and $(**)$ tells us that $\angle 1 = \angle 5$. Since this is a pair of corresponding angles, BF 5 tells us that ℓ and m are parallel.

For the first statement of part (b), we are given that the lines ℓ and m are parallel, and by BF 5 this tells us that

$$(*) \quad \angle 1 = \angle 5$$

Theorem 1(a) tells us that

$$(**) \quad \angle 1 + \angle 3 = 180^\circ$$

Combining $(*)$ and $(**)$ tells us that $\angle 3 + \angle 5 = 180^\circ$. An exactly similar argument tells us that the other two interior angles also add up to 180° .

For the second statement of part (b), we are given that a pair of interior angles on the same side of the transversal adds up to 180° . It's enough to give the proof when this pair is $\angle 3$ and $\angle 5$, since the argument is exactly similar if it is the other pair. So we are given

$$(*) \quad \angle 3 + \angle 5 = 180^\circ$$

Theorem 1(a) tells us

$$(**) \quad \angle 1 + \angle 3 = 180^\circ$$

Combining $(*)$ and $(**)$ we see that $\angle 1 = \angle 5$, and by BF 5 we conclude that ℓ and m are parallel.

QED

A convenient shortcut. When one part of an argument repeats an earlier part word for word, with only the names of the points and lines changed, you're allowed (and

encouraged) to skip the repetition and just say this part is “similar” This happened in the proof of Theorem 1 and several times in the proof of Theorem 2. But the part that’s skipped must be an exact repetition of an earlier part—if it merely resembles an earlier part you still have to give it in full.

2.2 The sum of the angles of a triangle.

Theorem 3. *The angles of a triangle add up to 180° .*

Proof Refer to Figure 3. BF 13 allows us to draw a line m through C which is parallel to AB . Now

$$(*) \quad \angle 1 + \angle 2 + \angle 3 = 180^\circ$$

by BF 6. Theorem 2(a) tells us that

$$(**) \quad \angle 1 = \angle A$$

and

$$(***) \quad \angle 3 = \angle B$$

Combining (*), (**) and (***) we see that $\angle A + \angle B + \angle 2 = 180^\circ$, which is what we were to show. QED

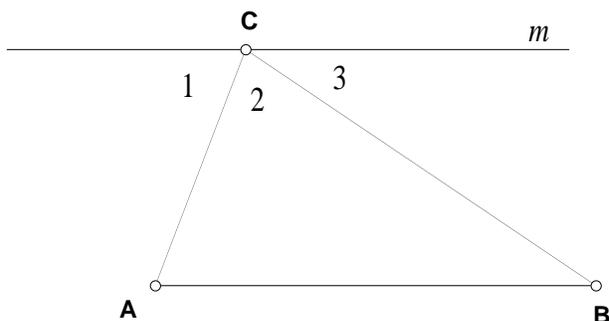


Figure 3

Note We don’t need to consider the different ways the picture might look because the *argument* works exactly the same way for all possible pictures.

Theorem 4. *If two triangles ABC and DEF have $\angle A = \angle D$ and $\angle B = \angle E$ then also $\angle C = \angle F$.*

Proof We know from Theorem 3 that

$$\angle A + \angle B + \angle C = 180^\circ$$

Combining this with the given, we have

$$(*) \quad \angle D + \angle E + \angle C = 180^\circ$$

But using Theorem 3 again we have

$$(**) \quad \angle D + \angle E + \angle F = 180^\circ$$

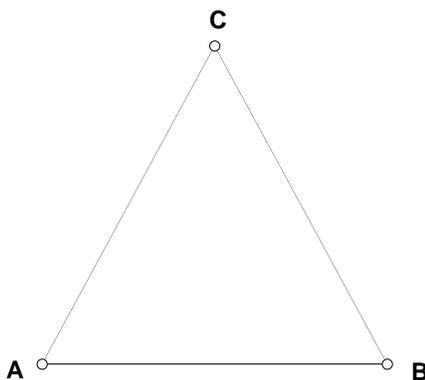
By (*), (**) and algebra we conclude that $\angle C = \angle F$. QED

Comment on the AAS criterion for congruence. Suppose that you have two triangles and you know that two pairs of corresponding angles and a *nonincluded* pair of corresponding sides are equal. You can't apply BF 3 directly to this situation, but Theorem 4 implies that *all* corresponding pairs of angles are equal, and then BF 3 does apply. So when you are in the AAS situation, you may conclude that the triangles are congruent, with the justification "Theorem 4 and BF 3." (For an example, see the proof of Theorem 5).

2.3 Isosceles triangles.

Theorem 5. (a) *If two sides of a triangle are equal then the opposite angles are equal.*

(b) *If two angles of a triangle are equal then the opposite sides are equal.*



Comment. This theorem is a two-way street: it says that if you are given either one of the statements "two sides are equal" or "two angles are equal" then the other statement

must also be true. There is a convenient abbreviation for this kind of situation: if we have a theorem that says “if A is true then B is true, and if B is true then A is true,” we can state it more briefly by saying “A is true *if and only if* B is true” or even more briefly by $A \iff B$. For example, we can restate Theorem 5 as “Two sides of a triangle are equal \iff the opposite angles are equal.” It’s important to note that, when we want to prove a theorem that has \iff , we have to give two proofs, one for each direction.

Proof of Theorem 5 We begin with part (a). This means that in Figure 4 we are given that $AC = BC$ and we want to prove $\angle A = \angle B$. Find the midpoint M of AB (which we are allowed to do by BF 10) and connect it to C . In triangles AMC and BMC we have $AC = BC$ (given), $AM = MB$ (definition of midpoint) and $MC = MC$. Now by BF 1 we have $\triangle AMC \cong \triangle BMC$, and from this we conclude that $\angle A = \angle B$ (definition of congruent triangles).

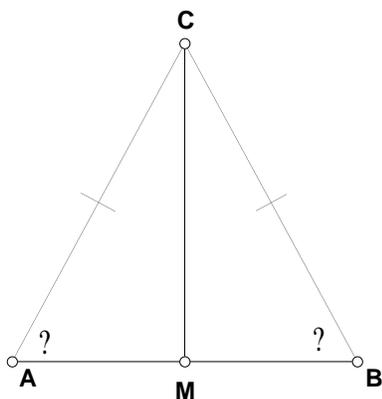


Figure 4

For part (b), we are given that $\angle A = \angle B$ and we want to prove that $AC = BC$. Draw the perpendicular line m from C to AB (allowed by BF 12) and give the intersection of m and AB the name D (see Figure 5). Then $\angle 1 = \angle 2$ (since both are right angles by the definition of perpendicular) and $\angle A = \angle B$ (given). Thus $\angle 3 = \angle 4$ (by Theorem 4). Furthermore, $CD = DC$, so $\triangle ADC \cong \triangle BDC$ (by BF 3) and from this we conclude that $AC = BC$ (definition of congruent triangles). QED

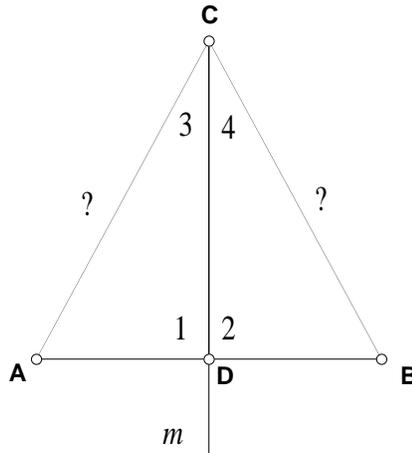


Figure 5

Note: In this proof, M and D are actually the same point. However, because they are constructed by different recipes, the information which is available for use is different in the two parts of the proof: for part (a) we are allowed to use the fact that $AM = MB$, whereas for part (b) what we are allowed to use is that $\angle ADC$ and $\angle BDC$ are right angles.

There is a standard name for the kind of triangle described in Theorem 5:

Isosceles A triangle with two equal sides is called isosceles.

The word “isosceles” is a Greek word meaning “equal sides.” Using this terminology, Theorem 5 says that a triangle is isosceles \iff it has two equal angles.

More about definitions. As I’ve mentioned, a frequently asked question about definitions is why they don’t include more information. For example, why doesn’t the definition of isosceles say that an isosceles triangle has *both* two equal sides and two equal angles? At first sight, this might seem to make Theorem 5 unnecessary. But more careful thought shows otherwise. Suppose we did define “isosceles” to mean two equal sides and two equal angles. We would still want to know that whenever two sides are equal we are guaranteed that two angles will be equal—that is, we would want to know that if a triangle has two equal sides then it is “isosceles” in the new sense. And we would also want to know that if it has two equal angles then it is “isosceles” in the new sense. In other words, we would still want to know both parts of Theorem 5, and the proof of Theorem 5 wouldn’t be any easier than before. The moral of this discussion is that by making the definition more complicated we wouldn’t actually have made anything else simpler. So we might as well at least make the definition as simple as possible, and this is what mathematicians usually do.

To put it another way, Theorem 5 is a fact about reality, so our only choice is to prove it or add it to the list of Basic Facts. We aren’t allowed to hide it inside a definition.

2.4 The area of a triangle.

Our goal in this section is to prove that the area of a triangle is one-half of the base times the height. We begin with a special case.

Theorem 6. *In triangle ABC , if $\angle B$ is a right angle then the area of the triangle is $\frac{1}{2}AB \cdot BC$.*

Proof Using BF 13, draw a line m through A parallel to BC and a line n through C parallel to AB (see Figure 6). Give the intersection of m and n the name D . Then $\angle 1 = \angle 4$ by Theorem 2(a) (using the fact that AD and BC are parallel) and $\angle 2 = \angle 3$ by Theorem 2(a) again (this time using the fact that AB and CD are parallel). Furthermore, $AC = AC$, so by BF 3 we have $\triangle ABC \cong \triangle CDA$. Now the area of $ABCD$ is equal to the sum of the areas of ABC and CDA (by BF 6), and the areas of these two triangles are the same (since they are congruent) so we conclude that the area of $ABCD$ is twice the area of ABC : in other words,

$$(*) \quad \text{area of } ABC = \frac{1}{2} \text{ area of } ABCD$$

Next we want to show that $ABCD$ is a rectangle. We are given that $\angle B$ is a right angle, and so by Theorem 2(b) we know that $\angle BCD$ is also a right angle. Using Theorem 2(b) again, this implies that $\angle D$ is a right angle, and this in turn implies (by one more use of Theorem 2(b)) that $\angle DAB$ is a right angle. Now we have shown that $ABCD$ is a rectangle (definition of rectangle) and so we know by BF 15 that

$$(**) \quad \text{area of } ABCD = AB \cdot BC.$$

Combining $(*)$ and $(**)$ gives us the formula which was to be proved. QED

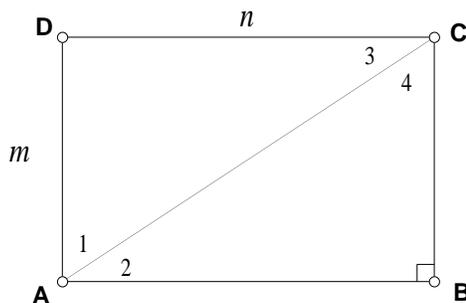
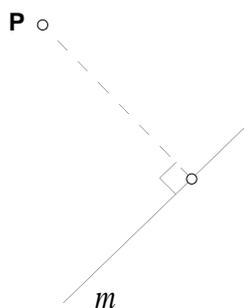


Figure 6

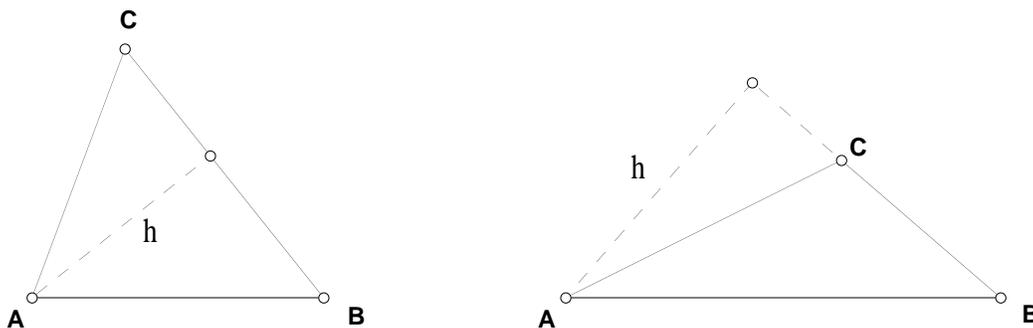
Next we need some definitions.

Distance from a point to a line The distance from a point P to a line m is defined to be the length of the line segment from P to m which is *perpendicular* to m .



Base and height of a triangle In triangle ABC , any side can be chosen as the base. Once we have chosen the base, the height is the distance from the remaining vertex to the line containing the base.

For example, if we choose BC as the base, then the height is the distance from A to \overleftrightarrow{BC} . (Remember that “distance” has just been defined to mean the *perpendicular* distance.) The perpendicular from A to BC will be inside of the triangle if the base angles are both less than 90° but if one of them is bigger than 90° it will be outside the triangle, as shown in the following picture.



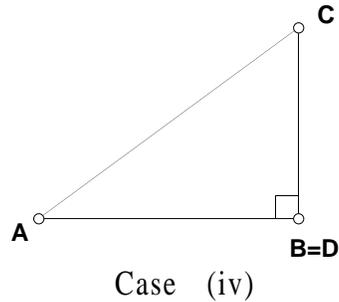
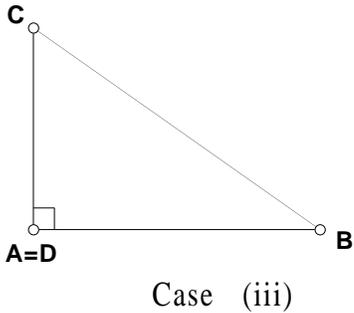
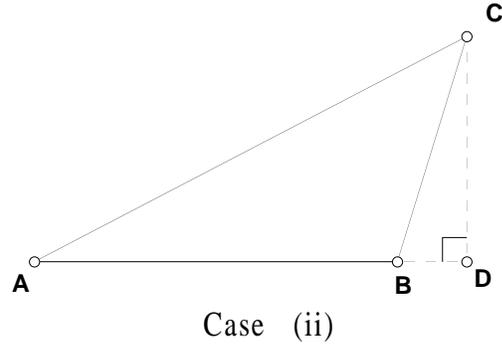
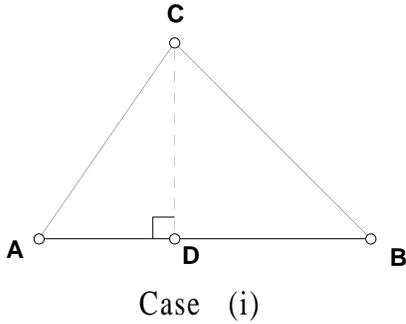
Theorem 7. *The area of a triangle is one-half of the base times the height.*

Note. For each triangle we are really getting *three* formulas for the area, because there are three ways of choosing the base.

In the proof of Theorem 7, for the first time, we have to consider *different* ways that the picture might look, because the argument will be different.

Proof of Theorem 7. It’s enough to give the proof when AB has been chosen as the base: the other two possibilities are proved by exactly similar arguments. Draw the line m that goes through C and is perpendicular to \overleftrightarrow{AB} (as we may by BF 12) and give the intersection of this line with \overleftrightarrow{AB} the name D . There are four cases to consider:

- Case (i). D is between A and B
- Case (ii). D is to the right of B or to the left of A .
- Case (iii). D is the same point as A .
- Case (iv). D is the same point as B .



In all four cases CD is the height of the triangle (by definition of height) and the angle formed by CD and line AB is a right angle (by definition of perpendicular).

In case (i) we have

$$\begin{aligned}
 \text{area of } ABC &= \text{area of } ADC + \text{area of } BDC && \text{by BF 6} \\
 &= \frac{1}{2}AD \cdot DC + \frac{1}{2}BD \cdot DC && \text{by Theorem 6} \\
 &= \frac{1}{2}(AD + BD)DC && \text{by algebra} \\
 &= \frac{1}{2}AB \cdot DC && \text{by BF 6}
 \end{aligned}$$

and this is what was to be shown for this case.

In case (ii), the area of ADC is equal to the area of ABC plus the area of BDC (by

BF 6) and so

$$\begin{aligned}\text{area of } ABC &= \text{area of } ADC - \text{area of } BDC \\ &= \frac{1}{2}AD \cdot DC - \frac{1}{2}BD \cdot DC \quad \text{by Theorem 6} \\ &= \frac{1}{2}(AD - BD)DC \quad \text{by algebra} \\ &= \frac{1}{2}AB \cdot DC \quad \text{by BF 6 and algebra}\end{aligned}$$

and this is what was to be shown for this case.

In case (iii) $\angle A$ must be a right angle, so we can apply Theorem 6. Case (iv) is similar. QED

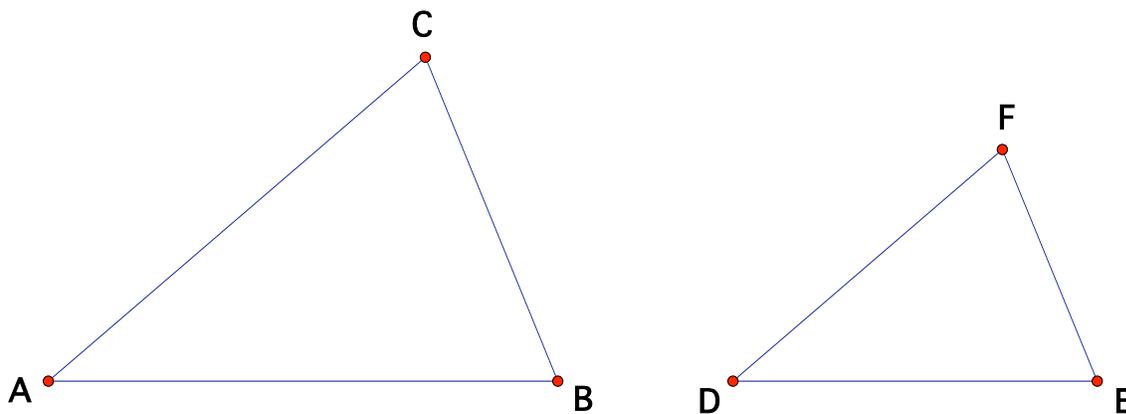
2.5 Areas of similar triangles.

What do you think happens to the area of a triangle when you double the sides? Most people who hear this question for the first time think that the area doubles. But thinking about right triangles shows that this can't be correct: if you double the sides of a right triangle then *both* the base and the height get doubled, so you create *two* factors of 2 in the area, making a total factor of 4. To sum up, when you double the sides of a right triangle, the area increases by a factor of 4.

A similar argument shows that when you multiply the sides of a right triangle by a factor r , the area gets multiplied by r^2 .

In fact, this is true for all triangles, not just for right triangles:

Theorem 8. *If $\triangle ABC \sim \triangle DEF$ and $\frac{AC}{DF} = r$ then the area of $\triangle ABC$ is r^2 times the area of $\triangle DEF$.*



The proof of Theorem 8 is a homework problem.

2.6 The Pythagorean Theorem and the Hypotenuse-Leg Theorem.

Theorem 9 (Pythagorean theorem). *In a right triangle the sum of the squares of the two legs is equal to the square of the hypotenuse.*

Proof In Figure 7, we are given that $\angle ACB$ is a right angle, and we want to prove that $a^2 + b^2 = c^2$ (where a , b and c are the lengths of BC , AC and AB respectively). We draw a perpendicular line ℓ from C to AB (which we are allowed to do by BF 12) and label the intersection of ℓ and AB by F . Let d and e stand for the lengths of AF and BF ; note that

$$(*) \quad d + e = c$$

by BF 6.

Now $\angle 1 = \angle ACB$ (because both are right angles) and $\angle A = \angle A$, so by Theorem 4 and the definition of similarity we see that $\triangle ABC \sim \triangle ACF$. Using BF 4, we see that

$$c/b = b/d$$

and algebra tells us

$$(**) \quad b^2 = cd$$

An exactly similar argument shows that $\triangle ABC \sim \triangle CBF$, and BF 4 tells us that

$$c/a = a/e$$

which by algebra gives

$$(***) \quad a^2 = ce$$

Combining $(*)$, $(**)$ and $(***)$ we see that

$$a^2 + b^2 = ce + cd = c(e + d) = c \cdot c = c^2$$

and this is what we wanted to show. QED

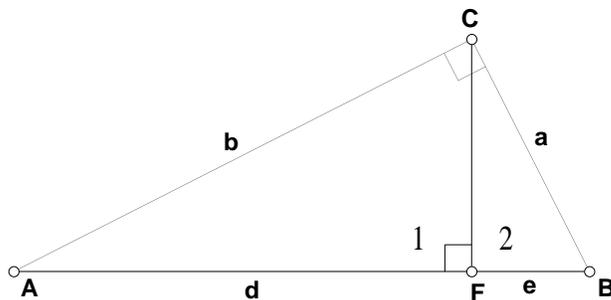
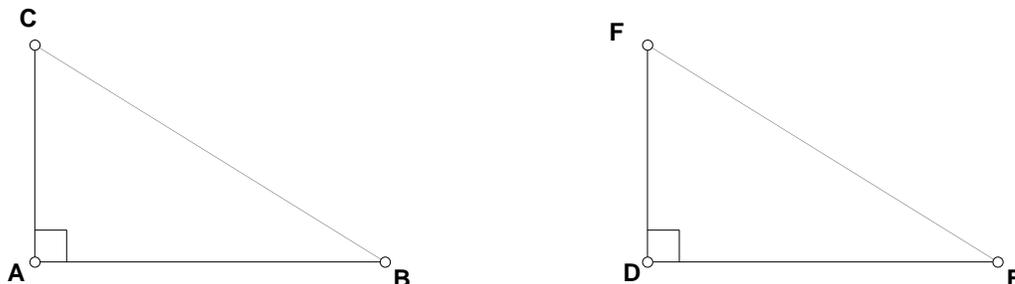


Figure 7

From the Pythagorean Theorem it is easy to prove the Hypotenuse-Leg criterion for congruence:

Theorem 10 (Hypotenuse-Leg Theorem). *In triangles ABC and DEF , if $\angle A$ and $\angle D$ are right angles, and if $BC = EF$ and $AB = DE$, then $\triangle ABC \cong \triangle DEF$. That is, if two right triangles have the hypotenuse and a leg matching then they are congruent.*



Proof The Pythagorean Theorem says that

$$AB^2 + AC^2 = BC^2$$

and

$$DE^2 + DF^2 = EF^2$$

Combining these equations with the given, we see that $AC^2 = DF^2$, and since AC and DF are positive numbers this implies $AC = DF$. Now BF 2 tells us that $\triangle ABC \cong \triangle DEF$. QED

2.7 Parallelograms.

Theorem 11. *If $ABCD$ is a parallelogram then opposite sides of $ABCD$ are equal.*

Proof In Figure 8, we are given that $ABCD$ is a parallelogram. Connect A and C by a line segment. Now AB is parallel to CD (by definition of parallelogram) so

$$(*) \quad \angle 1 = \angle 4$$

(Theorem 2(a)). Also, AD is parallel to BC (definition of parallelogram) so

$$(**) \quad \angle 2 = \angle 3$$

(Theorem 2(a)). Furthermore, $AC = AC$, and so $\triangle ACD \cong \triangle CAB$ (BF 3). From this we conclude that $AB = CD$ and $AD = BC$ (definition of \cong). QED

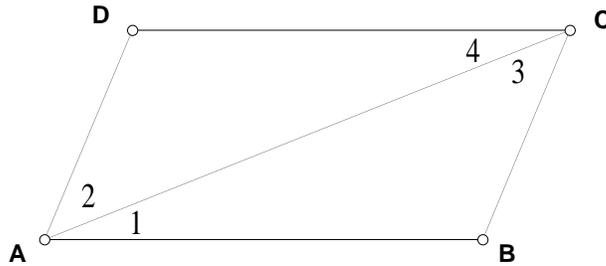


Figure 8

Theorem 12. *If $ABCD$ is a parallelogram then opposite angles of $ABCD$ are equal.*

Proof By Theorem 2(b), $\angle A + \angle B = 180^\circ$ and $\angle B + \angle C = 180^\circ$. Combining these two equations gives $\angle A = \angle C$. The proof that $\angle B = \angle D$ is similar. QED

Theorem 13. *If a quadrilateral has a pair of sides which are equal and parallel then it is a parallelogram.*

The proof of Theorem 13 is a homework problem.

Theorem 14. *A quadrilateral is a parallelogram \iff the diagonals bisect each other (that is, \iff the intersection of the two diagonals is the midpoint of each diagonal).*

The proof of Theorem 14 is a homework problem.

2.8 The distance between parallel lines.

Probably one of the first things you learned about parallel lines is that they stay at the same distance from each other, like train tracks. We can now prove this from other things we know. Recall (from Section 2.4) that the distance from a point to a line is the *perpendicular* distance.

Theorem 15. *Suppose that ℓ and m are parallel lines. If A and B are any points on ℓ , and if C and D are points on m with $AC \perp m$ and $BD \perp m$ then $AC = BD$.*

Proof See Figure 9. By the definition of perpendicular, $\angle 1$ and $\angle 2$ are right angles. Hence by BF 5 the lines AC and BD are parallel. Since we were given that AB is parallel to CD , we see that $ACDB$ is a parallelogram (definition of parallelogram). So now $AC = BD$ (Theorem 11). QED



Figure 9

Comment on the definition of parallel lines. The definition of parallel lines given in Section 1.1 is: two lines are parallel if they don't intersect. This is the standard definition in modern textbooks and also in Euclid. You might wonder why we don't use the property in Theorem 15 as the definition (as some Greek authors did before Euclid). The reason is that the definition in Section 1.1 is simpler. As I have mentioned before (at the end of Section 2.3) changing the definitions doesn't make the proofs any easier, so we may as well make the definition as simple as possible.¹

3 More about triangles.

3.1 The line through the midpoints of two sides of a triangle.

We begin with a convenient piece of algebra.

Theorem 16. (a) *Suppose that C is a point on the segment AB . C is the midpoint of $AB \iff AB = 2AC$.*

(b) *A line segment can have only one midpoint.*

Proof For (a), we begin with the \implies direction, so we are given that C is the midpoint of AB . Now $AB = AC + BC$ (by BF 6) and also $AC = BC$ (definition of midpoint). Combining these equations gives $AB = 2AC$.

In the \impliedby direction, we are given that $AB = 2AC$. But we also have $AB = AC + BC$ (by BF 6). Combining these equations gives $AC = BC$, so C is the midpoint of AB (definition of midpoint).

¹Interestingly, when non-Euclidean geometry was discovered it turned out that the definition of parallel in Section 1.1 was the correct definition to use in that situation, while Theorem 15 is no longer true in non-Euclidean geometry.

For (b), suppose that C and C' were both midpoints of AB . Then $AC = \frac{1}{2}AB$ and $AC' = \frac{1}{2}AB$ (both by part (a)) and so $AC = AC'$. Now BF 8 tells us that C and C' are the same point. QED

Theorem 17. *In triangle ABC , let D be the midpoint of AC and suppose that E is a point on BC with DE parallel to AB . Then E is the midpoint of BC and $DE = \frac{1}{2}AB$.*

Proof (See Figure 10.) By BF 5, $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$. Also, $\angle C = \angle C$. Thus $\triangle ABC \sim \triangle DEC$ (definition of similarity), and therefore

$$(*) \quad \frac{AC}{DC} = \frac{BC}{EC} = \frac{AB}{DE}$$

by BF 4. Since we are given that D is the midpoint of AC , we can apply Theorem 16(a) to get

$$(**) \quad \frac{AC}{DC} = 2$$

Combining $(*)$ and $(**)$ we see that $\frac{BC}{EC} = 2$, so that E is the midpoint of BC by Theorem 16(a).

Finally, $(*)$ and $(**)$ also tell us that

$$\frac{AB}{DE} = 2$$

so that $DE = \frac{1}{2}AB$. QED

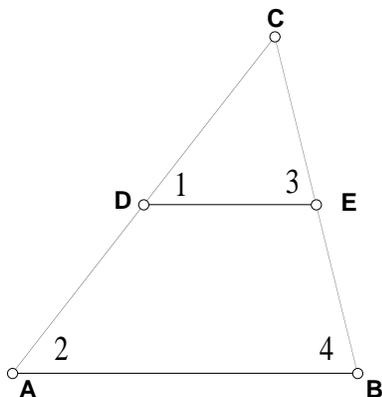


Figure 10

Our next theorem is closely related to Theorem 17 but considerably trickier to prove.

Theorem 18. *In triangle ABC , let D be the midpoint of AC and let E be the midpoint of BC . Then DE is parallel to AB and $DE = \frac{1}{2}AB$.*

Notice that the picture which illustrates this theorem is *the same* as that for Theorem 17, but the *information* we are given about this picture is different.

Proof Use BF 13 to draw a line m through D which is parallel to AB ; we record the fact that

$$(*) \quad m \text{ is parallel to } AB.$$

Give the intersection of m and BC the name F . Then we can apply Theorem 17 to conclude that F is the midpoint of BC . But then F is really the same point as E (by Theorem 16(b)). Now BF 7 tells us that DE is the same line as m , so DE is parallel to AB by (*). Finally, since we now know that E satisfies the hypothesis of Theorem 17, we can apply Theorem 17 to conclude $DE = \frac{1}{2}AB$. QED

The proof of Theorem 18 is the trickiest we have seen so far. You should study this proof carefully, because we will be using similar ideas in several other proofs and homework problems. Note that Theorem 18 would be rather easy if we had a Basic Fact that said that triangles with two sides proportional and the included angle equal are similar. But we don't include this fact among the Basic Facts, because we can *prove* it using the Basic Facts we already have. We will do so in the next section.

3.2 The SAS and SSS criteria for similarity.

Theorem 19 (SAS for similarity). *In triangles ABC and DEF , if $\angle C = \angle F$ and $\frac{AC}{DF} = \frac{BC}{EF}$ then $\triangle ABC \sim \triangle DEF$.*

The proof is rather tricky. The basic idea is to make a copy of $\triangle DEF$ on top of $\triangle ABC$, and then show that the copy is similar to $\triangle ABC$. But we won't know that the copy is actually congruent to $\triangle DEF$ until almost the end of the proof.

Proof (See Figure 11.) On ray \overrightarrow{CA} mark a point D' with $CD' = FD$ (this is allowed by BF 8). Draw the line m through D' which is parallel to AB (allowed by BF 13) and give the intersection of m and BC the name E' .

Our first goal is to show that $\triangle D'E'C \sim \triangle ABC$. Since m is parallel to AB , we can use BF 5 to see that $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$. Also, $\angle C = \angle C$, so

$$(*) \quad \triangle D'E'C \sim \triangle ABC$$

Our next goal is to see that $\triangle D'E'C \cong \triangle DEF$. Because of the way D' was constructed we know

$$(**) \quad D'C = DF$$

Next we use (*) and BF 4 to get

$$(***) \quad \frac{AC}{D'C} = \frac{BC}{E'C}$$

Combining (**) and (***) we have

$$\frac{AC}{DF} = \frac{BC}{E'C}$$

and comparing this with the given we obtain

$$\frac{BC}{E'C} = \frac{BC}{EF}$$

which by algebra gives $E'C = EF$. But we were given that $\angle C = \angle F$, so we can apply BF 2 to get

$$(\text{****}) \quad \triangle D'E'C \cong \triangle DEF$$

Now (*) tells us that $\angle 1 = \angle 2$, and (****) tells us that $\angle 1 = \angle D$, so we conclude that $\angle 2 = \angle D$. Similarly, $\angle 4 = \angle E$. And we were given $\angle C = \angle F$, so $\triangle ABC \sim \triangle DEF$ by definition of similarity. QED

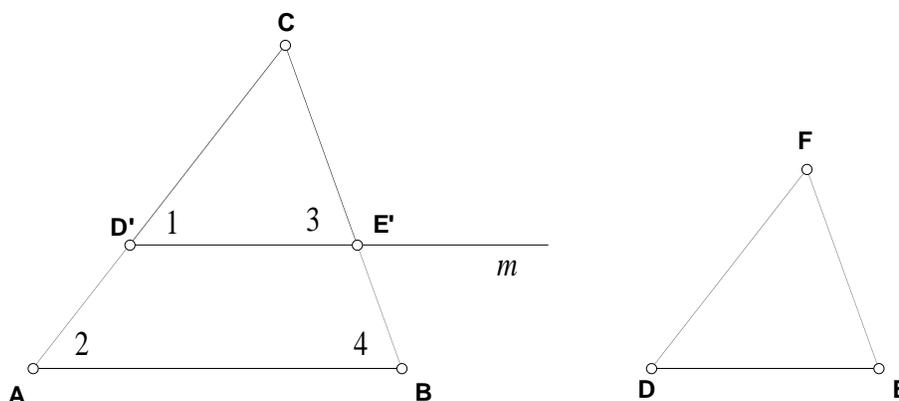


Figure 11

There is also a criterion for similarity analogous to the SSS criterion for congruence (it's the other direction of BF 4).

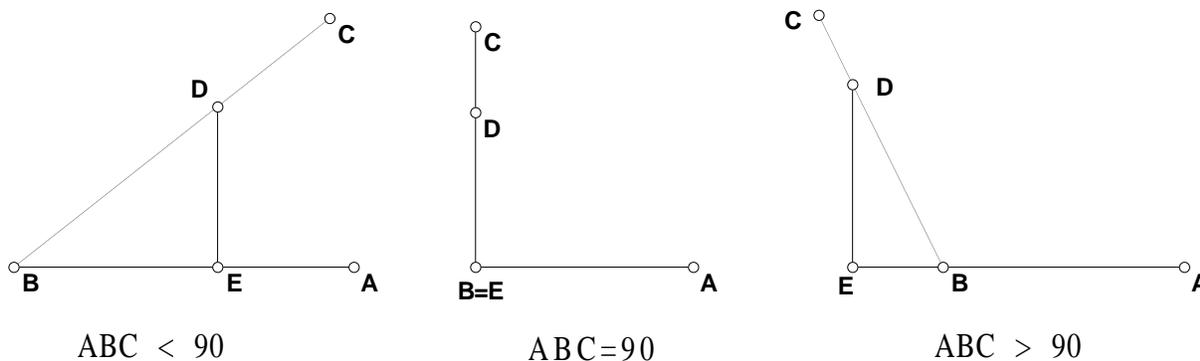
Theorem 20 (SSS for similarity). *In triangles ABC and DEF , if $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$ then $\triangle ABC \sim \triangle DEF$.*

The proof is a homework problem.

3.3 The sine of an angle.

In this section we introduce the sine of an angle and use it to give a new formula for the area of a triangle.

First we need an official definition for the sine, and before stating it we need to set the stage. Let $\angle ABC$ be any angle, and choose a point D on \overrightarrow{BC} . By BF 12 we can draw a line m through D perpendicular to \overleftrightarrow{AB} . Give the intersection of m and \overleftrightarrow{AB} the name E . Notice that there are three cases, depending on whether $\angle ABC$ is less than 90° , equal to 90° , or greater than 90° :



In all three cases the sine is given by the same formula:

Definition of sine. The sine of $\angle ABC$ is $\frac{DE}{DB}$.

Notice that for angles less than 90° this agrees with the formula you learned in high school:

$$\sin = \frac{\text{opposite}}{\text{hypotenuse}}$$

When applying the definition of sine to an angle you have the freedom to choose any point D you want (as long as D is on one of the sides of the angle). All choices are guaranteed to give the same answer, because if you choose D and your friend chooses D' the triangles DEB and $D'E'B$ will be similar, and so $\frac{DE}{DB}$ will be equal to $\frac{D'E'}{D'B'}$ by BF 4. See Figure 12.

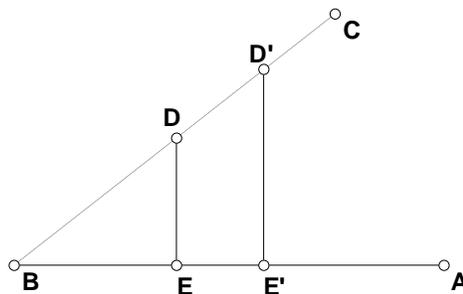


Figure 12

There is a useful relationship between the sines of angles that add up to 180° :

Theorem 21. $\sin(\angle ABC) = \sin(180^\circ - \angle ABC)$

Proof In Figure 13, $\angle FBC = 180^\circ - \angle ABC$ (by BF 6 and algebra). When we apply the definition of sine to $\angle ABC$ we get

$$\sin(\angle ABC) = \frac{DE}{DB}$$

and when we apply the definition of sine to $\angle FBC$ we get

$$\sin(\angle FBC) = \frac{DE}{DB}$$

so $\sin(\angle ABC) = \sin(\angle FBC)$ QED

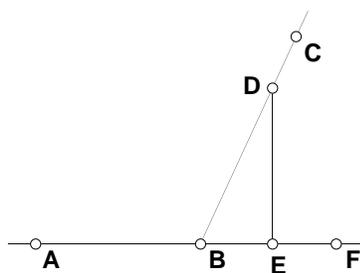


Figure 13

Theorem 22. *For any triangle ABC , the area can be calculated by any of the following three formulas:*

$$\text{area of } \triangle ABC = \frac{1}{2} AB \cdot AC \cdot \sin \angle A$$

$$\text{area of } \triangle ABC = \frac{1}{2} AB \cdot BC \cdot \sin \angle B$$

$$\text{area of } \triangle ABC = \frac{1}{2} AC \cdot BC \cdot \sin \angle C$$

that is, the area is one half the product of two sides times the sine of the included angle.

Proof It is enough to give the proof of the first formula—the proofs of the other two are completely similar. There are three cases: $\angle A < 90^\circ$, $\angle A = 90^\circ$, and $\angle A > 90^\circ$ (see Figure 14), but in fact the proof is the same in all three cases. Let us choose AB to be the base of the triangle, and (as usual) use BF 12 to draw a line through C perpendicular to \overleftrightarrow{AB} which intersects \overleftrightarrow{AB} at a point E . Then CE is the height of the triangle, and so we have

$$(*) \quad \text{area of } \triangle ABC = \frac{1}{2} AB \cdot CE$$

by Theorem 7. But the definition of sine says

$$\sin(\angle A) = \frac{CE}{CA}$$

and so

$$(**) \quad CE = CA \cdot \sin(\angle A)$$

Combining (*) and (**) gives the formula we wanted to prove. QED

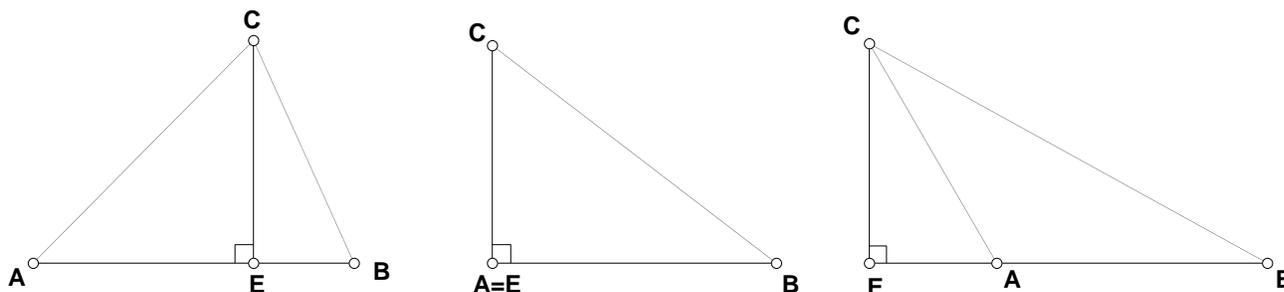


Figure 14

By comparing the three formulas in Theorem 22, we get an interesting relationship called the Law of Sines:

Theorem 23 (Law of Sines). *In any triangle ABC ,*

$$\frac{\sin(\angle A)}{BC} = \frac{\sin(\angle B)}{AC} = \frac{\sin(\angle C)}{AB}$$

Proof Theorem 22 tells us that

$$(*) \quad \frac{1}{2}AB \cdot AC \sin(\angle A) = \frac{1}{2}AB \cdot BC \sin(\angle B)$$

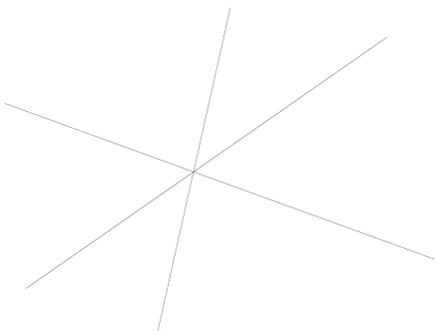
(since both sides of the equation are equal to the area of $\triangle ABC$). Now (*) and algebra give

$$\frac{\sin(\angle A)}{BC} = \frac{\sin(\angle B)}{AC}$$

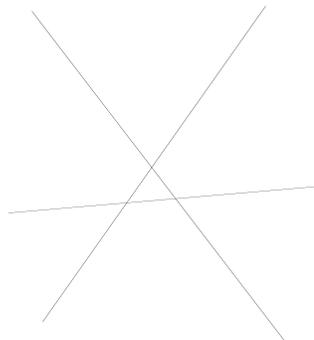
The proof of the other equality is completely similar. QED

4 Concurrency Theorems.

Definition of concurrent lines Three lines are *concurrent* if they meet at a single point.



concurrent lines



non-concurrent lines

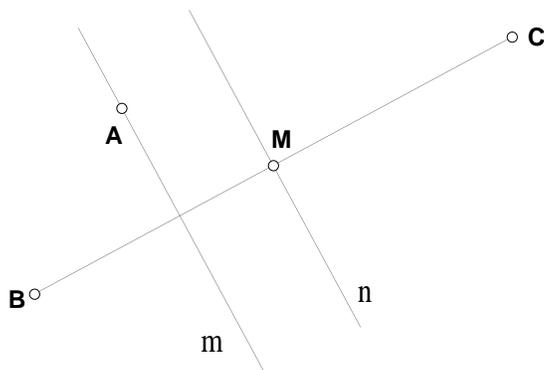
It's very unusual for three lines to be concurrent—ordinarily three lines will form a triangle. But in this chapter we will see that certain kinds of lines associated with a triangle are *forced* to be concurrent.

4.1 Concurrency of the perpendicular bisectors, angle bisectors, and altitudes.

Our first “concurrency theorem” concerns the perpendicular bisectors of the sides of a triangle.

Definition of perpendicular bisector The perpendicular bisector of a line segment is the line that goes through the midpoint and is perpendicular to the segment.

A common mistake: Suppose you have a point A and a line segment BC , and you put the following statement in a proof: “Draw the perpendicular bisector of BC through A ” This is wrong because the perpendicular bisector of BC may not go through A . It's a useful exercise to see why a step like this is not justified by BF 10 and BF 12. BF 10 allows us to find the midpoint M of BC , and BF 12 allows us to draw a line m through M which is perpendicular to BC ; then m is the perpendicular bisector of BC . On the other hand, BF 12 also allows us to draw a line n through A which is perpendicular to line BC . But m and n will not ordinarily be the same line, so it's not allowable to say “Draw the perpendicular bisector to BC through A ,” because ordinarily there is no such line.



Theorem 24. *For any triangle ABC , the perpendicular bisectors of AB , AC and BC are concurrent.*

It turns out that the obvious way of trying to prove Theorem 24 leads to a dead end. Anyone's first idea would be to draw the three perpendicular bisectors and see what happens. But then there would be two cases: the three perpendicular bisectors might meet in a point as in Figure 15:

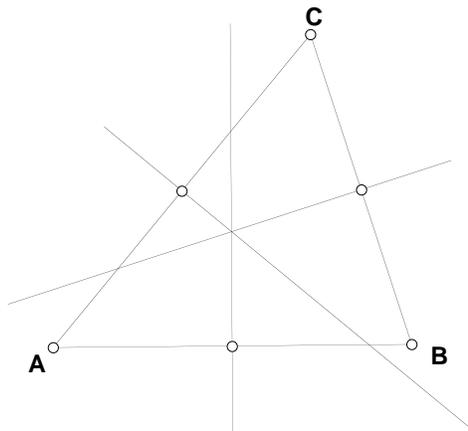


Figure 15

but since we don't yet *know* Theorem 24 we can't assume that the picture wouldn't look like Figure 16:

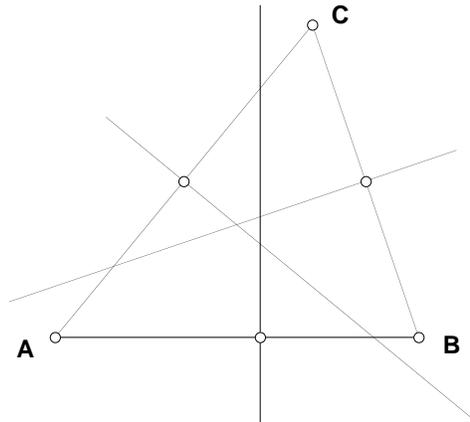


Figure 16

and Figure 16 is a dead end: there's nothing interesting we can say about it.

Instead, we do something trickier. We draw two of the perpendicular bisectors, then draw the line segment connecting their intersection to the third midpoint, and use congruent triangles to show that this line segment is the third perpendicular bisector. Here are the details:

Proof of Theorem 24 We are given the midpoints and the perpendicular bisectors of the three sides. Give the midpoints of AB , AC , and BC respectively the names M , N , and P .

The perpendicular bisectors of AB and AC intersect in a point which we will call X . Connect X to P . There are three cases:

- Case (i) X is inside the triangle.
- Case (ii) X is outside the triangle.
- Case (iii) X is on BC .

I'll give the proof of Case (i) here and ask you to do the other two on the homework.

(See Figure 17.) Our strategy is to show that XP is the perpendicular bisector of BC . First observe that $AM = MB$ (definition of midpoint), $MX = MX$, and $\angle AMX = \angle BMX$ (by definition of perpendicular). So $\triangle AMX \cong \triangle BMX$ by BF 2, and hence

$$(*) \quad AX = BX$$

(by the definition of congruent triangles). Similarly, $\triangle ANX \cong \triangle CNX$, so

$$(**) \quad AX = CX$$

Combining $(*)$ and $(**)$ gives

$$BX = CX$$

But also $CP = BP$ (definition of midpoint) and $PX = PX$ so $\triangle CPX \cong \triangle BPX$ (by BF 1) and therefore

$$(***) \quad \angle 1 = \angle 2$$

(definition of congruent triangles). On the other hand, Theorem 1(a) says that

$$(***) \quad \angle 1 + \angle 2 = 180^\circ$$

Combining $(***)$ and $(***)$ we see that $\angle 1 = 90^\circ$, so PX is perpendicular to BC (definition of perpendicular). Since we already knew that P is the midpoint of BC , we conclude that

(†) PX is the perpendicular bisector of BC

(definition of perpendicular bisector).

Now X is on the perpendicular bisectors of AB and AC (because of the way X was constructed) and also on the perpendicular bisector of BC (by (†)). That is, all three perpendicular bisectors contain X , and so they are concurrent (definition of concurrent). QED

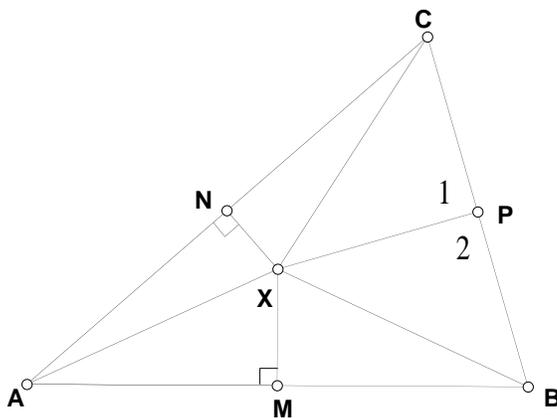


Figure 17

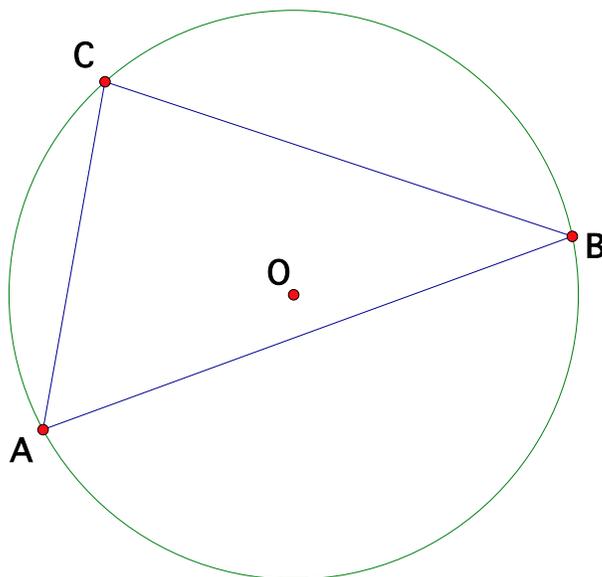
Definition of circumcenter The point where the three perpendicular bisectors of the sides of a triangle meet is called the circumcenter of the triangle.

It is traditional to give the circumcenter of a triangle the name O . The circumcenter has another important property. To state it, we first need

Definition of circle A circle consists of all of the points which are at a given distance (called the *radius*) from a given point (called the *center*).

Theorem 25. *Given a triangle ABC with circumcenter O , suppose that C is a circle with center O that goes through one of the vertices of the triangle. Then C also goes through the other two vertices.*

Proof O is the point that was called X in the proof of Theorem 24, and in that proof it was shown that $XA = XB = XC$. So if a circle \mathcal{C} with center at X goes through one of the vertices then it must go through all three vertices (definition of circle). QED



The circle in Theorem 25 is called the *circumscribed* circle of $\triangle ABC$. Notice that Theorem 25 gives a way to construct a circle going through any three given points (as long as they're not all on the same line).

We conclude this section with two more concurrence theorems.

Theorem 26. *For any triangle ABC , the bisectors of $\angle A$, $\angle B$ and $\angle C$ are concurrent.*

The proof is a homework problem.

Definition of incenter The point where the three angle bisectors meet is called the incenter of the triangle.

It is traditional to give the incenter of a triangle the name I . We will see in Section 8.5 that the incenter is the center of a circle which is tangent to all three sides of the triangle; this circle is called the *inscribed circle* of the triangle and this is the reason for the name “incenter.”

Definition of altitude An altitude of a triangle is a line that goes through a vertex of the triangle and is perpendicular to the opposite side.

Theorem 27. *For any triangle ABC , the three altitudes are concurrent.*

The proof is a homework problem.

Definition of orthocenter The point where the three altitudes meet is called the orthocenter of the triangle.

It is traditional to give the orthocenter of a triangle the name H . The name “orthocenter” is somewhat misleading since this point isn’t the center of an interesting circle. The name comes from the fact that “orthogonal” is another word for perpendicular.

4.2 Concurrence of the medians.

Definition of median A median of a triangle is a line that goes through a vertex of the triangle and through the midpoint of the opposite side.

We will show that the medians of a triangle are always concurrent. In order to prove this, we need a preliminary fact which is interesting for its own sake.

Theorem 28. *The point where two medians of a triangle intersect is $2/3$ of the way from each of the two vertices to the opposite midpoint.*

Proof In Figure 18, we are given that AP and BN are medians and that X is their intersection. We need to prove that $AX = \frac{2}{3}AP$ and $BX = \frac{2}{3}BN$.

Connect N and P . N and P are the midpoints of AC and BC (by definition of median) so by Theorem 18 we see that NP is parallel to AB . But then $\angle 1 = \angle 4$ and $\angle 2 = \angle 3$ (Theorem 2(a)), and so $\triangle ABX \sim \triangle PNX$ (Theorem 4 and the definition of similar triangles). Now BF 4 implies

$$(*) \quad \frac{AX}{XP} = \frac{BX}{XN} = \frac{AB}{NP}$$

But Theorem 18 also tells us that $\frac{AB}{NP} = 2$, so $(*)$ implies

$$(**) \quad AX = 2XP$$

On the other hand, BF 6 gives

$$(***) \quad AX + XP = AP$$

and combining $(**)$ and $(***)$ we see that $AX = \frac{2}{3}AP$. A similar argument shows that $BX = \frac{2}{3}BN$. QED

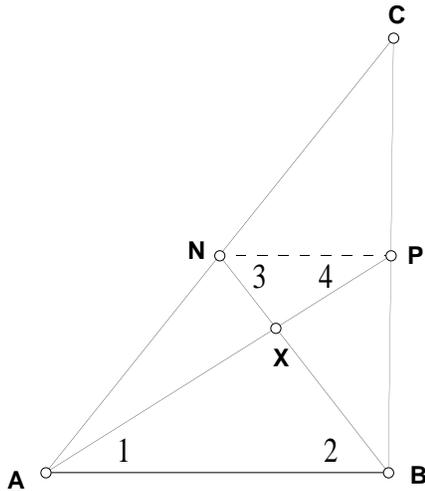


Figure 18

Theorem 29. *For any triangle ABC , the three medians are concurrent.*

Proof We are given the midpoints of sides AB , AC , and BC ; we name these midpoints M , N , and P respectively. Then AP , BN , and CM are the medians of $\triangle ABC$ (definition of median).

Give the intersection of AP and BN the name X (see the first picture in Figure 19). Then

$$(*) \quad AX = \frac{2}{3}AP$$

by Theorem 28.

Next, give the intersection of AP and CM (see the second picture in Figure 19) the name Y . Then

$$(**) \quad AY = \frac{2}{3}AP$$

by Theorem 28.

Combining $(*)$ and $(**)$ we see that $AX = AY$. But then X and Y are the same point, by BF 8. Now X is on AP and BN (because of the way it was constructed) and it is also on CM (because X is the same point as Y). So all three medians contain X , and hence they are concurrent (definition of concurrent). QED

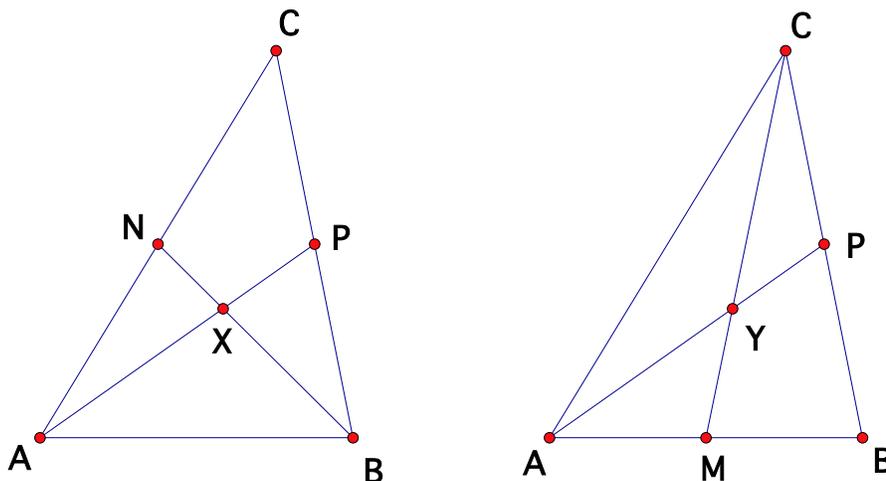


Figure 19

Definition of centroid The point where the three medians meet is called the centroid of the triangle.

It is traditional to give the centroid of a triangle the name G .

4.3 The Euler line.

Definition of collinear Three points are said to be collinear if they all lie on the same line.

In this section we will prove:

Theorem 30. *Let ABC be any triangle. Let O be the circumcenter of ABC , let G be the centroid of ABC , and let H be the orthocenter of ABC . Then O , G and H are collinear.*

This theorem was first proved by Euler in the eighteenth century by analytic geometry. The proof we will give was first discovered in the nineteenth century.

Proof See Figure 20. We will use an indirect approach, so we use BF 8 to construct a point H' on \overleftrightarrow{OG} , on the opposite side of G from O , with

$$(*) \quad H'G = 2OG$$

Our goal is to show that H' is the same point as H , and for this we only need to show that H' lies on all three altitudes of $\triangle ABC$.

Construct the midpoint M of AB (possible by BF 10) and connect CM . CM is a median of $\triangle ABC$ (definition of median) and so G is on CM (definition of centroid). Theorem 28 tells us that $CG = \frac{2}{3}CM$. Using BF 6 and algebra, we see that $MG = \frac{1}{3}CM$, so

$$(**) \quad \frac{CG}{MG} = 2$$

Now draw in \overleftrightarrow{OM} and $\overleftrightarrow{CH'}$. By Theorem 1(b) we have

$$(***) \quad \angle OGM = \angle H'GC$$

Combining (*), (**), (***) and Theorem 19 we obtain

$$(***) \quad \triangle OGM \sim \triangle H'GC$$

Now by definition of similar triangles we have $\angle OMG = \angle H'CG$, and by Theorem 2(a) we see that

$$(\dagger) \quad OM \text{ is parallel to } \overleftrightarrow{CH'}.$$

Next observe that the perpendicular bisector of AB contains O (by definition of circumcenter) and M (by definition of perpendicular bisector). It is therefore the same line as OM by BF 7; this tells us that OM is perpendicular to AB (by definition of perpendicular bisector) and so we have $\angle 1 = 90^\circ$ (by definition of perpendicular). Therefore $\angle 2 = 90^\circ$ by BF 5. This tells us that CH' is perpendicular to AB (definition of perpendicular) and therefore CH' is an altitude of $\triangle ABC$ (definition of altitude). That is, H' is on the altitude of $\triangle ABC$ that goes through C .

Similarly, H' is on the other two altitudes of $\triangle ABC$. Since H is also on all three altitudes (definition of orthocenter), we conclude that H and H' are the same point. Since we constructed H' to be on \overleftrightarrow{OG} , we now know that H is on \overleftrightarrow{OG} , so O , G and H are collinear (definition of collinear). QED

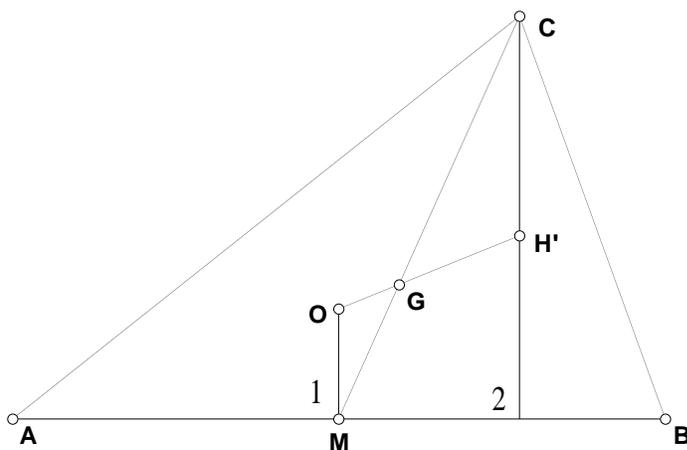


Figure 20

5 Euclid's Elements.

Introduction. Euclid's book was written around 300 BC. It was the finishing step in a developmental process that went back about 300 years, but unfortunately we have very little evidence of what the earlier stages of Greek geometry looked like: Euclid's text was so successful that the earlier versions ceased to be copied by hand (which was the only way for books to be preserved in those days). Greek research in geometry continued for several hundred more years, but Euclid's text was established as the definitive account of the foundations of geometry, and it retained this position until the early 19th century.

What was special about Greek geometry? Before the time of the Greeks there were already ancient peoples who knew many geometrical facts, for instance the Babylonians and the Egyptians, and since their time there have been other cultures, such as the Hindus, the Chinese and the Mayans, who rediscovered various facts (including the Pythagorean theorem) independently.

The new feature of Greek geometry, which had never existed before and which has never been discovered independently by other cultures, is the idea of proof. It is also true that the Greeks had factual knowledge about geometry which was vastly more extensive and sophisticated than of any other culture which pursued the subject. These two features of the Greek achievement were not unrelated, as we shall see.

What is proof? Briefly, proof means explaining why a given fact is true by showing how it follows logically from simpler facts.

When we do this, it is important to avoid circular argument, so we need to be sure that the simpler facts can themselves be explained without using the fact we are proving. (Here is a simple example of a circular argument: "There's no school today." "How do you know?" "My brother told me." "How does he know?" "I told him").

This in turn means that we have to have some way of keeping track of what has been used to prove what. Euclid does this by numbering the propositions and never using a later one to prove an earlier one.

What are postulates and "common notions?" It is important to realize that not every statement can be reduced to simpler pieces, since this would involve an infinite process of explaining each statement in terms of simpler ones, and then explaining those in terms of even simpler ones, and so on forever. We have to start somewhere, with statements whose truth is self-evident. In Euclid, these statements are the postulates and the axioms (*axioma* is the Greek word which

Heath translates as “common notions”). The difference between an axiom and a postulate in Euclid is that an axiom is a statement which could apply to other parts of mathematics besides geometry and a postulate is a statement specifically about geometry.

To sum up, a theorem or proposition is a true fact which can be explained in terms of simpler facts. A postulate or axiom is a true fact which cannot be explained in terms of simpler facts.

Incidentally, **a definition, unlike a postulate or axiom, is not a statement about reality. It is an agreement to use a certain word or short phrase to stand for a longer phrase.** For example, we use the word “parallelogram” to stand for “a four-sided figure with two pairs of parallel sides.”

Euclid’s system has two very remarkable features which may not be obvious at first glance. The first is that he has built *all* of geometry into a *single* unified system, where each statement follows ultimately from a few simple postulates and axioms. The second is the small number of the postulates and axioms and their simplicity. This can be illustrated with an analogy from chemistry: every chemical compound, no matter how complicated, can be broken down into the 92 naturally occurring elements. What Euclid showed was that in geometry the situation is even simpler: instead of 92 basic ingredients only 10 are needed.

The relation between proof and discovery. It is important to realize that proving and discovering are not unrelated activities: by sharpening their ability to explain a given fact in terms of known ones the Greeks also developed the skill of using thought to pass from known facts to new discoveries. The reason for this is that when the facts already known are organized and explained logically they can be used as a new basis for the imagination, and a strengthened imagination can explore new areas that would otherwise be out of reach. You have experienced something like this already in doing the homework problems.

Why does Euclid prove things that are obvious? For example, Proposition 5 is obvious from sight, and yet Euclid gives a rather elaborate proof of it. The question of why he does this is a very important one, which comes up even in high school mathematics, and you should decide for yourself what you think about it (it is worth knowing that even Isaac Newton, one of the greatest of mathematicians, had questions on this subject). Here are some relevant observations:

- If Euclid didn’t prove the facts in Book 1 which are obvious from sight, then he would have to add them to the list of postulates (because a true fact about geometry which can’t be explained in terms of simpler facts is a postulate) and this means that instead of having a simple list of 5 postulates we would have about 15.

- For the Greeks it was important not to rely on perceptions such as sight to establish the truth of things, because everyone knows that perceptions can be mistaken (as in optical illusions) and the Greeks wanted to discover truths about mathematics (and other things) that didn't suffer from human limitations but were eternally and universally valid.
- The exercise of proving something which is obvious from sight is one of the best ways to sharpen your sensitivity to subtle logical issues. This brings me to the topic of:

Cleverness and subtlety. Cleverness and subtlety are two different things. It is a little hard to describe each of them precisely, but here is an attempt. Some ways in which *cleverness* might be used in solving a homework problem are: finding the right pair of congruent or similar triangles, or drawing in the right extra line. *Subtlety* often seems like a logical trick; the main example we have seen so far is the indirect approach used in Theorems 18, 19, 24, 26 and 29. Proof by contradiction is also an example of subtlety.

The main point I want to make right now is that subtlety isn't just for logical tricksters but is a tool for solving problems which is just as important as cleverness.

So here is one reason why Euclid proves things that are obvious: it improves our ability to be subtle, and this in turn improves our ability to solve problems.

Flaws in Euclid. Since Euclid was a pioneer in organizing geometry as an axiomatic system, it is not surprising that he made some mistakes (more precisely, omissions, mainly in the form of missing axioms). I will point some of these out when we come to them, and you will notice others. It is interesting that almost none of these flaws was observed until the nineteenth century, when the discovery of non-Euclidean geometry caused mathematicians to take a very close look at Euclid. For the moment I want to observe that Euclid got it 90–95% right and that nineteenth-century mathematicians completed his work so that there are no flaws at all any more.

Learning from the historical process. You might ask why, since Euclid's treatment is not perfect, we use his version instead of some other. Here are some reasons:

- Euclid's version is a good place to start since it is closer to your own point of view than a more perfected version would be. He focuses on issues that you can learn from and does not obscure them with more technical issues that are necessary for complete perfection.
- A related point is that Euclid's language is closer to ordinary language than a perfected treatment would be.

- You can sharpen your own understanding by learning to spot the missing steps.

The influence of Euclid on modern mathematics. Until the nineteenth century, geometry was the only part of mathematics which was organized on an axiomatic basis. In the nineteenth century mathematicians began to encounter problems (involving Fourier series and other parts of calculus) which could not be answered confidently without the kind of close analysis provided by an axiomatic approach. They therefore undertook the project of reorganizing calculus on the Euclidean model. This led to the formal definition of limit and eventually to a completely axiomatic and rigorous approach to calculus. In the twentieth century all of mathematics has been organized on an axiomatic basis, ultimately resting on the axioms of set theory.

6 Comments on the definitions, postulates, common notions and propositions of Euclid Book I

The definitions. Many of the definitions (such as those for point and line) are not used at all in the proofs. A closer examination of some of these definitions shows that they are not really definitions in the mathematical sense: they don't describe the thing in terms of simpler things, as the definition of parallelogram does. The definitions that *are* used later are 10, 15, 16, 20, 22, and 23.

Note that in definition 20 an equilateral triangle is not considered to be isosceles, and in definition 22 a square is not considered to be an oblong (what does Euclid mean by oblong?) This is different from the modern practice. It is not uncommon for different versions of a subject to use somewhat different definitions; this is allowable because definitions are not statements about reality but are convenient abbreviations which the author and his or her readers agree on. Over the course of time it has been found to be more convenient to consider an equilateral triangle as a special kind of isosceles triangle, rather than to require that an isosceles triangle has two but not three equal sides, which is what Euclid does.

Within a given treatment of geometry the definitions can't be changed: the author has to use each word with the same meaning all the time.

Postulate 1 This is our BF 7, except that Euclid doesn't say explicitly that there is *only one* line through two given points.

Postulate 2 This is our BF 9.

Postulate 3 We didn't assume this as a Basic Fact, but Euclid will use it to prove some of the Basic Facts that we did assume.

It is important to realize that this Postulate allows us to do “Circle by center and point” but *not* “Circle by center and radius.” (This is the opposite of what you might expect from the way it is worded). I will explain this in class.

Postulate 4 This really *is* a statement about reality and not a definition, as I will explain in class.

Postulate 5 This postulate doesn’t seem nearly as self-evident as the others, so for about 2300 years people tried to prove it. Eventually they discovered that it cannot be proved from the other postulates and common notions—surprisingly, this led to a new subject, non-Euclidean geometry.

Common Notion 1. This is transitivity.

Common Notions 2 and 3. These are included in what we have been calling “algebra.” Euclid’s goal here is to state in advance what kinds of algebra he will allow, although he does sometimes use things not included in Common Notions 2 and 3, as we shall see.

Proposition 1 Frequently asked questions:

What does “circle BCD ” mean in line 8? Answer: Euclid names a circle by giving three points on it. Sometimes this means that he has to create a point whose only purpose is to name the circle it’s on—that’s what he’s done here with point D . He has to give three points instead of just two because there is more than one circle going through two given points but only one circle going through three given points.

What does “Post. 1” stand for in line 14? Answer: “Postulate 1” Incidentally, all of the reasons given in brackets were supplied by the modern editor, T.L. Heath, and were not in Euclid’s original manuscript. On quizzes and exams, you are responsible for giving all of the reasons, including those that Heath forgets to give, using only facts from Euclid and not from the course notes. (There is one exception: you don’t have to give citations for Postulate 1, since it occurs so often and is so elementary.)

What does “C.N. 1” stand for in line 22? Answer: “Common Notion 1”

Like all construction proofs, this one has two parts: a recipe for the construction (lines 1–14) and a verification that the recipe does what it is supposed to do (lines 15–26).

Notice that the proof ends with a summary: “Therefore the triangle ABC is equilateral”

Also notice the very last line: “what it was required to do”—this is how Euclid ends a construction proof (other proofs end with “what it was required to show.”)

Proposition 2 Frequently asked questions:

What does “I.1” mean in line 10? Answer: “Book I, Proposition 1”

What does “circle CGH ” mean in line 15? Answer: Here Euclid is being a bit sloppy in his explanation. What he really means is that we draw the circle that has center B and goes through C , then we find the intersection of this circle with the extended line DF and call this intersection point G . The point H plays no role except to be part of the name of the circle.

What does “circle GKL ” mean in line 16? Answer: Here we draw the circle with center D that goes through G , find the intersection of this circle with the extended line DE , and call this intersection L . The point K plays no role except to be part of the name of this circle.

The purpose of this proof is to show how one can carry out “Circle by center and radius” using only “Circle by center and point.”

This is a construction proof, so it has a recipe and a verification that the recipe works. The recipe is given in lines 1–17, and the verification is lines 18–32.

Proposition 3 This includes part of our BF 8. (It says that on a given ray there is a point at a given distance from the endpoint, but not that there is *only one* such point.)

Now that we have Proposition 3, we will never use Proposition 2 again. The only purpose of Proposition 2 was to prepare for the proof of Proposition 3.

Proposition 4 This is SAS. We have *assumed* it as BF 2, but Euclid gives a *proof* of it.

Frequently asked question: What does “Therefore etc.” mean in line 44? Answer: the editor has gotten tired of writing out the summary at the end of the proof, so he uses this for an abbreviation (see his note in the middle of page 249). Compare this to the last two lines in the proofs of Propositions 1, 2 and 3.

It is very important to note here that “the triangle will be equal to the triangle” (in line 4) does *not* mean “the triangles are congruent.” Euclid has no term corresponding to our term “congruent,” so whenever he wants to say that two triangles match up completely he has to mention all six parts, which gets tedious. When he says that two triangles are “equal” he is definitely not saying that they match up in all six parts, he is only saying that they have equal *area*. This can be seen most clearly by looking at the statement and picture for Proposition 37.

Proposition 5 This is one direction of our Theorem 5. Notice that we used BF 10 and BF 1 in our proof, and that Euclid doesn’t yet have these tools (eventually he proves BF 1 and BF 10 as Propositions 8 and 10).

Frequently asked question: What is happening in lines 15–25? Answer: Euclid is quoting Proposition 4 (SAS). Lines 15–19 show that the hypotheses of Proposition 4 are satisfied in his situation, and lines 20–25 spell out what the conclusion of Proposition 4 says in his situation. Note that he quotes Proposition 4 word for word instead of just referring to it—presumably this is to emphasize the point that he is specializing a general statement to a particular situation and that the hypotheses must match up exactly.

You should ignore the last part of the statement (“and, if the equal straight lines be produced further, the angles under the base will be equal to one another”) and its proof (lines 46–47).

Note that the proof ends with QED. This stands for the Latin phrase “Quod erat demonstrandum,” which is a translation of the Greek phrase which means “what was to be shown.” The editor ends construction proofs from now on with QEF, which stands for “quod erat faciendum” meaning “what was to be done” (see his note about line 48).

Proposition 6 This is the other direction of our Theorem 5.

Notice that in line 16 the phrase “the triangle DBC will be equal to the triangle ACB ” means that these triangles will have the same *area*. (Although in this case it is also true that the triangles will be congruent, that is not the point that Euclid is drawing our attention to here. The point of his argument is that if a triangle is contained inside another then its area must be smaller than that of the triangle it is contained in, so the areas of the two triangles cannot be equal, “the lesser to the greater.”)

This is our first example of a **proof by contradiction**.

Proposition 7 This proposition is only used to prove Proposition 8, and will never be used again. The modern name for a proposition which is used only as a tool to prove something else is “lemma.” You might find it easier to follow the proof of this proposition if you use numbers instead of three-letter names to label some of the angles.

What the proposition says is that, for a given point C , there can’t be a different point D on the same side of AB with *both* $AC = AD$ and $BC = BD$. (Notice that it’s perfectly possible to have a D with *either* $AC = AD$ or $BC = BD$).

Notice that in line 21 Euclid says that the angle CDB is “much greater” than the angle DCB . What is happening here is that Euclid is using the fact that if one thing is greater than a second, and the second is greater than a third, then the first is greater (in fact “much greater”) than the third. This sounds similar to Common Notion 1 but in fact it is a missing Common Notion that should have been added to the original list.

Proposition 8 This is SSS. We have *assumed* this, as BF 1, but Euclid *proves* it.

Propositions 9–12 These constructions are often taught in high school. Notice that (as usual for construction proofs) each of these proofs consists of a recipe followed by an explanation of why the recipe works.

Proposition 9 is our BF 11, Proposition 10 is our BF 10, and Propositions 11 and 12 are our BF 12.

Propositions 13 and 14 These are converses of each other, and Euclid uses Proposition 13, plus proof by contradiction, to prove Proposition 14. We will see this pattern, of a statement being used to prove its converse by means of proof by contradiction, several times. (But it isn't *always* possible to prove the converse this way, as we will see in discussing Propositions 27 and 29.) Proposition 14 will be used later to prove that certain lines are straight.

Proposition 16 Frequently asked questions:

Why doesn't Euclid just use the fact that the angles of a triangle add up to 180° ? Answer: Because it isn't yet available—the fact that the angles of a triangle add up to 180° is Proposition 32.

OK, then why doesn't Euclid just wait to prove Proposition 16 until after he has proved Proposition 32? Answer: He can't do this because it would be a circular argument! One of the steps in proving Proposition 32 is Proposition 31 (this is the fact that it is possible to construct a parallel line through a given point), and Proposition 31 in turn uses Proposition 27, and Proposition 27 uses Proposition 16. There is no way to prove Proposition 32 without proving Proposition 16 first. (It's true that we proved it in the course notes without using Proposition 16, but that's because we were assuming BF 13. Euclid *proves* BF 13, and to prove it he uses Proposition 16.)

This illustrates a basic difference between axiomatic systems and lists of facts: if we think of Proposition 16 purely as a fact then it is included in Proposition 32 and therefore not very important. But if we think about how to prove that Proposition 32 is true then Proposition 16 is an indispensable tool.

One more comment that is of some interest is that Proposition 16 (since it doesn't make use of Postulate 5) is true in both Euclidean and non-Euclidean geometry, whereas Proposition 32 is *not* true in non-Euclidean geometry.

Another frequently asked question: in line 10, how do we know where F is supposed to be? Answer: Euclid is cutting corners a bit here and should have said it more clearly. What he should have said is “let BE be extended (Postulate 2) and let a point F be marked on the extension with $EF = BE$ (Proposition 3).” This is the sort of thing that modern mathematicians are always very careful to spell out, which is one reason why modern mathematical writing often seems cluttered.

Propositions 18 and 19 This is another example where the statements are converses of each other and the second is proved from the first using proof by contradiction.

Euclid's statements of these propositions are somewhat confusing, so I've clarified them on the printed list that I'll hand out with quizzes and exams.

Proposition 20 Read the editor's note about the Epicureans on page 287. What the Epicureans were asking was "why do we prove things that are obvious?" The Epicureans were intelligent people and we shouldn't be surprised if our own students ask the same question. I have already given my own answer to this earlier in these notes, but at this point I can add to it: you can probably see now that one reason we prove things that are obvious is that the proofs are often exciting examples of the art of proof.

Propositions 24 and 25 This is yet another example where the second proposition is the converse of the first and is proved from it using proof by contradiction.

It is easier to follow the proof of Proposition 24 if you use numbers instead of three-letter names to label some of the angles.

Proposition 26 This has two parts: the first is ASA (proved on page 302) and the second is AAS (proved on page 303).

Frequently asked questions:

Why doesn't Euclid just use ASA to prove AAS like we usually do? Answer: because in order to do this you need Proposition 32 which he doesn't yet have.

OK, then why doesn't Euclid delay Proposition 26 until after Proposition 32? Answer: this is a harder question to answer. He could have done that, but we think that the reason he didn't was because he wanted to prove as many things as possible without using Postulate 5.

Does this mean there's something wrong with Postulate 5? Answer: No. This is another place where the difference between an axiomatic system and a list of facts becomes important. No one doubted that Postulate 5 was a true fact—the question was whether it was really a postulate (a true fact which cannot be proved) or a proposition (a true fact which can be proved). Euclid clearly felt that it was a postulate, but he was also willing to give people a chance to try to prove it if they wanted to and this seems to be why he showed how to prove Proposition 26 without it; now Proposition 26 could be used in trying to prove Postulate 5 without creating a circular argument. As a matter of historical fact, for the next 2300 years people made many attempts to prove Postulate 5, all of which turned out to be incorrect. We now know that it is impossible to prove Postulate 5, and this *impossibility* is closely related to the *possibility* of non-Euclidean geometry.

Proposition 27 This is the second part of our Theorem 2(a).

Proposition 28 This is the second part of our BF 5 and Theorem 2(b).

Proposition 29 This is the first parts of our BF 5, Theorem 2(a) and Theorem 2(b).

Proposition 29 is the converse of Proposition 27, but in this case Euclid does not follow the pattern of proving the converse from the original statement by contradiction. He could not follow the pattern if he wanted to because it is impossible (try it for yourself). Instead, he has to introduce a new ingredient that he hasn't used before, namely Postulate 5 (see line 24 of the proof).

Proposition 30 This is our BF 14.

Frequently asked question: Isn't this just transitivity? Answer: Transitivity is Common Notion 1 (things which are equal to the same thing are also equal to one another). In our case we are not asserting that the lines are "equal" but rather that they are "parallel," which is different. Common Notion 1 is used in the proof (line 18) because we can show that two lines are *parallel* by showing that certain angles are *equal*.

This proof is a very rare example of Euclid making a logical mistake: in line 5 he says "For let the straight line GK fall upon them" but it isn't obvious how to create such a line without *assuming* Proposition 30. I'll show in class how to fix the proof.

Proposition 31 This is our BF 13. We *assume* BF 13 and Euclid *proves* it.

Proposition 34 Notice that the word "parallelogram" (or rather "parallelogrammic area") is used here without having been defined. Euclid seems to assume that we know that the definition of parallelogram is "a four sided figure with two pairs of parallel sides." It is a mystery why he didn't include this definition in his list of definitions.

Proposition 35 This is related to the fact (which Euclid does not use or even mention) that the area of a parallelogram is base times height. In the case we are looking at the parallelograms have the same base, and the fact that they are "in the same parallels" forces them to have the same height.

Frequently asked question: Why doesn't he just say that? Answer: The Greeks had recently discovered the existence of irrational numbers, and because of this they regarded statements about measurement as uncertain and untrustworthy from a scientific point of view. Because of that, Euclid never talks about the length of a segment. The Greeks certainly knew our usual formulas for calculating area but they thought of these formulas as practical rules that didn't belong in a scientific account.

Proposition 36 The editor makes a truly awful mistake here: he gives Proposition 34 as the reason for the statement “Therefore EBCH is a parallelogram.” He should have said that this is because of the missing definition of parallelogram.

Proposition 37 This proof uses another missing common notion: “the halves of equal things are equal to one another.”

Proposition 42 This is the first step in proving Proposition 44. It is a good exercise to carry out the recipe in the proof of Proposition 42 on Geometer’s Sketchpad.

Proposition 43 Very frequently asked question: What does “EH” mean in line 4 and elsewhere? Answer: Euclid has suddenly introduced this as a convenient way of abbreviating “EKHA” as the name of a parallelogram. Similarly, he refers to the parallelograms FCGK, BGKE, and KFDH as FG, BK, and KD. I don’t know why he has decided to do this.

Proposition 44 This is a very powerful construction. To give an idea of how powerful it is, let us observe that we can use it to divide the length of one segment by the length of another using ruler and compass alone. To do this, we let the “given straight line” be the second segment, the “given rectilineal angle” be a right angle, and the “given triangle” be a right triangle with height two and base equal to the first segment. Then the side of the parallelogram (actually a rectangle in this case) which is constructed by Proposition 44 will be the quotient of the first segment by the second segment.

Again, it is a good exercise to carry out the recipe in this proof using Geometer’s Sketchpad (for this you need to have a script that does the recipe in Proposition 42).

I recommend that you read the note which starts at the bottom of page 342 about “the godlike men of old.”

Proposition 45 This proposition allows us to transform any given (polygonal) area into a rectangle with the same area, using ruler and compass alone.

Proposition 47 This is what most of Book I has been leading up to: Euclid’s proof of the Pythagorean theorem. The ancient commentators seem to agree that this proof is due to Euclid himself, which leads to the question of what Pythagoras’s original proof was. The answer seems to be that Pythagoras’s original proof was the same as the one in the course notes (using similar triangles). Between the time of Pythagoras and Euclid the Greeks discovered irrational numbers, which made the whole question of what is meant by “proportional sides” much more complicated—this is why Euclid doesn’t mention similar triangles anywhere in Book I. Eventually Euclid gives a full treatment of similarity (and proves our BF 4) in Book VI, but

he didn't want to delay the Pythagorean theorem until then so he gives a more "elementary" proof now.

Notice that in the diagram on page 349 the lines AL , FC and BK appear to be concurrent—they are in fact concurrent and this can be proved by Theorem 37.

Proposition 48 This is the converse to the Pythagorean Theorem. Euclid's proof uses the fact that if two squares have equal area then their sides are equal. Notice that he doesn't give any justification for this, which is a mistake on his part (maybe he was tired after the elaborate proof of Proposition 47). The missing statement sounds like a common notion (similar to "equals added to equals are equal") but in fact it can be proved, and the editor gives a proof at the bottom of page 348. This is a good example of how hard it can be to tell whether a simple-looking statement can be proved by using even simpler statements.

7 The theorems of Menelaus and Ceva.

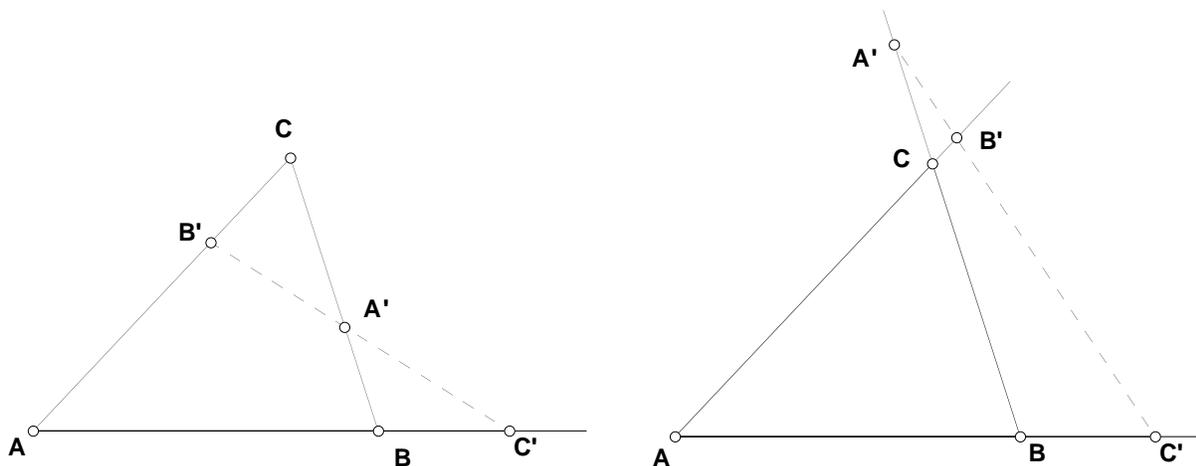
7.1 The theorem of Menelaus.

Our next goal is to prove:

Theorem 31 (Theorem of Menelaus). *Let ABC be any triangle. Let A' be a point of \overleftrightarrow{BC} other than B and C , let B' be a point of \overleftrightarrow{AC} other than A and C , and let C' be a point of \overleftrightarrow{AB} other than A and B . If A' , B' and C' are collinear then*

$$\frac{A'B}{A'C} \frac{B'C}{B'A} \frac{C'A}{C'B} = 1$$

The following picture illustrates the two cases of the theorem: either the line $A'B'C'$ crosses the triangle, or it stays entirely outside the triangle.



The theorem is named after Menelaus, who was a Greek mathematician in Alexandria in the second century AD. A line which crosses all three of the lines \overleftrightarrow{AB} , \overleftrightarrow{AC} and \overleftrightarrow{BC} , without going through any of the points A , B and C , is called a *Menelaus line*.

Proof We have to prove both cases:

Case (i) The Menelaus line $A'B'C'$ crosses the triangle.

Case (ii) The Menelaus line $A'B'C'$ is entirely outside the triangle.

I'll give the proof for Case (i); the other case is a homework problem.

(See Figure 21) Draw a line m through B parallel to AC (allowed by BF 13) and give the intersection of m with the line $A'B'C'$ the name X . Now $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$

(both by BF 5 applied to the parallel lines AC and m), and $\angle BC'X = \angle AC'B'$ because they are the same angle, so $\triangle BC'X \sim \triangle AC'B'$ by the definition of similarity. This allows us to apply BF 4 to get

$$(*) \quad \frac{C'B}{C'A} = \frac{BX}{B'A}$$

Next, we have $\angle 5 = \angle 6$ and $\angle 7 = \angle 8$ (both by Theorem 2(a) applied to the parallel lines AC and m) and so $\triangle BA'X \sim \triangle CA'B'$ by Theorem 4 and the definition of similarity. This allows us to apply BF 4 to get

$$(**) \quad \frac{A'B}{A'C} = \frac{BX}{B'C}$$

Solving (*) for BX gives

$$(***) \quad BX = \frac{C'B}{C'A} B'A$$

and solving (**) for BX gives

$$(***) \quad BX = \frac{A'B}{A'C} B'C$$

Combining (***) and (***) we have

$$(\dagger) \quad \frac{C'B}{C'A} B'A = \frac{A'B}{A'C} B'C$$

and multiplying both sides of (\dagger) by $\frac{C'A}{C'B} \frac{1}{B'A}$ gives

$$1 = \frac{A'B}{A'C} \frac{B'C}{B'A} \frac{C'A}{C'B}$$

which is what we were to prove. QED

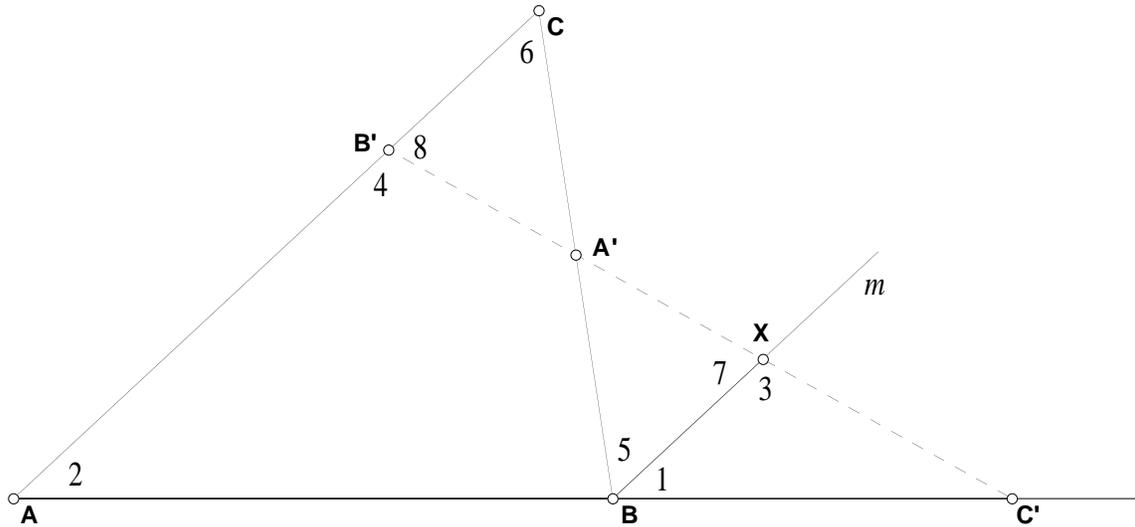
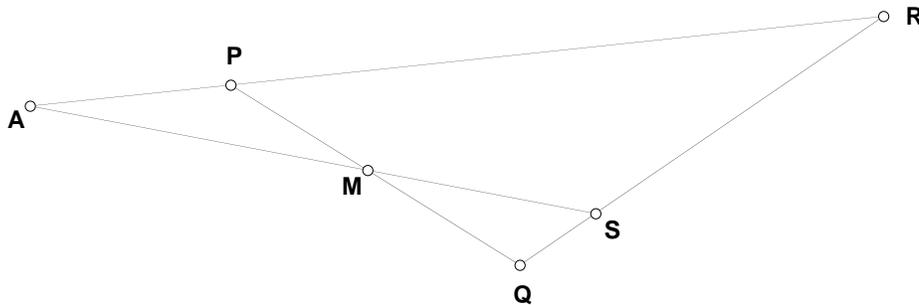


Figure 21

Note: The statement of Menelaus's theorem looks complicated, but there's an easy way to remember it. Suppose we want to apply it to the triangle PQR and the Menelaus line SMA in the following picture:



First write out a partial version of the Menelaus equation using just the points on the Menelaus line (in any order):

$$\frac{S}{S} \frac{M}{M} \frac{A}{A} = 1$$

Now fill in the rest by pairing each point on the Menelaus line with the two vertices of the side that contains it, *but make sure that each vertex appears once in the numerator and once in the denominator*. In our example, the point S gets paired with Q and R , the point M gets paired with P and Q , and the point A gets paired with P and R . We can choose whether to put SQ or SR in the numerator, but after that all choices

are determined. If we put SQ in the numerator, then in the second factor MQ has to go in the denominator (so that Q will appear once in the numerator and once in the denominator), and so on.

$$\frac{SQ}{SR} \frac{MP}{MQ} \frac{AR}{AP} = 1$$

If you go through the process with the points on the Menelaus line in a different order, or if you make a different choice about what goes in the numerator in the first factor, you get a different way of writing the same equation. For example, if we put the points on the Menelaus line in the order ASM and put AP in the numerator of the first factor, we get

$$\frac{AP}{AR} \frac{SR}{SQ} \frac{MQ}{MP} = 1$$

which is just the reciprocal of the previous equation.

7.2 The converse of a statement.

The word “converse” is often useful in understanding logical relationships in geometry. Here’s what it means: when you have a statement “if A then B”, the converse of the statement is “if B then A.” That is, in the converse the “given” and “to prove” are switched. The converse of a true statement may or may not be true. (When both the statement and its converse are true, we use the symbol \iff .)

Here is an example from ordinary life: the statement “if a person lives in West Lafayette then they are an Indiana resident” is true. But its converse says “if a person is an Indiana resident then they live in West Lafayette” and this is false.

7.3 The converse of Menelaus’s theorem: preliminary discussion.

Now let us consider Theorem 31 and its converse. We are given a triangle ABC and points A' , B' and C' on the lines \overleftrightarrow{BC} , \overleftrightarrow{AC} and \overleftrightarrow{AB} respectively. Theorem 31 says that if A' , B' and C' are collinear then the equation

$$(*) \quad \frac{A'B}{A'C} \frac{B'C}{B'A} \frac{C'A}{C'B} = 1$$

is satisfied. That is, the Theorem says that

$$A', B' \text{ and } C' \text{ collinear} \implies \text{equation } (*) \text{ is satisfied.}$$

The converse of Theorem 31 is the statement “whenever equation (*) is satisfied, A' , B' and C' will be collinear.” Rather surprisingly, this statement is not true. You can see this by doing an experiment with Geometer’s Sketchpad. If we choose A' and B' , there

will always be *two* points C' for which equation (*) is satisfied, but only *one* of these two points will be collinear with A' and B' .

On the other hand, the converse of Theorem 31 is close to being true, since for each choice of A' and B' there is only one bad (i.e., noncollinear) point C' which satisfies equation (*). This suggests that we should be able to get a true statement by making a small modification to the converse of Theorem 31.

The first thing to notice is that, of the two points C' on the line \overleftrightarrow{AB} which satisfy equation (*), one of them is always inside the line segment AB and the other one is always outside of AB . This suggests that it's important to keep track of whether the points are on the original (nonextended) sides of the triangle or not. Experimenting with Geometer's Sketchpad, we arrive at the following corrected version of the converse of Menelaus's theorem.

Let ABC be any triangle. Let A' be a point of \overleftrightarrow{BC} other than B and C , let B' be a point of \overleftrightarrow{AC} other than A and C , and let C' be a point of \overleftrightarrow{AB} other than A and B . Suppose that

(1) the equation

$$\frac{A'B}{A'C} \frac{B'C}{B'A} \frac{C'A}{C'B} = 1$$

is satisfied

and

one of the following is true:

(2a) Two of the points A' , B' and C' are on the original (nonextended) sides of the triangle and the third is not

or

(2b) None of the three points is on the original (nonextended) sides of the triangle

Then A' , B' and C' are collinear.

So far we have only experimental evidence for the statement in the box and we still need to prove it. But there's another issue that must be dealt with first: the statement in the box is messy, and this makes it both hard to prove and hard to use in proofs. We need to find a way to say the same thing in a more convenient way. The way to do this is to incorporate a + or - sign in ratios like $\frac{C'A}{C'B}$ to keep track of whether C' is inside the segment AB or not. We will explain how to do this in the next section.

7.4 Signed ratios.

Definition of signed ratio Let A , B and C be three points on a line. The symbol

$\frac{\overrightarrow{CA}}{\overrightarrow{CB}}$ is equal to $\frac{CA}{CB}$ if C is outside of the segment AB and is equal to $-\frac{CA}{CB}$ if C is inside the segment AB .

(One way to think about the signed ratio $\frac{\overrightarrow{CA}}{\overrightarrow{CB}}$ is that it is the quotient of the *vector*

\overrightarrow{CA} by the *vector* \overrightarrow{CB} . It isn't usually possible to divide one vector by another, but it is possible in our situation since the vectors are on the same line and must therefore be scalar multiples of each other. The signed ratio is positive if the vectors point in the same direction and negative if they point in opposite directions. However, we will not be using vectors in this course.)

We can now restate conditions (2a) and (2b) in the box on page 54 in a simple and convenient way: condition (2a) says that two of the signed ratios $\frac{\overrightarrow{A'B}}{\overrightarrow{A'C}}$, $\frac{\overrightarrow{B'C}}{\overrightarrow{B'A}}$ and $\frac{\overrightarrow{C'A}}{\overrightarrow{C'B}}$ are negative, and condition (2b) says that none of them are negative. So (2a) and (2b) *together* can be combined into the single statement that the product

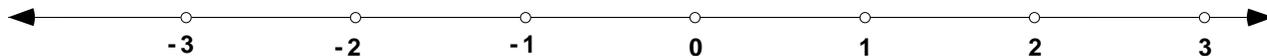
$$\frac{\overrightarrow{A'B}}{\overrightarrow{A'C}} \frac{\overrightarrow{B'C}}{\overrightarrow{B'A}} \frac{\overrightarrow{C'A}}{\overrightarrow{C'B}}$$

is positive. And combining this with condition (1) we see that the box on page 54 simply says that if

$$\frac{\overrightarrow{A'B}}{\overrightarrow{A'C}} \frac{\overrightarrow{B'C}}{\overrightarrow{B'A}} \frac{\overrightarrow{C'A}}{\overrightarrow{C'B}} = 1$$

then the points A' , B' and C' will be collinear. Of course, we still need to prove this (which will be done in Section 7.5).

Our main goal in this section is Theorem 33. Before we get to that we need to think about how signed ratios work when the points are on a number line:



Every point on the number line has a *coordinate*, which is a real number that specifies the exact location of the point.

If X and Y are two points on the number line, with coordinates x and y , then the length of XY is given by the formula $|x - y|$. Now notice that if we leave out the absolute

value symbol, we get a number $x - y$ which contains two pieces of information: its *size* (that is, its absolute value) is the length of XY , and its *sign* tells whether X is to the right or to the left of Y (X is to the right of Y when the sign is positive, and to the left of Y when the sign is negative).

Theorem 32. *Let A , B , and C be three points on a number line, and let a , b and c be their coordinates. Then*

$$\frac{\overrightarrow{CA}}{\overrightarrow{CB}} = \frac{c - a}{c - b}.$$

Proof First notice that $\frac{\overrightarrow{CA}}{\overrightarrow{CB}}$ and $\frac{c - a}{c - b}$ have the same absolute value (because the absolute value of $\frac{\overrightarrow{CA}}{\overrightarrow{CB}}$ is $\frac{CA}{CB}$, and this is equal to $\frac{|c - a|}{|c - b|}$), so we just need to check that they have the same sign.

There are four cases to consider:

Case (i): C is to the right of both A and B .

Case (ii): C is to the left of both A and B .

Case (iii): C is between A and B and A is to the left of B .

Case (iv): C is between A and B and A is to the right of B .

In Case (i), $\frac{\overrightarrow{CA}}{\overrightarrow{CB}}$ is positive, and $\frac{c - a}{c - b}$ is also positive because the numerator and denominator are both positive. In Case (ii), $\frac{\overrightarrow{CA}}{\overrightarrow{CB}}$ is positive, and $\frac{c - a}{c - b}$ is also positive because the numerator and denominator are both negative. In Case (iii), $\frac{\overrightarrow{CA}}{\overrightarrow{CB}}$ is negative, and $\frac{c - a}{c - b}$ is also negative because the numerator is positive and the denominator is negative. In Case (iv), $\frac{\overrightarrow{CA}}{\overrightarrow{CB}}$ is negative, and $\frac{c - a}{c - b}$ is also negative because the numerator is negative and the denominator is positive. QED

Now we can state and prove the main fact of this section.

Theorem 33. *Let ℓ be any line, and let A , B , C' and C'' be points on ℓ , with A not the same point as B and with C' and C'' different from both A and B . If $\frac{\overrightarrow{C'A}}{\overrightarrow{C'B}} = \frac{\overrightarrow{C''A}}{\overrightarrow{C''B}}$ then C' is the same point as C'' .*

Proof Make ℓ into a number line and let c' , c'' , a and b stand for the coordinates of C' , C'' , A and B . We are given

$$\frac{\overrightarrow{C'A}}{\overrightarrow{C'B}} = \frac{\overrightarrow{C''A}}{\overrightarrow{C''B}}$$

and by Theorem 32 this tells us that

$$(*) \quad \frac{c' - a}{c' - b} = \frac{c'' - a}{c'' - b}$$

Cross-multiplying in (*) gives

$$(c' - a)(c'' - b) = (c' - b)(c'' - a)$$

and simplifying this we obtain

$$c'c'' - ac'' - c'b + ab = c'c'' - bc'' - c'a + ab$$

Cancellation gives

$$-ac'' - c'b = -bc'' - c'a.$$

Moving the terms with c' to the left and the terms with c'' to the right we get

$$(**) \quad c'(a - b) = c''(a - b)$$

Now we are given that A and B are different points, so their coordinates a and b are different numbers, and therefore $a - b \neq 0$. This allows us to cancel $a - b$ in (**), so we have

$$c' = c''$$

This means that the coordinates of C' and C'' are the same number, so C' and C'' are the same point. QED

Notice that Theorem 33 would not be true if we used ordinary ratios instead of signed ratios. Whenever C' , A and B are points on a line, there will always be a point C'' *different* from C' for which the ordinary ratios $\frac{C'A}{C'B}$ and $\frac{C''A}{C''B}$ are the same. When this happens, one of the points C' and C'' will be inside the segment AB and the other will be outside, so the *signed* ratios $\frac{\overrightarrow{C'A}}{\overrightarrow{C'B}}$ and $\frac{\overrightarrow{C''A}}{\overrightarrow{C''B}}$ will be different.

7.5 The converse of Menelaus's theorem.

First let us check that Theorem 31 remains true with signed ratios.

Theorem 34. Let ABC be a triangle. Let A' be a point of \overleftrightarrow{BC} other than B and C , let B' be a point of \overleftrightarrow{AC} other than A and C , and let C' be a point of \overleftrightarrow{AB} other than A and B . If A' , B' and C' are collinear then

$$\frac{\overrightarrow{A'B}}{\overrightarrow{A'C}} \frac{\overrightarrow{B'C}}{\overrightarrow{B'A}} \frac{\overrightarrow{C'A}}{\overrightarrow{C'B}} = 1.$$

Proof Theorem 31 tells us that the equation is true if we ignore the signs, so we know that the left-hand side is equal to either $+1$ or -1 , and we want it to be $+1$. Thus all we have left to show is that the left-hand side is positive. There are two cases to consider:

Case (i) The Menelaus line $A'B'C'$ crosses the triangle.

Case (ii) The Menelaus line $A'B'C'$ is entirely outside the triangle.

For Case (i), see Figure 22. In this case two of the points are on the (nonextended) sides of the triangle and one is on an extended side, so two of the signed ratios are negative and one is positive (by the definition of signed ratio). Thus the product of all three is positive.

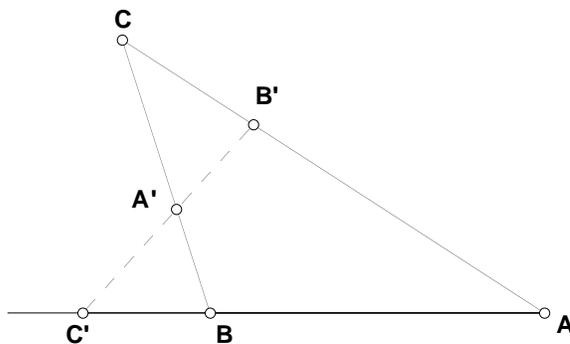


Figure 22

For Case (ii), see Figure 23. In this case all three points are on extended sides of the triangle, so all three signed ratios are positive (by the definition of signed ratio) and the product is also positive. QED

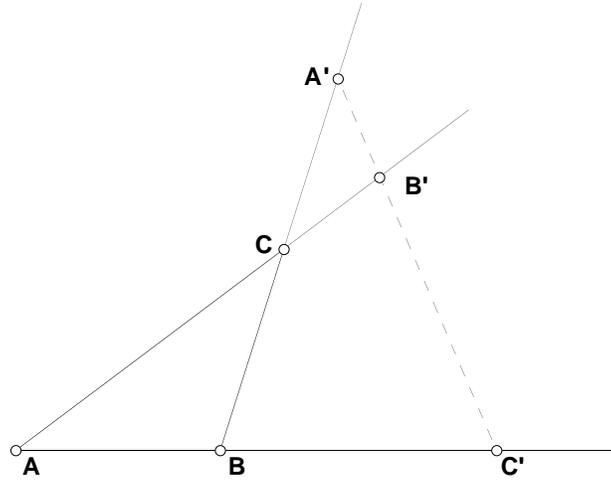


Figure 23

Our next theorem is the converse of Theorem 34; this is the corrected converse of Menelaus's theorem.

Theorem 35. Let ABC be a triangle. Let A' be a point of \overleftrightarrow{BC} other than B and C , let B' be a point of \overleftrightarrow{AC} other than A and C , and let C' be a point of \overleftrightarrow{AB} other than A and B . If

$$\frac{\overrightarrow{A'B}}{\overrightarrow{A'C}} \frac{\overrightarrow{B'C}}{\overrightarrow{B'A}} \frac{\overrightarrow{C'A}}{\overrightarrow{C'B}} = 1$$

then A' , B' and C' are collinear.

Proof (See Figure 24) Draw the line $\overleftrightarrow{A'B'}$, and give the intersection of this line with \overleftrightarrow{AB} the name C'' . Then

$$(*) \quad A', B' \text{ and } C'' \text{ are collinear.}$$

We want to show that C'' is the same point as C' . Since A' , B' and C'' are collinear, we can apply Theorem 34 to get

$$\frac{\overrightarrow{A'B}}{\overrightarrow{A'C}} \frac{\overrightarrow{B'C}}{\overrightarrow{B'A}} \frac{\overrightarrow{C''A}}{\overrightarrow{C''B}} = 1$$

Combining this with the given, we obtain

$$\frac{\overrightarrow{C'A}}{\overrightarrow{C'B}} = \frac{\overrightarrow{C''A}}{\overrightarrow{C''B}}$$

Now Theorem 33 tells us that C'' is the same point as C' , and combining this fact with (*) we see that A' , B' and C' are collinear. QED

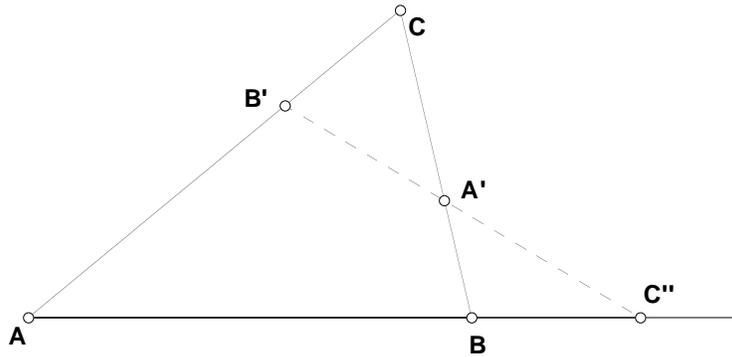


Figure 24

7.6 The Theorem of Ceva and its converse.

Ceva proved this theorem in the late 1600's:

Theorem 36 (Theorem of Ceva). *Let ABC be a triangle. Let A' be a point of \overleftrightarrow{BC} other than B and C , let B' be a point of \overleftrightarrow{AC} other than A and C , and let C' be a point of \overleftrightarrow{AB} other than A and B . If $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$ and $\overleftrightarrow{CC'}$ are concurrent then*

$$\frac{\overrightarrow{A'B}}{\overrightarrow{A'C}} \frac{\overrightarrow{B'C}}{\overrightarrow{B'A}} \frac{\overrightarrow{C'A}}{\overrightarrow{C'B}} = -1$$

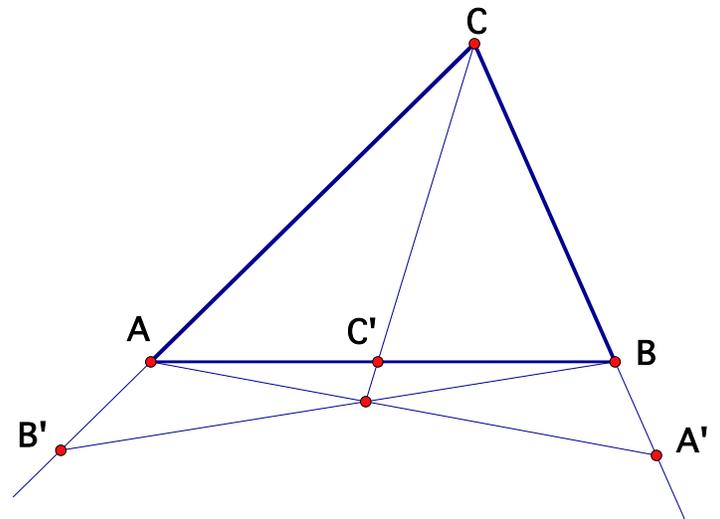
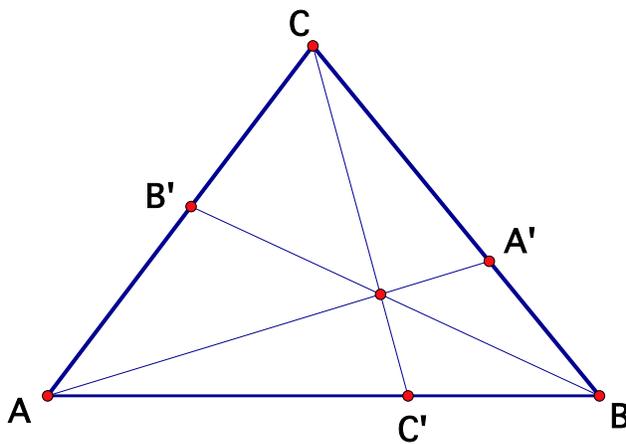


Figure 25

Proof There are two cases, shown in Figure 25. You have shown in homework problems that the equation is true if we ignore the signs, so we know that the left-hand side is equal to either $+1$ or -1 , and we want it to be -1 , so we just need to show that the left-hand side is negative. In the first picture of Figure 25, all three factors are negative (by the definition of signed ratio) so the product is negative. In the second picture of Figure 25, two factors are positive and one is negative (by the definition of signed ratio) so the product is negative. QED

The converse of Theorem 36 is also true.

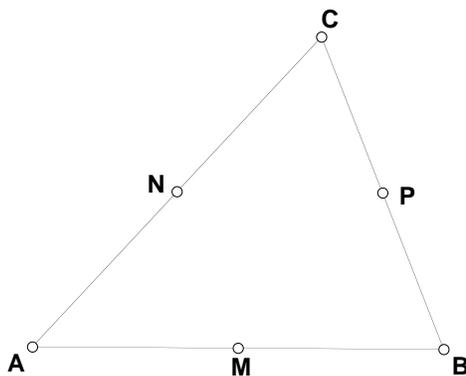
Theorem 37. Let ABC be a triangle. Let A' be a point of \overleftrightarrow{BC} other than B and C , let B' be a point of \overleftrightarrow{AC} other than A and C , and let C' be a point of \overleftrightarrow{AB} other than A and B . If

$$\frac{\overrightarrow{A'B}}{\overrightarrow{A'C}} \frac{\overrightarrow{B'C}}{\overrightarrow{B'A}} \frac{\overrightarrow{C'A}}{\overrightarrow{C'B}} = -1$$

then $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$ and $\overleftrightarrow{CC'}$ are concurrent.

The proof is a homework problem.

As an example of how useful Theorem 37 can be, let us use it to give a new proof of Theorem 29 (concurrency of the medians):



Given: M , N and P are midpoints. To prove: AP , BN and CM are concurrent.

Proof We know $PB = PC$ by definition of midpoint, so $\frac{PB}{PC} = 1$. Also, P is between B and C , so by the definition of signed ratio we have

$$\frac{\overrightarrow{PB}}{\overrightarrow{PC}} = -\frac{PB}{PC} = -1$$

Similarly, $\frac{\overrightarrow{NC}}{\overrightarrow{NA}} = -1$ and $\frac{\overrightarrow{MA}}{\overrightarrow{MB}} = -1$. Combining these equations gives

$$\frac{\overrightarrow{PB}}{\overrightarrow{PC}} \frac{\overrightarrow{NC}}{\overrightarrow{NA}} \frac{\overrightarrow{MA}}{\overrightarrow{MB}} = -1$$

so we may apply Theorem 37 (with $A' = P$, $B' = N$, and $C' = M$) to conclude that \overleftrightarrow{AP} , \overleftrightarrow{BN} and \overleftrightarrow{CM} are concurrent. QED

8 Circles.

Recall that I gave the definition of circle in Section 4.1.

8.1 Inscribed angles and central angles.

Definition of inscribed angle Given three points A , B and C on a circle, the angle $\angle ABC$ is said to be inscribed in the circle

Definition of central angle Given a circle with center O , a central angle is an angle with its vertex at O .

If A , B and C are points on a circle with center O , there is a relationship between the inscribed angle $\angle ABC$ and the central angle $\angle AOC$. However, the relationship is somewhat complicated and depends on the positions of A , B , C and O . The following pictures show the possible cases.

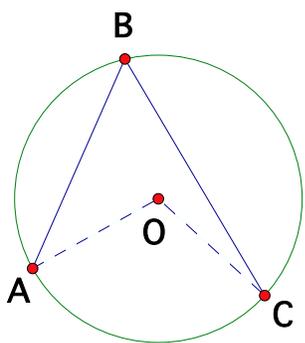


Figure 26

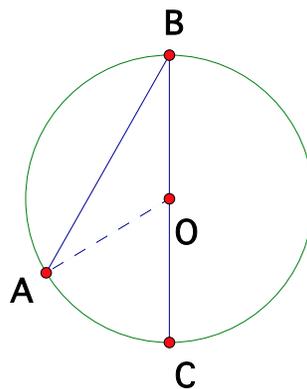


Figure 27

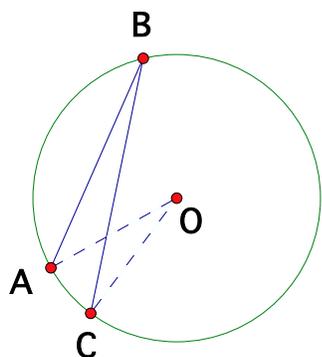


Figure 28

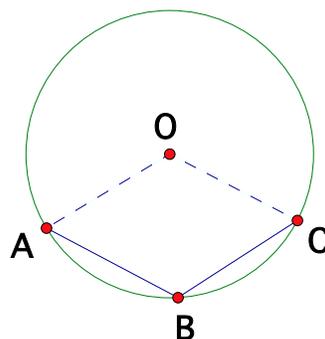


Figure 29

Experimenting with Geometer's Sketchpad shows that for pictures like Figures 26, 27, and 28, $\angle ABC = \frac{1}{2}\angle AOC$, but for pictures like Figure 29, $\angle ABC = 180^\circ - \frac{1}{2}\angle AOC$. It turns out that in order to know which formula applies we just have to know the location of point B :

Theorem 38. *Let A , B and C be points on a circle with center O .*

- (a) *If B is outside of $\angle AOC$, then $\angle ABC = \frac{1}{2}\angle AOC$.*
- (b) *If B is inside of $\angle AOC$, then $\angle ABC = 180^\circ - \frac{1}{2}\angle AOC$.*

Proof For part (a) there are four cases:

- (i) O is on ray BA
- (ii) O is on ray BC
- (iii) O is inside $\angle ABC$
- (iv) O is outside $\angle ABC$

For case (i) see Figure 30. We know that $OB = OC$, by definition of circle, so $\angle OCB = \angle OBC$ by Theorem 5. Let us call both of these angles x , and let $y = \angle AOC$. Then

$$(*) \quad y + \angle BOC = 180^\circ$$

by Theorem 1(a), and

$$(**) \quad 2x + \angle BOC = 180^\circ$$

by Theorem 3. Combining equations (*) and (**) gives $y = 2x$, which is what we were to prove in this case.

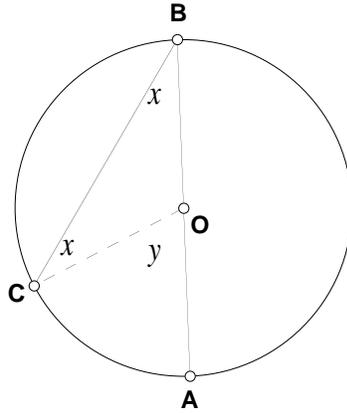


Figure 30

Case (ii) is precisely similar to case (i).

For Case (iii), see Figure 31. We can apply case (ii) to $\angle ABD$ to get

$$(*) \quad \angle ABD = \frac{1}{2}\angle AOD$$

and we can apply case (ii) to $\angle CBD$ to get

$$(**) \quad \angle CBD = \frac{1}{2}\angle COD$$

Finally, we have

$$\begin{aligned} \angle ABC &= \angle ABD + \angle CBD && \text{by BF 6} \\ &= \frac{1}{2}\angle AOD + \frac{1}{2}\angle COD && \text{by } (*) \text{ and } (**) \\ &= \frac{1}{2}(\angle AOD + \angle COD) && \text{by algebra} \\ &= \frac{1}{2}\angle AOC && \text{by BF 6} \end{aligned}$$

so $\angle ABC = \frac{1}{2}\angle AOC$, which is what we were to prove in this case.

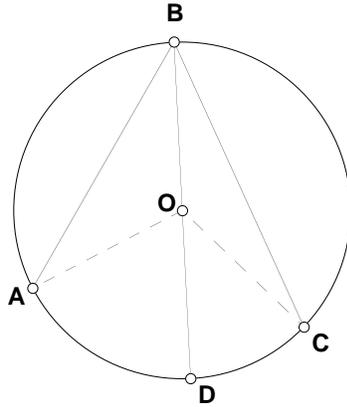


Figure 31

For case (iv), see Figure 32. We can apply case (ii) to $\angle ABD$ to get

$$(*) \quad \angle ABD = \frac{1}{2}\angle AOD$$

and we can apply case (ii) to $\angle CBD$ to get

$$(**) \quad \angle CBD = \frac{1}{2}\angle COD$$

Then we have

$$\begin{aligned} \angle ABC &= \angle ABD - \angle CBD && \text{by BF 6 and algebra} \\ &= \frac{1}{2}\angle AOD - \frac{1}{2}\angle COD && \text{by } (*) \text{ and } (**) \\ &= \frac{1}{2}(\angle AOD - \angle COD) && \text{by algebra} \\ &= \frac{1}{2}\angle AOC && \text{by BF 6 and algebra} \end{aligned}$$

so $\angle ABC = \frac{1}{2}\angle AOC$, which is what we were to prove in this case.

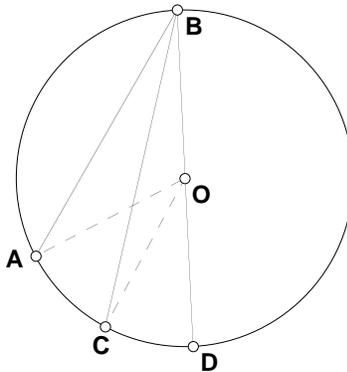


Figure 32

It remains to prove part (b) of the Theorem. For this, see Figure 33. We know that $OA = OB$, by definition of circle, so $\angle OAB = \angle OBA$ by Theorem 5. Let us call both of these angles x . Similarly, $\angle OBC = \angle OCB$; let us call both of these angles y . Next we apply Theorem 3 to triangles AOB and COB to get

$$(*) \quad \angle AOB = 180^\circ - 2x$$

and

$$(**) \quad \angle COB = 180^\circ - 2y$$

Now

$$\begin{aligned} \angle AOC &= \angle AOB + \angle COB && \text{by BF 6} \\ &= (180^\circ - 2x) + (180^\circ - 2y) && \text{by } (*) \text{ and } (**) \\ &= 360^\circ - 2(x + y) && \text{by algebra} \\ &= 360^\circ - 2\angle ABC && \text{by BF 6} \end{aligned}$$

So we have shown that $\angle AOC = 360^\circ - 2\angle ABC$. Solving for $\angle ABC$ gives

$$\angle ABC = 180^\circ - \frac{1}{2}\angle AOC$$

which is what we were to prove in this case. QED

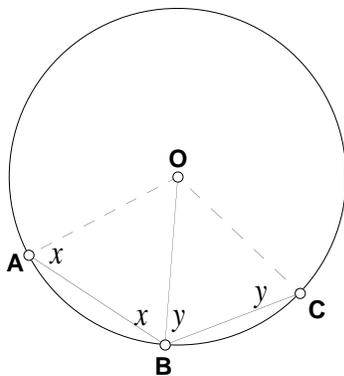


Figure 33

Some comments on the proof. Notice that in all four parts of the proof of Theorem 38 the basic strategy is to draw line OB . Also notice that once case (ii) of part (a) has been proved, we are allowed to use it to prove the remaining cases without having to worry about a circular argument. Finally, notice the general resemblance between the proof of part (a) and the proof of Theorem 7.

Because Theorem 38 contains two different formulas which are valid in different situations, it can be hard to remember. In fact there is a way to unify the two formulas into a single formula, using the idea of *arc measure* which we discuss in the next section.

8.2 Arcs and arc measurement.

An arc is a piece of a circle. We can measure an arc in degrees: here is the official definition.

Degree of arc-measure A degree of arc on a circle is the $\frac{1}{360}$ th part of the full circle.

Note. Although angles can never be bigger than 180° , arcs can be anything up to 360° .

In order to use this definition we need to have a way of naming arcs. First notice that the endpoints of an arc do *not* determine the arc; there are always two different arcs with these endpoints (for example, in Figure 34 there are two different arcs with endpoints AB , one of which goes through C and the other through D). In order to specify an arc unambiguously, we have to give the two endpoints and a point in the middle. For example, in Figure 34 the two arcs with endpoints A and B are denoted ACB and ADB . One thing that may cause confusion is that occasionally we will create a point on an arc whose only purpose is to help name the arc.

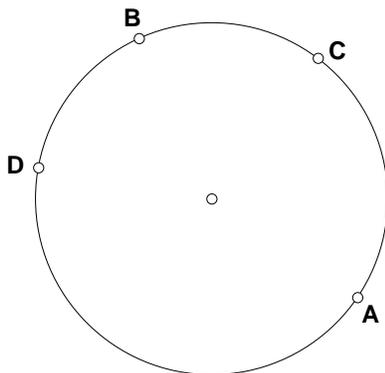


Figure 34

Next we will consider the arc cut off by an angle. We begin with the case of a central angle.

Definition of arc cut off by a central angle The arc cut off by a central angle is the part of the circle inside the angle (see Figure 35).

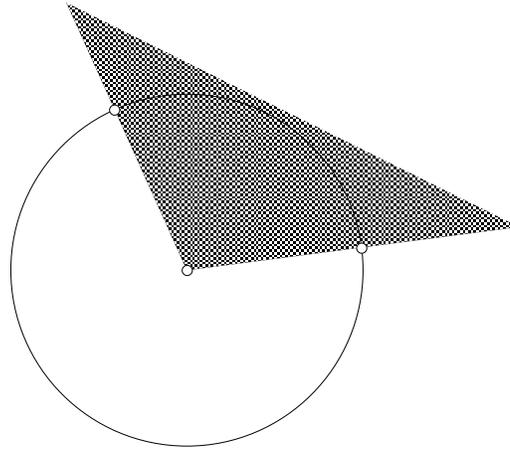


Figure 35

The most important fact about the arc cut off by a central angle is:

Theorem 39. *The arc cut off by a central angle is the same number of degrees as the angle.*

Proof It suffices to show that a 1° central angle cuts off a 1° arc. In Section 1.1 we defined a 1° angle to be the $\frac{1}{180}$ th part of a straight angle, so it suffices to check that a central straight angle cuts off a 180° arc. But this is true because a central straight angle divides the circle into two equal pieces, and we have defined the whole circle to be 360° of arc. QED

Next we consider the arc cut off by an inscribed angle.

Definition of the arc cut off by an inscribed angle The arc cut off by an inscribed angle $\angle ABC$ is the part of the circle inside the angle. In Figure 36 the part of the plane inside $\angle ABC$ is shaded, and the arc cut off by $\angle ABC$ is the shaded arc.

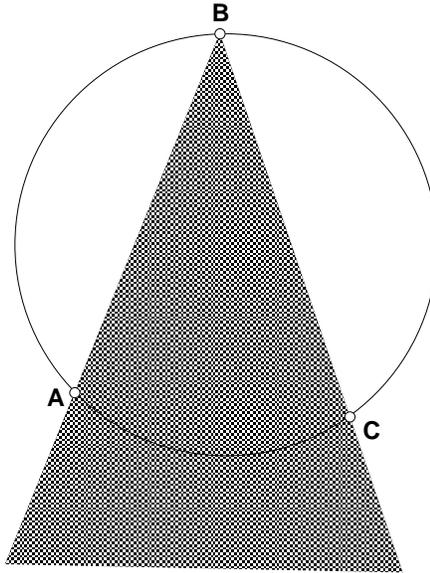


Figure 36

By analogy with Theorem 39, it's natural to ask whether there is a simple relationship between the size of an inscribed angle and the size of the arc which it cuts off. Experimenting with Geometer's Sketchpad shows that the angle is always $\frac{1}{2}$ of the arc. We will prove this in the next section.

8.3 Inscribed angles and arcs.

We can use the idea of the arc cut off by an angle to give a simpler, unified version of Theorem 38:

Theorem 40. *Let A , B and C be points on a circle, and let ADC be the arc cut off by $\angle ABC$. Then*

$$\angle ABC = \frac{1}{2} \text{arc } ADC$$

that is, the number of degrees in angle ABC is half the number of degrees in arc ADC .

(Notice that the only role of the point D in this theorem is to allow us to name the arc ADC .)

Proof There are four possible cases, shown in Figures 37, 38, 39 and 40.

In Figures 37, 38 and 39, the arc cut off by $\angle AOC$ is ADC , so in these three figures the arc-measure of ADC is equal to the degree measure of $\angle AOC$ by Theorem 39. But we also know in these three cases that $\angle ABC = \frac{1}{2} \angle AOC$ (by part (a) of Theorem 38), so we conclude that $\angle ABC = \frac{1}{2} \text{arc } ADC$ for these three cases.

In Figure 40, the arc ADC is the part of the circle *outside* $\angle AOC$, so in this figure the arc-measure of ADC is equal to $360^\circ - \angle AOC$ (by Theorem 39, BF 6 and algebra). But we also know that $\angle ABC = 180^\circ - \frac{1}{2}\angle AOC$ in this case (by part (b) of Theorem 38), so we conclude that $\angle ABC = \frac{1}{2}\text{arc } ADC$ in this case also. QED

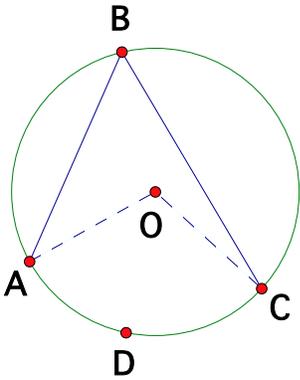


Figure 37

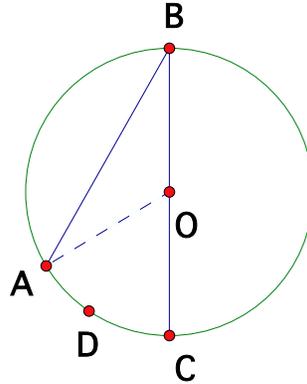


Figure 38

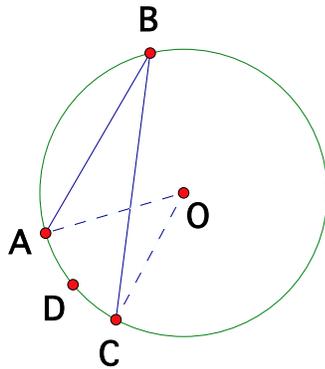


Figure 39

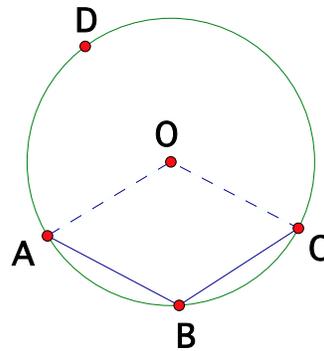


Figure 40

Note. The reason why Theorem 38 is more complicated than Theorem 40 is that the angles $\angle ABC$ and $\angle AOC$ don't always cut off the same arc, as you can see from Figures 37, 38, 39 and 40. Since the relation between $\angle ABC$ and the arc it cuts off is simple, this forces the relation between $\angle ABC$ and $\angle AOC$ to be complicated.

Next consider the situation where we have *four* points A , B , C and D on a circle and we are interested in the angles $\angle ABC$ and $\angle ADC$. The arcs cut off by $\angle ABC$ and $\angle ADC$ will always have the same endpoints (namely A and C), but they won't always

be the same arc (see Figure 41: in the first picture the two angles do cut off the same arc but in the second picture they don't).

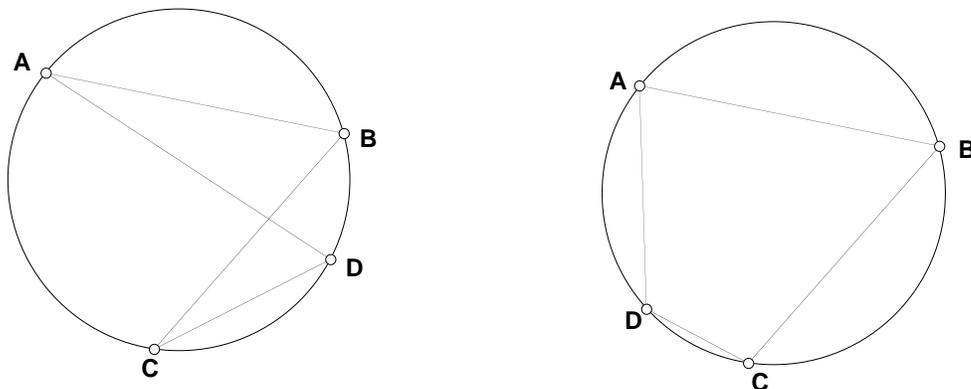


Figure 41

Theorem 41. *Let A, B, C and D be points on a circle and consider the angles ABC and ADC .*

- (a) *If $\angle ABC$ and $\angle ADC$ cut off the same arc, then $\angle ABC = \angle ADC$.*
 (b) *If $\angle ABC$ and $\angle ADC$ do not cut off the same arc, then $\angle ABC = 180^\circ - \angle ADC$.*

Proof Part (a) is immediate from Theorem 40.

For part (b), see the second picture in Figure 41. Observe that the arc cut off by $\angle ABC$ is arc ADC and the arc cut off by $\angle ADC$ is arc ABC . By *BF 6* and algebra,

$$(*) \quad \text{arc } ADC \text{ is } 360^\circ \text{ minus arc } ABC.$$

Now we have

$$\begin{aligned} \angle ABC &= \frac{1}{2} \text{arc } ADC && \text{by Theorem 40} \\ &= \frac{1}{2}(360^\circ - \text{arc } ABC) && \text{by } (*) \\ &= 180^\circ - \frac{1}{2} \text{arc } ABC && \text{by algebra} \\ &= 180^\circ - \angle ADC && \text{by Theorem 40} \end{aligned}$$

which is what we were to show. QED

Theorem 41 is often useful, and in particular it can be used to prove the following:

Theorem 42. *Let A, B, C and D be points on a circle, and suppose that the lines AB and CD meet at a point P . Then $PA \cdot PB = PC \cdot PD$.*

There are two cases, which are illustrated in Figures 42 and 43. The proofs of the two cases are homework problems.

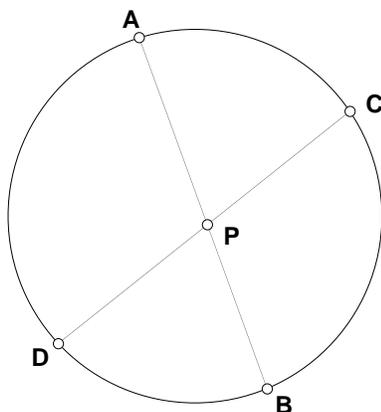


Figure 42

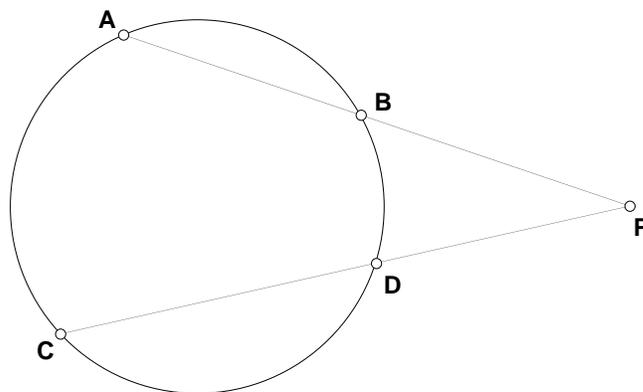


Figure 43

8.4 The tangent line to a circle.

Definition of tangent line A line is tangent to a circle \iff it intersects the circle in exactly one point.

Theorem 43. Let C be a circle with center O , let A be a point on the circle, and let m be a line through A . Then m is tangent to the circle \iff m is perpendicular to OA .

Proof For the \Leftarrow direction, see Figure 44. Choose a point of m other than A and call it B . Since m is perpendicular to OA , we can apply Theorem 9 to get

$$OB^2 = OA^2 + AB^2$$

Since $AB^2 > 0$, this tells us that $OB^2 > OA^2$, and this implies $OB > OA$. But OA is the radius of the circle, so B cannot be on the circle (by definition of circle). Therefore A is the only point of m which is on the circle, so m is tangent to the circle by definition of tangent.

The \implies direction is a homework problem. QED

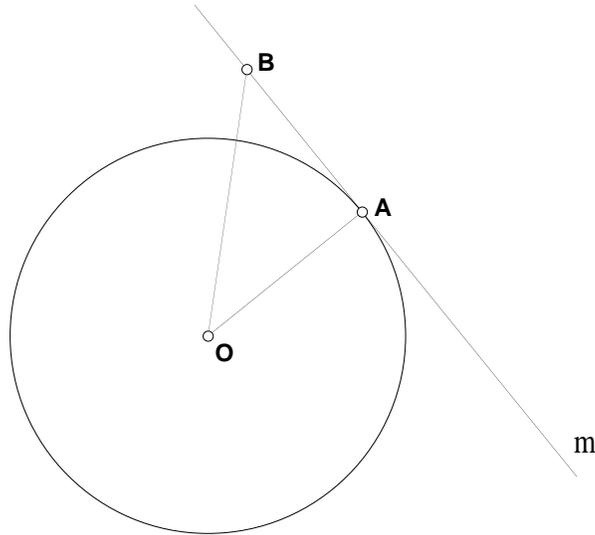


Figure 44

8.5 The inscribed circle of a triangle.

We can now prove a fact that was stated in Section 4.1: the incenter of a triangle (that is, the intersection of the angle bisectors) is the center of a circle that is tangent to all three sides.

We need a preliminary fact.

Theorem 44. *Given a line ℓ and a point P not on it, it is possible to construct a circle with center at P that is tangent to ℓ .*

Proof Draw the line m that goes through P and is perpendicular to ℓ (BF 12), and give the intersection of ℓ and m the name Q . Then the circle with center P that goes through Q is tangent to ℓ by Theorem 43. QED

Theorem 45. *Given a triangle ABC with incenter I , the circle with center at I that is tangent to one of the sides is also tangent to the other two sides.*

Proof (See Figure 45.) Draw the lines ℓ , m and n that go through I and are perpendicular to AB , BC , and AC respectively (BF 12). Give the intersection of ℓ with AB the name D , the intersection of m with BC the name E , and the intersection of n with AC the name F . The circle with center I that goes through D is tangent to AB by Theorem 43. It was shown in the proof of Theorem 26 that $ID = IE = IF$, so this circle also goes through E and F , and is tangent to BC and AC by Theorem 43. QED

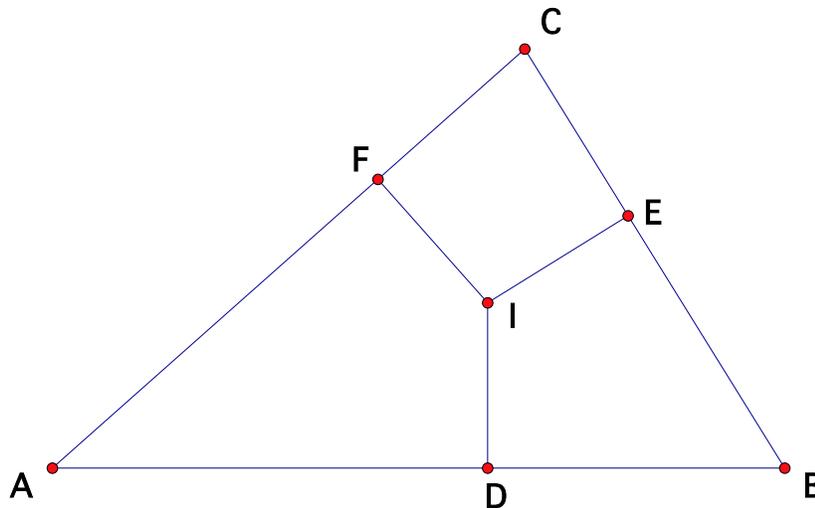


Figure 45

8.6 Angles with a side tangent to a circle.

Next we have an analog of Theorem 40 when one of the sides of the angle is tangent to the circle.

Arc cut off by an angle with one side tangent to the circle at the vertex If A and B are points on a circle and C is a point on the tangent line at B , then the arc cut off by $\angle ABC$ is the part of the circle inside $\angle ABC$ (see Figure 46).

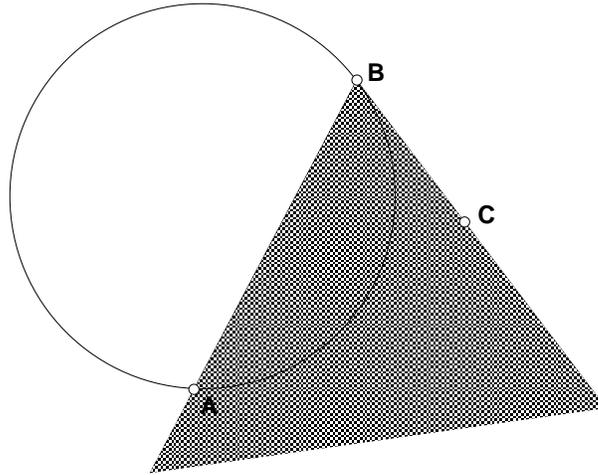


Figure 46

Theorem 46. *Let A and B be points on a circle, let C be a point on the tangent line at B , and let ADB be the arc cut off by $\angle ABC$. Then*

$$\angle ABC = \frac{1}{2} \text{arc } ADB$$

Proof There are three cases:

- (i) O is on ray BA
- (ii) O is inside $\angle ABC$
- (iii) O is outside $\angle ABC$

For case (i) see Figure 47. Here arc ADB is a semicircle, so it measures 180° , and $\angle ABC$ is a right angle (by Theorem 43) so $\angle ABC$ is half of arc ADB in this case.

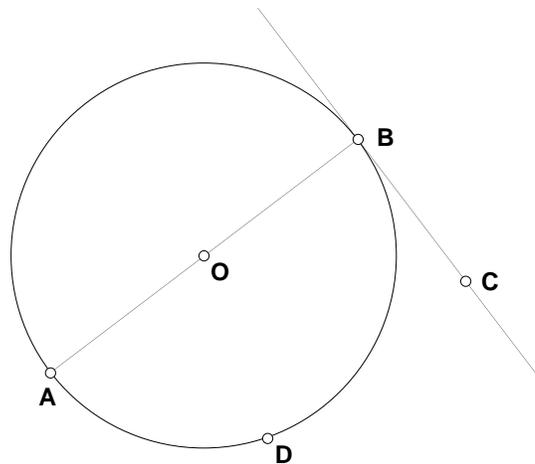


Figure 47

For case (ii) see Figure 48. Draw line OB and give the intersection of this line with the circle the name E . Then

$$\begin{aligned}
 \angle ABC &= \angle 1 + \angle 2 && \text{by BF 6} \\
 &= \frac{1}{2}\text{arc } AFE + \frac{1}{2}\text{arc } EDB && \text{by Theorem 40 and Case (i)} \\
 &= \frac{1}{2}(\text{arc } AFE + \text{arc } EDB) && \text{algebra} \\
 &= \frac{1}{2}\text{arc } ADB && \text{by BF 6 (applied to arcs)}
 \end{aligned}$$

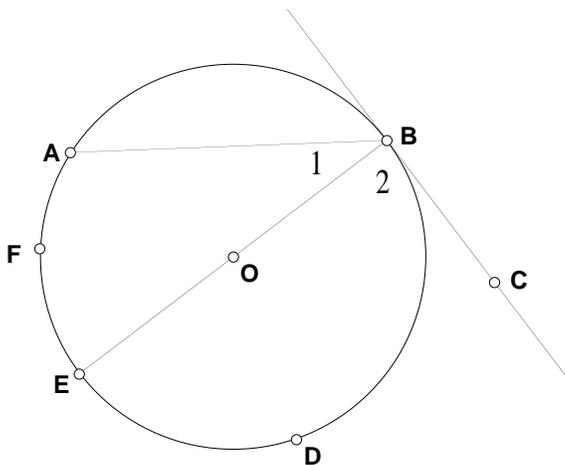


Figure 48

For Case (iii) see Figure 49. Draw line OB and give the intersection of this line with the circle the name E . Then

$$\begin{aligned}
 \angle ABC &= \angle EBC - \angle 1 && \text{by BF 6 and algebra} \\
 &= \frac{1}{2}\text{arc } EDB - \frac{1}{2}\text{arc } EFA && \text{by Case (i) and Theorem 40} \\
 &= \frac{1}{2}(\text{arc } EDB - \text{arc } EFA) && \text{algebra} \\
 &= \frac{1}{2}\text{arc } ADB && \text{by BF 6 (applied to arcs) and algebra}
 \end{aligned}$$

QED

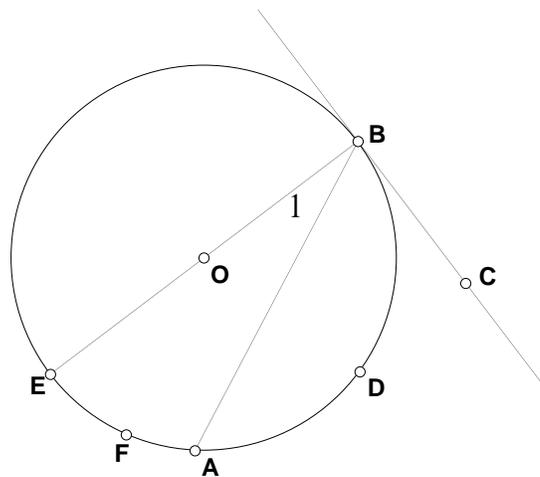


Figure 49

8.7 The circumscribed circle of a quadrilateral.

We have seen in Theorem 25 that for any triangle there is a circle (called the circumscribed circle) that goes through all three vertices of the triangle. It is natural to ask whether the same thing is true for a quadrilateral. A little thought shows that not every quadrilateral $ABCD$ can have a circumscribed circle, because there is only one circle that goes through the vertices A , B and C , and the vertex D might or might not be on this circle. We therefore modify the question (as mathematicians often do) to ask *which* quadrilaterals have circumscribed circles.

Theorem 41(b) gives us some useful information: if the four vertices of a quadrilateral $ABCD$ lie on a circle (as in Figure 50) then $\angle ABC + \angle ADC = 180^\circ$ and $\angle DAB + \angle DCB = 180^\circ$.

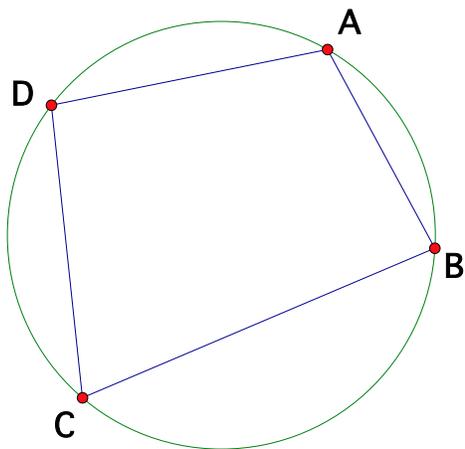


Figure 50

This shows again that not every quadrilateral can have a circumscribed circle: a necessary condition is that the opposite pairs of angles add to 180° . Our next theorem shows that this is the *only* condition that is needed.

Theorem 47. *If the opposite pairs of angles in a quadrilateral add to 180° then there is a circle going through all four vertices.*

The proof is a nice example of proof by contradiction. We will use some facts from Euclid in the proof.

Proof Let $ABCD$ be the quadrilateral and let \mathcal{C} be the circle through A , B and C . There are three possibilities:

- (i) D is on \mathcal{C} ,
- (ii) D is inside \mathcal{C} , or
- (iii) D is outside \mathcal{C} .

We will show that (ii) and (iii) are impossible, so that (i) must be true, which is what we wanted to prove.

So assume that (ii) happens (see Figure 51).

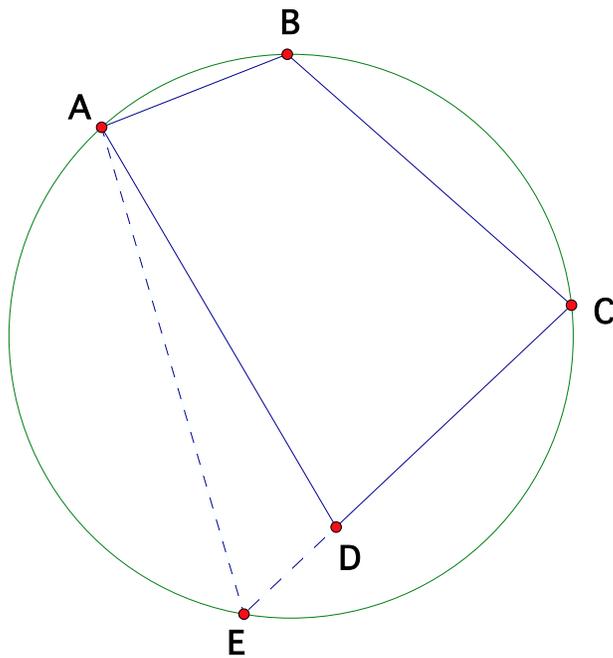


Figure 51

Let E be the point where ray \overrightarrow{CD} intersects the circle. Then

$$\angle ABC + \angle AEC = 180^\circ$$

by Theorem 41(b), and

$$\angle ABC + \angle ADC = 180^\circ$$

by the given, so

$$\angle AEC = \angle ADC$$

by algebra. But this contradicts Euclid Proposition 16 (applied to triangle AED) so (ii) was impossible.

Now assume that (iii) happens (see Figure 52).

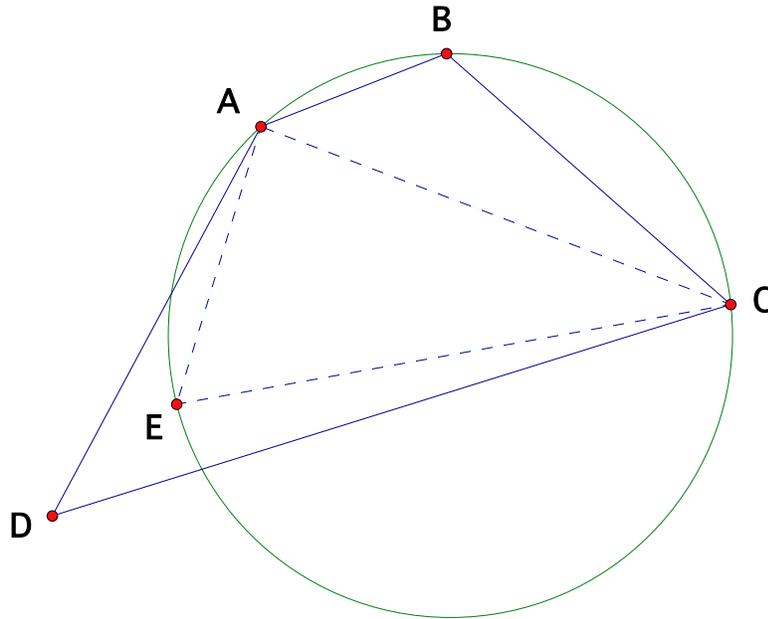


Figure 52

Let E be a point on the circle which is inside $\angle ADC$ and on the same side of AC as D (as shown in the Figure). Then

$$\angle ABC + \angle AEC = 180^\circ$$

by Theorem 41(b), and

$$\angle ABC + \angle ADC = 180^\circ$$

by the given, so

$$\angle AEC = \angle ADC$$

by algebra. But this contradicts Euclid Proposition 21 (applied to triangle ADC with AE and CE meeting within it). So (iii) was impossible.

Therefore (i) must be true. QED